GEOMETRIC PSEUDODIFFERENTIAL CALCULUS WITH APPLICATIONS TO QFT ON CURVED SPACETIMES

JAN DEREZIŃSKI

Dep. of Math. Meth. in Phys.
Faculty of Physics, University of Warsaw

in collaboration with

WOJCIECH KAMIŃSKI, ADAM LATOSIŃSKI
and DANIEL SIEMSSEN

1. Balanced Geometric Weyl Quantization
2. Schrödinger operators on Riemannian manifolds, the asymptotics of their inverse
3. Klein-Gordon operators on Lorentzian manifolds, their self-adjointness, distinguished inverses and bisolutions (propagators).
4. Asymptotics of propagators

BALANCED GEOMETRIC WEYL QUANTIZATION

The usual Weyl quantization of $b \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^d)$ is the operator $\mathrm{Op}(b) : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}'(\mathbb{R}^d)$ with the kernel

$$\mathrm{Op}(b)(x,y) := \int b\Big(\frac{x+y}{2}, p\Big) \mathrm{e}^{\frac{\mathrm{i}}{h}(y-x)p} \frac{\mathrm{d}p}{(2\pi h)^d}.$$

Hilbert-Schmidt operators correspond to square integrable symbols:

$$(2\pi h)^{-d} \mathrm{Tr}\mathrm{Op}(a)^* \mathrm{Op}(b) = \int \overline{a(z,p)} b(z,p) \mathrm{d}z \mathrm{d}p.$$

Consider a (pseudo-)Riemannian manifold M.

There exists a neighborhood of the diagonal $\Omega \subset M \times M$ with the property that every pair $(x, y) \in \Omega$ is joined by a unique geodesics $[0, 1] \ni \tau \mapsto \gamma_{x,y}(\tau)$ such that $\gamma_{x,y} \times \gamma_{x,y} \subset \Omega$. It is called a geodesically convex neighborhood of the diagonal.

Let $x \in M$ and $u \in \mathrm{T}_x M$. We will write

$$x + u := \exp_x(u).$$

Let $(x, y) \in \Omega$. The symbol $y - x$ will denote the unique vector in $T_x M$ tangent to the geodesics $\gamma_{x,y}$ such that

$$x + (y - x) = y.$$

$(y - x)_\tau$ will denote the vector in $\mathrm{T}_{x+\tau(y-x)} M$ such that

$$\big(x + \tau(y - x)\big) + (1 - \tau)(y - x)_\tau = y.$$

The Van Vleck–Morette determinant is defined as

$$\Delta(x,y) := \left|\frac{\partial(y-x)}{\partial y}\right| \frac{|g(x)|^{\frac{1}{2}}}{|g(y)|^{\frac{1}{2}}}.$$

Note that

$$\Delta(x,y) = \Delta(y,x), \quad \Delta(x,x) = 1.$$

If B is an operator $C_{\mathrm{c}}^{\infty}(M) \to \mathcal{D}'(M)$ then its kernel is a distribution in $\mathcal{D}'(M \times M)$ such that

$$\langle f|Bg\rangle = \int f(x)B(x,y)g(y)\mathrm{d}x\mathrm{d}y, \quad f,g \in C_{\mathrm{c}}^{\infty}(M).$$

We will treat elements of $C_{\mathrm{c}}^{\infty}(M)$ not as scalar functions, but as half-densities. With this convention, the kernel of an operator is a half-density on $M \times M$.

We will say that M is geodesically simple if each pair of points is joined by a unique geodesics, so that $\Omega = M \times M$.

Assume first that M is geodesically simple. Consider a function on the phase space, often called a symbol

$$\mathrm{T}^* M \ni (z, p) \ni b(z, p).$$

The balanced geometric Weyl quantization of b, denoted $\mathrm{Op}(b)$, is the operator with the kernel

$$\begin{aligned}\mathrm{Op}(b)(x,y) :=& \Delta(x,y)^{\frac{1}{2}} \frac{|g(x)|^{\frac{1}{4}}|g(y)|^{\frac{1}{4}}}{\big|g\big(z\big)\big|^{\frac{1}{2}}} \\ &\times \int b\big(z,p\big) \mathrm{e}^{\frac{\mathrm{i}}{h}up} \frac{\mathrm{d}p}{(2\pi h)^d},\end{aligned}$$

where

$$z := x + \frac{y-x}{2}, \quad u := (y-x)_{\frac{1}{2}}.$$

Note that T^*M possesses a natural density, hence there is a natural identification of scalars with half-densities.

Up to a coefficient, the quantization that we defined is unitary from $L^2(T^*M)$ to operators on $L^2(M)$ equipped with the Hilbert-Schmidt scalar product:

$$\frac{1}{(2\pi h)^d}\int_{T^*M}\overline{c(x,p)}b(x,p)\mathrm{d}x\mathrm{d}p = \mathrm{Tr}\mathrm{Op}(c)^*\mathrm{Op}(b).$$

Define the star product

$$\mathrm{Op}(a \star b) = \mathrm{Op}(a)\mathrm{Op}(b).$$

Here is its asymptotic expansion in Planck's constant:

$$\begin{aligned}
(a \star b) \sim\ & ab + h\tfrac{\mathrm{i}}{2}\big(a_\alpha b^\alpha - a^\alpha b_\alpha\big) \\
& + h^2\Big(-\tfrac{1}{8}\big(a_{\alpha_1\alpha_2} b^{\alpha_1\alpha_2} - 2a_{\alpha_1}^{\alpha_2} b_{\alpha_2}^{\alpha_1} + a^{\alpha_1\alpha_2} b_{\alpha_1\alpha_2}\big) \\
+ \tfrac{1}{12} & R_{\alpha_1\alpha_2} a^{\alpha_2} b^{\alpha_1} - \tfrac{1}{24} R^\beta{}_{\alpha_1\alpha_2\alpha_3} p_\beta \big(a^{\alpha_2} b^{\alpha_1\alpha_3} + a^{\alpha_1\alpha_3} b^{\alpha_2}\big)\Big) \\
& + \dots
\end{aligned}$$

Lower indices—horizontal (spatial) derivatives.
Upper indices—vertical (momentum) derivatives.

If M is not geodesically simple, in the definition of $\mathrm{Op}(b)$ we need to put a cutoff χ equal 1 in a neighborhood of the diagonal and supported in Ω. This does not affect the semiclassical expansion of the starproduct.

SCHRÖDINGER OPERATORS ON A RIEMANNIAN MANIFOLD AND THE ASYMPTOTICS OF THEIR INVERSE

Consider a symbol quadratic in the momenta, with the principal part given by the Riemannian metric:

$$k(z,p) = g^{\mu\nu}(z)\big(p_\mu - A_\mu(z)\big)\big(p_\nu - A_\nu(z)\big) + Y(z).$$

Its quantization is a magnetic Schrödinger operator

$$K := \mathrm{Op}(k) = |g|^{-\frac{1}{4}}(\mathrm{i}h\partial_\mu + A_\mu)|g|^{\frac{1}{2}}g^{\mu\nu}(\mathrm{i}h\partial_\nu + A_\nu)|g|^{-\frac{1}{4}} + \frac{1}{6}R + Y.$$

K is a self-adjoint operator on $L^2(M)$. We are interested in the corresponding

heat semigroup $\quad W(t) := \mathrm{e}^{-tK}, \; \mathrm{Re} t > 0$

and Green's operator (inverse) $\quad G := \frac{1}{K}.$

They are closely related:

$$G = \int_0^\infty W(t) \mathrm{d}t.$$

We would like to compute the asymptotics of their kernels. We make the ansatz

$$W(t) = \mathrm{Op}\big(w(t)\big),$$

$$w(t, z, p) \sim \mathrm{e}^{-tk(z,p)} \sum_{n=0}^{\infty} \frac{t^n}{n!} w_n(z, p),$$

$$w_0(z, p) = 1.$$

By applying the geometric pseudodifferential calculus one can iteratively find

$$w_n(z,p) = \sum w_{n,\alpha}(z)\big(p - A(z)\big)^{\alpha}.$$

It is easy to see that w_n is a polynomial in $\big(p - A(z)\big)$ of degree $\leq \frac{3}{2}n$. Using the fact that the principal symbol is given by the metric we show that degree $\leq n$.

From this one obtains

$$W(t,x,y) \sim \mathcal{W}(t,x,y) := \frac{\Delta^{\frac{1}{2}}(x,y)|g(x)|^{\frac{1}{4}}|g(y)|^{\frac{1}{4}}}{|g(z)|^{\frac{1}{2}}(4\pi t h^2)^{\frac{d}{2}}}$$

$$\times \exp\Big(-\frac{1}{4t} v g^{-1}(z) v - tY(z)\Big) \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} v^\beta \mathcal{B}_{k,\beta}(z) \mathrm{e}^{-\mathrm{i}vA(z)},$$

where as usual

$$z := x + \frac{y-x}{2}, \quad u := (y-x)_{\frac{1}{2}}, \quad v = \frac{u}{h}.$$

What is the meaning of $\sim$? We can write

$$W(t,x,y) := \frac{\Delta^{\frac{1}{2}}(x,y)|g(x)|^{\frac{1}{4}}|g(y)|^{\frac{1}{4}}}{|g(z)|^{\frac{1}{2}}(4\pi t h^2)^{\frac{d}{2}}}$$
$$\times \exp\Big(-\frac{1}{4t} v g^{-1}(z) v - tY(z) \Big) B(t,z,v) \mathrm{e}^{-\mathrm{i}vA(z)}.$$

Then formally

$$B(t,z,v) = \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} v^\beta \mathcal{B}_{k,\beta}(z) + O(h^\infty).$$

Maybe we can fix $h = 1$ and replace $O(h^{\infty})$ with $O(t^{\infty})$.

For geodesically simple manifolds, perhaps we can replace it by $O(|v|^{\infty})$.

In the literature

$$W(t,x,y)\frac{\Delta^{\frac{1}{2}}(x,y)|g(x)|^{\frac{1}{4}}|g(y)|^{\frac{1}{4}}}{|g(z)|^{\frac{1}{2}}(4\pi t)^{\frac{d}{2}}}$$
$$\times \exp\Big(-\frac{1}{4t}(x-y)^2\Big)B(t,x,y)$$
$$B(t,x,y) \sim \sum_{n=0}^{\infty}\frac{t^n}{n!}B_n(x,y),$$

is called the Minackshisundaram-Pleijel expansion or the Schwinger-De Witt expansion.

The usual way to find this expansion is to solve recursively the differential equation

$$(\partial_t + K)W(t, x, y) = 0, \qquad t > 0,$$
$$W(0, x, y) = \delta(x, y).$$

This method does not give a unique answer for all coefficients, unlike the pseudodifferential calculus.

Assume that $Y > 0$. By integrating the heat kernel we obtain an asymptotics of Green's operator:

$$\mathcal{G}(x,y) := \frac{\Delta^{\frac{1}{2}}(x,y)|g(x)|^{\frac{1}{4}}|g(y)|^{\frac{1}{4}}}{(4\pi)^{\frac{d}{2}}} \sum_{k=0}^{\infty} u^{\beta}\mathcal{W}_{k,\beta}(z)\mathrm{e}^{-\mathrm{i}uA(z)}$$

$$\times\, 2K_{k+1-\frac{d}{2}}\Big(\sqrt{ug^{-1}(z)uY(z)}\Big)\left(\frac{ug^{-1}(z)u}{4Y(z)}\right)^{\frac{k+1-\frac{d}{2}}{2}},$$

where K_m are the MacDonald functions.

Using the well-known expansions of the MacDonald functions we obtain a version of the Hadamard expansion

$$\begin{aligned} G(x,y) \sim \mathcal{G}(x,y) &= \Delta^{\frac{1}{2}}(x,y)|g(x)|^{\frac{1}{4}}|g(y)|^{\frac{1}{4}}\mathrm{e}^{-\mathrm{i}uA(z)} \\ &\times \Big(\big(ug^{-1}(z)u\big)^{1-\frac{d}{2}}\sum_{\alpha} u^{\alpha}w_{\alpha}(z) \\ &+ \log\big(ug^{-1}(z)u\big)\sum_{\alpha} u^{\alpha}v_{\alpha}(z)\Big). \end{aligned}$$

(In odd dimensions the term with the logarithm is absent).

KLEIN-GORDON OPERATORS, THEIR INVERSES AND BISOLUTIONS (PROPAGATORS)

Assume that M is equipped with the metric tensor g, the electromagnetic potential A and the scalar potential (or "mass squared") Y. Consider the operator

$$K := |g|^{-\frac{1}{4}}(\mathrm{i}\partial_\mu + A_\mu)|g|^{\frac{1}{2}}g^{\mu\nu}(\mathrm{i}\partial_\nu + A_\nu)|g|^{-\frac{1}{4}} + Y$$

If M is a Riemannian manifold, then K would be called a Schrödinger operator.

We consider a globally hyperbolic Lorentzian manifold, and then K is called a Klein-Gordon operator. Its mathematical theory is much more complicated!

We say that G is a bisolution of K if

$$GK = KG = 0.$$

We say that G is an inverse (Green's operator if

$$GK = KG = \mathbb{1}.$$

In quantum field theory an important role is played by certain distinguished bisolutions and inverses. We will call them propagators.

The most important propagators on the Minkowski space:

the forward/backward or advanced/retarded propagator

$$G^{\vee/\wedge}(p) := \int \frac{\mathrm{d}p\mathrm{e}^{\mathrm{i}xp}}{(2\pi)^4(p^2+m^2 \mp \mathrm{i}0\mathrm{sgn}p^0)},$$

the Feynman/anti-Feynman propagator

$$G^{\mathrm{F}/\overline{\mathrm{F}}}(p) := \int \frac{\mathrm{d}p\mathrm{e}^{\mathrm{i}xp}}{(2\pi)^4(p^2+m^2 \mp \mathrm{i}0)},$$

the Pauli-Jordan propagator

$$G^{\mathrm{PJ}}(p) := \int \frac{\mathrm{d}p\mathrm{e}^{\mathrm{i}xp}\mathrm{sgn}(p^0)\delta(p^2+m^2)}{(2\pi)^4},$$

and the positive/negative frequency bisolution

$$G^{(+)/(-)}(p) := \int \frac{\mathrm{d}p\mathrm{e}^{\mathrm{i}xp}\theta(\pm p^0)\delta(p^2+m^2)}{(2\pi)^4}.$$

In QFT textbooks, the Pauli-Jordan propagator expresses commutation relations of fields, and hence it is often called the commutator function.

The positive frequency bisolution is the vacuum expectation of a product of two fields and is often called the 2-point function.

The Feynman propagator is the vacuum expectation of the time-ordered product of fields and is used to evaluate Feynman diagrams.

It is well-known that on an arbitrary globally hyperbolic spacetime one can define the forward propagator (inverse) $G^{\vee}$ and the backward propagator (inverse) $G^{\wedge}$.

Their difference is a bisolution called sometimes the Pauli-Jordan propagator (bisolution)

$$G^{\mathrm{PJ}} := G^{\vee} - G^{\wedge}.$$

All of them have a causal support. We will jointly call them classical propagators. They are relevant for the Cauchy problem.

We are however more interested in "non-classical propagators", typical for quantum field theory. They are less known to pure mathematicians and more difficult to define on curved spacetimes:

- the Feynman propagator G^{F},
- the anti-Feynman propagator $G^{\overline{\mathrm{F}}}$,
- the positive frequency bisolution $G^{(+)}$,
- the negative frequency bisolutions $G^{(-)}$.

There exists a well-known paper of Duistermat-Hörmander, which defined Feynman parametrices (a parametrix is an approximate inverse in appropriate sense).

There exists a large literature devoted to the so-called Hadamard states, which can be interpreted as bisolutons with approximately positive frequencies. These are however large classes of bisolutions. We would like to have distinguished choices.

It is possible and helpful to introduce a time variable t, so that the spacetime is $M = \mathbb{R}\times\Sigma$. We can assume that there are no time-space cross terms so that the metric can be written as

$$-g_{00}(t,\vec{x})\mathrm{d}^2t + g_{ij}(t,\vec{x})\mathrm{d}x^i\mathrm{d}x^j.$$

By conformal rescaling we can assume that $g_{00} = 1$, so that, setting $V := A^0$, we have

$$\begin{aligned} K &= -(\mathrm{i}\partial_t + V)^2 + L, \\ L &= -|g|^{-\frac{1}{4}}(\mathrm{i}\partial_i + A_i)|g|^{\frac{1}{2}}g^{ij}(\mathrm{i}\partial_j + A_j)|g|^{-\frac{1}{4}} + Y. \end{aligned}$$

We rewrite the Klein-Gordon equation $Ku = 0$ as a 1st order equation for the Cauchy data

$$\Big(\partial_t + \mathrm{i}B(t)\Big)\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix} = 0,$$

$$\begin{bmatrix}u_1(t)\\u_2(t)\end{bmatrix} := \begin{bmatrix}u(t)\\ \mathrm{i}\partial_t u(t) - W(t)u(t)\end{bmatrix}$$

$$B(t) := \begin{bmatrix}W(t) & \mathbb{1}\\ L(t) & \overline{W}(t)\end{bmatrix},$$

$$W(t) := V(t) + \frac{\mathrm{i}}{4}|g|(t)^{-1}\partial_t|g|(t).$$

Denote by $U(t,t')$ the dynamics defined by $B(t)$, that is

$$\begin{aligned}\partial_t U(t,t') &= -\mathrm{i}B(t)U(t,t'),\\ U(t,t) &= \mathbb{1}.\end{aligned}$$

Note that if

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$$

is a bisolution/inverse of $\partial_t + \mathrm{i}B(t)$, then E_{12} is a bisolution/inverse of K.

The classical propagators can be easily expressed in terms of the dynamics:

$$\begin{aligned} E^{\mathrm{PJ}}(t,t') &:= U(t,t'), & E^{\mathrm{PJ}}_{12} &= -\mathrm{i}G^{\mathrm{PJ}}; \\ E^{\vee}(t,t') &:= \theta(t-t')\,U(t,t'), & E^{\vee}_{12} &= -\mathrm{i}G^{\vee}; \\ E^{\wedge}(t,t') &:= -\theta(t'-t)\,U(t,t'), & E^{\wedge}_{12} &= -\mathrm{i}G^{\wedge}. \end{aligned}$$

The dynamics preserves the pseudounitary structure (a complexification of the symplectic structure) given by charge matrix

$$Q := \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Let us introduce the classical Hamiltonian

$$H(t) := QB(t) = \begin{pmatrix} L(t) & \overline{W}(t) \\ W(t) & \mathbb{1} \end{pmatrix}.$$

In a physically realistic case the Hamiltonian $H(t)$ is positive. Mathematically it is often convenient to assume that it is invertible (has a mass gap).

Assume now for a moment that the problem is stationary, so that L, V, B, H do not depend on time t. Clearly,

$$U(t,t') = \mathrm{e}^{-\mathrm{i}(t-t')B}.$$

The quadratic form H defines the so-called energy scalar product. It is easy to see that B can be interpreted as a self-adjoint operator with a gap in its spectrum around 0. Let $\Pi^{(\pm)}$ be the projections onto the positive/negative part of the spectrum of B.

We define the positive and negative frequency bisolutions and the Feynman and anti-Feynman inverse on the level of $\partial_t + \mathrm{i}B(t)$:

$$E^{(\pm)}(t,t') := \pm \mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(\pm)},$$

$$E^{\mathrm{F}}(t,t') := \theta(t-t')\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(+)} - \theta(t'-t)\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(-)},$$

$$E^{\overline{\mathrm{F}}}(t,t') := \theta(t-t')\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(-)} - \theta(t'-t)\,\mathrm{e}^{-\mathrm{i}(t-t')B}\Pi^{(+)}.$$

They lead to corresponding propagators on the level of K:

$$G^{(\pm)} := E_{12}^{(\pm)},$$
$$G^{\mathrm{F}} := -\mathrm{i}E_{12}^{\mathrm{F}},$$
$$G^{\overline{\mathrm{F}}} := -\mathrm{i}E_{12}^{\overline{\mathrm{F}}}.$$

They satisfy the relations

$$G^{\mathrm{F}} - G^{\overline{\mathrm{F}}} = \mathrm{i}G^{(+)} + \mathrm{i}G^{(-)},$$
$$G^{\mathrm{F}} + G^{\overline{\mathrm{F}}} = G^{\vee} + G^{\wedge}.$$

In the static case in QFT there is a distinguished state given by the vacuum Ω. As on the Minkowski space, we have the relations

$$\begin{aligned} -\mathrm{i}G^{\mathrm{PJ}}(x,y) &= [\hat{\phi}(x), \hat{\phi}(y)], \\ G^{(+)}(x,y) &= \big(\Omega|\hat{\phi}(x)\hat{\phi}(y)\Omega\big), \\ -\mathrm{i}G^{\mathrm{F}}(x,y) &= \big(\Omega|\mathrm{T}\big(\hat{\phi}(x)\hat{\phi}(y)\big)\Omega\big). \end{aligned}$$

Can one generalize non-classical propagators to non-static spacetimes? We claim that the answer is yes if the spacetime is asymptotically stationary and the Hamiltonians in the far future and past are positive.

We also need to make some mild technical asumptions, which allow us to use the setting of Hilbertizable spaces (In the stationary case there was a natural scale of Hilbert spaces, which is not available when the generator is time-dependent. Instead, we assume that the evolution preserves a class of equivalent scalar products—a Hilbertizable structure).

Let $B(-\infty)$, resp. $B(+\infty)$ be the generator of the dynamics in the far past and future. We define the incoming positive/negative frequency bisolution $E_-^{(\pm)}$, resp. the outgoing positive/negative frequency bisolution $E_+^{(\pm)}$ by transporting the projections

$$\mathbb{1}_{[0,\infty[}\big(\pm B(-\infty)\big) = \Pi_-^{(\pm)},$$
$$\mathbb{1}_{[0,\infty[}\big(\pm B(+\infty)\big) = \Pi_+^{(\pm)}.$$

with help of the evolution.

We obtain bisolutions $G_-^{(\pm)}$ and $G_+^{(\pm)}$ with a clear physical meaning.

$G_-^{(+)}$ defines the incoming vacuum state in the distant past given by a vector Ω_-. It corresponds to a preparation of an experiment.

$G_+^{(+)}$ corresponds to the outgoing vacuum state in the remote future given by a vector Ω_+. This vector is related to the future measurements.

The projection $\Pi_-^{(+)}$ can be transported by the dynamics to any time t, obtaining the projection $\Pi_-^{(+)}(t)$. Similarly we obtain the projection $\Pi_+^{(-)}(t)$. Using the fact that the dynamics is pseudounitary, one can show that for for all t the subspaces

$$\mathrm{Ran}\Pi_-^{(+)}(t), \quad \mathrm{Ran}\Pi_+^{(-)}(t)$$

are complementary.

Define $\Pi_{+-}^{(+)}(t)$, $\Pi_{+-}^{(-)}(t)$ to be the unique pair of projections corresponding to the pair of spaces

$$\mathrm{Ran}\Pi_{-}^{(+)}(t), \quad \mathrm{Ran}\Pi_{+}^{(-)}(t)$$

The (in-out) Feynman propagator is defined as

$$\begin{aligned} E^{\mathrm{F}}(t_2, t_1) &:= \theta(t_2 - t_1)U(t_2, t_1)\Pi_{+-}^{(+)}(t_1) \\ &\quad -\theta(t_1 - t_2)U(t_2, t_1)\Pi_{+-}^{(-)}(t_1), \\ G^{\mathrm{F}} &:= -\mathrm{i}E_{12}^{\mathrm{F}}. \end{aligned}$$

In a somewhat different setting, the construction of G^{F} was given by A.Vasy et al and by Gerard-Wrochna. But it seems that the naturalness and simplicity of the above construction was realized only recently.

Here is the physical meaning of the Feynman propagator: it is the expectation value of the time-ordered product of fields between the in-vacuum and the out-vacuum:

$$G^{\mathrm{F}}(x,y) = \frac{\big(\Omega_+|\mathrm{T}\big(\hat{\phi}(x)\hat{\phi}(y)\big)\Omega_-\big)}{\big(\Omega_+|\Omega_-\big)}.$$

It is easy to see that on a general spacetime the Klein-Gordon operator K is Hermitian (symmetric) on $C_{\mathrm{c}}^{\infty}(M)$ in the sense of the Hilbert space $L^2(M)$. It seems natural to ask whether it is essentially self-adjoint on, say $C_{\mathrm{c}}^{\infty}(M)$. This question turns out to be a surprisingly difficult.

Theorem. [D., Siemssen] Assume the spacetime is stationary.
(1) K is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(M)$.
(2) For $s > \frac{1}{2}$, the operator G^{F} is bounded from the space $\langle t\rangle^{-s}L^2(M)$ to $\langle t\rangle^{s}L^2(M)$. Besides, in the sense of these spaces,

$$\mathrm{s}-\lim_{\epsilon\searrow 0}(K-\mathrm{i}\epsilon)^{-1}=G^{\mathrm{F}}.$$

Conjecture. On a large class of asymptotically stationary spacetimes
(1) the Klein-Gordon operator K is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(M)$,
(2) in the sense $\langle t\rangle^{-s}L^2(M)\to\langle t\rangle^{s}L^2(M)$,

$$\mathrm{s-}\lim_{\epsilon\searrow 0}(K-\mathrm{i}\epsilon)^{-1}=G^{\mathrm{F}}.$$

In a recent paper of A. Vasy this conjecture is proven for asymptotically Minkowskian spaces. It is true if the spatial dimension is zero (when the Klein-Gordon operator reduces to the 1-dimensional Schrödinger operator). It is also true on a large class of cosmological spacetimes. Presumably, one can prove it on symmetric spacetimes.

Surprisingly, we have not found a trace of this question in the older mathematical literature. Many respected mathematicians and mathematical physicists react with disgust to this question, claiming that it is completely non-physical.

However, in the physical literature there are many papers that take the self-adjointness of the Klein-Gordon operator for granted. The method of computing the Feynman propagator with external fields and possibly on curved spacetimes based on the identity

$$\frac{1}{(K - \mathrm{i}0)} = \mathrm{i}\int_0^\infty \mathrm{e}^{-\mathrm{i}tK}\mathrm{d}t \qquad (*)$$

has even a name: the Fock–Schwinger or Schwinger–DeWitt method. Of course, without the self-adjointness of K, $(*)$ does not make sense.

ASYMPTOTICS OF PROPAGATORS AROUND THE DIAGONAL

In the Lorentzian case, even if we can interpret K as self-adjoint, the heat semigroup does not exists and instead one should consider the so-called proper time dynamics $W(\mathrm{i}t) = \mathrm{e}^{-\mathrm{i}tK}$.

One can apply the balanced geometric pseudodifferential calculus to find the asymptotics of $W(\mathrm{i}t)$ around the diagonal:

$$W(\mathrm{i}t, x, y) \sim \mathcal{W}(\mathrm{i}t, x, y) := \frac{\Delta^{\frac{1}{2}}(x, y)|g(x)|^{\frac{1}{4}}|g(y)|^{\frac{1}{4}}}{|g(z)|^{\frac{1}{2}}(4\pi\mathrm{i}t)^{\frac{d}{2}}}$$

$$\times \exp\Big(-\frac{1}{4\mathrm{i}t} u g^{-1}(z) u - \mathrm{i}t Y(z)\Big) \sum_{k=0}^{\infty} (\mathrm{i}t)^k u^\beta \mathcal{W}_{k,\beta}(z) \mathrm{e}^{-\mathrm{i}uA(z)}.$$

One can obtain the Feynman and the anti-Feynman propagator by integration:

$$G_F := (K - \mathrm{i}0)^{-1} = \mathrm{i}\int_0^\infty W(\mathrm{i}t)\mathrm{d}t,$$
$$G_{\overline{F}} := (K + \mathrm{i}0)^{-1} = -\mathrm{i}\int_0^\infty W(-\mathrm{i}t)\mathrm{d}t.$$

Here is the asymptotics of the Feynman and anti-Feynman propagator:

$$G^{\mathrm{F}/\overline{\mathrm{F}}}(x,y) \sim \mathcal{G}^{\mathrm{F}/\overline{\mathrm{F}}}(x,y)$$
$$:= \frac{\Delta^{\frac{1}{2}}(x,y)|g(x)|^{\frac{1}{4}}|g(y)|^{\frac{1}{4}}}{(4\pi)^{\frac{d}{2}}} \sum_{k=0}^{\infty} u^{\beta}\mathcal{W}_{k,\beta}(z)\mathrm{e}^{-\mathrm{i}uA(z)}$$
$$\times \pm \mathrm{i}2K_{k+1-\frac{d}{2}}\Big(\sqrt{ug^{-1}(z)uY(z)\pm \mathrm{i}0}\Big)\left(\frac{ug^{-1}(z)u\pm \mathrm{i}0}{4Y(z)}\right)^{\frac{k+1-\frac{d}{2}}{2}}.$$

Note that for spacelike u we can drop $\pm\mathrm{i}0$.

For timelike u they are obtained by an appropriate analytic continuation:
Instead of the MacDonald function

$$\pm 2\mathrm{i}K_{k+1-\frac{d}{2}}\Big(\sqrt{ug^{-1}(z)uY(z)\pm \mathrm{i}0}\Big),$$

we need to put the Hankel functions of the first and second kind

$$-\pi H^{\pm}_{k+1-\frac{d}{2}}\Big(\sqrt{-ug^{-1}(z)uY(z)\mp \mathrm{i}0}\Big).$$

Note that $\frac{1}{2}(\mathcal{G}_{\mathrm{F}} + \mathcal{G}_{\overline{F}})$ vanishes for spacelike separated points. The same property is shared by $\frac{1}{2}(G^{\vee} + G^{\wedge})$. Indeed, on the level of full asymptotic expansions we have

$$\mathcal{G}_{\mathrm{F}} + \mathcal{G}_{\overline{\mathrm{F}}} \sim \mathcal{G}^{\vee} + \mathcal{G}^{\wedge}.$$

Let us stress that this does not imply

$$G_{\mathrm{F}} + G_{\overline{\mathrm{F}}} = G^{\vee} + G^{\wedge},$$

except for some special spacetimes.

By multiplying $\mathcal{G}_{\mathrm{F}} + \mathcal{G}_{\overline{\mathrm{F}}}$ with $\theta(\pm x^0)$ we can compute the asymptotics of the retarded and advanced propagators:

$$\mathcal{G}^{\vee/\wedge}(x,y) := \frac{\Delta^{\frac{1}{2}}(x,y)|g(x)|^{\frac{1}{4}}|g(y)|^{\frac{1}{4}}}{(4\pi)^{\frac{d}{2}}} \sum_{k=0}^{\infty} u^{\beta} \mathcal{W}_{k,\beta}(z) \mathrm{e}^{-\mathrm{i}uA(z)}$$

$$\times \pi J_{k+1-\frac{d}{2}}\left(\sqrt{ug^{-1}(z)uY(z)}\right)\left(\frac{ug^{-1}(z)u}{4Y(z)}\right)^{\frac{k+1-\frac{d}{2}}{2}},$$

$$u^2 < 0, \quad \pm u^0 > 0,$$

where J_m are the Bessel functions.