Quantum Massless Field in 1+1 Dimensions

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Abstract. We present a construction of the algebra of operators and the Hilbert space for a quantum massless field in 1+1 dimensions.

1 Introduction

It is usually stated that quantum massless bosonic fields in 1+1 dimensions (with noncompact space dimension) do not exist. With massive fields the correlation function

$$\langle \Omega|\phi(f_1)\phi(f_2)\Omega\rangle = \int \frac{\mathrm{d}p}{2\pi 2E_p}\hat{f}_1^*(E_p,p)\hat{f}_2(E_p,p) \tag{1}$$

(where $E_p = \sqrt{p^2+m^2}$) is well defined but in the limit $m \to 0$ diverges because of the infrared problem. The limit exists only after adding an additional nonlocal constraint on the smearing functions:

$$\hat{f}(0,0) = \int \mathrm{d}t\mathrm{d}x f(t,x) = 0\ . \tag{2}$$

Under this constraint it is not difficult to construct massless fields in 1+1 dimension (see eg. [15], where the framework of the Haag-Kastler axioms is used).

Massless fields are extensively used for example in string theory (albeit most often after Wick rotation to the space with Euclidean signature). They also appear as the scaling limit of massive fields [6]. Usually, in these applications, the constraint (2) appears to be present at least implicitly. e.g. in string amplitudes one imposes the condition that sum of all momenta is equal to 0. Nevertheless, it seems desirable to have a formalism for massless 1+1-dimensional fields free of this constraint.

In the literature there are many papers that propose to use an indefinite metric Hilbert space for this purpose [4, 9–13]. Clearly, an indefinite metric is not physical and in order to determine physical observables one needs to perform a reduction similar to that of the Gupta-Bleuler formalism used in QED. The outcome of this Gupta-Bleuler-like procedure is essentially equivalent to

J. Dereziński and K.A. Meissner: *Quantum Massless Field in 1+1 Dimensions*, Lect. Notes Phys. **690**, 107–127 (2006)
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imposing the constraint (2) [11]. Therefore, we do not find the indefinite metric approach appropriate.

In this paper we present two explicit constructions of (positive definite) Hilbert spaces with representations of the massless Poincaré algebra in 1+1 dimensions and local fields (or at least their exponentials). We allow all test functions f that belong to the Schwartz class on the 1+1 dimensional Minkowski space, without the constraint (2). We try to make sure that as many Wightman axioms as possible are satisfied.

In the first construction we obtain a separable Hilbert space and well defined fields, however we do not have a vacuum vector. In the second construction, the Hilbert space is non-separable, only exponentiated fields are well defined, but there exists a vacuum vector. Thus, neither of them satisfies all Wightman axioms. Nevertheless, we believe that both our constructions are good candidates for a physically correct massless quantum field theory in 1+1 dimensions.

Our constructions have supersymmetric extensions, which we describe at the end of our article.

In the literature known to us the only place where one can find a treatment of massless fields in 1+1 dimension similar to ours is [1,2] by Acerbi, Morchio and Strocchi. Their construction is equivalent to our second (nonseparable) construction. We have never seen our first (separable) construction of massless fields in the literature.

Acerbi, Morchio and Strocchi start from the C^*-algebra associated to the CCR over the symplectic space of solutions of the wave equation parametrized by the initial conditions. Then they apply the GNS construction to the Poincaré invariant quasi-free state obtaining a non-regular representation of CCR.

In our presentation we prefer to use the derivatives of right and left movers to parametrize fields, rather than the initial conditions. We also avoid, as long as possible, to invoke abstract constructions from the theory of C^*-algebras, which may be less transparent to some of the readers. We explain the relationship between our formalism and that of [1, 2]. The symmetry structure of this theory is surprisingly rich. Some of the objects are covariant only under Poincaré group but there are others that are covariant under larger groups: $A_+(1,\mathbb{R})\times A_+(1,\mathbb{R})$, $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$, $\mathrm{Diff}_+(\mathbb{R})\times\mathrm{Diff}_+(\mathbb{R})$, $\mathrm{Diff}_+(S^1)\times\mathrm{Diff}_+(S^1)$.

2 Fields

The action of the 1+1 dimensional free real scalar massless field theory reads

$$S = \frac{1}{2}\int \mathrm{d}t\mathrm{d}x \ \left((\partial_t\phi)^2 - (\partial_x\phi)^2\right) \ . \tag{3}$$

This leads to the equations of motion

$$(-\partial_t^2 + \partial_x^2)\phi = 0 \,. \tag{4}$$

The solution of (4) is the sum of right and left movers, i.e. functions of $(t-x)$ and $(t+x)$ respectively:

$$\phi(t,x) = \phi_R(t-x) + \phi_L(t+x) \,. \tag{5}$$

We will often used "smeared" fields in the sense

$$\phi(f) = \int \mathrm{d}t\mathrm{d}x\, \phi(t,x) f(t,x) \,,$$

where we assume that f are real Schwartz functions. Because of (5), they can be written in the form

$$\phi(f) = \phi(g_R, g_L) \,,$$

where

$$\hat{g}_R(k) = \hat{f}(k,k), \quad \hat{g}_L(k) = \hat{f}(k,-k) \,,$$

$(k \geq 0)$ and the Fourier transforms of the test function f and g are defined as

$$\hat{f}(E,p) := \int \mathrm{d}t\mathrm{d}x\; f(t,x)\; \mathrm{e}^{\mathrm{i}Et-\mathrm{i}px} \,, \tag{6}$$

$$\hat{g}(k) := \int_{-\infty}^{\infty} \mathrm{d}t g(t) \mathrm{e}^{-\mathrm{i}kt} \,. \tag{7}$$

The function g_R corresponds to right movers and g_L to left movers. Note that

$$\hat{g}_R(0) = \hat{g}_L(0) =: \hat{g}(0) \,. \tag{8}$$

$\hat{g}(0)$ is real, because function f is real.

We introduce the notation

$$(g_1|g_2) := \quad \frac{1}{2\pi} \lim_{\epsilon \searrow 0} \left(\int_{\epsilon}^{\infty} \frac{\mathrm{d}k}{k} \hat{g}_1^*(k)\hat{g}_2(k) + \ln(\epsilon/\mu)\hat{g}_1^*(0)\hat{g}_2(0) \right) \,, \tag{9}$$

where μ is a positive constant having the dimension of mass. For functions that satisfy $\hat{g}(0) = 0$, $(g_1|g_2)$ is a (positive) scalar product – otherwise it is not positive definite and therefore cannot be used directly in the construction of a Hilbert space. Such a scalar product corresponds to quantization of the theory in a constant compensating background.

In view of the infrared divergence we factorize the Hilbert space into two parts – one that is infrared safe and the second that in some sense regularizes the divergent part.

We introduce the creation $a_R^\dagger(k)$, $a_L^\dagger(k)$ and annihilation $a_R(k)$, $a_L(k)$ operators as well as pair of operators (χ, p). They satisfy the commutation relations

$$
\begin{aligned}
\left[a_R(k), a_R^\dagger(k')\right] &= 2\pi k\delta(k-k')\ , \\
\left[a_L(k), a_L^\dagger(k')\right] &= 2\pi k\delta(k-k')\ , \\
[\chi, p] &= \mathrm{i}
\end{aligned} \tag{10}
$$

with all other commutators vanishing.

To proceed we choose two real functions $\sigma_R(x)$ and $\sigma_L(x)$ satisfying

$$
\hat{\sigma}_R(0) = \hat{\sigma}_L(0) = 1 \tag{11}
$$

and otherwise arbitrary. To simplify further formulae we define the combinations

$$
\begin{aligned}
a_{\sigma R}(k) &:= a_R(k) - \mathrm{i}\hat{\sigma}_R(k)\chi \\
a_{\sigma R}^\dagger(k) &:= a_R^\dagger(k) + \mathrm{i}\hat{\sigma}_R^*(k)\chi \\
a_{\sigma L}(k) &:= a_L(k) - \mathrm{i}\hat{\sigma}_L(k)\chi \\
a_{\sigma L}^\dagger(k) &:= a_L^\dagger(k) + \mathrm{i}\hat{\sigma}_L^*(k)\chi
\end{aligned} \tag{12}
$$

and therefore

$$
\begin{aligned}
\left[a_{\sigma R}(k), a_{\sigma R}^\dagger(k')\right] &= 2\pi k\delta(k-k')\ , \\
\left[a_{\sigma L}(k), a_{\sigma L}^\dagger(k')\right] &= 2\pi k\delta(k-k')\ , \\
[a_{\sigma R}(k), p] &= \hat{\sigma}_R(k)\ , \\
\left[a_{\sigma R}^\dagger(k), p\right] &= -\hat{\sigma}_R^*(k)\ , \\
[a_{\sigma L}(k), p] &= \hat{\sigma}_L(k)\ , \\
\left[a_{\sigma L}^\dagger(k), p\right] &= -\hat{\sigma}_L^*(k).
\end{aligned} \tag{13}
$$

Now we are in a position to introduce the field operator $\phi(g_R, g_L)$, depending on a pair of functions g_R, g_L satisfying (8).

$$
\begin{aligned}
\phi(g_R, g_L) = \int \frac{\mathrm{d}k}{2\pi k}\Big(&(\hat{g}_R(k) - \hat{g}(0)\hat{\sigma}_R(k))a_{\sigma R}^\dagger(k) \\
&+(\hat{g}_R^*(k) - \hat{g}(0)\hat{\sigma}_R^*(k))a_{\sigma R}(k) + (\hat{g}_L(k) - \hat{g}(0)\hat{\sigma}_L(k))a_{\sigma L}^\dagger(k) \\
&+(\hat{g}_L^*(k) - \hat{g}(0)\hat{\sigma}_L^*(k))a_{\sigma L}(k)\Big) + \hat{g}(0)p\ .
\end{aligned} \tag{14}
$$

The field $\phi(g_R, g_L)$ is hermitian and satisfies the commutation relation

$$
\begin{aligned}
&[\phi(g_{R1}, g_{L1}), \phi(g_{R2}, g_{L2})] \\
&= (g_{R1}|g_{R2}) - (g_{R2}|g_{R1}) + (g_{L1}|g_{L2}) - (g_{L2}|g_{L1}) \\
&= \mathrm{i}2\mathrm{Im}(g_{R1}|g_{R2}) + \mathrm{i}2\mathrm{Im}(g_{L1}|g_{L2})\ .
\end{aligned} \tag{15}
$$

The commutator in (15) does not depend on the functions σ_R, σ_L.

3 Poincaré Covariance

Let $A_+(1,\mathbb{R})$ denote the group of orientation preserving affine transformations of the real line, that is the group of maps $t \mapsto at+b$ with $a>0$. The proper Poincaré group in 1+1 dimension can be naturally embedded in the direct product of two copies of $A_+(1,\mathbb{R})$, one for the right movers and one for the left movers.

The infinitesimal generators of the right $A_+(1,\mathbb{R})$ group will be denoted H_R (the right Hamiltonian) and D_R (the right generator of dilations) and they satisfy the commutation relations

$$[D_R, H_R] = \mathrm{i}H_R \ .$$

The representation of these operators in terms of the creation and annihilation operators is given by

$$\begin{aligned} H_R &= \int \frac{\mathrm{d}k}{2\pi}\, a^\dagger_{\sigma R}(k) a_{\sigma R}(k) \ , \\ D_R &= \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}k}{2\pi} \Big(a^\dagger_{\sigma R}(k) \partial_k a_{\sigma R}(k) - \big(\partial_k a^\dagger_{\sigma R}(k)\big) a_{\sigma R}(k) \Big) \ . \end{aligned} \tag{16}$$

Their action on fields is given by

$$\begin{aligned} [H_R, \phi(g_R, g_L)] &= -\mathrm{i}\phi(\partial_t g_R, 0) \ , \\ [D_R, \phi(g_R, g_L)] &= \mathrm{i}\phi(\partial_t t g_R, 0) \ , \end{aligned}$$

and in the exponentiated form by

$$\begin{aligned} \mathrm{e}^{\mathrm{i}sH_R}\phi(g_R, g_L)\mathrm{e}^{-\mathrm{i}sH_R} &= \phi(g_R(\cdot - s), g_L \\ \mathrm{e}^{\mathrm{i}sD_R}\phi(g_R, g_L)\mathrm{e}^{-\mathrm{i}sD_R} &= \phi(\mathrm{e}^{-s} g_R(\mathrm{e}^{-s}\cdot), g_L) \ . \end{aligned}$$

For $(a,b) \in A_+(1,\mathbb{R})$ we set $r_{a,b}g(t) := a^{-1}g(a^{-1}(t-b))$ and

$$R_R(a,b) = \mathrm{e}^{\mathrm{i}\ln a D_R}\mathrm{e}^{\mathrm{i}bH_R} \ .$$

R_R is a unitary representation of $A_+(1,\mathbb{R})$, which acts naturally on the fields:

$$R_R(a,b)\phi(g_R, g_L)R_R(a,b)^\dagger = \phi(r_{a,b}g_R, g_L) \ . \tag{17}$$

Note, however, that $r_{a,b}$ does not preserve the indefinite scalar product (9) unless we impose the constraint $\hat{g}(0)$=0:

$$(r_{a,b}g_1 | r_{a,b}g_2) = (g_1|g_2) - \ln a \hat{g}_1^*(0)\hat{g}_2(0) \ .$$

Similarly we introduce the left Hamiltonian H_L and the left generator of dilations D_L satisfying analogous commutation relations and the representation of the left $A_+(1,\mathbb{R})$.

The Poincaré group generators are the Hamiltonian $H = H_R + H_L$, the momentum $P = H_R - H_L$ and the boost operator $\Lambda = D_R - D_L$ (the only Lorentz generator in 1+1 dimensions). The elements of the Poincaré group are of the form

$$(a, b_R), (a^{-1}, b_L) \in A_+(1, \mathbb{R}) \times A_+(1, \mathbb{R}) \ .$$

The scalar product $(g_{R1}|g_{R2}) + (g_{L1}|g_{L2})$ is invariant wrt the proper Poincaré group.

4 Changing the Compensating Functions

It is important to discuss the dependence of the whole construction on the choice of compensating functions σ_R and σ_L.

Let $\tilde{\sigma}_R$ and $\tilde{\sigma}_L$ be another pair of real functions satisfying (11). Set $\xi_R(x) := \tilde{\sigma}_R(x) - \sigma_R(x)$, $\xi_L(x) := \tilde{\sigma}_L(x) - \sigma_L(x)$. Note that $\hat{\xi}_R(0) = \hat{\xi}_L(0) = 0$. Define

$$\begin{aligned} U(\xi_R, \xi_L) = \exp\Bigg(\int \frac{\mathrm{d}k}{2\pi k} \Big(& \mathrm{i}\chi \hat{\xi}_R^*(k) a_R(k) + \mathrm{i}\chi \hat{\xi}_R(k) a_R^\dagger(k) \\ & + \frac{1}{2}\chi^2 (\hat{\xi}_R^*(k)\hat{\sigma}_R(k) - \hat{\xi}_R(k)\hat{\sigma}_R^*(k)) + R \to L \Big) \Bigg) \ . \end{aligned} \tag{18}$$

Using the formula

$$\mathrm{e}^A B \mathrm{e}^{-A} = B + [A, B] + \frac{1}{2}[A, [A, B]] + \cdots \tag{19}$$

we have

$$\begin{aligned} U a_R(k) U^{-1} &= a_R(k) - \mathrm{i}\chi \hat{\xi}_R(k) \ , \\ U a_R^\dagger(k) U^{-1} &= a_R^\dagger(k) + \mathrm{i}\chi \hat{\xi}_R^*(k) \ , \\ U a_L(k) U^{-1} &= a_L(k) - \mathrm{i}\chi \hat{\xi}_L(k) \ , \\ U a_L^\dagger(k) U^{-1} &= a_L^\dagger(k) + \mathrm{i}\chi \hat{\xi}_L^*(k) \ , \\ U \chi U^{-1} &= \chi \end{aligned} \tag{20}$$

and

$$\begin{aligned} U p U^{-1} = p + \int \frac{\mathrm{d}k}{2\pi k} \Big(& - \hat{\xi}_R^*(k) a_R(k) - \hat{\xi}_R(k) a_R^\dagger(k) \\ & - \mathrm{i}\chi \hat{\xi}_R(k)\hat{\sigma}_R^*(k) + \mathrm{i}\chi \hat{\xi}_R^*(k)\hat{\sigma}_R(k) + R \to L \Big) . \end{aligned} \tag{21}$$

Using these relations we get for example

$$\begin{aligned} U \phi_\sigma(g_R, g_L) U^{-1} &= \phi_{\tilde{\sigma}}(g_R, g_L) \ , \\ U a_{\sigma R}^\dagger U^{-1} &= a_{\tilde{\sigma} R}^\dagger \ , \\ U H_{\sigma R} U^{-1} &= H_{\tilde{\sigma} R} \ , \end{aligned} \tag{22}$$

where we made explicit the dependence of ϕ and H_R on σ and $\tilde{\sigma}$. Thus the two constructions – with σ and with $\tilde{\sigma}$ – are unitarily equivalent.

5 Hilbert Space

The Hilbert space of the system is the product of three spaces: $\mathcal{H} = \mathcal{H}_R \otimes \mathcal{H}_L \otimes \mathcal{H}_0$. $\mathcal{H}_R$ is the bosonic Fock space spanned by the creation operators $a_R^\dagger(k)$ acting on the vacuum vector $|\Omega_R\rangle$. Analogously $\mathcal{H}_L$ is the bosonic Fock space spanned by the creation operators $a_L^\dagger(k)$ acting on the vacuum vector $|\Omega_L\rangle$. With the third sector $\mathcal{H}_0$ we have essentially two options. If we take the usual choice $\mathcal{H}_0 = L^2(\mathbb{R}, \mathrm{d}\chi)$ then we can define the vacuum state (vacuum expectation value) but there does not exist a vacuum vector. On the other hand, we can take $\mathcal{H}_0 = l^2(\mathbb{R})$, i.e. the space with the scalar product

$$(f|g) = \sum_{\chi \in \mathbb{R}} f^*(\chi) g(\chi) \; , \tag{23}$$

which is a nonseparable space. It may sound as a nonstandard choice, it has however the advantage of possessing a vacuum vector. The orthonormal basis in the latter space consists of the Kronecker delta functions δ_χ for each $\chi \in \mathbb{R}$. In the nonseparable case, the operator p, and therefore also $\phi(g_R, g_L)$, cannot be defined. But there exist operators $\mathrm{e}^{\mathrm{i}sp}$, for $s \in \mathbb{R}$, and also $\mathrm{e}^{\mathrm{i}\phi(g_R, g_L)}$. The commutation relations for these exponential operators follow from the commutation relations for p and $\phi(g_R, g_L)$ described above.

In such a space the vacuum vector is given by

$$|\Omega\rangle = |\Omega_R \otimes \Omega_L \otimes \delta_0\rangle \; . \tag{24}$$

This vector is invariant under the action of the Poincaré group and the action of the gauge group U. We now prove that it is the unique vector with the lowest energy. Note first that H is diagonal in $\chi \in \mathbb{R}$. Now for an arbitrary $\Phi \in \mathcal{H}_R \otimes \mathcal{H}_L$ and $\chi_1 \in \mathbb{R}$,

$$\begin{aligned} &\langle \Phi \otimes \delta_{\chi_1} | H | \Phi \otimes \delta_{\chi_1} \rangle \\ &= \int \Big(\Big\langle \Phi | \big(a_R^\dagger(k) + \mathrm{i}\chi_1 \hat{\sigma}_R^*(k)\big)\big(a_R(k) - \mathrm{i}\chi_1 \hat{\sigma}_R(k)\big) \Phi \Big\rangle \frac{\mathrm{d}k}{2\pi} + R \to L \Big) \end{aligned} \tag{25}$$

For any χ_1, the expression (25) is nonnegative. If $\chi_1 = 0$, it has a unique ground state $|\Omega_R \otimes \Omega_L\rangle$.

If $\chi_1 \neq 0$, then (25) has no ground state. In fact, it is well known that a ground state of a quadratic Hamiltonian is a coherent state, that is given by a vector of the form

$$|\beta_R, \beta_L\rangle = C \exp\left(\int \frac{\mathrm{d}k}{2\pi k} \Big(\beta_R(k) a_R^\dagger(k) + \beta_L(k) a_L^\dagger(k) \Big) \right) |\Omega_R \otimes \Omega_L\rangle \tag{26}$$

and C is the normalizing constant

$$C = \exp\left(-\frac{1}{2} \int \frac{\mathrm{d}k}{2\pi k} \big(|\beta_R(k)|^2 + |\beta_L(k)|^2 \big) \right) . \tag{27}$$

If we set $\Phi = |\beta_R, \beta_L\rangle$ in (25), then we obtain

$$\langle \Phi \otimes \delta_{\chi_1} | H | \Phi \otimes \delta_{\chi_1} \rangle = |\beta_R(k) + \mathrm{i}\chi_1 \sigma_R(k)|^2 + R \to L \ .$$

that takes the minimum for

$$\beta_R(k) = -\mathrm{i}\chi_1 \hat{\sigma}_R(k), \quad \beta_L(k) = -\mathrm{i}\chi_1 \hat{\sigma}_L(k) \ .$$

But for $\chi_1 \neq 0$, $|\beta_R, \beta_L\rangle$ is not well defined as a vector in the Hilbert space. To see this we can note that the normalizing constant C equals zero, because then

$$\chi_1^2 \int \frac{\mathrm{d}k}{2\pi k} \left(|\hat{\sigma}_R(k)|^2 + |\hat{\sigma}_L(k)|^2 \right) = \infty \ .$$

(The fact that operators of the form (25) have no ground state is well known in the literature, see eg. [7]).

In the nonseparable case $\mathcal{H}_0 = l^2(\mathbb{R})$, the expectation value

$$\langle \Omega | \cdot | \Omega \rangle =: \omega(\cdot)$$

is a Poincaré-invariant state (positive linear functional) on the algebra of observables. If we take the separable case $\mathcal{H}_0 = L^2(\mathbb{R}, \mathrm{d}\chi)$, the state ω can also be given a meaning, even though the vector Ω does not exist (since then δ_0 is not well defined).

Note that in the nonseparable case the state ω can act on an arbitrary bounded operator on $\mathcal{H}$. In the separable case we have to restrict ω to a smaller algebra of operators, say, the algebra (or the C^*-algebra) spanned by the operators of the form $\mathrm{e}^{\mathrm{i}\phi(g_R, g_L)}$.

The expectation values of the exponentials of the 1+1-dimensional massless field make sense and can be computed, both in the separable and nonseparable case:

$$\omega\left(\exp\left(\mathrm{i}\phi(g_R, g_L)\right)\right) = \exp\left(-\frac{1}{2} \int \frac{\mathrm{d}k}{2\pi k} \left(|\hat{g}_R(k)|^2 + |\hat{g}_L(k)|^2 \right) \right) \ . \tag{28}$$

Note that the integral in the exponent of (28) is the usual integral of a positive function, and not its regularization as in (9). Therefore, if $\hat{g}(0) \neq 0$, then this integral equals $+\infty$ and (28) equals zero.

The "two-point functions" of massless fields in 1+1 dimension, even smeared out ones, are not well defined. Formally, they are introduced as

$$\omega\left(\phi(g_{R,1}, g_{L,1})\phi(g_{R,2}, g_{L,2})\right) \ . \tag{29}$$

If we use the nonseparable $\mathcal{H}_0 = l^2(\mathbb{R})$, then field operators $\phi(g_R, g_L)$ is not well defined if $\hat{g}(0) \neq 0$, and thus (29) is not defined. If we use the separable Hilbert space $\mathcal{H}_0 = L^2(\mathbb{R}, \mathrm{d}\chi)$, then $\phi(g_{R,1}, g_{L,1})\phi(g_{R,2}, g_{L,2})$ are unbounded operators and there is no reason why the state ω could act on them. Thus (29) a priori does not make sense. In the usual free quantum field theory, if mass is positive or dimension more than 2, the expectation values the exponentials depend on the smeared fields analytically, and by taking their second derivative at $(g_R, g_L) = (0, 0)$, one can introduce the 2-point function. This is not the case for massless field in 1+1 dimension.

The above discussion shows that the problem of the non-positive definiteness of the two-point function, so extensively discussed in the literature [4, 9, 11–13] does not exist in our formalism.

It should be noted that massless fields in 1+1 dimension do not satisfy the Wightman axioms [14]. In the separable case there is no vacuum vector in the Hilbert space; in the nonseparable case there is a vacuum vector, but there are no fields $\phi(f)$, only the "Weyl operators" $\mathrm{e}^{\mathrm{i}\phi(f)}$.

6 Fields in Position Representation

So far in our discussion we found it convenient to use the momentum representation. The position representation is, however, better suited for many purposes.

Let $\mathcal{W}(t)$ denote the Fourier transform of the appropriately regularized distribution $\frac{\theta(k)}{2\pi k}$, that is

$$\begin{aligned}\mathcal{W}(t) &= \frac{1}{2\pi}\lim_{\epsilon\to 0}\left(\int_{k>\epsilon}\frac{\mathrm{d}k}{k}\mathrm{e}^{\mathrm{i}kt} + \ln(\epsilon/\mu)\right) \\ &= \frac{1}{2\pi}\left(-\gamma_E - \ln|\mu t| + \frac{\mathrm{i}\pi}{2}\mathrm{sgn}(t)\right) = \mathcal{W}^*(-t)\end{aligned} \tag{30}$$

where γ_E is the Euler's constant. We can rewrite (9) as

$$(g_1|g_2) = \int_{-\infty}^{\infty} \mathrm{d}t\mathrm{d}s g_1^*(t)\mathcal{W}(t-s)g_2(s) \tag{31}$$

To describe massless field in the position representation we introduce the operators $\psi_R(t)$ defined as

$$\psi_R(t) = \int \frac{\mathrm{d}k}{2\pi k}\left(a_R^\dagger(k)\mathrm{e}^{-\mathrm{i}kt} + a_R(k)\mathrm{e}^{\mathrm{i}kt}\right) ,$$

and similarly for $R \to L$. Note that it is allowed to smear $\psi_R(t)$ and $\psi_L(t)$ only with test functions satisfying

$$\int g(t)\mathrm{d}t = 0 .$$

Note that $\mathcal{W}(t-s)$ and $\frac{\mathrm{i}}{2}\mathrm{sgn}(t-s)$ are the correlator and the commutator functions for $\psi_R(t)$:

$$\langle \Omega_R | \psi_R(t)\psi_R(s)\Omega_R\rangle = \mathcal{W}(t-s)\ ,$$

$$[\psi_R(t), \psi_R(s)] = \frac{\mathrm{i}}{2}\mathrm{sgn}(t-s)\ ,$$

and similarly for $R \to L$.

We introduce also

$$\begin{aligned}\psi_{\sigma R}(t) &= \int \frac{\mathrm{d}k}{2\pi k} a^\dagger_{\sigma R}(k)\mathrm{e}^{-ikt} + \int \frac{\mathrm{d}k}{2\pi k} a_{\sigma R}(k)\mathrm{e}^{ikt} \\ &= \psi_R(t) + \frac{\chi}{2}\int \mathrm{d}s \sigma_R(s)\mathrm{sgn}(t-s)\ ,\end{aligned}$$

as well as $R \to L$.

It is perhaps useful to note that formally we can write

$$\psi_{\sigma R}(t) = Y_R \psi_R(t) Y_R^\dagger\ ,$$

where

$$Y_R := \exp\left(\mathrm{i}\chi \int \mathrm{d}t \sigma_R(t)\psi_R(t)\right)$$

Note that Y_R is not a well defined operator, since $\hat{\sigma}_R(0) \neq 0$.

Expressed in position representation the fields are given by

$$\phi(g_R, g_L) = \int \mathrm{d}t (g_R(t) - \hat{g}(0)\sigma_R(t))\psi_{\sigma R}(t) + R \to L + \hat{g}(0)p\ . \qquad (32)$$

Since $\int (g_R(t) - \hat{g}(0)\sigma_R(t))\mathrm{d}t = 0$ the whole expression is well defined.

The commutator of two fields equals

$$\begin{aligned}[\phi(f_1), \phi(f_2)] = \mathrm{i}\int &\mathrm{d}t_1 \mathrm{d}t_2 \mathrm{d}x_1 \mathrm{d}x_2 f_1(t_1, x_1) f_2(t_2, x_2) \\ &\times (\mathrm{sgn}(t_1 - t_2 + x_1 - x_2) + \mathrm{sgn}(t_1 - t_2 - x_1 + x_2))\end{aligned}$$

Note that the commutator of fields is causal – it vanishes if the supports of f_1 and f_2 are spatially separated.

7 The $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ Covariance

Massless fields in 1+1 dimension satisfying the constraint (2) actually possess much bigger symmetry than just the $A_+(1,\mathbb{R}) \times A_+(1,\mathbb{R})$ symmetry, they are covariant wrt the action of $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ (for right and left movers).

We will restrict ourselves to the action of $SL(2,\mathbb{R})$ for, say, right movers. First we consider it on the level of test functions.

We assume that test functions satisfy $\hat{g}(0) = 0$ and

$$g(t) = O(1/t^2), \quad |t| \to \infty\ . \qquad (33)$$

Let

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2, \mathbb{R}) \tag{34}$$

(i.e. $ad - bc = 1$). We define the action of C on g by

$$(r_C g)(t) = (-ct + a)^{-2} g\left(\frac{dt - b}{-ct + a}\right) . \tag{35}$$

Note that (35) preserves (33) and the scalar product

$$(r_C g_1 | r_C g_2) = (g_1 | g_2) ,$$

and is a representation, that is $r_{C_1} r_{C_2} = r_{C_1 C_2}$.

We second quantize r_C by introducing the unitary operator $R_R(C)$ on $\mathcal{H}_R$ fixed uniquely by the conditions

$$\begin{aligned} R_R(C)\Omega_R &= \Omega_R , \\ R_R(C)\psi_R(t)R_R(C)^\dagger &= \psi_R\left(\frac{at + b}{ct + d}\right) . \end{aligned} \tag{36}$$

Note that

$$R_R(C)\left(\int \mathrm{d}t g_R(t)\psi_R(t)\right)R_R(C)^\dagger = \int \mathrm{d}t (r_C g_R)(t)\psi_R(t). \tag{37}$$

$C \mapsto R_R(C)$ is a representation in $\mathcal{H}_R$. Thus the operators $R_R(C)$ act naturally on fields satisfying (2) (and hence also $\hat{g}_R(0) = 0$).

The fields without the constraint (2) are not covariant with respect to $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$, since this symmetry fails even at the classical level. What remains is the $A_+(1, \mathbb{R}) \times A_+(1, \mathbb{R})$ symmetry described in (17). Note that $A_+(1, \mathbb{R})$ can be viewed as a subgroup of $SL(2, \mathbb{R})$:

$$A_+(1, \mathbb{R}) \ni (a, b) \mapsto C \begin{bmatrix} a^{1/2} & ba^{-1/2} \\ 0 & a^{-1/2} \end{bmatrix} \in SL(2, \mathbb{R}) . \tag{38}$$

Clearly, on the restricted Hilbert space, under the identification (38), $R_R(a, b)$ coincides with $R_R(C)$.

8 Normal Ordering

In the theory without the compensating sector the normal ordering can be introduced in a standard way. In particular we have

$$:\mathrm{e}^{\mathrm{i}\phi(g_R, g_L)}: \; = \; \mathrm{e}^{\frac{1}{2}(g_R|g_R) + \frac{1}{2}(g_L|g_L)} \mathrm{e}^{\mathrm{i}\phi(g_R, g_L)} \tag{39}$$

If the compensating sector is present then the theory does not act in the Fock space any longer and we do not have an invariant particle number operator. It is however possible (and useful) to introduce the notion of the normal ordering. For Weyl operators it is by definition given by (39). For an arbitrary operator, we first decompose it in terms of Weyl operators, and then we apply (39). Note that our definition has an invariant meaning wrt the change of the compensating function: in the notation of (22) we have

$$U : \mathrm{e}^{\mathrm{i}\phi_\sigma(g_R,g_L)} : U^{-1} =: \mathrm{e}^{\mathrm{i}\phi_{\tilde{\sigma}}(g_R,g_L)} : \ .$$

Normal ordering is Poincaré invariant but suffers anomalies under remaining $A_+(1,\mathbb{R}) \times A_+(1,\mathbb{R})$ transformations (because the prefactor on the rhs of (39) is invariant only under the Poincaré group). If the constraint (2) is satisfied, then normal ordering is $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$ covariant.

9 Classical Fields

In order to better understand massless quantum fields in 1+1 dimension it is useful to study the underlying classical system, that is the wave equation in 1+1 dimension (4).

From the general representation of any classical solution

$$\phi(t,x) = \phi_R(t-x) + \phi_L(t+x) \tag{40}$$

we get (in notation where $f(\pm\infty)$ stands for $\lim_{t\to\pm\infty} f(t)$)

$$\begin{aligned} \phi(t,\infty) + \phi(t,-\infty) &= \phi_R(-\infty) + \phi_L(\infty) + \phi_R(\infty) + \phi_L(-\infty) \\ &= \phi(\infty,x) + \phi(-\infty,x) \end{aligned} \tag{41}$$

It will be convenient to denote by the space of Schwartz functions on $\mathbb{R}$ by $\mathcal{S}$ and by $\partial_0^{-1}\mathcal{S}$ the space of functions whose derivatives belong to $\mathcal{S}$ and satisfy the condition $f(\infty) = -f(-\infty)$.

We are interested only in those solutions that restricted to lines of constant time and lines of constant position belong to $\partial_0^{-1}\mathcal{S}$ (we will denote them as $\mathcal{F}_{11}$). Neglecting a possible global constant shift we therefore assume that they satisfy

$$\phi(t,\infty) + \phi(t,-\infty) = \phi(\infty,x) + \phi(-\infty,x) = 0 \tag{42}$$

$\mathcal{F}_{11}$ is characterized by two numbers

$$\begin{aligned} \lim_{t\to\infty} \phi(t,x) &= -\lim_{t\to-\infty} \phi(t,x) =: c_0 \ , \\ \lim_{x\to\infty} \phi(t,x) &= -\lim_{x\to-\infty} \phi(t,x) =: c_1 \ . \end{aligned}$$

It is natural to distinguish the following subclasses of solutions to (4):

– $\mathcal{F}_{00}$ – solutions that restricted to lines of constant time and to lines of constant position belong to $\mathcal{S}$ i.e. $c_0 = c_1 = 0$.

– $\mathcal{F}_{10}$ – solutions that restricted to lines of constant time belong to $\mathcal{S}$ and restricted to lines of constant position belong to $\partial_0^{-1}\mathcal{S}$ i.e. $c_1 = 0$.
– $\mathcal{F}_{01}$ – solutions that restricted to lines of constant position belong to $\mathcal{S}$ and restricted to lines of constant time belong to $\partial_0^{-1}\mathcal{S}$ i.e. $c_0 = 0$.

There are several useful ways to parametrize elements of $\mathcal{F}_{11}$.

1. **Initial conditions at $t = 0$:**

$$\begin{aligned} f_0(x) &= \phi(0,x) , \\ f_1(x) &:= \partial_t \phi(0,x) . \end{aligned} \tag{43}$$

Here, $f_0 \in \partial_0^{-1}\mathcal{S}$, $f_1 \in \mathcal{S}$. Note that

$$c_0 = \frac{1}{2}\int f_1(x)\mathrm{d}x, \qquad c_1 = f_0(\infty) .$$

2. **Derivatives of right/left movers:**

$$\begin{aligned} g_R(t) &:= -\frac{1}{2} f_0'(-t) + \frac{1}{2} f_1(-t) , \\ g_L(t) &:= \frac{1}{2} f_0'(t) + \frac{1}{2} f_1(t) . \end{aligned}$$

Note that $g_R, g_L \in \mathcal{S}$ and they satisfy

$$\int g_R(t)\mathrm{d}t = c_0 - c_1, \qquad \int g_L(t)\mathrm{d}t = c_0 + c_1 . \tag{44}$$

3. **Right/left movers:**

$$\begin{aligned} \phi_R(t) &= \frac{1}{2}\int g_R(t-u)\mathrm{sgn}(u)\mathrm{d}u \\ \phi_L(t) &= \frac{1}{2}\int g_L(t-u)\mathrm{sgn}(u)\mathrm{d}u . \end{aligned} \tag{45}$$

Note that $\phi_R, \phi_L \in \partial_0^{-1}\mathcal{S}$ and they satisfy

$$\begin{aligned} \phi_R(\infty) &= -\phi_R(-\infty) = \frac{1}{2}(c_0 - c_1) , \\ \phi_L(\infty) &= -\phi_L(-\infty) = \frac{1}{2}(c_0 + c_1) . \end{aligned} \tag{46}$$

We can go back from (g_R, g_L) to (f_0, f_1) by

$$\begin{aligned} f_0(x) &= \frac{1}{2}\int g_R(s-x)\mathrm{sgn}(-s)\mathrm{d}s + \frac{1}{2}\int g_R(s+x)\mathrm{sgn}(-s)\mathrm{d}s , \\ f_1(x) &= g_R(-x) + g_L(x) . \end{aligned}$$

We can go back from (ϕ_R, ϕ_L) to (g_R, g_L) by

$$g_R = \phi'_R, \quad g_L = \phi'_L \ .$$

The unique solution of (4) with the initial conditions (43) equals

$$\phi(t, x) = \phi_R(t - x) + \phi_L(t + x) \ .$$

It will be sometimes denoted by $\phi(g_R, g_L)$.

In the literature, one can find all three parametrizations of solutions of the wave equation. In particular, note that 3. is especially useful in the case of $\mathcal{F}_{00}$, since then $\phi_R, \phi_L \in \mathcal{S}$.

Note that in our paper we use 2. as the standard parametrization of solutions of the wave equation. We are interested primarily in the space $\mathcal{F}_{10}$. Note that $\mathcal{F}_{10}$ are the solutions to the wave equation with $f_0, f_1 \in \mathcal{S}$. Equivalently, for $\mathcal{F}_{10}$, the functions g_R, g_L satisfy

$$\int g_R(t)\mathrm{d}t = \int g_L(t)\mathrm{d}t \ . \tag{47}$$

We equip the space $\mathcal{F}_{11}$ with the Poisson bracket, which we write for all three parametrizations:

$$\{\phi(g_{R1}, g_{L1}), \phi(g_{R2}, g_{L2})\} = \int f_{01}(x) f_{12}(x)\mathrm{d}x - \int f_{02}(x) f_{11}(x)\mathrm{d}x \tag{48}$$

$$= \int g_{R1}(t)\mathrm{sgn}(s - t) g_{R1}(s)\mathrm{d}t\mathrm{d}s + \int g_{L1}(t)\mathrm{sgn}(s - t) g_{L1}(s)\mathrm{d}t\mathrm{d}s$$

$$= \mathrm{Im}(g_{R1}|g_{R2}) + \mathrm{Im}(g_{L1}|g_{L2}) \tag{49}$$

$$= \frac{1}{2}\int \partial_t\phi_{R1}(t)\phi_{R2}(t)\mathrm{d}t + \frac{1}{2}\int \partial_t\phi_{L1}(t)\phi_{L2}(t)\mathrm{d}t \ . \tag{50}$$

Above, (f_{0i}, f_{1i}) and (ϕ_{Ri}, ϕ_{Li}) correspond to (g_{Ri}, g_{Li}). The formula in (48) is the usual Poisson bracket for the space of solutions of relativistic 2nd order equations (both wave and Klein-Gordon equations). (49) we have already seen in (15).

The Poisson bracket in $\mathcal{F}_{11}$ is invariant wrt to the conformal group fixing the infinities preserving separately the orientation of right and left movers, that is $\mathrm{Diff}_+(\mathbb{R}) \times \mathrm{Diff}_+(\mathbb{R})$. In the case of $\mathcal{F}_{00}$ we can extend this action to the full orientation preserving conformal group, that is $\mathrm{Diff}_+(S^1) \times \mathrm{Diff}_+(S^1)$, where we identify $\mathbb{R}$ together with the point at infinity with the unit circle.

10 Algebraic Approach

Among mathematical physicists, it is popular to use the formalism of C^*-algebras to describe quantum systems. A description of massless fields in 1+1 dimension within this formalism is sketched in this section.

To quantize the space $\mathcal{F}_{11}$, we consider formal expressions

$$\mathrm{e}^{\mathrm{i}\phi(g_R,g_L)} \tag{51}$$

equipped with the relations

$$\begin{aligned} \mathrm{e}^{\mathrm{i}\phi(g_{R1},g_{L1})}\mathrm{e}^{\mathrm{i}\phi(g_{R2},g_{L2})} &= \mathrm{e}^{\mathrm{iIm}(g_{R1}|g_{R2})+\mathrm{iIm}(g_{L1}|g_{L2})}\mathrm{e}^{\mathrm{i}\phi(g_{R1}+g_{R2},g_{L1}+g_{L2})}; \\ (\mathrm{e}^{\mathrm{i}\phi(g_R,g_L)})^\dagger &= \mathrm{e}^{\mathrm{i}\phi(-g_R,-g_L)} \ . \end{aligned}$$

Linear combinations of (51) form a $*$-algebra, which we will denote Weyl $(\mathcal{F}_{11})$. (If we want, we can take its completion in the natural norm and obtain a C^*-algebra).

Note that the group $\mathrm{Diff}_+(\mathbb{R})\times\mathrm{Diff}_+(\mathbb{R})$ acts on Weyl $(\mathcal{F}_{11})$ by $*$-automorphisms. In other words, we have two actions

$$\begin{aligned} \mathrm{Diff}_+(\mathbb{R}) \ni F &\mapsto \alpha_R(F) \in \mathrm{Aut}(\mathrm{Weyl}(\mathcal{F}_{11}))\ , \\ \mathrm{Diff}_+(\mathbb{R}) \ni F &\mapsto \alpha_L(F) \in \mathrm{Aut}(\mathrm{Weyl}(\mathcal{F}_{11}))\ , \end{aligned}$$

commuting with one another given by

$$\begin{aligned} \alpha_R(F)\left(\mathrm{e}^{\mathrm{i}\phi(g_R,g_L)}\right) &= \mathrm{e}^{\mathrm{i}\phi(r_Fg_R,g_L)}\ , \\ \alpha_L(F)\left(\mathrm{e}^{\mathrm{i}\phi(g_R,g_L)}\right) &= \mathrm{e}^{\mathrm{i}\phi(g_R,r_Fg_L)}\ . \end{aligned} \tag{52}$$

Above, $\mathrm{Aut}(\mathrm{Weyl}(\mathcal{F}_{11}))$ denotes the group of $*$-automorphisms of the algebra $\mathrm{Weyl}(\mathcal{F})$) and $r_Fg(t) := \frac{1}{F'(t)}g(F^{-1}(t))$.

Similarly $\mathrm{Diff}_+(S^1)\times\mathrm{Diff}_+(S^1)$ acts on $\mathrm{Weyl}(\mathcal{F}_{00})$ by $*$-automorphisms.

The state ω given by (28) is invariant wrt $A_+(1,\mathbb{R})\times A_+(1,\mathbb{R})$ on $\mathrm{Weyl}(\mathcal{F}_{11})$ and wrt $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$ on $\mathrm{Weyl}(\mathcal{F}_{00})$.

In our paper we restricted ourselves to $\mathrm{Weyl}(\mathcal{F}_{10})$.

The constructions presented in this paper give representations of $\mathrm{Weyl}(\mathcal{F}_{10})$ in a Hilbert space $\mathcal{H}$ and two commuting with one another strongly continuous unitary representations

$$\begin{aligned} A_+(1,\mathbb{R}) \ni (a,b) &\mapsto R_R(a,b) \in U(\mathcal{H})\ , \\ A_+(1,\mathbb{R}) \ni (a,b) &\mapsto R_L(a,b) \in U(\mathcal{H})\ . \end{aligned}$$

implementing the automorphisms (52):

$$\alpha_R(a,b)(A) = R_R(a,b)AR_R(a,b)^\dagger\ , \tag{53}$$

$$\alpha_L(a,b)(A) = R_L(a,b)AR_L(a,b)^\dagger. \tag{54}$$

In the case of the algebra $\mathrm{Weyl}(\mathcal{F}_{00})$ the same is true for $SL(2,\mathbb{R})$.

In Sect. 5 we described two representations that satisfy the above mentioned conditions. The first, call it π_{I}, represents $\mathrm{Weyl}(\mathcal{F}_{10})$ in a separable Hilbert space. Its drawback is the absence of a vacuum vector – a Poincaré invariant vector. The second, call it π_{II}, represents $\mathrm{Weyl}(\mathcal{F}_{10})$ in a non-separable Hilbert space. It has an invariant vector $|\Omega\rangle$.

We can perform the GNS construction with ω. As a result we obtain the representation π_{II} together with the cyclic invariant vector Ω. The description of this construction for massless fields in 1+1 dimension can be found in [1], Sect. III D, and [2] Sect. 4. Note, however, that we have not seen the representation π_{I} in the literature, even though one can argue that it is in some ways superior to π_{II}.

Let us make a remark concerning the role played by the functions (σ_R, σ_L). We note that $\mathcal{F}_{00}$ is a subspace of $\mathcal{F}_{10}$ of codimension 1. Fixing (σ_R, σ_L) satisfying (11) allows us to identify $\mathcal{F}_{10}$ with $\mathcal{F}_{00} \oplus \mathbb{R}$. Thus any (g_R, g_L) satisfying (47) is decomposed into the direct sum of $(g_R - \hat{g}(0)\sigma_R, g_L - \hat{g}(0)\sigma_L)$ and $\hat{g}(0)(\sigma_R, \sigma_L)$.

Of course, similar constructions can be performed for the algebra $\mathrm{Weyl}(\mathcal{F}_{11})$ or $\mathrm{Weyl}(\mathcal{F}_{01})$. In the literature, algebras of observables based on $\mathcal{F}_{01}$ appear in the context of "Doplicher-Haag-Roberts charged sectors" in [5, 6, 15].

11 Vertex Operators

Finally, let us make some comments about the so-called vertex operators, often used in string theory [8]. Let δ_y denote the delta function at $y \in \mathbb{R}$.

Let $t_{R1}, \ldots, t_{Rn} \in \mathbb{R}$ correspond to insertions for right movers and $t_{L1}, \ldots, t_{Lm} \in \mathbb{R}$ correspond to insertions for left movers. Suppose that the complex numbers $\beta_{R1}, \ldots, \beta_{Rn}$, and $\beta_{L1}, \ldots, \beta_{Lm}$ denote the corresponding insertion amplitudes and satisfy

$$\sum \beta_{Ri} = \sum \beta_{Lj} \ .$$

Then the corresponding vertex operator is formally defined as

$$V(t_{R1}, \beta_{R1}; \ldots; t_{Rn}, \beta_{Rn}; t_{L1}, \beta_{L1}; \ldots; t_{Lm}, \beta_{Lm}) = \ :\exp\left(\mathrm{i}\phi(g_R, g_L)\right): , \tag{55}$$

where

$$g_R = \beta_{R1}\delta_{t_{R1}} + \cdots + \beta_{Rn}\delta_{t_{Rn}}, \tag{56}$$

$$g_L = \beta_{L1}\delta_{t_{L1}} + \cdots + \beta_{Lm}\delta_{t_{Lm}}. \tag{57}$$

Strictly speaking, the rhs of (55) does not make sense as an operator in the Hilbert space. In fact, in order that $\mathrm{e}^{\mathrm{i}\phi(g_R, g_L)}$ be a well defined operator, we need that

$$\int \frac{\mathrm{d}k}{2\pi k} \left|\hat{g}_R(k) - \hat{g}(0)\hat{\sigma}_R(k)\right|^2 + \int \frac{\mathrm{d}k}{2\pi k} \left|\hat{g}_L(k) - \hat{g}(0)\hat{\sigma}_L(k)\right|^2 < \infty. \tag{58}$$

This is not satisfied if g_R or g_L are as in (56) and (57).

Nevertheless, proceeding formally, we can deduce various identities. For instance, we have the Poincaré covariance:

$$
\begin{aligned}
& R_R(a, b_R) R_L(a^{-1}, b_L) \\
& \times V(t_{R1}, \beta_{R1}; \dots; t_{Rn}, \beta_{Rn}; t_{L1}, \beta_{L1}; \dots) \\
& \times R_L^\dagger(a^{-1}, b_L) R_R^\dagger(a, b_R) \\
& = V(a t_{R1} + b_L, \beta_{R1}; \dots; a^{-1} t_{Rn} + b_R, \beta_{Rn}; a^{-1} t_{L1} + b_L, \beta_{L1}; \dots) \,.
\end{aligned}
$$

If in addition $\sum \beta_{Ri} = 0$, then a similar identity is true for $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$.

Clearly, we have

$$
\begin{aligned}
& \omega \left(V(t_{R1}, \beta_{R1}; \dots; t_{Rn}, \beta_{Rn}; t_{L1}, \beta_{L1}; \dots; t_{Lm}, \beta_{Lm}) \right) \\
& = \begin{cases} 1, & \sum \beta_{Ri} = 0; \\ 0, & \sum \beta_{Ri} \neq 0. \end{cases} \,.
\end{aligned} \tag{59}
$$

The following identities are often used in string theory for the calculation of on-shell amplitudes. Suppose that $t_{R1}, \dots, t_{Rn}$ are distinct, and the same is true for $t_{L1}, \dots, t_{Ln}$. Then, using (30), we obtain

$$
\begin{aligned}
& V(t_{R1}, \beta_{R1}; t_{L1}, \beta_{L1}) \cdots V(t_{Rn}, \beta_{Rn}; t_{Ln}, \beta_{Ln}) \\
& = \mathrm{e}^{\left(\sum\limits_{i<j} \mathcal{W}(t_{Ri} - t_{Rj}))\beta_{Ri}\beta_{Rj} + \mathcal{W}(t_{Li} - t_{Lj})\beta_{Li}\beta_{Lj} \right)} \\
& \times V(t_{R1}, \beta_{R1}; \dots; t_{Rn}, \beta_{Rn}; t_{L1}, \beta_{L1}; \dots; t_{Ln}, \beta_{Ln}) \\
& = \prod_{i<j} \left(\frac{t_{Ri} - t_{Rj}}{\mathrm{i}\mu \mathrm{e}^{\gamma_E}} \right)^{-\beta_{Ri}\beta_{Rj}/2\pi} \left(\frac{t_{Li} - t_{Lj}}{\mathrm{i}\mu \mathrm{e}^{\gamma_E}} \right)^{-\beta_{Li}\beta_{Lj}/2\pi} \\
& \times V(t_{R1}, \beta_{R1}; \dots; t_{Rn}, \beta_{Rn}; t_{L1}, \beta_{L1}; \dots; t_{Ln}, \beta_{Ln}) \,.
\end{aligned}
$$

12 Fermions

Massless fermions in 1+1 dimension do not pose such problems as bosons. The fields are spinors, they will be written as $\begin{bmatrix} \lambda_R(t,x) \\ \lambda_L(t,x) \end{bmatrix}$. They satisfy the Dirac equation

$$
\begin{bmatrix} \partial_t - \partial_x & 0 \\ 0 & \partial_t + \partial_x \end{bmatrix} \begin{bmatrix} \lambda_R(t,x) \\ \lambda_L(t,x) \end{bmatrix} = 0 \,.
$$

We will also use the fields smeared with real functions f, where the condition (2) is not needed any more:

$$
\begin{bmatrix} \lambda_R(f) \\ \lambda_L(f) \end{bmatrix} = \int \begin{bmatrix} \lambda_R(t,x) \\ \lambda_L(t,x) \end{bmatrix} f(t,x) \mathrm{d}t \mathrm{d}x \,.
$$

Because of the Dirac equation, they can be written as

$$\lambda_R(f) = \lambda_R(g_R), \quad \lambda_L(f) = \lambda_L(g_L) \ ,$$

where g_R and g_L where introduced when we discussed bosons.

For $k > 0$, we introduce fermionic operators (for right and left sectors) $b_R(k)$ and $b_L(k)$ satisfying the anticommutation relations

$$\begin{aligned} \{b_R(k), b_R^\dagger(k')\} &= 2\pi\delta(k-k') \ , \\ \{b_L(k), b_L^\dagger(k')\} &= 2\pi\delta(k-k'), \end{aligned} \tag{60}$$

with all other anticommutators vanishing. Now

$$\begin{aligned} \lambda_R(g_R) &= \int \frac{\mathrm{d}k}{2\pi} \left(g_R^*(k) b_R(k) + g_R(k) b_R^\dagger(k) \right) \\ \lambda_L(g_L) &= \int \frac{\mathrm{d}k}{2\pi} \left(g_L^*(k) b_L(k) + g_L(k) b_L^\dagger(k) \right) \end{aligned} \tag{61}$$

The anticommutation relations for the smeared fields read

$$\{\lambda_R^\dagger(g_{R1}), \lambda_R(g_{R2})\} = \int g_{R1}^*(t) g_{R2}(t) \mathrm{d}t = \int \frac{\mathrm{d}k}{2\pi} \hat{g}_{R1}^*(k) \hat{g}_{R2}(k) \tag{62}$$

and similarly for the left sector. Note the difference of the fermionic scalar product (62) and the bosonic one $(\cdot|\cdot)$.

In terms of space-time smearing functions these anticommutation relations read

$$\begin{aligned} \{\lambda_R^\dagger(f_1), \lambda_R(f_R)\} &= 2 \int \mathrm{d}t \mathrm{d}x \delta(t+x) f_1(t,x) f_2(t,x) \ , \\ \{\lambda_L^\dagger(f_1), \lambda_R(f_L)\} &= 2 \int \mathrm{d}t \mathrm{d}x \delta(t-x) f_1(t,x) f_2(t,x) \ . \end{aligned}$$

Fermionic fields are covariant with respect to the group $A_+(1,\mathbb{R}) \times A_+(1,\mathbb{R})$. We will restrict ourselves to discussing the covariance for say, right movers. The right Hamiltonian and the right dilation generator are

$$\begin{aligned} H_R^{\mathrm{f}} &= \int \frac{\mathrm{d}k}{2\pi} k b_R^\dagger(k) b_R(k) \\ D_R^{\mathrm{f}} &= \frac{\mathrm{i}}{2} \int \frac{\mathrm{d}k}{2\pi} \left(b_R^\dagger(k) k \partial_k b_R(k) - \big(k \partial_k b_R^\dagger(k) \big) b_R(k) \right) \ . \end{aligned}$$

We have the usual commutation relations for H_R^{f} and D_R^{f} and their action on the fields is anomaly-free:

$$\begin{aligned} \left[H_R^{\mathrm{f}}, \lambda_R(g_R) \right] &= -\mathrm{i}\lambda(\partial_t g_R) \ , \\ \left[D_R^{\mathrm{f}}, \lambda_R(g_R) \right] &= \mathrm{i}\lambda\big((t\partial_t + 1/2) g_R \big). \end{aligned}$$

We have also the covariance with respect to the conformal group $SL(2,\mathbb{R})\times SL(2,\mathbb{R})$. We need to assume that test functions satisfy

$$g(t) = O(1/t), \quad |t| \to \infty \; . \tag{63}$$

We define the action of

$$C = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{R}) \tag{64}$$

on g by

$$(r^{\mathrm{f}}_C g)(t) = (-ct+a)^{-1} g\left(\frac{dt-b}{-ct+a}\right) \; . \tag{65}$$

Note that (65) has a different power than (35). It is a unitary representation for the scalar product $(\cdot|\cdot)_{\mathrm{f}}$.

We second quantize r^{f}_C on the fermionic Fock space by introducing the unitary operator $R^{\mathrm{f}}_R(C)$ fixed uniquely by the conditions

$$\begin{aligned} R^{\mathrm{f}}_R(C)\Omega_R &= \Omega_R \; , \\ R^{\mathrm{f}}_R(C)\lambda_R(t)R^{\mathrm{f}}_R(C)^\dagger &= (ct+d)^{-1}\lambda_R\left(\frac{at+b}{ct+d}\right) \; . \end{aligned} \tag{66}$$

Note that $C \mapsto R^{\mathrm{f}}_R(C)$ is a unitary representation and it acts naturally on fields:

$$R^{\mathrm{f}}_R(C)\lambda_R(g_R)R^{\mathrm{f}}_R(C)^\dagger = \lambda_R(r^{\mathrm{f}}_C g_R), \tag{67}$$

13 Supersymmetry

In this section we consider both bosons and fermions. Thus our Hilbert space is the tensor product of the bosonic and fermionic part. We assume that the bosonic and fermionic operators commute with one another. Clearly, our theory is $A_+(1,\mathbb{R}) \times A_+(1,\mathbb{R})$ covariant. In fact, the right Hamiltonian and the generator of dilations for the combined theory are equal to $H_R + H^{\mathrm{f}}_R$ and $D_R + D^{\mathrm{f}}_R$.

In the case of the theory with the constraint (2), we have also the $SL(2,\mathbb{R}) \times SL(2,\mathbb{R})$. covariance.

On top of that, the combined theory is supersymmetric. The supersymmetry generators Q_R, Q_L are defined as

$$\begin{aligned} Q_R &= \int \frac{\mathrm{d}k}{2\pi}\left(a^\dagger_{\sigma R}(k)b_R(k) + a_{\sigma R}(k)b^\dagger_R(k)\right) \; , \\ Q_L &= \int \frac{\mathrm{d}k}{2\pi}\left(a^\dagger_{\sigma L}(k)b_L(k) + a_{\sigma L}(k)b^\dagger_L(k)\right) \; . \end{aligned} \tag{68}$$

They satisfy the basic supersymmetry algebra relations without the central charge

$$\begin{aligned}\{Q_R, Q_R\} &= 2(H_R + H_R^{\mathrm{f}}) ,\\ \{Q_L, Q_L\} &= 2(H_L + H_L^{\mathrm{f}}) ,\\ \{Q_R, Q_L\} &= 0. \end{aligned} \tag{69}$$

The action of the supersymmetric charge transforms bosons into fermions and vice versa:

$$\begin{aligned}[][Q_R, \phi(g_R, g_L)] &= \lambda_R(g_R) ,\\ [Q_L, \phi(g_R, g_L)] &= \lambda_L(g_L) ,\\ [Q_R, \lambda_R(g_R)] &= \phi(\partial_t g_R, 0) ,\\ [Q_L, \lambda_L(g_L)] &= \phi(0, \partial_t g_L) .\end{aligned}$$

The pair of operators $\begin{bmatrix} Q_R \\ Q_L \end{bmatrix}$ behaves like a spinor under the Poincaré group. Even more is true: we have the covariance under the group $A_+(1,\mathbb{R}) \times A_+(1,\mathbb{R})$, which for the right movers can be expressed in terms of the following commutation relations:

$$[H_R, Q_R] = 0 , \qquad [D_R, Q_R] = -\frac{\mathrm{i}}{2} Q_R .$$

Acknowledgement

J.D. was partly supported by the European Postdoctoral Training Program HPRN-CT-2002-0277, the Polish KBN grant SPUB127 and 2 P03A 027 25. K.A.M. was partially supported by the Polish KBN grant 2P03B 001 25 and the European Programme HPRN–CT–2000–00152.

J.D. would like to thank S. DeBièvre, C. Gérard and C. Jäkel for useful discussions.

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