1-DIMENSIONAL SCHRÖDINGER OPERATORS WITH COMPLEX POTENTIALS

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JAN DEREZIŃSKI AND VLADIMIR GEORGESCU

ABSTRACT. We discuss realizations of $L := -\partial_x^2 + V(x)$ as closed operators on $L^2]a,b[$, where V is complex, locally integrable and may have an arbitrary behavior at (finite or infinite) endpoints a and b. The main tool of our analysis are Green's operators, that is, various right inverses of L.

CONTENTS

1. Introduction

The paper is devoted to operators of the form

$$L = -\partial_x^2 + V(x) \tag{1.1}$$

on $]a, b[$, where $a < b$, a can be $-\infty$ and b can be ∞. The potential V can be complex, have low regularity, and a rather arbitrary behavior at the boundary of the domain: we assume that $V \in L^1_{\mathrm{loc}}]a, b[$. We study realizations of L as closed operators on $L^2]a, b[$.

Operators of the form (1.1) are commonly called *1-dimensional Schrödinger operators*. The name *Sturm-Liouville operators* is also used, and is probably historically better justified.

Sturm-Liouville operators is a classic subject with a lot of literature. Most of the literature is devoted to the real case, when L can be realized as self-adjoint operator. It is, however, quite striking that the usual theory well-known from the real (self-adjoint) case works almost equally well in the complex case. In particular, essentially the same theory for boundary conditions and the same formulas for *Green's operators* (right inverses of (1.1)) hold as in the real case. We will describe these topics in detail in this paper.

A large part of the literature on Sturm-Liouville operators assumes that potentials are L^1 near finite endpoints. Under this condition one can impose the so called *regular boundary conditions* (Dirichlet, Neumann or Robin). In this case, it is natural to use the so-called *Weyl-Titchmarsh function* and the formalism of the so-called so called *boundary triplets*, see e.g. [2] and references therein. We are interested in general boundary conditions, such as those considered in [4, 9, 10], where the above approach does not directly apply. See the discussion at the end of Subsect. 4.2.

One of the motivations of the present work is the study of exactly solvable Schrödinger operators, such as those given by the Bessel equation [4, 9], or the Whittaker equation [10]. Analysis of those operators indicates that non-real potentials are as good from the point of view of the exact solvability as real ones. It is also natural to organize exactly solvable Schrödinger operators in holomorphic families, whose elements are self-adjoint only in exceptional cases. Therefore, a theory for Sturm-Liouville operators with complex potentials and general boundary conditions provides a natural framework for the study of exactly solvable Hamiltonians.

As we mentioned above, we suppose that $V \in L^1_{\mathrm{loc}}]a, b[$. The theory is much easier if $V \in L^2_{\mathrm{loc}}]a, b[$, because one could then assume that the operator acts on $C^2_{\mathrm{c}}]a, b[$. Dealing with potentials in L^1_{loc} causes of a lot of trouble—this is however a rather natural assumption. We think that handling a more general case forces us to better understand the problem. Actually, one could consider even more singular potentials: it is easy to generalize our results to potentials V being a Borel measures on $]a, b[$.

In the first, preliminary section, we study the inhomogeneous problem given by the operator (1.1) by basic ODE methods. We introduce some distinguished Green's operators: The *two-sided Green's operators* are related to boundary conditions on both sides. The *forward* and *backward Green's operators* are related to the Cauchy problem at the endpoints of the interval. These operators belong to the most often used objects in mathematics. Usually they appear under the guise of *Green's functions*, which are the integral kernels of Green's operators.

The remaining sections are devoted to realizations of L as closed operators on the Hilbert space $L^2]a, b[$. The most obvious realizations are the *minimal* one $L_{\min}$ and the *maximal* one $L_{\max}$. We prove that these operators are closed and densely defined. Under the assumption $V \in L^1_{\mathrm{loc}}]a, b[$ the proof is qute long and technical but, in our opinion, instructive. If we assumed $V \in L^2_{\mathrm{loc}}]a, b[$, the proof would be easy.

At this point it is helpful to recall basic theory of Sturm-Liouville operators for real potentials. One is usually interested in self-adjoint extensions of the Hermitian operator $L_{\min}$. They are are situated "half-way" between $L_{\min}$ and $L_{\max}$. More precisely, we have 3 possibilities:

(1) $L_{\min} = L_{\max}$: then $L_{\min}$ is already self-adjoint.
(2) The codimension of $\mathcal{D}(L_{\max})$ in $\mathcal{D}(L_{\min})$ is 2: if $L_\bullet$ is a self-adjoint extension of $L_{\min}$, the inclusions $\mathcal{D}(L_{\min}) \subset \mathcal{D}(L_\bullet) \subset \mathcal{D}(L_{\max})$ are of codimension 1.

(3) The codimension of $\mathcal{D}(L_{\max})$ in $\mathcal{D}(L_{\min})$ is 4: if $L_\bullet$ is a self-adjoint extension of $L_{\min}$, the inclusions $\mathcal{D}(L_{\min}) \subset \mathcal{D}(L_\bullet) \subset \mathcal{D}(L_{\max})$ are of codimension 2.

Note that in the literature it is common to use the theory of *deficiency indices.* The cases (1), (2), resp. (3) correspond to $L_{\min}$ having the deficiency indices $(0,0)$, $(1,1)$ and $(2,2)$. However, the deficiency indices do not have a straightforward generalization to the complex case.

Let us go back to complex potentials. Note that the Hermitian conjugation of an operator A, denoted A^*, is no longer very useful. Instead, one often uses the *transposition* $A^\# := \overline{A^*}$, where the bar denote the complex conjugation. In particular, the role of self-adjoint operators is taken up by self-transposed operators, satisfying $A^\# = A$.

By choosing a subspace of $\mathcal{D}(L_{\max})$ closed in the graph topology and restricting $L_{\max}$ to this subspace we can define a closed operator. Such operators will be called *closed realizations of* L. We will show that in the complex case closed realizations of L possess a theory quite analogous to that of the real case.

We are mostly interested in realizations of L whose domain contains $\mathcal{D}(L_{\min})$. Such realizations are defined by specifying boundary conditions. Similarly as in the real case, boundary conditions are given by functionals on $\mathcal{D}(L_{\max})$ that vanish on $\mathcal{D}(L_{\min})$. For each of endpoints, a and b, there is a space of functionals describing boundary conditions. We call the dimension of this space the *boundary index at* a, resp. b, and denote it $\nu_a(L)$, resp. $\nu_b(L)$. They can take the values 0 or 2 only. Therefore, we have the following classification of operators L:

$$\begin{aligned} &(1) \dim \mathcal{D}(L_{\max})/\mathcal{D}(L_{\min}) = 0, \text{ or } L_{\min} = L_{\max} \\ &(2) \dim \mathcal{D}(L_{\max})/\mathcal{D}(L_{\min}) = 2, \\ &(3) \dim \mathcal{D}(L_{\max})/\mathcal{D}(L_{\min}) = 4. \end{aligned} \tag{1.2}$$

Let $\lambda \in \mathbb{C}$. It is natural to consider the space of solutions of $(L-\lambda)u = 0$ that are square integrable near a, resp. b. We denote these spaces by $\mathcal{U}_a(\lambda)$, resp. $\mathcal{U}_a(\lambda)$. In the real case we have a relationship:

$$\nu_a(L) = 2 \Leftrightarrow \dim \mathcal{U}_a(\lambda) = 2 \ \forall \lambda \in \mathbb{C}. \tag{1.3}$$

In the complex case we can show $\Leftarrow$ in (1.3). We conjecture that also $\Rightarrow$ holds (Conjecture 5.9).

The most useful realizations of L are those possessing non-empty resolvent set. Not all L possess such realizations. One can classify such L's as follows. If L possesses a realization $L_\bullet$ with a non-empty resolvent set, then one of the following conditions holds:

$$\begin{aligned} &(1) \dim \mathcal{D}(L_{\max})/\mathcal{D}(L_\bullet) = 0, \text{ or } L_{\max} = L_\bullet \\ &(2) \dim \mathcal{D}(L_{\max})/\mathcal{D}(L_\bullet) = 1, \\ &(3) \dim \mathcal{D}(L_{\max})/\mathcal{D}(L_\bullet) = 2. \end{aligned} \tag{1.4}$$

In the real case there is a strict correspondence between (1), (2) and (3) of (1.2) and (1), (2) and (3) of (1.4). In the complex case this correspondence holds if Conjecture 5.9 is true. Without Conjecture 5.9 we only have the relations

$$\begin{aligned} (1.2)(1) &\quad \Leftrightarrow \quad (1.4)(1), \\ (1.2)(2) &\quad \Rightarrow \quad (1.4)(2), \\ (1.2)(3) &\quad \Rightarrow \quad (1.4)(2) \text{ or } (3), \end{aligned} \tag{1.5}$$

In cases (1) and (2) from Table (1.4) we describe all realizations with nonempty resolvent set and their resolvents. We prove that if $L_\bullet$ is such a realization, then we can find $u \in \mathcal{U}_a(\lambda)$ and $v \in \mathcal{U}_b(\lambda)$ with the Wronskian equal to 1, so that the integral kernel of $(L_\bullet - \lambda)^{-1}$ can then be easily expressed in terms of u and v.

The case (3) is much richer. We describe all realizations of L that are separated (given by independent boundary conditions at a and b). If in addition they are self-transposed, then essentialy the same formula as in (1) and (2) gives $(L_\bullet - \lambda)^{-1}$. There are however two other separated realizations of L, which are denoted L_a and L_b, with boundary conditions only at a, resp. b. They are not self-transposed, in fact, they satisfy $L_a^\# = L_b$. Their resolvents are given by what we call *forward* and *backward Green's operators*, which incidentally are cousins of the *retarded* and *advanced Green's functions*, well-known from the theory of the wave equation.

In the last section we discuss potentials with a negative imaginary part. We show that under some weak conditions they define dissipative Sturm-Liouville operators. We also describe Weyl's limit point–limit cricle method for such potentials. For real potentials, this method allows us to determine the dimension of $\mathcal{U}_a(\lambda)$ for $\mathrm{Im}(\lambda) > 0$: if a is limit point, then $\dim\mathcal{U}_a(\lambda) = 1$; if a is limit circle then $\dim\mathcal{U}_a(\lambda) = 2$. The picture is more complicated if the potential is complex: there are examples where the endpoint a is limit point and $\mathcal{U}_a(\lambda)$ is two-dimensional.

Sturm-Liouville operators is one of the most classic topics in mathematics. Already in the first half of 19 century Sturm and Liouville considered second order differential operators on a finite interval with various boundary consitions. The theory was extended to a half-line and a line in a celebrated work by Weyl.

2nd order ODE's and Sturm-Liouville operators are considered in many textbooks, including Coddington-Levinson [5], Dunford-Schwartz [13, 14], Naimark [20], Pryce [21], de Alfaro-Regge [7], Reed-Simon [22], Stone [24], Titchmarsh [26], Teschl [25]. However, in the literature complex potentials are rarely studied in detail, and if so, then one does not pay attention to nontrivial boundary conditions.

The present manuscript grew out of the Appendix of [4] devoted to Sturm-Liouville operators with the potential $\frac{1}{x^2}$. [4] and its follow-up papers [9, 10] illustrated that Sturm-Liouville operators with complex potentials and unusual boundary conditions appear naturally in various situations.

We decided to make the exposition as complete and self-contained as possible, explaining things that are perhaps obvious to experts, but often difficult to many readers. We use freely the modern operator theory—this is not the case of a large part of literature, which often sticks to old-fashioned terminology.

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2. Basic ODE theory

2.1. **Notations.** Recall that $a < b$, a can be $-\infty$ and b can be ∞. The notation $[a, b]$ stands for the interval including the enpoints a and b, while $]a, b[$ for the interval without endpoints. $[a, b[$ and $]a, b]$ have the obvious meaning.

In some cases one could use the notation involving either $[a, b]$ or $]a, b[$ without a change of the meaning. For instance, $L^p[a, b] = L^p]a, b[$. For esthetic reasons, we try to use a uniform notation—we always write $L^p]a, b[$.

In other cases, the choice of either $[a, b]$ or $]a, b[$ influences the meaning of a symbol. For instance, $C[a, b] \subsetneq C]a, b[$. For example, $f \in C[-\infty, b]$ implies that $\lim_{x\to-\infty} f(x) =: f(-\infty)$ exists.

$f \in L^p_{\mathrm{loc}}]a, b[$ iff for any $a < a_1 < b_1 < b$, we have $f\Big|_{[a_1,b_1]} \in L^p]a_1, b_1[$.

$f \in L^p_{\mathrm{c}}]a, b[$ iff $f \in L^p]a, b[$ and $\mathrm{supp} f$ is a compact subset of $]a, b[$. The analogous meaning has the subscript c in different situations.

2.2. **Absolutely continuous functions.** We will denote by $AC]a, b[$ the space of *absolutely continuous functions* on $]a, b[$, that is, distributions on $]a, b[$ whose derivative is in $L^1_{\mathrm{loc}}]a, b[$. We will denote f' or ∂f the derivative of a distribution f. We have $AC]a, b[\subset C]a, b[$. If $f, g \in AC]a, b[$, then $fg \in AC]a, b[$ and the Leibniz rule holds:

$$(fg)' = f'g + fg'. \tag{2.1}$$

$AC^n]a, b[$ will denote the space of distributions whose nth derivative is in $AC]a, b[$.

Lemma 2.1. *Let $f_n \in AC]a, b[$ be a sequence such that for any $a < a_1 < b_1 < b$, $f_n \to f$ uniformly on $[a_1, b_1]$ and $f_n' \to g$ in $L^1[a_1, b_1]$. Then $f \in AC]a, b[$ and $g = f'$.*

We will denote by $AC[a,b]$ the space of functions on $[a,b]$ whose (distributional) derivative is in $L^1]a,b[$. Clearly, $AC[a,b] \subset C[a,b]$. If $f \in AC[a,b]$, then

$$\int_a^b f'(x)\mathrm{d}x = f(b) - f(a). \tag{2.2}$$

Note that a can be $-\infty$ and b can be ∞.

Obviously, if $f \in AC]a,b[$ and $a < a_1 < b_1 < b$ then $f\Big|_{[a_1,b_1]}$ belongs to $AC[a_1,b_1]$.

2.3. **Choice of funtional-analytic setting.** Throughout the section, we assume that $V \in L^1_{\mathrm{loc}}]a,b[$ and we consider the differential expression

$$L := -\partial^2 + V. \tag{2.3}$$

Sometimes we restrict our operator to a smaller interval, say $]c,d[$, where $a \leq c < d \leq b$. Then (2.3) restricted to $]c,d[$ is denoted $L^{c,d}$.

Eventually, we would like to study operators in $L^2]a,b[$ associated to L, which in many respects seems the most natural setting for Sturm-Liouville operators. There are however situations, where it is preferable to use a different functional-analytic formalism for (2.3).

Suppose that we choose $L^1_{\mathrm{loc}}]a,b[$ as the target space for (2.3), which seems to be a rather general function space. Note that if $f \in L^\infty_{\mathrm{loc}}]a,b[$, the product fV is well defined and belongs to $L^1_{\mathrm{loc}}]a,b[$. Moreover, this is the best we can do if V is an arbitrary locally integrable function, i.e. we cannot replace L^∞_{loc} by a larger space. Then, if we consider L^∞_{loc} as the initial space for (2.3) and we require that the target space for (2.3) is $L^1_{\mathrm{loc}}]a,b[$, we are forced to work with functions $f \in L^\infty_{\mathrm{loc}}]a,b[$ such that $Lf \in L^1_{\mathrm{loc}}]a,b[$. But then $f'' \in L^1_{\mathrm{loc}}]a,b[$, and hence $f \in AC^1]a,b[$.

Therefore it is natural to consider (2.3) as an an operator $L : AC^1]a,b[\to L^1_{\mathrm{loc}}]a,b[$, which we will do throughout this paper. Restrictions of L to subspaces of $AC^1]a,b[$ which are sent into $L^2]a,b[$ by L are the objects of main interest in our study.

We equip $L^1_{\mathrm{loc}}]a,b[$ with the topology of local uniform convergene, i.e. a sequence $\{f_n\}$ converges to f if and only if $\lim_{n\to\infty} \|f_n - f\|_{L^1(J)} = 0$ for any compact $J \subset]a,b[$. Clearly this is a complete space. It is convenient to think of L as an operator in $L^1_{\mathrm{loc}}]a,b[$ with domain $AC^1]a,b[$. Then L is densely defined and later on we will prove that it is is closed (see Corollary 2.16).

2.4. **The Cauchy problem.** For $g \in L^1_{\mathrm{loc}}]a,b[$ we consider the problem

$$Lf = g. \tag{2.4}$$

Proposition 2.2. *Let $a < d < b$. Then for any p_0, p_1 there exists a unique $f \in AC^1]a,b[$ satisfying (2.4) and*

$$f(d) = p_0, \quad f'(d) = p_1. \tag{2.5}$$

Proof. Define the operators Q_d and T_d by their integral kernels

$$Q_d(x,y) := \begin{cases} (y-x)V(y), & x < y < d, \\ (x-y)V(y), & x > y > d, \\ 0 & \text{otherwise}; \end{cases}$$

$$T_d(x,y) := \begin{cases} (y-x), & x < y < d, \\ (x-y), & x > y > d, \\ 0 & \text{otherwise}. \end{cases}$$

The Cauchy problem can be rewritten as $F(f) = f$ where F is a map on $C]a,b[$ given by

$$F(f)(x) := p_0 + p_1(x-d) + Q_d f(x) + T_d g(x). \tag{2.6}$$

If $a \leq a_1 < d < b_1 \leq b$ and we view Q_d as an operator on $C[a_1,b_1]$ with the supremum norm, then

$$\|Q_d\| \leq \max\Big\{ \int_{a_1}^{d} |V(y)(y-a_1)|\mathrm{d}y,\ \int_d^{b_1} |V(y)(y-b_1)|\mathrm{d}y \Big\}. \tag{2.7}$$

If the interval $[a_1, b_1]$ is finite, the operator T_d is bounded from $L^1]a_1, b_1[$ into $C[a_1, b_1]$.

Thus, by choosing a sufficiently small interval $[a_1, b_1]$ containing d, we can make F well defined and contractive on $C[a_1, b_1]$. (F is contractive iff $\|Q_d\| < 1$). By Banach's Fixed Point Theorem (or the convergence of an appropriate Neumann series) there exists $f \in C[a_1, b_1]$ such that $f = F(f)$. Then note that we have

$$f'(x) = F(f)'(x) = p_1 + \int_d^x V(y)f(y)\mathrm{d}y + \int_d^x g(y)\mathrm{d}y$$

hence $f' \in AC[a_1, b_1]$ and $f \in AC^1[a_1, b_1]$.

Thus for every $d \in]a, b[$ we can find an open interval containing d on which there exists a unique solution to the Cauchy problem. We can cover $]a, b[$ with intervals $]a_j, b_j[$ containing d_j with the analogous property. This allows us to extend the solution with initial conditions at any $d \in]a, b[$ to the whole $]a, b[$. □

2.5. **Regular and semiregular endpoints.** Sturm-Liouville operators possess the simplest theory when $-\infty < a < b < \infty$ and $V \in L^1]a, b[$. Then we say that L *is a regular operator.* Most of the classical Sturm-Liouville theory is devoted to such operators. More generally, the following standard terminology will be convenient.

Definition 2.3. *The end point a is called* regular*, or L is called* regular at a*, if a is finite and $\int_a^d |V(x)|\mathrm{d}x < \infty$ if $a < d < b$. Similarly for b. Hence L is* regular *if both endpoints are regular.*

Sturm-Liouville operators satisfying the following conditions are also relatively well behaved:

Definition 2.4. *The end point a is called* semiregular *if a is finite and $\int_a^d (x-a)|V(x)|\mathrm{d}x < \infty$ if $a < d < b$. Similarly for b.*

Proposition 2.5. *Let $g \in L^1]a, d[$*

(1) *Let a be a regular endpoint. Let p_0, p_1 be given. Then there exists a unique $f \in AC^1[a, b[$ satisfying (2.4) and*
$$f(a) = p_0, \quad f'(a) = p_1. \tag{2.8}$$
(2) *Let a be a semiregular endpoint. Then all solutions f of (2.4) have a limit at a.*

Proof. (1) is proven as Prop. 2.2, choosing $d = a$.

To prove (2) we put d inside $]a, b[$, demanding in addition that $\int_a^d V(y)|y - a|\mathrm{d}y < 1$. This guarantees that the operator Q_d is contractive on $C[a, d]$. □

An example of a potential with a finite point which is not semiregular is $V(x) = \frac{c}{x^2}$ on $]0, \infty[$. For its theory see [4, 9].

2.6. **Wronskian.**

Definition 2.6. *The* Wronskian *of two derivable functions u, v is $W(u, v) = uv' - u'v$. We set*
$$W(u, v; x) = W_x(u, v) = u(x)v'(x) - u'(x)v(x). \tag{2.9}$$

Proposition 2.7. *Let $u, v \in AC^1]a, b[$. Then the* Lagrange identity *holds:*
$$\partial_x W(u, v; x) = -(Lu)(x)v(x) + u(x)(Lv)(x). \tag{2.10}$$
Consequently, if $Lu = Lv = 0$, then $W(u, v)$ is a constant function.

Proof. Since $u, u', v, v' \in AC]a, b[$, the Wronskian can be differentiated and a simple computation yields (2.10). □

The set of solutions in $AC^1]a, b[$ of the homogeneous equation $Lf = 0$ is a two dimensional complex space $\operatorname{Ker} L$ and the map $W : \operatorname{Ker} L \times \operatorname{Ker} L \to \mathbb{C}$ is bilinear and antisymmetric. Two solutions $u, v \in \operatorname{Ker} L$ are linearly independent if and only if $W(u, v) \neq 0$. If $u_2 = \alpha u_1 + \beta v_1, v_2 = \gamma u_1 + \delta v_1$ then

$$W(u_2, v_2) = W(\alpha u_1, \delta v_1) + W(\beta v_1, \gamma u_1) = (\alpha\delta - \beta\gamma)W(u_1, v_1).$$

hence if $W(u_1, v_1) = 1$ then $W(u_2, v_2) = 1$ if and only if $\alpha\delta - \beta\gamma = 1$, and in this case a simple computation gives

$$u_2(x)v_2(y) - u_2(y)v_2(x) = u_1(x)v_1(y) - u_1(y)v_1(x), \quad x, y \in]a, b[. \tag{2.11}$$

Thus the function

$$G_{\leftrightarrow}(x, y) = u(x)v(y) - u(y)v(x) \tag{2.12}$$

is independent of the choice of the solutions u, v of the homogeneous equation $Lf = 0$ if they satisfy $W(u, v) = 1$. (2.12) can be interpreted as the integral kernel of an operator $G_{\leftrightarrow} : L^1_{\mathrm{c}}]a, b[\to AC^1]a, b[$, and will be called the *canonical bisolution of* L. It satisfies

$$LG_{\leftrightarrow} = 0, \quad G_{\leftrightarrow}L = 0, \quad G_{\leftrightarrow}(x, y) = -G_{\leftrightarrow}(y, x). \tag{2.13}$$

2.7. **Green's operators.** The expression "Green's function" is commonly used to denote the integral kernel of a right inverse of a differential operator, usually of a second order. We will use the expression "Green's operator" for a right inverse of L.

Definition 2.8. *An operator* $G_{\bullet} : L^1_{\mathrm{c}}]a, b[\to AC^1]a, b[$ *is called a* Green's operator of L *if*

$$LG_{\bullet}g = g, \quad g \in L^1_{\mathrm{c}}]a, b[. \tag{2.14}$$

Note that we do not require that $G_{\bullet}L = \mathbb{1}$. Note also that $G_{\leftrightarrow}$ is not Green's operator—it is a bisolution. However, it is so closely related to various Green's operators that we use the same letter G to denote it.

There are many Green's operators. If $G_{\bullet}$ is a Green's operator, u, v are two solutions of the homogeneous equation, and $\phi, \psi \in L^{\infty}_{\mathrm{loc}}]a, b[$ are arbitrary, then

$$G_{\bullet} + |u\rangle\langle\phi| + |v\rangle\langle\psi|$$

is also a Green's operator. Recall that if E, F are vector spaces, g belongs to the dual of E, and $f \in F$, then $|f\rangle\langle g|$ is the linear map $E \to F$ defined by $e \mapsto g(e)f$.

Let us define some distinguished Green's operators. Let u, v be two solutions of the homogeneous equation such that

$$W(v, u) = 1.$$

We easily check that the operators $G_{u,v}$, G_a and G_b defined below are Green's operators in the sense of Def. 2.8:

Definition 2.9. Green's operator associated to u at a and v at b, *denoted* $G_{u,v}$, *is defined by its integral kernel*

$$G_{u,v}(x, y) := \begin{cases} u(x)v(y), & x < y, \\ v(x)u(y), & x > y. \end{cases}$$

Operators of the form $G_{u,v}$ will be sometimes called *two-sided Green's operators.*

Definition 2.10. Forward Green's operator $G_{\rightarrow}$ *has the integral kernel*

$$G_{\rightarrow}(x, y) := \begin{cases} 0, & x < y, \\ v(x)u(y) - u(x)v(y), & x > y. \end{cases} \tag{2.15}$$

Definition 2.11. Backward Green's operator $G_{\leftarrow}$ *has the integral kernel*

$$G_{\leftarrow}(x, y) := \begin{cases} u(x)v(y) - v(x)u(y), & x < y, \\ 0, & x > y. \end{cases}$$

By the comment after (2.11), the operators $G_{\rightarrow}$ and $G_{\leftarrow}$ are independent of the choice of u, v. For $a < b_1 < b$, $G_{\rightarrow}$ maps $L^1_{\mathrm{c}}]b_1, b[$ into functions that are zero on $]a, b_1]$. Similarly, for $a < a_1 < b$, $G_{\rightarrow}$ maps $L^1_{\mathrm{c}}]a, a_1[$ into functions that are zero on $]a_1, b]$.

Note also some formulas for differences of two kinds of Green's operators:

$$G_{u,v} - G_{\rightarrow} = |u\rangle\langle v|, \tag{2.16}$$

$$G_{u,v} - G_{\leftarrow} = |v\rangle\langle u|, \tag{2.17}$$

$$G_{\rightarrow} - G_{\leftarrow} = |v\rangle\langle u| - |u\rangle\langle v| = G_{\leftrightarrow}, \tag{2.18}$$

$$G_{u,v} - G_{u_1,v_1} = |u\rangle\langle v| - |u_1\rangle\langle v_1|, \tag{2.19}$$

The following definition introduces another class of Green's operators in the sense of Def. 2.8, which are generalizations of forward and backward Green's operators.

Definition 2.12. Green's operator associated to $d \in]a, b[$ *is defined by the integral kernel*

$$G_d(x,y) := \begin{cases} u(x)v(y) - v(x)u(y), & x < y < d, \\ v(x)u(y) - u(x)v(y), & x > y > d, \\ 0, & \textit{otherwise}. \end{cases}$$

As in the case of $G_{\rightarrow}$ and $G_{\leftarrow}$, these operators are independent of the choice of u, v. Note that if $a < a_1 < d < b_1 < b$, then G_d maps $L^1_{\mathrm{c}}]a, a_1[$ on functions that are zero on $[a_1, b[$, and $L^1_{\mathrm{c}}]b_1, b[$ on functions that are zero on $]a, b_1]$.

The proof of Prop. 2.2 suggests how to construct G_d without knowing v, u using the operators Q_d and T_d defined there. We have, at least formally,

$$G_d = (1\!\!1 - Q_d)^{-1}T_d. \tag{2.20}$$

If we choose $a \leq a_1 < d < b_1 \leq b$ with a finite $[a, b]$ and

$$\max\left\{ \int_{a_1}^{d} |V(x)(x - a_1)|\mathrm{d}x, \int_{d}^{b_1} |V(x)(x - b_1)|\mathrm{d}x \right\} < 1, \tag{2.21}$$

then (2.20) is given by a convergent Neumann series in the sense of an operator from $L^1]a_1, b_1[$ to $C[a_1, b_1]$.

Remark 2.13. *The 1-dimensional Schrödinger equation can be interpreted as the Klein Gordon equation on a* 1+0 *dimensional spacetime (no spacial dimensions, only time). The operators* $G_{\leftrightarrow}$, $G_{\rightarrow}$ *and* $G_{\leftarrow}$ *have important generalizations to globally hyperbolic spacetimes of any dimension—they are then usually called the Pauli-Jordan, retarded, resp. advanced propagator, see e.g.* [11].

2.8. **Some estimates.**

The following elementary estimates will be useful later on.

Lemma 2.14. *Let* J *be an open interval of length* $\nu < \infty$ *and* $f \in L^1(J)$ *with* $f'' \in L^1(J)$*. Then* f *and* f' *are continuous functions on the closure of* J *and if* a *is an end point of* J *then*

$$|f(a)| + \nu|f'(a)| \leq C \int_J \left(\nu|f''(x)| + \nu^{-1}|f(x)|\right) \mathrm{d}x, \tag{2.22}$$

where C *is a real number independent of* f *and* J.

Proof. By a scaling argument we may assume $\nu = 1$. It suffices to assume that f is a distribution on $\mathbb{R}$ such that $f'' = 0$ outside J. Let $\theta : \mathbb{R} \to \mathbb{R}$ be of class C^∞ outside of 0 and such that $\theta(x) = 0$ if $x \leq 0$, $\theta(x) = x$ if $0 < x < 1/2$, $\theta(x) = 0$ if $x \geq 1$. Define η by $\theta'' = \delta - \eta$ where δ is the Dirac measure at the origin. Clearly η is of class C^∞ with support in $[1/2, 1]$. For any distribution f we have

$$f = \delta * f = \theta'' * f + \eta * f = \theta * f'' + \eta * f \quad \text{hence also} \quad f' = \theta' * f'' + \eta' * f.$$

This clearly implies (2.22) for $\nu = 1$ and a the right end point of J. □

Lemma 2.15. *Assume that* $\ell := \sup_J \int_J |V(x)|\mathrm{d}x < \infty$ *where* J *runs over all the intervals* $J \subset]a, b[$ *of length* ≤ 1*. Then there are numbers* $C, \nu_0 > 0$ *such that*

$$\|f\|_{L^\infty(J)} + \nu\|f'\|_{L^\infty(J)} \leq C\nu\|Lf\|_{L^1(J)} + C\nu^{-1}\|f\|_{L^1(J)} \tag{2.23}$$

for all $f \in L^\infty_{\mathrm{loc}}]a, b[$*, all* $\nu \leq \nu_0$*, and all intervals* $J \subset]a, b[$ *of length* ν*.*

Proof. Note that for a continuous f we have $f'' \in L^1_{\rm loc}$ if and only $Lf \in L^1_{\rm loc}$ and then f' is absolutely continuous. We take $\nu_0 \leq 1$ and strictly less than half the length of $]a,b[$. If $\nu \leq \nu_0$, then (2.22) gives for x such that $]x, x+\nu[\subset]a,b[$:

$$\begin{aligned} |f(x)| + \nu|f'(x)| &\leq C\int_x^{x+\nu} \big(\nu|Lf| + \nu|Vf| + \nu^{-1}|f(y)|\big)\,\mathrm{d}y \\ &\leq C\|\nu|Lf| + \nu^{-1}|f|\|_{L^1(x,x+\nu)} + C\ell\nu\|f\|_{L^\infty(x,x+\nu)} \\ &\leq C\nu\|Lf\|_{L^1(J)} + C\nu^{-1}\|f\|_{L^1(J)} + C\ell\nu\|f\|_{L^\infty(J)}. \end{aligned}$$

If $x \in]a,b[$ and $]x, x+\nu[\not\subset]a,b[$ then $]x-\nu, x[\subset]a,b[$ and we have an estimate as above with $]x, x+\nu[$ replaced by $]x-\nu, x[$. Hence

$$\|f\|_{L^\infty(J)} + \nu\|f'\|_{L^\infty(J)} \leq C\nu\|Lf\|_{L^1(J)} + C\nu^{-1}\|f\|_{L^1(J)} + C\ell\nu\|f\|_{L^\infty(J)}.$$

If ν_0 is such that $C\ell\nu_0 < 1$ we get the required estimate. □

Recall (see page 6) that $L^1_{\rm loc}]a,b[$ is equipped with the topology of local uniform convergence and that we think of L as an operator in $L^1_{\rm loc}]a,b[$ with domain $AC^1]a,b[$. The next result says that this operator is closed.

Corollary 2.16. *Let $\{f_n\}$ be a sequence in $AC^1]a,b[$ such that the sequences $\{f_n\}$ and $\{Lf_n\}$ are Cauchy in $L^1_{\rm loc}]a,b[$. Then the limits $f := \lim\limits_{n\to\infty} f_n$ and $g := \lim\limits_{n\to\infty} Lf_n$ exist in $L^1_{\rm loc}]a,b[$ and we have $f \in AC^1]a,b[$ and $Lf = g$.*

Proof. The estimate (2.23) implies that on every compact interval J we have uniform convergence of f_n to f (and also of f_n' to f'). Therefore, $Vf_n\big|_J \to Vf\big|_J$ in $L^1(J)$ for any such J. Hence, $-f_n'' = Lf_n - Vf_n$ converges in $L^1(J)$ to $g - Vf$. Therefore, by Lemma 2.1, $-f'' = g - Vf$. We know that $g - Vf \in L^1_{\rm loc}]a,b[$, hence $f \in AC^1]a,b[$. □

3. Basic L^2 theory

3.1. **Bilinear scalar product.**

We consider the Hilbert space $L^2]a,b[$ with the scalar product

$$(f|g) := \int_a^b \overline{f(x)}g(x)\mathrm{d}x. \tag{3.1}$$

In addition, it is also equipped with the bilinear form

$$\langle f|g\rangle := \int_a^b f(x)g(x)\mathrm{d}x. \tag{3.2}$$

Thus we use round brackets for the sesquilinear scalar product and angular brackets for the closely related bilinear form. Note that in some sense the latter plays a more important role in our paper (and in similar exactly solvable problems) than the former. See e.g. [8, 10], where the same notation is used.

$L^2]a,b[$ has also a natural complex conjugation. Clearly, $\langle f|g\rangle = (\overline{f}|g)$. If B is an operator, then $\overline{B}$ denotes the *complex conjugation of* B

$$\overline{B}f := \overline{B\overline{f}}. \tag{3.3}$$

B^* denotes the usual Hermitian adjoint of B, whereas $B^\# = \overline{B^*}$ denotes the *transpose* of B, that is, its adjoint w.r.t. the (3.2).

Clearly, if B is a bounded linear operator with

$$\big(Bf\big)(x) := \int_a^b B(x,y)f(y)\mathrm{d}y,$$

then

$$\big(B^*f\big)(x) = \int_a^b \overline{B(y,x)}f(y)\mathrm{d}y, \tag{3.4}$$

$$\big(B^\# f\big)(x) = \int_a^b B(y,x)f(y)\mathrm{d}y, \tag{3.5}$$

$$\big(\overline{B}f\big)(x) = \int_a^b \overline{B(x,y)}f(y)\mathrm{d}y. \tag{3.6}$$

An operator B is *self-adjoint* if $B = B^*$. We will say that it is *self-transposed* if $B^\# = B$. It is useful to note that a holomorphic function of a self-transposed operator is self-transposed.

If $\mathcal{G} \subset L^2]a,b[$, we will write

$$\mathcal{G}^\perp := \{f \in L^2]a,b[\mid (f|g) = 0,\ g \in \mathcal{G}\}, \tag{3.7}$$

$$\mathcal{G}^{\mathrm{perp}} := \{f \in L^2]a,b[\mid \langle f|g\rangle = 0,\ g \in \mathcal{G}\} = \overline{\mathcal{G}^\perp}. \tag{3.8}$$

3.2. **The maximal and minimal operator.** As before, we assume that $V \in L^1_{\mathrm{loc}}]a,b[$.

Definition 3.1. *The* maximal operator $L_{\max}$ *is defined by*

$$\mathcal{D}(L_{\max}) := \big\{f \in L^2]a,b[\cap AC^1]a,b[\mid Lf \in L^2]a,b[\big\}, \tag{3.9}$$

$$L_{\max}f := Lf, \quad f \in \mathcal{D}(L_{\max}). \tag{3.10}$$

We equip $\mathcal{D}(L_{\max})$ *with the graph norm*

$$\|f\|_L^2 := \|f\|^2 + \|Lf\|^2.$$

Remark 3.2. *Note that* $L^2]a,b[\subset L^1_{\mathrm{loc}}]a,b[$. *Therefore, as explained in Subsect. 2.3,* $f \in L^\infty_{\mathrm{loc}}]a,b[$ *and* $Lf \in L^2]a,b[$ *implies* $f \in AC^1]a,b[$. *Therefore, in (3.9) we can replace* $AC^1]a,b[$ *with* $L^\infty_{\mathrm{loc}}]a,b[$ *(or* $C]a,b[$*, or* $C^1]a,b[$*).*

Recall that $AC^1_{\mathrm{c}}]a,b[$ are once absolutely differentiable functions of compact support.

Definition 3.3. *We set*

$$\mathcal{D}(L_{\mathrm{c}}) := AC^1_{\mathrm{c}}]a,b[\cap \mathcal{D}(L_{\max}).$$

Let L_{c} *be the restriction of* $L_{\max}$ *to* $\mathcal{D}(L_{\mathrm{c}})$*. Finally,* $L_{\min}$ *is defined as the closure of* L_{c}*.*

The next theorem is the main result of this subsection:

Theorem 3.4. *The operators* $L_{\min}, L_{\max}$ *have the following properties.*

(1) The operators $L_{\max}$ *and* $L_{\min}$ *are closed, densely defined and* $L_{\min} \subset L_{\max}$*.*

(2) $L^\#_{\max} = L_{\min}$ *and* $L^\#_{\min} = L_{\max}$*.*

(3) Suppose that $f_1, f_2 \in \mathcal{D}(L_{\max})$*. Then there exist*

$$W(f_1,f_2;a) := \lim_{d \searrow a} W(f_1,f_2;d), \tag{3.11}$$

$$W(f_1,f_2;b) := \lim_{d \nearrow b} W(f_1,f_2;d), \tag{3.12}$$

and the so-called Green's identity *(the integrated form of the Lagrange identity) holds:*

$$\langle L_{\max}f_1|f_2\rangle - \langle f_1|L_{\max}f_2\rangle = W(f_1,f_2;b) - W(f_1,f_2;a). \tag{3.13}$$

(4) We set $W_d(f_1,f_2) = W(f_1,f_2;d)$ *for any* $d \in [a,b]$ *and* $f_1, f_2 \in \mathcal{D}(L_{\max})$*. Then for any* $d \in [a,b]$ *the map* $W_d : \mathcal{D}(L_{\max}) \times \mathcal{D}(L_{\max}) \to \mathbb{C}$ *is a continuous bilinear antisymmetric form, in particular*

$$|W_d(f_1,f_2)| \le C_d\|f_1\|_L\|f_2\|_L. \tag{3.14}$$

(5) $\mathcal{D}(L_{\min})$ *coincides with*

$$\big\{f \in \mathcal{D}(L_{\max}) \ \mid\ W(f,g;a) = 0 \text{ and } W(f,g;b) = 0 \text{ for all } g \in \mathcal{D}(L_{\max})\big\}. \tag{3.15}$$

One of the things we will need to prove is the density of $\mathcal{D}(L_{\mathrm{c}})$ in $L^2]a,b[$. This is easy if $V \in L^2_{\mathrm{loc}}]a,b[$ (see Prop. 3.11), but with our assumptions on the potential the proof is not so trivial, because the idea of approximating an $f \in L^2(I)$ with smooth functions does not work: $\mathcal{D}(L_{\max})$ may not contain any "nice" function, as the example described below shows.

Example 3.5. Let $V(x) = \sum_\sigma c_\sigma |\mathrm{x}-\sigma|^{-1/2}$ where σ runs over the set of rational numbers and $c_\sigma \in \mathbb{R}$ satisfy $c_\sigma > 0$ and $\sum_\sigma c_\sigma < \infty$. Then $V \in L^1_{\mathrm{loc}}(\mathbb{R})$ but V is not square integrable on any nonempty open set. Hence there is no C^2 nonzero function in the domain of L in $L^2(\mathbb{R})$.

Before proving Thm 3.4, we first state an immediate consequence of Lemma 2.15:

Lemma 3.6. (1) *Let J be a finite interval whose closure is contained in $]a,b[$. Then*

$$\left\| f\big|_J \right\| \le C_J \|f\|_L, \tag{3.16}$$

$$\left\| f'\big|_J \right\| \le C_J \|f\|_L. \tag{3.17}$$

(2) *Let $\chi \in C^\infty]a,b[$ with $\chi' \in C^\infty_{\mathrm{c}}]a,b[$. Then*

$$\|\chi f\|_L \le C_\chi \|f\|_L. \tag{3.18}$$

As in the previous section, we fix $u, v \in AC^1]a,b[$ that span $\operatorname{Ker} L$ and satisfy $W(v,u) = 1$.

Our proof of Thm 3.4 uses ideas from [24, Theorem 10.11] and [20, Sect. 17.4] and is based on an abstract result described in Lemma A.1. The following lemma contains the key arguments of the proof of (1) and (2) of Thm 3.4:

Lemma 3.7. *If L is a regular operator (cf. Definition 2.3) then*

(1) $\operatorname{Ker} L_{\max} = \operatorname{Ker} L$.
(2) $\operatorname{Ran} L_{\max} = L^2]a,b[$.
(3) $\langle L_{\mathrm{c}} f | g\rangle = \langle f | L_{\max} g\rangle$, $f \in \mathcal{D}(L_{\mathrm{c}})$, $g \in \mathcal{D}(L_{\max})$.
(4) $\operatorname{Ran} L_{\mathrm{c}} = L^2_{\mathrm{c}}]a,b[\cap (\operatorname{Ker} L)^{\mathrm{perp}}$.
(5) $(\operatorname{Ran} L_{\mathrm{c}})^{\mathrm{perp}} = \operatorname{Ker} L$.
(6) $\mathcal{D}(L_{\mathrm{c}})$ *is dense in* $L^2]a,b[$.

Proof. Clearly, $\operatorname{Ker} L = \operatorname{Span}(u,v) \subset AC^1[a,b] \subset L^2]a,b[$. Therefore, $\operatorname{Ker} L \subset \mathcal{D}(L_{\max})$. This proves (1).

Recall that in (2.15) we defined the forward Green's operator $G_\to$. Under the assumptions of the present lemma, it maps $L^2]a,b[$ into $AC^1[a,b]$. Therefore, for any $g \in L^2]a,b[$, $\alpha, \beta \in \mathbb{C}$,

$$f = \alpha u + \beta v + G_\to g$$

belongs to $AC^1[a,b]$ and verifies $Lf = g$. Therefore, $f \in \mathcal{D}(L_{\max})$. Hence $L_{\max}$ is surjective. This proves (2).

To obtain (3) we integrate twice by parts. This is allowed by (2.1) and (2.2), since $f, g \in AC^1[a,b]$.

It is obvious that $\operatorname{Ran} L_{\mathrm{c}} \subset L^2_{\mathrm{c}}]a,b[$. $\operatorname{Ran} L_{\mathrm{c}} \subset (\operatorname{Ker} L_{\max})^{\mathrm{perp}}$ follows from (3).

Let us prove the converse inclusions. Let $g \in L^2_{\mathrm{c}}]a,b[\cap (\operatorname{Ker} L)^{\mathrm{perp}}$. Set $f := G_\to g$. Clearly, $Lf = g$. Using $\int_a^b gu = \int_a^b gv = 0$ we see that f has compact support. Hence $f \in \mathcal{D}(L_{\mathrm{c}})$. This proves (4).

$L^2_{\mathrm{c}}]a,b[$ is dense in $L^2]a,b[$ and $(\operatorname{Ker} L)^{\mathrm{perp}}$ has a finite codimension. Therefore, by Lemma A.2, $L^2_{\mathrm{c}}]a,b[\cap (\operatorname{Ker} L)^{\mathrm{perp}}$ is dense in $\operatorname{Ker} L$. This implies (5).

By applying Lemma A.1 with $T := \overline{L}_{\max}$ and $S := L_{\mathrm{c}}$, we obtain (6). □

Proof of Thm 3.4. It follows from Lemma 3.7 (6) that $\mathcal{D}(L_{\mathrm{c}})$ is dense in $L^2]a,b[$. We have

$$L_{\mathrm{c}}^\# \supset L_{\max} \tag{3.19}$$

by integration by parts, as in the proof of (3), Lemma 3.7.

Suppose that $h, k \in L^2]a,b[$ such that

$$\langle L_{\mathrm{c}} f | h\rangle = \langle f | k\rangle, \quad f \in \mathcal{D}(L_{\mathrm{c}}). \tag{3.20}$$

In other words, $h \in \mathcal{D}(L_{\mathrm{c}})$ and $L_{\mathrm{c}}^{\#}h = k$. Choose $d \in]a,b[$. We set $h_d := G_d k$, where G_d is defined in Def. 2.12. Clearly, $Lh_d = k$. For $f \in \mathcal{D}(L_{\mathrm{c}})$, set $g := L_{\mathrm{c}}f$. We can assume that $\mathrm{supp} f \subset [a_1, b_1]$ for $a < a_1 < b_1 < b$. Now

$$\langle g|h_d\rangle = \langle L_{\mathrm{c}}f|h_d\rangle = \langle f|Lh_d\rangle = \langle f|k\rangle = \langle L_{\mathrm{c}}f|h\rangle = \langle g|h\rangle.$$

By Lemma 3.7 (4) applied to $[a_1.b_1]$,

$$h = h_d + \alpha u + \beta v \tag{3.21}$$

on $[a_1, b_1]$. But since a_1, b_1 were arbitrary under the condition $a < a_1 < b_1 < b$, (3.21) holds on $]a,b[$. Hence $Lh = k$. Therefore, $h \in \mathcal{D}(L_{\max})$ and $L_{\max}h = k$. This proves that

$$L_{\mathrm{c}}^{\#} \subset L_{\max}. \tag{3.22}$$

From (3.19) and (3.22) we see that $L_{\mathrm{c}}^{\#} = L_{\max}$. In particular, $L_{\max}$ is closed and L_{c} is closable. We have

$$L_{\min} = L_{\mathrm{c}}^{\#\#} = L_{\max}^{\#}. \tag{3.23}$$

This ends the proof of (1) and (2).

For $f, g \in \mathcal{D}(L_{\max})$ and $a < a_1 < b_1 < b$ we have

$$\begin{aligned}\int_{a_1}^{b_1} (Lf(x)g(x) - f(x)Lg(x))\mathrm{d}x &= \int_{a_1}^{b_1} (f(x)g'(x) - f'(x)g(x))'\mathrm{d}x \\ &= W(f,g;a_1) - W(f,g;b_1).\end{aligned} \tag{3.24}$$

The lhs of (3.24) clearly converges as $a_1 \searrow a$. Therefore, the limit (3.11) exists. Similarly, by taking $b_1 \nearrow b$ we show that the limit (3.12) exists. Taking both limits we obtain (3.13). This proves (3).

If $d \in]a,b[$, then (3.14) is an immediate consequence of (3.16) and (3.17). We can rewrite (3.24) as

$$W(f,g;a) = -\int_a^d \big((Lf)(x)g(x) - f(x)Lg(x)\big)\mathrm{d}x + W(f,g;d). \tag{3.25}$$

Now both terms on the right of (3.25) can be estimated by $C\|f\|_L \|g\|_L$. This shows (3.14) for $d = a$. The proof for $d = b$ is analogous.

Let L_w be L restricted to (3.15). By (3.14), (3.15) is a closed subspace of $\mathcal{D}(L_{\max})$. Hence, L_w is closed. Obviously, $L_{\mathrm{c}} \subset L_w$. By (3.13), $L_w \subset L_{\max}^{\#}$. By (2), we know that $L_{\max}^{\#} = L_{\min}$. But $L_{\min}$ is the closure of L_{c}. Hence $L_w = L_{\min}$. This proves (5). □

Remark 3.8. Here is an alternative, more direct proof of the closedness of $L_{\max}$. Let $f_n \in \mathcal{D}(L_{\max})$ be a Cauchy sequence wrt the graph norm. This means that f_n and Lf_n are Cauchy sequences wrt $L^2]a,b[$. Let $f := \lim\limits_{n\to\infty} f_n$, $g := \lim\limits_{n\to\infty} Lf_n$. Let J be an arbitrary sufficiently small closed interval in $]a,b[$. We have

$$\|f_n - f_m\|_{L^1(J)} \le \sqrt{|J|}\|f_n - f_m\|_{L^2(J)}, \tag{3.26}$$

$$\|Lf_n - Lf_m\|_{L^1(J)} \le \sqrt{|J|}\|Lf_n - Lf_m\|_{L^2(J)}. \tag{3.27}$$

Hence f_n satisfies the conditions of Cor. 2.16. Hence $f \in AC^1]a,b[$ and $g = Lf$. Hence $f \in \mathcal{D}(L_{\max})$ and it is the limit of f_n in the sense of the graph norm. Therefore, $\mathcal{D}(L_{\max})$ is complete. Hence $L_{\max}$ and $L_{\min}$ are closed.

3.3. **Smooth functions in the domain of $L_{\max}$.** We point out a certain pathology of the operators $L_{\max}$ and $L_{\min}$ if V is only locally integrable.

Lemma 3.9. *(1) The imaginary part of V is locally square integrable if and only if $\mathcal{D}(L_{\mathrm{c}})$ is stable under conjugation and in this case $\mathcal{D}(L_{\min})$ and $\mathcal{D}(L_{\max})$ are also stable under conjugation.*

(2) *If the imaginary part of V is not square integrable on any open set, then for $f \in \mathcal{D}(L_{\max})$ we have $\overline{f} \in \mathcal{D}(L_{\max})$ only if $f = 0$; in particular, $\mathcal{D}(L_{\max})$ does not contain any nonzero real function.*

Proof. (1): Write $Lf = -f'' + V_1 f + iV_2 f$ if $V = V_1 + iV_2$ with V_1, V_2 real. Then if $V_2 \in L^2_{\mathrm{loc}}]a, b[$ and $f \in AC^1_{\mathrm{c}}]a, b[$ we have $V_2 f \in L^2]a, b[$ so $-f''+Vf \in L^2]a, b[$ if and only if $-f''+V_1 f \in L^2]a, b[$ hence $-\overline{f}'' + V_1\overline{f} \in L^2]a, b[$ so we get $-\overline{f}'' + V\overline{f} \in L^2]a, b[$, thus $\mathcal{D}(L_{\mathrm{c}})$ is stable under conjugation. The corresponding assertion concerning $\mathcal{D}(L_{\min})$ follows by taking the completion, and that concerning $\mathcal{D}(L_{\max})$ follows by taking the transposition.

Reciprocally, assume that $\mathcal{D}(L_{\mathrm{c}})$ is stable under conjugation and let $x_0 \in]a, b[$. Then there is $f \in \mathcal{D}(L_{\mathrm{c}})$ such that $f(x_0) \neq 0$ and we may assume that its real part $g = (f + \overline{f})/2$ does not vanish on a neighbourhood of x_0. Then $g \in \mathcal{D}(L_{\mathrm{c}})$ hence $-g'' + V_1 g + iV_2 g \in L^2]a, b[$ and so must be the imaginary part of this function hence V_2 is square integrable on a neighbourhood of x_0.

(2): Assume now that V_2 is not square integrable on any open set. If $f \in AC^1$ is real then $-f'' + Vf \in L^2$ if and only if $-f'' + V_1 f \in L^2$ and $V_2 f \in L^2$ and if $f \neq 0$ then the second condition implies $f = 0$. Finally, if $f \in \mathcal{D}(L_{\max})$ and $\overline{f} \in \mathcal{D}(L_{\max})$ then the functions $f + \overline{f}$ and $f - \overline{f}$ will be zero by (1). $\square$

Remark 3.10. If $\overline{L} := -\partial^2 + \overline{V}$ then $L^*_{\min} = \overline{L}_{\max}$, where $L^*_{\min}$ is the Hermitian adjoint of $L_{\min}$, and clearly $\mathcal{D}(\overline{L}_{\max}) = \{\overline{f} \mid f \in \mathcal{D}(L_{\max})\}$. Thus if the imaginary part of V is not square integrable on any open set then $\mathcal{D}(L_{\min}) \cap \mathcal{D}(L^*_{\min}) = \{0\}$. On the other hand, if the imaginary part of V is locally square integrable, then $\mathcal{D}(L_{\min}) \subset \mathcal{D}(L^*_{\min})$.

If $V \in L^2_{\mathrm{loc}}$, many things simplify:

Proposition 3.11. *If $V \in L^2_{\mathrm{loc}}]a, b[$ then $C^\infty_{\mathrm{c}}]a, b[$ is a dense subspace of $\mathcal{D}(L_{\min})$.*

Proof. Clearly $C^\infty_{\mathrm{c}}]a, b[\subset \mathcal{D}(L_{\mathrm{c}})$. Let $f \in C_{\mathrm{c}}]a, b[$. Then $Lf \in L^2]a, b[$ if and only if $f'' \in L^2]a, b[$. Fix some $\theta \in C^\infty_{\mathrm{c}}(\mathbb{R})$ with $\int \theta = 1$ and let $\theta_n(x) := n\theta(nx)$ with $n \geq 1$. Then for n large $f_n := \theta_n * f \in C^\infty_{\mathrm{c}}]a, b[$ and has support in a fixed small neighbourhood of supp$f$. Moreover, $f_n \to f$ in $C^1_{\mathrm{c}}]a, b[$, in particular $f_n \to f$ uniformly with supports in a fixed compact, which clearly implies $Vf_n \to Vf$ in $L^2]a, b[$. Moreover $f''_n \to f''$ in $L^2]a, b[$. $\square$

3.4. **Closed operators contained in $L_{\max}$.** If $\mathcal{D}(L_\bullet)$ is a subspace of $\mathcal{D}(L_{\max})$ closed in the $\|\cdot\|_L$ norm, then the operator

$$L_\bullet := L_{\max}\Big|_{\mathcal{D}(L_\bullet)} \tag{3.28}$$

is closed and contained in $L_{\max}$. We can call such an operator $L_\bullet$ a *closed realization of L.*

We will be mostly interested in operators $L_\bullet$ that satisfy

$$\mathcal{D}(L_{\min}) \subset \mathcal{D}(L_\bullet) \subset \mathcal{D}(L_{\max}) \tag{3.29}$$

so that

$$L_{\min} \subset L_\bullet \subset L_{\max}. \tag{3.30}$$

They are automatically densely defined.

One can easily check if a realization of L contains $L_{\min}$ with help of the following criterion:

Proposition 3.12. *Suppose that $L_\bullet$ is a closed densely defined operator contained in $L_{\max}$. Then $L^\#_\bullet$ is contained in $L_{\max}$ if and only if $L_{\min} \subset L_\bullet$. In particular, if $L^\#_\bullet = L_\bullet$, then $L_{\min} \subset L_\bullet$.*

Proof. Since $L_\bullet$ is densely defined, the operator $L^\#_\bullet$ is well defined and clearly $L_{\min} \subset L_\bullet$ is equivalent to $L^\#_\bullet \subset L^\#_{\min} = L_{\max}$. $\square$

The most obvious examples of such operators are given by one-sided boundary conditions:

Definition 3.13. *Set*

$$\mathcal{D}(L_a) := \{f \in \mathcal{D}(L_{\max}) \mid W(f, g; a) = 0 \text{ for all } g \in \mathcal{D}(L_{\max})\}, \tag{3.31}$$

$$\mathcal{D}(L_b) := \{f \in \mathcal{D}(L_{\max}) \mid W(f, g; b) = 0 \text{ for all } g \in \mathcal{D}(L_{\max})\}. \tag{3.32}$$

Let L_a, resp. L_b be $L_{\max}$ restricted to $\mathcal{D}(L_a)$, resp. $\mathcal{D}(L_b)$.

Proposition 3.14. *L_a and L_b are closed and densely defined operators satisfying*

$$L_a^{\#} = L_b, \quad L_b^{\#} = L_a, \tag{3.33}$$

$$L_{\min} \subset L_a \subset L_{\max}, \quad L_{\min} \subset L_b \subset L_{\max}. \tag{3.34}$$

4. Boundary conditions

4.1. **Regular endpoints.**

Proposition 4.1. *If L is regular at a then any function $f \in \mathcal{D}(L_{\max})$ extends to a function of class C^1 on the left closed interval $[a, b[$, hence $f(a)$ and $f'(a)$ are well defined, and for $f, g \in \mathcal{D}(L_{\max})$ we have $W_a(f, g) = f(a)g'(a) - f'(a)g(a)$. Similarly if L is regular at b. Thus if L is regular then $\mathcal{D}(L_{\max}) \subset C^1[a, b]$ and Green's identity (3.13) has the classical form*

$$\langle L_{\max} f_1 | f_2 \rangle - \langle f_1 | L_{\max} f_2 \rangle = \big(f_1(b)f_2'(b) - f_1'(b)f_2(b)\big) - \big(f_1(a)f_2'(a) - f_1'(a)f_2(a)\big).$$

Thus if L is a regular operator then we have four continuous linear functionals on $f \in \mathcal{D}(L_{\max})$

$$f \mapsto f(a), \quad f \mapsto f'(a), \tag{4.1}$$

$$f \mapsto f(b), \quad f \mapsto f'(b), \tag{4.2}$$

which give a convenient description of the closed operators $L_\bullet$ such that $L_{\min} \subset L_\bullet \subset L_{\max}$. In particular, $\mathcal{D}(L_{\min})$ is the intersection of the kernels of (4.1) and (4.2), $\mathcal{D}(L_a)$ is the intersection of the kernels of (4.1) and $\mathcal{D}(L_b)$ is the intersection of the kernels of (4.2).

4.2. **Boundary functionals.**

It is possible to extend the strategy described above to the case of an arbitrary L by using an abstract version of the notion of boundary value of a function. We shall do it in this section.

The abstract theory of boundary value functionals goes back to J. W. Calkin's thesis [6] who used it for the classification of the self-adjoint extensions of symmetric operators. The theory was adapted to symmetric differential operators of any order by Naimark [20] and to operators with complex coefficients of class C^∞ by Dunford and Schwarz in [13, ch. XIII]. In this section we shall use this technique in the case of second order operators with potentials which are only locally integrable: this loss of regularity is a problem for some arguments in [13].

Recall that $\mathcal{D}(L_{\max})$ is equipped with the Hilbert space structure associated to the norm $\|f\|_L = \sqrt{\|f\|^2 + \|Lf\|^2}$. Following [13, §XXX.2], we introduce the following notions.

Definition 4.2. *A* boundary functional for L *is any linear continuous form on $\mathcal{D}(L_{\max})$ which vanishes on $\mathcal{D}(L_{\min})$. A* boundary functional at a *is a boundary functional ϕ such that $\phi(f) = 0$ for all $f \in \mathcal{D}(L_{\max})$ with $f(x) = 0$ near a;* boundary functionals at b *are defined similarly. $\mathcal{B}(L)$ is the set of boundary functionals for L and $\mathcal{B}_a(L), \mathcal{B}_b(L)$ the subsets of boundary functionals at a and b.*

$\mathcal{B}(L)$ is a closed linear subspace of the topological dual $\mathcal{D}(L_{\max})'$ of $\mathcal{D}(L_{\max})$ and $\mathcal{B}_a(L), \mathcal{B}_b(L)$ are closed linear subspaces of $\mathcal{B}(L)$. By using a partition of unity on $]a, b[$ it is easy to prove that

$$\mathcal{B}(L) = \mathcal{B}_a(L) \oplus \mathcal{B}_b(L), \tag{4.3}$$

a topological direct sum.

Definition 4.3. *We define*

$$\begin{aligned} &\textit{the } \text{boundary index for } L \text{ at } a, && \nu_a(L) := \dim \mathcal{B}_a(L), \\ &\textit{the } \text{boundary index for } L \text{ at } b, && \nu_b(L) := \dim \mathcal{B}_b(L), \\ &\textit{and the } \text{total boundary index for } L, && \nu(L) := \dim \mathcal{B}(L) = \nu_a(L) + \nu_b(L). \end{aligned}$$

By definition, the subspace $\mathcal{B}(L) \subset \mathcal{D}(L_{\max})'$ is the polar set of the closed subspace $\mathcal{D}(L_{\min})$ of $\mathcal{D}(L_{\max})$. Hence it is canonically identified with the dual space of $\mathcal{D}(L_{\max})/\mathcal{D}(L_{\min})$:

$$\mathcal{B}(L) = \big(\mathcal{D}(L_{\max})/\mathcal{D}(L_{\min})\big)'. \tag{4.4}$$

Clearly one may also define $\mathcal{B}_a(L)$ as the set of continuous linear forms on $\mathcal{D}(L_{\max})$ which vanish on the closed subspace $\mathcal{D}(L_a)$, and similarly for $\mathcal{B}_b(L)$. Thus

$$\mathcal{B}_a(L) = \big(\mathcal{D}(L_{\max})/\mathcal{D}(L_a)\big)'. \tag{4.5}$$

Definition 4.4. *For each $f \in \mathcal{D}(L_{\max})$ and $x \in [a, b]$, we introduce the functional*

$$\vec{f}_x : \mathcal{D}(L_{\max}) \to \mathbb{C} \quad \textit{defined by} \quad \vec{f}_x(g) = W_x(f, g). \tag{4.6}$$

By Theorem 3.4 it is a well defined linear continuous form on $\mathcal{D}(L_{\max})$.

Remember that if $x \in]a, b[$, then we can write

$$\vec{f}_x(g) = f(x)g'(x) - f'(x)g(x) = W_x(f, g) \quad \forall g \in C^1]a, b[. \tag{4.7}$$

If $x = a$, in general we cannot write (4.7) (unless a is regular). However we know that for all $x \in [a, b]$ (4.6) depends weakly continuously on x. Thus in general

$$\mathrm{w} - \lim_{x \to a} \vec{f}_x = \vec{f}_a. \tag{4.8}$$

It is easy to see that $\vec{f}_a \in \mathcal{B}_a$, cf. (3.31) for example. We shall prove below that any boundary value functional at the endpoint a is of this form.

Theorem 4.5. (i) *$f \mapsto \vec{f}_a$ is a linear surjective map $\mathcal{D}(L_{\max}) \to \mathcal{B}_a(L)$.*
(ii) *$W_a(f, g) = 0$ for all $f, g \in \mathcal{D}(L_{\max})$ if and only if $\mathcal{B}_a(L) = \{0\}$.*
(iii) *$W_a(f, g) \neq 0$ if and only if the functionals $\vec{f}_a, \vec{g}_a$ are linearly independent.*
(iv) *If $W_a(f, g) \neq 0$ then $\{\vec{f}_a, \vec{g}_a\}$ is a basis in $\mathcal{B}_a(L)$; then $\forall h \in \mathcal{D}(L_{\max})$ we have*

$$\vec{h}_a = cW_a(g, h)\vec{f}_a + cW_a(h, f)\vec{g}_a \qquad \textit{with } c = -1/W_a(f, g). \tag{4.9}$$

Proof. Let $\mathcal{W}_a$ be the set of linear forms of the form $\vec{f}_a$, this is a vector subspace of $\mathcal{B}_a(L)$ and we shall prove later that $\mathcal{W}_a = \mathcal{B}_a(L)$. For the moment, note that $\mathcal{W}_a$ separates the points of $\mathcal{Y}_a := \mathcal{D}(L_{\max})/\mathcal{D}(L_a)$, i.e. we have $W_a(f, g) = 0$ for all f if and only if $g \in \mathcal{D}(L_a)$, cf. (3.15) and (3.31). On the other hand, (4.5) implies that $\mathcal{B}_a(L) = \{0\}$ is equivalent to $\mathcal{D}(L_{\max}) = \mathcal{D}(L_a)$ which in turn is equivalent to $W_a(f, g) = 0$ for all $f, g \in \mathcal{D}(L_{\max})$ by (3.31). This proves (ii).

For the rest of the proof we need *Kodaira's identity* [21, pp. 151–152], namely: if f, g, h, k are C^1 functions on $]a, b[$ then

$$W(f, g)W(h, k) + W(g, h)W(f, k) + W(h, f)W(g, k) = 0, \tag{4.10}$$

with the usual definition $W(f, g) = fg' - f'g$. The relation obviously holds pointwise on $]a, b[$. If $f, g, h, k \in \mathcal{D}(L_{\max})$, then the relation extends to $[a, b]$, in particular

$$W_a(f, g)W_a(h, k) + W_a(g, h)W_a(f, k) + W_a(h, f)W_a(g, k) = 0, \tag{4.11}$$

and similarly at b. This implies (4.9) if $W_a(f, g) \neq 0$ from which it follows that $\{\vec{f}_a, \vec{g}_a\}$ is a basis in the vector space $\mathcal{W}_a$, in particular $\mathcal{W}_a$ has dimension 2. But $\mathcal{W}_a \subset \mathcal{Y}_a'$ separates the points of $\mathcal{Y}_a$ hence $\mathcal{W}_a = \mathcal{Y}_a' = \mathcal{B}_a(L)$ which proves the surjectivity of the map $f \mapsto \vec{f}(a)$. This proves (i) and (iv) completely and also one implication in (iii). It remains to prove that $\vec{f}_a, \vec{g}_a$ are linearly dependent if $W_a(f, g) = 0$.

We prove this but with different notations which allow us to use what we have already shown. Let f such that $\vec{f}_a \neq 0$. Then $\vec{f}(a)$ is part of a basis in $\mathcal{W}_a = \mathcal{B}_a(L)$, hence there is g such that $\{\vec{f}_a, \vec{g}_a\}$ is a basis in $\mathcal{B}_a(L)$. Then $W_a(f, g) \neq 0$ and we have (4.9). Thus if $W_a(h, f) = 0$ then $\vec{h}_a = cW_a(g, h)\vec{f}_a$, so $\vec{h}_a, \vec{f}_a$ are linearly dependent. □

The space $\mathcal{B}_a$ is naturally a symplectic space. In fact, if $\mathcal{B}_a$ is nontrivial, then we can find k, h with $W_a(k,h) \neq 0$. By the Kodaira identity,

$$\begin{aligned} W_a(f,g) &= \frac{-W_a(f,k)W_a(g,h) + W_a(f,h)W_a(g,k)}{W_a(h,k)} \\ &= \frac{-\vec{f}_a(k)\vec{g}_a(h) - \vec{f}_a(h)\vec{g}_a(k)}{W_a(h,k)}. \end{aligned} \tag{4.12}$$

Thus if we set for $\phi, \psi \in \mathcal{B}_a$ with $\vec{f}_a = \phi$, $\vec{g}_a = \psi$,

$$\sigma_a(\phi,\psi) := W_a(f,g), \tag{4.13}$$

then σ_a is a well defined symplectic form on $\mathcal{B}_a$. Moreover, $f \mapsto \vec{f}_a$ maps the form W_a onto σ_a. If $\sigma_a(\phi,\psi) = 1$, then by the Kodaira identity

$$W(h,k) = \phi(h)\psi(k) - \psi(h)\phi(k). \tag{4.14}$$

In the literature boundary functionals are usually described using the notion of *boundary triplet*. Let us make a comment on this concept. Suppose, for definiteness, that $\nu_a = \nu_b = 2$. Choose bases

$$\phi_a, \psi_a, \text{ of } \mathcal{B}_a \text{ and } \phi_b, \psi_b \text{ of } \mathcal{B}_b \tag{4.15}$$

such that $\sigma_a(\phi_a,\psi_a) = \sigma_b(\phi_b,\psi_b) = 1$. We have the maps

$$\mathcal{D}(L_{\max}) \ni f \mapsto \phi(f) := \big(\phi_a(f), \phi_b(f)\big) \in \mathbb{C}^2; \tag{4.16}$$

$$\mathcal{D}(L_{\max}) \ni f \mapsto \psi(f) := \big(\psi_a(f), \psi_b(f)\big) \in \mathbb{C}^2. \tag{4.17}$$

Then we can rewrite Green's formula (3.13) as

$$\langle L_{\max} f | g\rangle - \langle f | L_{\max} g\rangle = \langle \psi(f)|\phi(g)\rangle - \langle \phi(f)|\psi(g)\rangle. \tag{4.18}$$

The triplet $(\mathbb{C}^2, \phi, \psi)$ is often called in the literature a *boundary triplet*, see e.g. [2] and references therein. It can be used to characterize operators in between $L_{\min}$ and $L_{\max}$.

Thus a boundary triplet is essentially a choice of a basis (4.15) in the space of boundary functionals. Such a choice is often natural: in particular this is the case of regular boundary conditions, see (4.1), (4.2). In our paper we consider rather general potentials for which there may be no natural choice for (4.15). Therefore, we do not use the boundary triplet formalism.

4.3. **Classification of endpoints and of realizations of L.** The next fact is a consequence of Theorem 4.5. One may think of the assertion "$\nu_a(L)$ can only take the values 0 or 2" as a version of Weyl's dichotomy, cf. §5.2.

Theorem 4.6. *$\nu_a(L)$ can be 0 or 2: we have $\nu_a(L) = 0 \Leftrightarrow W_a = 0$ and $\nu_a(L) = 2 \Leftrightarrow W_a \neq 0$. Similarly for $\nu_b(L)$, hence $\nu(L) \in \{0, 2, 4\}$.*

Remark 4.7. According to the terminology in [13], we might say that L has *no boundary values at a* if $\nu_a(L) = 0$ and that L has *two boundary values at a* if $\nu_a(L) = 2$.

Example 4.8. If L is regular at the endpoint a then $\nu_a(L) = 2$. It is clear that $f \mapsto f(a)$ and $f \mapsto f'(a)$ are linearly independent and Theorem 4.6 implies that they form a basis in $\mathcal{B}_a(L)$.

Example 4.9. If L is semiregular at a then we also have $\nu_a(L) = 2$. However, in general we have only one naturally distinguished boundary functional: $f \mapsto f(a)$.

As a consequence of Theorem 4.6 we get the following classification of Sturm-Liouville operators in terms of the boundary functionals.

(1) $\nu_a(L) = \nu_b(L) = 0$. This is equivalent to $L_{\min} = L_a = L_b = L_{\max}$.
(2) $\nu_a(L) = 0$, $\nu_b(L) = 2$, Then $\mathcal{D}(L_{\min})$ is a subspace of codimension 2 in $\mathcal{D}(L_{\max})$. This is equivalent to $L_a = L_{\max}$, and to $L_{\min} = L_b$.
(3) $\nu_a(L) = 2$, $\nu_b(L) = 0$. Then $\mathcal{D}(L_{\min})$ is a subspace of codimension 2 in $\mathcal{D}(L_{\max})$. This is equivalent to $L_b = L_{\max}$, and to $L_{\min} = L_a$.
(4) $\nu_a(L) = \nu_b(L) = 2$. Then $\mathcal{D}(L_{\min})$ is a subspace of codimension 4 in $\mathcal{D}(L_{\max})$.

In case (2) the operators $L_\bullet$ with $L_{\min} \subsetneq L_\bullet \subsetneq L_{\max}$ are defined by nonzero boundary value functionals ϕ at a: $\mathcal{D}(L_\bullet) = \{f \in \mathcal{D}(L_{\max}) \mid \phi(f) = 0\}$. Similarly in case (3).

Consider now the case (4). The domain of nontrivial realizations $L_\bullet$ could be then of codimension $1, 2$, or 3 in $\mathcal{D}(L_{\max})$. We will see that realizations of codimension 2 are the most important.

Each realization of L extending $L_{\min}$ is defined by a subspace $\mathcal{C}_\bullet \subset \mathcal{B}_a \oplus \mathcal{B}_b$.

$$\mathcal{D}(L_\bullet) := \{f \in \mathcal{D}(L_{\max}) \ \mid \phi(f) = 0, \quad \phi \in \mathcal{C}_\bullet\} \tag{4.19}$$

The space $\mathcal{C}_\bullet$ is called the *space of boundary conditions for* $L_\bullet$. The dimension of $\mathcal{C}_\bullet$ coincides with the codimension of $\mathcal{D}(L_\bullet)$ in $\mathcal{D}(L_{\max})$.

Definition 4.10. *We say that the boundary conditions $\mathcal{C}_\bullet$ are* separated *if*

$$\mathcal{C}_\bullet = \mathcal{C}_\bullet \cap \mathcal{B}_a \oplus \mathcal{C}_\bullet \cap \mathcal{B}_b. \tag{4.20}$$

For instance, L_a and L_b are given by separated boundary conditions $\mathcal{B}_a$, resp. $\mathcal{B}_b$.

Definition 4.11. *Let $\phi \in \mathcal{B}_a$ and $\psi \in \mathcal{B}_b$ such that $\phi \neq 0$ if $\mathcal{B}_a \neq \{0\}$ and similarly for ψ. Then the realization of L with the boundary condition $\mathbb{C}\phi \oplus \mathbb{C}\psi$ will be denoted $L_{\phi\psi}$.*

Clearly, $L_{\phi\psi}$ has separated boundary conditions, more explicitly $L_{\phi\psi}$ is the restriction of $L_{\max}$ to $\mathcal{D}(L_{\phi\psi}) = \{f \in \mathcal{D}(L_{\max}) \mid \phi(f) = \psi(f) = 0\}$. $L_{\phi\psi}$ depends only on the complex lines determined by ϕ and ψ and if $\phi, \psi \neq 0$ then the relations $\phi(f) = \psi(f) = 0$ can be stated as:

$$\text{there are complex numbers } c_a(f), c_b(f) \text{ such that } \quad \vec{f_a} = c_a(f)\phi \text{ and } \vec{f_b} = c_b(f)\psi.$$

If for example $\mathcal{B}_b = \{0\}$ then we set $L_\phi = L_{\phi 0}$ and there is no boundary condition at b, so

$$\mathcal{D}(L_\phi) = \{f \in \mathcal{D}(L_{\max}) \mid \phi(f) = 0\} = \{f \in \mathcal{D}(L_{\max}) \mid \exists c(f) \text{ such that } \vec{f_a} = c(f)\phi\}.$$

4.4. **Properties of boundary functionals.** The next proposition is a version of [13, XIII.2.27] in our context.

Proposition 4.12. *If $\phi \in \mathcal{B}_a(L)$ then there are continuous functions $\alpha, \beta :]a, b[\to \mathbb{C}$ such that*

$$\phi(f) = \lim_{x \to a} \big(\alpha(x)f(x) + \beta(x)f'(x)\big) \quad \forall f \in \mathcal{D}(L_{\max}).$$

Reciprocally, if α, β are complex functions on $]a, b[$ and $\lim_{x\to a} \big(\alpha(x)f(x) + \beta(x)f'(x)\big) =: \phi(f)$ exists $\forall f \in \mathcal{D}(L_{\max})$, then $\phi \in \mathcal{B}_a(L)$.

Proof. The first assertion follows from Theorem 4.5-(i) and relations (4.8), (4.7) while the second one is a consequence of Banach-Steinhaus theorem. □

Recall that for $d \in [a, b]$ the symbol $L^{a,d}$ denotes the operator $-\partial^2 + V$ on the interval $]a, d[$.

Lemma 4.13. *Let $d \in]a, b[$. Then*

$$\dim \mathcal{B}_a(L) = \dim \mathcal{B}_a(L^{a,d}). \tag{4.21}$$

Proof. Since d is a regular endpoint for L^d, the maximal operator $L^{a,d}_{\max}$ associated to $L^{a,d}$ has the property $\mathcal{D}(L^{a,d}_{\max}) \subset C^1]a, d]$. Thus the restriction map $R : f \mapsto f\big|_{]a,d[}$ is a surjective map $\mathcal{D}(L_{\max}) \to \mathcal{D}(L^{a,d}_{\max})$ such that $R\mathcal{D}(L_a) = \mathcal{D}(L^{a,d}_a)$. If ϕ is a boundary value functional at a for $L^{a,d}$ then clearly $\phi \circ R$ is a boundary value functional at a for $L^{a,d}$ and the map $\phi \mapsto \phi \circ R$ is a bijective map $\mathcal{B}_a(L) \to \mathcal{B}_a(L^{a,d})$. □

We note that the space $\mathcal{B}(L)$ and its subspaces $\mathcal{B}_a(L), \mathcal{B}_b(L)$ depend on L only through the domains $\mathcal{D}(L_{\max})$ and $\mathcal{D}(L_{\min})$. So in order to compute them one can sometimes change the potential and consider an operator $L^U := -\partial^2 + U$ instead of $L := -\partial^2 + V$. This is especially useful if U is real: for example, U could be the real part of V, if its imaginary part is bounded.

Proposition 4.14. *Let $U :]a,b[\to \mathbb{C}$ measurable such that $\|(U-V)f\| \le \alpha\|Lf\| + \beta\|f\|$ for some real numbers α, β with $\alpha < 1$ and all $f \in \mathcal{D}(L_{\max})$. Then $\mathcal{D}(L_{\max}) = \mathcal{D}(L^U_{\max})$ and $\mathcal{D}(L_{\min}) = \mathcal{D}(L^U_{\min})$. Hence $\mathcal{B}(L) = \mathcal{B}(L^U)$ and*

$$\nu_a(L) = \nu_a(L^U), \quad \nu_b(L) = \nu_b(L^U). \tag{4.22}$$

Proof. We have

$$(1-\alpha)\|Lf\| - \beta\|f\| \le \||L^U f\| \le (1+\alpha)\|Lf\| + \beta\|f\|$$

so the norms $\|\cdot\|_L$ and $\|\cdot\|_{L^U}$ are equivalent. Then we use (4.4). $\square$

4.5. **Infinite endpoints.** Suppose now that our interval is right-infinite. We will show that if the potential stays bounded in average at infinity, then all elements of the maximal domain converge to zero at ∞ together with their derivative, which obviously implies that their Wronskian converges to zero.

Proposition 4.15. *Suppose that $b = \infty$ and*

$$\limsup_{c\to\infty} \int_c^{c+1} |V(x)|\mathrm{d}x < \infty. \tag{4.23}$$

Then

$$f \in \mathcal{D}(L_{\max}) \quad \Rightarrow \quad \lim_{x\to\infty} f(x) = 0, \ \lim_{x\to\infty} f'(x) = 0. \tag{4.24}$$

Hence $\nu_b = 0$.

Of course, an analogous statement is true for $a = -\infty$ on left-infinite intervals.
Proof of Prop. 4.15. Let $\nu < \nu_0$ and let $J_n := [a+n\nu, a+(n+1)\nu]$. Then, using first (2.23) and then the Schwarz inequality, we obtain

$$\begin{aligned}\|f\|_{L^\infty(J_n)} + \nu\|f'\|_{L^\infty(J_n)} &\le C_1\|Lf\|_{L^1(J_n)} + C_2\|f\|_{L^1(J_n)} \\ &\le C_1\sqrt{\nu}\|Lf\|_{L^2(J_n)} + C_2\sqrt{\nu}\|f\|_{L^2(J_n)} \underset{n\to\infty}{\to} 0.\end{aligned}$$

This implies (4.24). $\square$

5. Solutions square integrable near endpoints

5.1. **Spaces $\mathcal{U}_a(\lambda)$ and $\mathcal{U}_b(\lambda)$.** In the real case one can compute the boundary indices with help of eigenfunctions of the operator L which are square integrable around a given endpoint. The space of such eigenfunctions is interesting in its own right, and we devote this section to the study of its properties.

Definition 5.1. *If $\lambda \in \mathbb{C}$ then $\mathcal{U}_a(\lambda)$ is the set of $f \in AC^1]a,b[$ such that $(L-\lambda)f = 0$ and f is L^2 on $]a,d[$ for some, hence for all d such that $a < d < b$. Similarly we define $\mathcal{U}_b(\lambda)$.*

Proposition 5.2. *If a is a semiregular endpoint for L, then $\dim\mathcal{U}_a(\lambda) = 2$ for all $\lambda \in \mathbb{C}$. Besides, if a is regular, we can choose $u, v \in \mathrm{Ker}(L-\lambda)$ such that*

$$u(a) = 1, \quad u'(a) = 0, \tag{5.1}$$
$$v(a) = 0, \quad v'(a) = 1. \tag{5.2}$$

Similarly for b.

Proof. We apply Prop. 2.5. $\square$

5.2. **Two-dimensional $\mathcal{U}_a(\lambda)$.** The next proposition contains the main technical fact about the dimensions of the $\mathcal{U}_a(\lambda)$.

Proposition 5.3. *Assume that all the solutions of $Lf = 0$ are square integrable near a. If $f \in C^1]a,b[$ and $|Lf| \le B|f|$ for some $B > 0$, then f is square integrable near a. In particular, if $U \in L^\infty]a,b[$ then all the solutions of $(L+U)f = 0$ are square integrable near a.*

Proof. We may clearly assume that b is a regular endpoint and $f \in C^1]a,b]$. Let G_b be the right-sided Green's operator of L (Definition 2.11). If $Lf = g$, then $L(f - G_b g) = 0$. Therefore

$$f(x) = \alpha u(x) + \beta v(x) + \int_x^b \big(u(x)v(y) - v(x)u(y)\big) g(y)\mathrm{d}y, \tag{5.3}$$

for some α, β. Set $A := \sqrt{|\alpha|^2 + |\beta|^2}$ and $\mu(x) := \sqrt{|u(x)|^2 + |v(x)|^2}$. Then

$$|f(x)| \le A\mu(x) + \mu(x)\int_x^b \mu(y)|g(y)|\mathrm{d}y \le \mu(x)\Big(A + B\int_x^b \mu(y)|f(y)|\mathrm{d}y\Big),$$

and the Gronwall Lemma applied to $|f|/\mu$ imples

$$|f(x)| \le A\mu(x)\exp\Big(B\int_x^b \mu^2(y)\mathrm{d}y\Big). \tag{5.4}$$

Clearly the right hand side of (5.4) is square integrable. □

The above proposition has the following important consequence.

Proposition 5.4. *If $\dim\mathcal{U}_a(\lambda) = 2$ for some $\lambda \in \mathbb{C}$ then $\dim\mathcal{U}_a(\lambda) = 2$ for all $\lambda \in \mathbb{C}$. Besides, if this is the case, then $\nu_a(L) = 2$.*

5.3. **The kernel of $L_{\max}$.** Let us describe the relationship between the dimension of the kernel of $L_{\max} - \lambda$ and the dimensions of spaces $\mathcal{U}_a(\lambda)$ and $\mathcal{U}_b(\lambda)$.

The first proposition is a corrolary of Prop. 5.4:

Proposition 5.5. *The following statements are equivalent:*

(1) $\dim\mathrm{Ker}(L_{\max} - \lambda) = 2$ *for some* $\lambda \in \mathbb{C}$.
(2) $\dim\mathrm{Ker}(L_{\max} - \lambda) = 2$ *for all* $\lambda \in \mathbb{C}$.
(3) $\dim\mathcal{U}_a(\lambda_a) = \dim\mathcal{U}_b(\lambda_b) = 2$ *for some* $\lambda_a, \lambda_b \in \mathbb{C}$.
(4) $\dim\mathcal{U}_a(\lambda) = \dim\mathcal{U}_b(\lambda) = 2$ *for all* $\lambda \in \mathbb{C}$.

Besides, if this is the case, then $\nu_a(L) = \nu_b(L) = 2$.

The next two propositions are essentially obvious:

Proposition 5.6. *Let $\lambda \in \mathbb{C}$. We have $\dim\mathrm{Ker}(L_{\max} - \lambda) = 1$ if and only if one of the following statements is true:*

(1) $\dim\mathcal{U}_a(\lambda) = \dim\mathcal{U}_b(\lambda) = 1$ *and* $\mathcal{U}_a(\lambda) = \mathcal{U}_b(\lambda)$.
(2) $\dim\mathcal{U}_a(\lambda) = 2$ *and* $\dim\mathcal{U}_b(\lambda) = 1$.
(3) $\dim\mathcal{U}_a(\lambda) = 1$ *and* $\dim\mathcal{U}_b(\lambda) = 2$.

Proposition 5.7. *Let $\lambda \in \mathbb{C}$ and $\mathcal{U}_a(\lambda) \ne \{0\}$, $\mathcal{U}_b(\lambda) \ne \{0\}$. Then $\dim\mathrm{Ker}(L_{\max} - \lambda) = 0$ if and only if $\dim\mathcal{U}_a(\lambda) = \dim\mathcal{U}_b(\lambda) = 1$ and $\mathcal{U}_a(\lambda) \ne \mathcal{U}_b(\lambda)$.*

5.4. **Conjecture.** If V is real then there is a well known and simple relation between $\nu_a(L)$ and the dimension of the spaces $\mathcal{U}_a(\lambda)$, cf. Proposition 5.13. This is quite a convenient way of computing $\nu_a(L)$. In this subsection we explore what can be said on this question for arbitrary complex potentials. The difficulty is related to the fact that in the complex case there is no simple relation between the (geometric) limit point/circle method and the dimension of the spaces $\mathcal{U}_a(\lambda)$, see Subsect. 7.5.

The following relationship is easy to see:

Proposition 5.8. *For any $\lambda \in \mathbb{C}$ we have $\dim\mathcal{U}_a(\lambda) = 2 \Rightarrow \nu_a(L) = 2$.*

Proof. Indeed, we may choose two solutions u, v of the equation $(L-\lambda)f = 0$ such that $W(u, v) = 1$. Hence if all the solutions of $(L-\lambda)f = 0$ are square integrable near a, then $W_a(u, v) = 1$, Thus $W_a \neq 0$, or $\nu_a(L) = 2$. □

Now we state a conjecture which, if true, allows us to compute $\nu_a(L)$ by estimating the behaviour near a of the solutions of $Lu = \lambda u$ for certain complex λ.

Conjecture 5.9. *If $\nu_a(L) = 2$ then $\dim \mathcal{U}_a(\lambda) = 2$ for some complex λ.*

If the conjecture is true, we have in fact much more:

Lemma 5.10. *The conjecture 5.9 is equivalent to each of the following statements:*

$$\nu_a(L) = 0 \Leftrightarrow \dim \mathcal{U}_a(\lambda) \leq 1 \ \forall \lambda \in \mathbb{C}, \tag{5.5}$$

$$\nu_a(L) = 2 \Leftrightarrow \dim \mathcal{U}_a(\lambda) = 2 \ \forall \lambda \in \mathbb{C}. \tag{5.6}$$

Proof. By Proposition 5.4 one may replace in (5.5) and (5.6) $\forall\lambda$ by $\exists\lambda$, in particular the statements (5.5) and (5.6) are equivalent. Hence, by Prop. 5.8 the conjecture is equivalent to (5.6). □

The Conjecture 5.9 can be restated as a boundary value problem. Below, by "f is square integrable near a" we mean $\int_a^d |f|^2 < \infty$ for some $a < d < b$.

Lemma 5.11. *The Conjecture 5.9 is equivalent to the following property: if $\nu_a(L) = 2$ and $\phi, \psi \in \mathcal{B}_a(L)$ are linearly independent, then $\forall \alpha, \beta \in \mathbb{C}$ there is a unique $f \in AC^1]a, b[$ such that f is square integrable near a, $Lf = 0$, and $\phi(f) = \alpha, \psi(f) = \beta$.*

Proof. Assume that the conjecture is true and $\nu_a(L) = 2$. Then there are $u, v \in \mathcal{U}_a(0)$ such that $W(u, v) = 1$. Then, due to Theorem 4.5, the boundary value functionals $\vec{u}(a), \vec{v}(a) \in \mathcal{B}_a(L)$ are linearly independent. It is clear that it suffices to prove the property stated in the lemma for a unique couple ϕ, ψ and we may take $\phi = \vec{u}_a$ and $\psi = \vec{v}_a$. Then we have to find a solution f of $Lf = 0$ such that $W(u, f) = \alpha$ and $W(v, f) = \beta$. Since $W(u, cu+dv) = d$ and $W(v, cu+dv) = -c$ hence it suffices to take $f = -\beta u + \alpha v$. This f is uniquely defined because u, v is a basis in $\mathcal{U}_a(0)$ hence any element of this space can be written as $f = cu + dv$ with a unique couple of numbers c, d. Reciprocally, if $\nu_a(L) = 2$ and the property stated in the lemma is true, then the map $(\alpha, \beta) \mapsto f \in \mathcal{U}_a(0)$ is bijective, hence $\dim \mathcal{U}_a(0) = 2$. □

5.5. **Von Neumann decomposition.** Von Neumann's theory for the classification of self-adjoint extensions of a Hermitian operators is well known, cf. [24, 13]. In the present subsection we will investigate how to adapt it to our situation.

For this recall that the differential operator associated to the complex conjugate $\overline{V}$ is denoted $\overline{L} = -\partial^2 + \overline{V}$. The maximal and minimal operators associated to $\overline{L}$ are denoted $\overline{L}_{\max}$ and $\overline{L}_{\min}$. If J is the operator of complex conjugation, we clearly have $\overline{L}_{\max} = JL_{\max}J$ and $\overline{L}_{\min} = JL_{\min}J$, in particular

$$\mathcal{D}(\overline{L}_{\max}) = \{\overline{f} \mid f \in \mathcal{D}(L_{\max})\}, \quad \text{and} \quad \operatorname{Ker}(\overline{L} - \overline{z}) = J \operatorname{Ker}(L - z), \quad \forall z \in \mathbb{C}.$$

Thus $\mathcal{D}(L_{\max}) \cap \mathcal{D}(\overline{L}_{\max})$ could be $\{0\}$, cf. Lemma 3.9. Finally, $L_{\min}^* = \overline{L}_{\max}$ hence

$$(\operatorname{Ran} L_{\min})^{\perp} = \operatorname{Ker} \overline{L}_{\max}. \tag{5.7}$$

Lemma 5.12. *There is a canonical linear isomorphism*

$$\mathcal{B}(L) \simeq \{f \in \mathcal{D}(\overline{L}_{\max}) \mid \overline{L}f \in \mathcal{D}(L_{\max}) \text{ and } L\overline{L}f + f = 0\}. \tag{5.8}$$

Proof. The space $\mathcal{D}(L_{\max})$ has a natural Hilbert space structure inherited from its graph which is a closed subspace of $L^2]a, b[\oplus L^2]a, b[$, namely

$$(f|g)_L = (f|g) + (Lf|Lg) = \langle \overline{f}|g\rangle + \langle \overline{Lf}|Lg\rangle = \langle \overline{f}|g\rangle + \langle \overline{L}\,\overline{f}|Lg\rangle \tag{5.9}$$

with the notations (3.1) and (3.2). It follows that the bilinear form

$$\langle\cdot|\cdot\rangle_L : \mathcal{D}(\overline{L}_{\max}) \times \mathcal{D}(L_{\max}) \to \mathbb{C} \quad \text{given by} \quad \langle f|g\rangle_L := (\overline{f}|g)_L = \langle f|g\rangle + \langle \overline{f}u|Lg\rangle \tag{5.10}$$

allows us to identify the topological dual of $\mathcal{D}(L_{\max})$ with $\mathcal{D}(\overline{L}_{\max})$ as follows: if we denote $\langle f|\cdot\rangle_L$ the continuous linear form $g \mapsto \langle f|g\rangle_L$, then the map $f \mapsto \langle f|\cdot\rangle_L$ is a linear bijective map of $\mathcal{D}(\overline{L}_{\max})$ onto the topological dual $\mathcal{D}(L_{\max})'$. Since $\langle f|\overline{f}\rangle_L = \|\overline{f}\|_L^2$, this map is also isometric. This identification $\mathcal{D}(L_{\max})' = \mathcal{D}(\overline{L}_{\max})$ forces us to set

$$\mathcal{B}(L) = \{f \in \mathcal{D}(\overline{L}_{\max}) \mid \langle f|g\rangle_L = 0 \ \forall g \in \mathcal{D}(L_{\min})\}. \tag{5.11}$$

More explicitly the condition on f is $\langle f|g\rangle + \langle \overline{L}f|Lg\rangle = 0$ if $g \in \mathcal{D}(L_{\min})$ and this is equivalent to $\overline{L}f \in \mathcal{D}(L_{\max})$ and $L\overline{L}f = -f$. □

Formally $L\overline{L}f + f = 0$ is a fourth order differential equation but, since V is only locally L^1, with very singular coefficients. We may, however, write it as a second order system of equations as follows: if we set $f_1 = f$ and $f_2 = \overline{L}f_1$ then $Lf_2 + f_1 = 0$ hence

$$\begin{pmatrix} \overline{L} & -1 \\ 1 & L \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus by using the $\mathbb{C}^2$-valued function $F = \left(\begin{smallmatrix} f_1 \\ f_2 \end{smallmatrix}\right) \in \mathcal{D}(\overline{L}_{\max}) \oplus \mathcal{D}(L_{\max})$ and the matrix valued potential $W = \left(\begin{smallmatrix} V & -1 \\ 1 & \overline{V} \end{smallmatrix}\right)$ we see that $L\overline{L}f + f = 0$ may be written

$$-F'' + WF = 0. \tag{5.12}$$

The operator $\mathscr{L} = -\partial^2 + W = \left(\begin{smallmatrix} \overline{L} & -1 \\ 1 & L \end{smallmatrix}\right)$ acts in $L^2]a,b[\oplus L^2]a,b[$ and (5.12) means $F \in \operatorname{Ker} \mathscr{L}$.

With the help of this formalism we now prove, in the case of real potentials, a stronger version of Conjecture 5.9.

Proposition 5.13. *If V is a real function, then:*

(1) $\nu_a(L) = 0 \Leftrightarrow \dim \mathcal{U}_a(\lambda) = 1 \ \forall \lambda \in \mathbb{C} \setminus \mathbb{R}$;
(2) $\nu_a(L) = 2 \Leftrightarrow \dim \mathcal{U}_a(\lambda) = 2 \ \forall \lambda \in \mathbb{C}$.

Proof. We give a complete proof. The von Neumann's formalism is particularly efficient for real potentials. Indeed, if V is real then $\overline{L} = L$, $L_{\min}$ is Hermitian, and $L_{\max} = L_{\min}^*$. Then

$$\mathcal{B}(L) \simeq \{u \in \mathcal{D}(L_{\max}) \mid Lu \in \mathcal{D}(L_{\max}) \text{ and } L_{\max}^2 u + u = 0\}. \tag{5.13}$$

But we have

$$\operatorname{Ker}(L_{\max}^2 + 1) = \operatorname{Ker}(L_{\max} - \mathrm{i}) + \operatorname{Ker}(L_{\max} + \mathrm{i}). \tag{5.14}$$

Indeed, the inclusion $\supset$ is obvious. To prove the inclusion $\subset$ we use

$$(L^2 + 1) = (L - \mathrm{i})(L + \mathrm{i}) = (L + \mathrm{i})(L - \mathrm{i}).$$

Thus if $f \in \operatorname{Ker}(L_{\max}^2 + 1)$ and $f_\pm = (L \pm \mathrm{i})f$ then $f = (f_+ - f_-)/2\mathrm{i}$ and $(L \mp \mathrm{i})f_\pm = 0$, or $f_\pm \in \operatorname{Ker}(L_{\max} \mp \mathrm{i})$, hence (5.14) is proved. So under the identification (5.13) we have

$$\mathcal{B}(L) \simeq \operatorname{Ker}(L_{\max} - \mathrm{i}) + \operatorname{Ker}(L_{\max} + \mathrm{i}). \tag{5.15}$$

The last sum is obviously algebraically direct but also orthogonal for the scalar product (5.9) hence, due to (5.11), we have an orthogonal direct sum decomposition

$$\mathcal{D}(L_{\max}) = \mathcal{D}(L_{\min}) \oplus \mathcal{B}(L) = \mathcal{D}(L_{\min}) \oplus \operatorname{Ker}(L_{\max} - \mathrm{i}) \oplus \operatorname{Ker}(L_{\max} + \mathrm{i}). \tag{5.16}$$

The map $f \mapsto \overline{f}$ is a real linear isomorphism of $\operatorname{Ker}(L_{\max} - \mathrm{i})$ onto $\operatorname{Ker}(L_{\max} + \mathrm{i})$ hence these spaces have equal dimension ≤ 2 and so $\dim \mathcal{B}(L) = 2 \dim \operatorname{Ker}(L_{\max} - \mathrm{i}) \in \{0, 2, 4\}$. Of course, we have already proved this in a much simpler way, but (5.16) will be useful below.

From (4.21) it follows that we may assume that b is a regular endpoint. Then $\dim \mathcal{B}(L) = \dim \mathcal{B}_a(L) + 2$ due to (4.3). Then Theorem 4.6 gives $W_a = 0 \Leftrightarrow \dim \mathcal{B}(L) = 2$ and $W_a \neq 0 \Leftrightarrow \dim \mathcal{B}_a(L) = 4$. Then the preceding discussion gives $W_a = 0 \Leftrightarrow \dim \operatorname{Ker}(L_{\max} \mp \mathrm{i}) = 1$ and $W_a = 2 \Leftrightarrow \dim \operatorname{Ker}(L_{\max} \mp \mathrm{i}) = 2$. If $W_a = 0$ then one may deduce that $\dim \operatorname{Ker}(L_{\max} - \lambda) = 1$ for any $\lambda \in \mathbb{C} \setminus \mathbb{R}$ by an easy analytic continuation argument (see any text on symmetric operators). Proposition 5.3 (or 5.4) implies then that $W_a \neq 0$ if and only if for any $\lambda \in \mathbb{C}$ all the solutions of $Lf = \lambda f$ are square integrable near a. □

Corollary 5.14. *If the imaginary part of V is bounded then Conjecture 5.9 is true, i.e*

(1) $\nu_a(L) = 0 \Leftrightarrow \forall\lambda \in \mathbb{C}$ *at least one solution of* $Lf = \lambda f$ *is not square integrable near* a;
(2) $\nu_a(L) = 2 \Leftrightarrow \forall\lambda \in \mathbb{C}$ *all the solutions of* $Lf = \lambda f$ *are square integrable near* a.

Proof. We will prove that $\nu_a(L) = 2 \Rightarrow \dim \mathcal{U}_a(\lambda) = 2$ for any $\lambda \in \mathbb{C}$. Let $V_{\rm R}$ and $V_{\rm I}$ be the real and imaginary part of V and $L_{\rm R} = -\partial^2 + V_{\rm R}$. From Proposition 4.14 we get $\nu_a(L_{\rm R}) = 2$. Hence by Proposition 5.13 all the solutions of $L_{\rm R}f = \lambda f$ are square integrable near a. Finally, Proposition 5.3 implies that the solutions of $Lf = \lambda f$ are square integrable near a. □

6. Spectrum and Green's operators

6.1. **General L^2 Green's operators.** In this section we investigate Green's operators that are bounded on $L^2]a,b[$. Clearly, such Green's operators are inverses of closed realizations of L on $L^2]a,b[$. Their existence means that zero belongs to the resolvent set of this realization. By replacing L with $L - z$ we can this way study spectral properties of realizations of L.

Definition 6.1. *We say that* $G_\bullet$ *is an* L^2 Green's operator of L *if it is a bounded operator on* $L^2]a,b[$ *such that* $\operatorname{Ran} G_\bullet \subset \mathcal{D}(L_{\max})$ *and*

$$L_{\max}G_\bullet = \mathbb{1}. \tag{6.1}$$

Proposition 6.2. *The following conditions are equivalent:*
(1) L *has an* L^2 *Green's operator,*
(2) $L_{\max} : \mathcal{D}(L_{\max}) \to L^2]a,b[$ *is surjective,*
(3) $L_{\min} : \mathcal{D}(L_{\min}) \to L^2]a,b[$ *is injective and has closed range.*
If these conditions are satisfied then L^2 *Green's operators are in bijective correspondence with closed subspaces* $\mathcal{L}$ *of* $\mathcal{D}(L_{\max})$ *such that* $\mathcal{D}(L_{\max}) = \mathcal{L} \oplus \operatorname{Ker} L_{\max}$ *(topological direct sum).*

Proof. If $G_\bullet$ is an L^2 Green's operator for L then $L_{\max}$ is surjective due to (6.1). Reciprocally, assume $L_{\max}$ is surjective. Since $\operatorname{Ker} L_{\max}$ is a finite dimensional subspace of $\mathcal{D}(L_{\max})$, there is a closed subspace $\mathcal{L}$ of $\mathcal{D}(L_{\max})$ such that $\mathcal{D}(L_{\max}) = \mathcal{L} \oplus \operatorname{Ker} L_{\max}$. Then $L_\bullet = L_{\max}\big|_{\mathcal{L}}$ is a bijective map $\mathcal{L} \to L^2]a,b[$ and $G_\bullet := L_\bullet^{-1}$ is an L^2 Green's operator for L. This proves (1) $\Leftrightarrow$ (2). To prove the equivalence with (3), we use the closed range theorem (see Theorem A.3) after identifying the dual (not the anti-dual) space of $L^2]a,b[$ with itself with the help of (3.2). Thus $L_{\max}$ has closed range if and only if $L_{\max}^{\#} = L_{\min}$ has closed range and $\operatorname{Ker} L_{\min}$ is the orthogonal of $\operatorname{Ran} L_{\max}$, so $L_{\max}$ is surjective if and only if $L_{\min}$ is injective with closed range. □

Observe that under the conditions of Proposition 6.2 we have (with the notation of (3.8))

$$\operatorname{Ran} L_{\min} = (\operatorname{Ker} L_{\max})^{\rm perp} \quad \text{and} \quad \operatorname{Ker} L_{\max} = (\operatorname{Ran} L_{\min})^{\rm perp}. \tag{6.2}$$

Proposition 6.3. *If* $G_\bullet$ *is an* L^2 *Green's operator of* L *and* $K : L^2]a,b[\to \operatorname{Ker} L_{\max}$ *is a linear continuous map, then* $G_\bullet + K$ *is also an* L^2 *Green's operator. If* G_1, G_2 *are two* L^2 *Green's operators of* L*, then there are 3 possibilities:*

(1) $\dim \operatorname{Ker} L_{\max} = 0$. *Then* $G_1 = G_2$*, so there is at most one* L^2 *Green operator.*
(2) $\dim \operatorname{Ker} L_{\max} = 1$. *Then if* $u \in L^2]a,b[$ *is a nonzero solution of* $Lg = 0$,

$$G_1 - G_2 = |u\rangle\langle\phi|, \qquad \textit{for some} \quad \phi \in L^2]a,b[.$$

(3) $\dim \operatorname{Ker} L_{\max} = 2$. *Then if* u, v *are linearly independent solutions in* $L^2]a,b[$ *of* $Lg = 0$,

$$G_1 - G_2 = |u\rangle\langle\phi| + |v\rangle\langle\psi|, \qquad \textit{for some} \quad \phi, \psi \in L^2]a,b[.$$

Proof. We have $L_{\max}(G_1 - G_2) = 0$ on $L^2]a,b[$ and $\operatorname{Ker} L_{\max} = \operatorname{Ker} L \cap L^2]a,b[$. But $\dim \operatorname{Ker} L = 2$, therefore $\dim \operatorname{Ker} L_{\max}$ can be 0, 1 or 2. □

Proposition 6.4. *Let* $G_\bullet$ *be an* L^2 *Green's operator. Then*

(1) $\operatorname{Ker} G_\bullet = \{0\}$.
(2) $G_\bullet$ *is bounded from* $L^2]a,b[$ *to* $\mathcal{D}(L_{\max})$.

(3) $P_\bullet := G_\bullet L_{\max}$ *is a bounded projection on the space* $\mathcal{D}(L_{\max})$ *such that*

$$\operatorname{Ran} P_\bullet = \operatorname{Ran} G_\bullet, \quad \operatorname{Ker} P_\bullet = \operatorname{Ker} L_{\max}.$$

(4) $\operatorname{Ran} G_\bullet$ *is closed in* $\mathcal{D}(L_{\max})$.
(5) $\mathcal{D}(L_{\max}) = \operatorname{Ran} G_\bullet \oplus \operatorname{Ker} L_{\max}$ *where* $\oplus$ *means the topological direct sum.*

Proof. (1) is obvious and

$$\|G_\bullet f\|_L^2 = \|L_{\max} G_\bullet f\|^2 + \|G_\bullet f\|^2 \leq \big(1 + \|G_\bullet\|^2\big)\|f\|^2 \tag{6.3}$$

implies (2). Since $L_{\max} : \mathcal{D}(L_{\max}) \to L^2]a,b[$ is bounded, $P_\bullet$ is bounded on $\mathcal{D}(L_{\max})$. Then

$$P_\bullet^2 = G_\bullet(L_{\max} G_\bullet)L_{\max} = G_\bullet L_{\max} = P_\bullet$$

hence $P_\bullet$ is a projection.

It is obvious that $\operatorname{Ran} P_\bullet \subset \operatorname{Ran} G_\bullet$. If $g \in L^2]a,b[$ then

$$G_\bullet g = G_\bullet L_{\max} G_\bullet g = P_\bullet G_\bullet g.$$

Hence $\operatorname{Ran} G_\bullet \subset \operatorname{Ran} P_\bullet$. This shows that $\operatorname{Ran} G_\bullet = \operatorname{Ran} P_\bullet = \mathcal{L}$ (cf. Proposition 6.2).

It is obvious that $\operatorname{Ker} L_{\max} \subset \operatorname{Ker} P_\bullet$. If $0 = P_\bullet f$, then

$$0 = L_{\max} P_\bullet f = (L_{\max} G_\bullet) L_{\max} f = L_{\max} f.$$

Hence $\operatorname{Ker} P_\bullet \subset \operatorname{Ker} L_{\max}$. This shows that $\operatorname{Ker} P_\bullet = \operatorname{Ker} L_{\max}$.

Thus we have shown (3), which implies immediately (4) and (5). $\square$

Proposition 6.5. *Suppose that* $G_\bullet$ *is a bounded everywhere defined operator on* $L^2]a,b[$ *and* $\mathcal{D} \subset L^2]a,b[$ *a dense subspace such that* $G_\bullet \mathcal{D} \subset \mathcal{D}(L_{\max})$ *and*

$$L_{\max} G_\bullet g = g, \quad g \in \mathcal{D}. \tag{6.4}$$

Then $G_\bullet$ *is an* L^2 *Green's operator of* L.

Proof. Let $f \in L^2]a,b[$ and $(f_n) \subset \mathcal{D}$ such that $f_n \underset{n\to\infty}{\to} f$. Then $G_\bullet f_n \underset{n\to\infty}{\to} G_\bullet f$ and $L_{\max} G_\bullet f_n = f_n \underset{n\to\infty}{\to} f$. By the closedness of $L_{\max}$, $G_\bullet f \in \mathcal{D}(L_{\max})$ and $L_{\max} G_\bullet f = f$. $\square$

Let $G_\bullet$ be a Green's operator in the sense of Definition 2.8. Clearly, $L^2_{\mathrm{c}}]a,b[$ is contained in $L^1_{\mathrm{c}}]a,b[$. Besides, $L^2_{\mathrm{c}}]a,b[$ is dense in $L^2]a,b[$. Therefore, if the restriction of $G_\bullet$ to $L^2_{\mathrm{c}}]a,b[$ is bounded, then it has a unique extension to a bounded operator on $L^2]a,b[$. This extension, which by Prop. 6.5 is an L^2 Green's operator, will be denoted by the same symbol $G_\bullet$.

Definition 6.6. *Let* $G_\bullet$ *be an* L^2 *Green's operator of* L. *Then we define* $L_\bullet$ *to be the restriction of* $L_{\max}$ *to*

$$\mathcal{D}(L_\bullet) := \operatorname{Ran} G_\bullet \tag{6.5}$$

Observe that $\mathcal{D}(L_\bullet)$ is the subspace denoted $\mathcal{L}$ in the proof of Proposition 6.2. Since this subspace is closed in $\mathcal{D}(L_{\max})$, we have:

Proposition 6.7. $L_\bullet$ *is a closed operator such that* $L_\bullet G_\bullet = \mathbb{1}$ *on* $L^2]a,b[$ *and* $G_\bullet L_\bullet = \mathbb{1}$ *on* $\mathcal{D}(L_\bullet)$. *Thus* 0 *belongs to the resolvent set of* $L_\bullet$ *and* $L_\bullet^{-1} = G_\bullet$. *Besides,*

$$\mathcal{D}(L_{\max}) = \mathcal{D}(L_\bullet) \oplus \operatorname{Ker} L_{\max}. \tag{6.6}$$

Thus, for every L^2 Green's operator $G_\bullet$, its inverse is a closed operator $L_\bullet$ contained in $L_{\max}$. Obviously, if $\operatorname{Ker} L_{\max} \neq 0$, its domain is not dense in $\mathcal{D}(L_{\max})$. But it is often dense in $L^2]a,b[$.

6.2. L^2 **Green's operators whose inverses contain** $L_{\min}$. We are mostly interested in closed operators that lie between $L_{\min}$ and $L_{\max}$. Such operators are automatically densely defined. The corresponding Green's operators are characterized in the following proposition:

Proposition 6.8. *Let $G_\bullet$ be an L^2 Green's operator and $L_\bullet$ its inverse. Then $L_{\min} \subset L_\bullet$ if and only if $G_\bullet^\#$ is an L^2 Green's operator.*

Proof. Recall that $L_{\max} = L_{\min}^\#$. Thus $L_{\min} \subset L_\bullet$ if and only if

$$L_{\min} \subset L_\bullet \subset L_{\min}^\#. \tag{6.7}$$

Applying the transposition reverses the inclusion. Therefore,

$$L_{\min}^\# \supset L_\bullet^\# \supset L_{\min}. \tag{6.8}$$

But

$$G_\bullet = L_\bullet^{-1}, \quad G_\bullet^\# = (L_\bullet^{-1})^\# = (L_\bullet^\#)^{-1} \tag{6.9}$$

Therefore, $G_\bullet^\#$ is the L^2 Green's operator associated with $L_\bullet^\#$. This proves $\Rightarrow$. Clearly, the above argument can be reversed. □

Proposition 6.9. *If L has an L^2 Green's operator then it has self-transposed L^2 Green's operators. If $G_\bullet$ is a self-transposed L^2 Green's operator of L and $G_\bullet = L_\bullet^{-1}$, then $L_{\min} \subset L_\bullet = L_\bullet^\#$.*

Proof. If L has an L^2 Green's operator then $L_{\min}$ is injective (see Proposition 6.2) and this is equivalent to $\mathcal{D}(L_{\min}) \cap \operatorname{Ker} L_{\max} = 0$. Since $\mathcal{D}(L_{\min})$ is of finite codimension in $\mathcal{D}(L_{\max})$, from the last assertion of Proposition 6.2 it follows that there are L^2 Green's operators with $\mathcal{D}(L_{\min}) \subset \mathcal{L} = \mathcal{D}(L_\bullet)$. Then a self-transposed L^2 Green's operator exists, for example $(G_\bullet + G_\bullet^\#)/2$. By the Proposition 6.8, all self-transposed L^2 Green's operators are such that $L_{\min} \subset L_\bullet = L_\bullet^\#$. □

The following proposition should be compared with Prop. 6.3.

Proposition 6.10. *If $L_{\max}$ has an L^2 Green's operator, then there are 3 possibilities:*

(1) $\dim \operatorname{Ker} L_{\max} = 0$. *Then the L^2 Green's operator is unique, self-transposed, and its inverse is $L_{\max} = L_{\min}$.*

(2) $\dim \operatorname{Ker} L_{\max} = 1$. *Then all L^2 Green's operators whose inverses contain $L_{\min}$ are self-transposed. If G_1, G_2 are two such Green's operators and $u \in \operatorname{Ker} L_{\max}$ is nonzero, then*

$$G_1 - G_2 = \alpha |u\rangle\langle u|, \quad \textit{for some } \alpha \in \mathbb{C}.$$

(3) $\dim \operatorname{Ker} L_{\max} = 2$. *Then if G_1, G_2 are two L^2 Green's operators whose inverses contain $L_{\min}$ and $u_1, u_2 \in \operatorname{Ker} L_{\max}$ are linearly independent, then*

$$G_1 - G_2 = \sum_{i,j} \alpha_{ij} |u_i\rangle\langle u_j| \quad \textit{for some matrix} \quad [\alpha_{ij}].$$

Proof. If $\dim \operatorname{Ker} L_{\max} = 0$ and there is a Green operator then this operator is unique by (1) of Proposition 6.3 and its range is equal to $\mathcal{D}(L_{\max})$ by (5) of Proposition 6.4. Thus $G_\bullet = L_{\max}^{-1}$ and $L_{\max} = L_{\min}$ by Proposition 6.2, for example. Then

$$G_\bullet^\# = (L_{\max}^{-1})^\# = (L_{\max}^\#)^{-1} = (L_{\min})^{-1} = G_\bullet \tag{6.10}$$

which finishes the proof of assertion (1).

If $\dim \operatorname{Ker} L_{\max} = 1$ then Proposition 6.9 shows that L has a self-transposed L^2 Green's operator G_1. Then, by (2) of Proposition 6.3, if G_2 is another L^2 Green's operator, we have $G_1 - G_2 = |u\rangle\langle\phi|$ for some $\phi \in L^2]a,b[$ hence we also have $G_1 - G_2^\# = |\phi\rangle\langle u|$. If the inverse of G_2 contains $L_{\min}$ then by Proposition 6.8 $G_2^\#$ is a Green's operator of L, hence $L_{\max}(G_1 - G_2^\#) = 0$, which clearly is equivalent to $L_{\max}\phi = 0$. Since $\dim \operatorname{Ker} L_{\max} = 1$ we get $\phi = \alpha u$ for some $\lambda \in \mathbb{C}$ and then $G_2 = G_1 - \alpha|u\rangle\langle u|$ so that G_2 is self-transposed.

Finally, let us assume $\dim \operatorname{Ker} L_{\max} = 2$ and let G_1, G_2 be Green's operators whose inverses contain $L_{\min}$. Then $G_1^\#, G_2^\#$ are also Green's operators, due to Proposition 6.8. By (3) of Proposition 6.3 we get $G_1 - G_2 = |u_1\rangle\langle\phi_1| + |u_2\rangle\langle\phi_2|$ for some $\phi_1, \phi_2 \in L^2]a,b[$. This implies

$G_1^\# - G_2^\# = |\phi_1\rangle\langle u_1| + |\phi_2\rangle\langle u_2|$ and since the range of the operator $G_1^\# - G_2^\#$ must also be included in $\operatorname{Ker} L_{\max}$ we get $\phi_1\langle u_1|v\rangle + \phi_2\langle u_2|v\rangle \in \operatorname{Ker} L_{\max}$ for all $v \in L^2]a,b[$. Since u_1, u_2 are linearly independent, there are vectors v_1, v_2 such that $\langle u_i|v_j\rangle = \delta_{ij}$ hence $\phi_1, \phi_2 \in \operatorname{Ker} L_{\max}$. □

In the next proposition we describe the integral kernel of an L^2 Green's operator $G_\bullet$ whose inverse contains $L_{\min}$. Recall that for any $x \in]a,b[$ we denote by $L^{a,x}$, resp. $L^{x,b}$ the restriction of L to $L^2]a,x[$, resp. $L^2]x,b[$. We also can define $L^{a,x}_{\max}$ and $L^{x,b}_{\max}$, etc. Note that x is a regular point of both $L^{a,x}$ and $L^{x,b}$ (V is integrable on a neighbourhood of x).

Proposition 6.11. *Let $G_\bullet$ be an L^2 Green's operator for L such that $G_\bullet^\#$ is also an L^2 Green's operator for L. Then $G_\bullet$ is an integral operator whose integral kernel*

$$]a,b[\times]a,b[\ni (x,y) \mapsto G_\bullet(x,y) \in \mathbb{C}$$

is a function separately continuous in x and y which has the following properties:
(1) for each $a < x < b$ the function $G_\bullet(x,\cdot)$ restricted to $]a,x[$, resp. $]x,b[$ belongs to $\mathcal{D}(L^{a,x}_{\max})$, resp. $\mathcal{D}(L^{x,b}_{\max})$ and satisfies $LG_\bullet(x,\cdot) = 0$ outside x. Besides, $G_\bullet(x,\cdot)$ and its derivative have limits at x from the left and the right satisfying

$$\begin{aligned} G_\bullet(x,x-0) - G_\bullet(x,x+0) &= 0, \\ \partial_2 G_\bullet(x,x-0) - \partial_2 G_\bullet(x,x+0) &= 1; \end{aligned}$$

(2) for each $a < y < b$ the function $G_\bullet(\cdot,y)$ restricted to $]a,y[$, resp. $]y,b[$ belongs to $\mathcal{D}(L^{a,y}_{\max})$, resp. $\mathcal{D}(L^{y,b}_{\max})$ and satisfies $LG_\bullet(\cdot,y) = 0$ outside y. Besides, $G_\bullet(\cdot,y)$ and its derivative have limits at y from the left and the right satisfying

$$\begin{aligned} G_\bullet(y-0,y) - G_\bullet(y+0,y) &= 0, \\ \partial_1 G_\bullet(y-0,y) - \partial_1 G_\bullet(y+0,y) &= 1; \end{aligned}$$

Proof. We shall use ideas from the proof of Lemma 4 p. 1315 in [13]. $G_\bullet$ is a continuous linear map $G_\bullet : L^2]a,b[\to \mathcal{D}(L_{\max})$ and for each $x \in]a,b[$ we have a continuous linear form $\varepsilon_x : f \mapsto f(x)$ on $\mathcal{D}(L_{\max})$, hence we get a continuous linear form $\varepsilon_x \circ G_\bullet : L^2]a,b[\to \mathbb{C}$. Thus for each $x \in]a,b[$ there exists a unique $\phi_x \in L^2]a,b[$ such that

$$(G_\bullet f)(x) = \int_a^b \phi_x(y) f(y)\mathrm{d}y, \quad \forall f \in L^2]a,b[\,.$$

We get a map $\phi :]a,b[\to L^2]a,b[$ which is continuous, and even locally Lipschitz, because if $J \subset]a,b[$ is compact and $x,y \in J$ then

$$\begin{aligned} \left| \int_a^b (\phi_x(z) - \phi_y(z)) f(z)\mathrm{d}z \right| &= |(G_\bullet f)(x) - (G_\bullet f)(y)| \le \|(G_\bullet f)'\|_{L^\infty(J)}|x-y| \\ &\le C_1 \|G_\bullet f\|_{\mathcal{D}(L_{\max})}|x-y| \le C_2\|f\||x-y|, \end{aligned}$$

hence $\|\phi_x - \phi_y\| \le C_2|x-y|$. By taking $f = L_\bullet g$, $g \in \mathcal{D}(L_\bullet)$, we get

$$g(x) = \int_a^b \phi_x(y)(L_\bullet g)(y)\mathrm{d}y. \tag{6.11}$$

Set $\phi_x^a := \phi_x\big|_{]a,x[}$ and $\phi_x^b := \phi_x\big|_{]x,b[}$. (6.11) can be rewritten as

$$g(x) = \int_a^x \phi_x^a(y)(L_\bullet g)(y)\mathrm{d}y + \int_x^b \phi_x^b(y)(L_\bullet g)(y)\mathrm{d}y. \tag{6.12}$$

Since $G_\bullet^\#$ is also an L^2 Green's operator, we have $L_{\min} \subset L_\bullet \subset L_{\max}$. Assuming that $g \in \mathcal{D}(L_{\min})$ and $g(y) = 0$ in a neighborhood of x, we can rewrite (6.12) as

$$0 = \int_a^x \phi_x^a(y)(L^{a,x}_{\min} g)(y)\mathrm{d}y + \int_x^b \phi_x^b(y)(L^{a,x}_{\min} g)(y)\mathrm{d}y. \tag{6.13}$$

Such functions g are dense in $\mathcal{D}(L^{a,x}_{\min}) \oplus \mathcal{D}(L^{x,b}_{\min})$. Therefore, ϕ^a_x belongs to $\mathcal{D}(L^{a,x}_{\max})$ and ϕ^b_x belongs to $\mathcal{D}(L^{x,b}_{\max})$. Since x is a regular end of both intervals $]a,x[$ and $]x,b[$ the function ϕ_x and its derivative ϕ'_x extend to continuous functions on $]a,x]$ and $[x,b[$. However, these extensions are not necessarily continuous on $]a,b[$, i.e. we must distinguish the left and right limits at x, denoted $\phi_x(x \pm 0)$ and $\phi'_x(x \pm 0)$.

We now take $g \in \mathcal{D}(L_{\min})$ in (6.11). By taking into account (5) of Theorem 3.4 and what we proved above we have $W(\phi_x, g; a) = 0$ and $W(\phi_x, g; b) = 0$. Denote ϕ^a_x and ϕ^b_x the restrictions of ϕ_x to the intervals $]a,x[$ and $]x,b[$. Then by using Green's identity on $]a,x[$ and $]x,b[$ in (6.12) we get

$$g(x) = -W(\phi^a_x, g; x) + W(\phi^b_x, g; x).$$

We may compute the last two terms explicitly because x is a regular end of both intervals:

$$\begin{aligned} W(\phi^a_x, g; x) &= \phi_x(x-0)g'(x) - \phi'_x(x-0)g(x), \\ W(\phi^b_x, g; x) &= \phi_x(x+0)g'(x) - \phi'_x(x+0)g(x). \end{aligned}$$

Thus we get

$$g(x) = (\phi_x(x+0) - \phi_x(x-0))g'(x) + (\phi'_x(x-0) - \phi'_x(x+0))g(x).$$

The values $g(x)$ and $g'(x)$ may be specified in an arbitrary way under the condition $g \in \mathcal{D}(L_{\min})$ so we get $\phi_x(x+0) - \phi_x(x-0) = 0$ and $\phi'_x(x-0) - \phi'_x(x+0) = 1$. Thus ϕ_x must be a continuous function which is continuously derivable outside x and its derivative has a jump $\phi'_x(x+0) - \phi'_x(x-0) = -1$ at x.

Thus $G_\bullet$ is an integral operator with kernel $G_\bullet(x,y) = \phi_x(y)$. But $G^\#_\bullet$ is also an L^2 Green's operator and clearly $G^\#_\bullet$ has kernel $G^\#_\bullet(x,y) = \phi_y(x)$. Repeating the above arguments applied to $G^\#_\bullet$ we obtain the remaining statements of the proposition. □

Let us describe one consequence of the above proposition, where we use the notation of Definition 5.1:

Proposition 6.12. *If there exists a realization of L such that $\lambda \in \mathbb{C}$ is in its resolvent set, then* $\dim \mathcal{U}_a(\lambda) \geq 1$ *and* $\dim \mathcal{U}_b(\lambda) \geq 1$.

Proof. Suppose that L possesses a realization and $\lambda \in \mathbb{C}$ is contained in its resolvent set. This means that $L - \lambda$ possesses an L^2 Green's operator $G_\bullet$. By Proposition 6.9 it can be chosen to satisfy $G_\bullet = G^\#_\bullet$. Then Proposition 6.11 implies that for any $x \in]a,b[$ the function $G_\bullet(x,\cdot) \in L^2]a,b[$ satisfies $LG_\bullet(x,\cdot) = 0$ on $]a,x[$ and $]x,b[$. And we have $G_\bullet(x,\cdot)\big|_{]a,x[} \neq 0$ and $G_\bullet(x,\cdot)\big|_{]x,b[} \neq 0$ due to (6.11) for example. □

6.3. **Forward and backward Green's operators.**

Let us study the L^2 theory of the forward Green's operator $G_\to$. Recall that if u, v span $\operatorname{Ker} L$ with $W(v,u) = 1$, then $G_\to$ is given by

$$G_\to g(x) = v(x)\int_a^x u(y)g(y)\mathrm{d}y - u(x)\int_a^x v(y)g(y)\mathrm{d}y. \tag{6.14}$$

Of course, similar results are valid for the backward operator $G_\leftarrow$.

Proposition 6.13. *Assume* $\dim \operatorname{Ker} L_{\max} = 2$. *Then*

(1) $G_\to$ is Hilbert-Schmidt. In particular, it is an L^2 Green's operator of L.
(2) Let L_a be the operator defined in Def. 3.13. L_a has an empty spectrum and $(L_a - \lambda)^{-1}$ is compact for avery $\lambda \in \mathbb{C}$. We have $L_a^{-1} = G_\to$.
(3) Every $f \in \mathcal{D}(L_{\max})$ has a unique decomposition as

$$f = \alpha u + \beta v + f_a, \quad f_a \in G_\to L^2]a,b[. \tag{6.15}$$

(4) We can also define $G_\leftarrow$ with analogous properties. We have

$$G^\#_\to = G_\leftarrow, \quad L_b^{-1} = G_\leftarrow. \tag{6.16}$$

Proof. By hypothesis, $u, v \in L^2]a,b[$. The Hilbert-Schmidt norm of $G_\to$ is clearly bounded by $\sqrt{2}\|u\|_2\|v\|_2$. Then by Proposition 6.7 zero belongs to the resolvent set of L_a, $L_a^{-1} = G_a$, and

$$\mathcal{D}(L_{\max}) = \mathcal{D}(L_a) \oplus \operatorname{Ker} L_{\max}, \tag{6.17}$$

which can be restated as the decomposition (6.15). If $\lambda \in \mathbb{C}$ and V is replaced by $V - \lambda$ then the new $G_\to$ will be the resolvent at λ of L_a, which proves the second assertion in (2). Finally, (6.16) is proved by a simple computation. □

There is also a one-sided version of Prop. 6.13:

Proposition 6.14. *Assume that* $\dim \mathcal{U}_a(0) = 2$. *Then* $G_\to$ *extends as a map from* $L^2]a,b[$ *to* $C^1]a,b[$ *satisfying the bounds*

$$|G_\to g(x)| \leq \Big(|u(x)|\|v\|_x + |v(x)|\|u\|_x\Big)\|g\|_x, \tag{6.18}$$

$$|\partial_x G_\to g(x)| \leq \Big(|u'(x)|\|v\|_x + |v'(x)|\|u\|_x\Big)\|g\|_x, \tag{6.19}$$

where $\|g\|_x := \Big(\int_a^x |g(y)|^2 \mathrm{d}y\Big)^{\frac12}$. *If* $\chi \in C_c^\infty[a,b[$, $\chi = 1$ *around* a, *then every* $f \in \mathcal{D}(L_{\max})$ *has a unique decomposition as*

$$f = \alpha\chi u + \beta\chi v + f_a, \quad f_a \in \mathcal{D}(L_>). \tag{6.20}$$

Proof. Let $a < d < b$. Then we can restrict our problem to $]a,d[$. Now $\dim \mathcal{U}_a(0) = \dim \mathcal{U}_d(0) = 2$. Therefore, we can apply Prop. 6.13, using the fact that $G_\to$ restricted to $L^2]a,d[$ is an L^2 Green's operator of $L^{a,d}$. □

Note that in Prop. 6.14 we do not claim that $G_\to$ is the inverse of L_a, nor that it is bounded.

Proposition 6.15. $G_\to$ *is bounded if and only if* $\dim \operatorname{Ker} L_{\max} = 2$ *(so that the assumptions of Prop. 6.13 are valid).*

Proof. Let $G_\to$ be bounded. Then so is $G_\to^\# = G_\leftarrow$. Let us recall the identity (2.18):

$$G_\to - G_\leftarrow = |v\rangle\langle u| - |u\rangle\langle v|. \tag{6.21}$$

But the boundedness of the rhs of (6.21) implies $v, u \in L^2]a,b[$. □

6.4. **Green's operators with two-sided boundary conditions.** In this subsection we study Green's operators having two-sided boundary conditions. Suppose that $u, v \in \operatorname{Ker}(L)$ are linearly independent. Without a loss of generality we can suppose that $W(v,u) = 1$. Recall that the two-sided Green's operator $G_{u,v}$ was defined in Def. 2.9:

$$G_{u,v}g(x) := \int_x^b u(x)v(y)g(y)\mathrm{d}y + \int_a^x v(x)u(y)g(y)\mathrm{d}y.$$

Let us start with the following simple fact:

Proposition 6.16. *Let* $G_{u,v}$ *be bounded on* $L^2]a,b[$. *Then* $u \in \mathcal{U}_a(0)$ *and* $v \in \mathcal{U}_b(0)$.

Proof. Let $a < d < b$. If $G_{u,v}$ is bounded, then so is $\mathbb{1}_{]a,d]}(x)G_{u,v}\mathbb{1}_{]d,b]}(x)$, where x denotes the operator of multiplication by the variable in $]a,b[$. But its integral kernel is

$$u(x)\mathbb{1}_{]a,d]}(x)v(y)\mathbb{1}_{]d,b]}(y)$$

where x and y denote the variables in $]a,b[$. This is a rank one operator with the norm

$$\Big(\int_a^d |u|^2(x)\mathrm{d}x\Big)^{\frac12}\Big(\int_d^b |v|^2(x)\mathrm{d}x\Big)^{\frac12}.$$ □

Motivated partly by the above proposition, until the end of this subsection we assume that $u \in \mathcal{U}_a(0)$ and $v \in \mathcal{U}_b(0)$.

Recall from Def. 4.11 that if $\phi \in \mathcal{B}_a$ and $\psi \in \mathcal{B}_b$ be are nonzero functionals, then we define $L_{\phi,\psi}$ as the operator satisfying $L_{\phi,\psi} \subset L_{\max}$ and

$$\mathcal{D}(L_{\phi,\psi}) := \{f \in \mathcal{D}(L_{\max}) \ \mid \phi(f) = \psi(f) = 0\}.$$

Let us assume that the functionals ϕ, ψ have the form

$$\phi = \vec{u}_a, \quad \psi = \vec{v}_b. \tag{6.22}$$

with the notation introduced in Def. 4.4. If (6.22) holds, we will often write $L_{u,v}$ instead of $L_{\phi,\psi}$.

Proposition 6.17. *Let $\chi \in C^\infty]a,b[$ such that $\chi = 1$ close to a and $\chi = 0$ close to b. Then,*

$$\mathcal{D}(L_{u,v}) = \mathcal{D}(L_{\min}) + \operatorname{Span}\{\chi u, (1-\chi)v\}, \tag{6.23}$$

$$L_{u,v}^{\#} = L_{u,v}, \tag{6.24}$$

$$G_{u,v}L^2_{\mathrm{c}}]a,b[\subset \mathcal{D}(L_{u,v}), \tag{6.25}$$

$$G_{u,v}L^2]a,b[\subset AC^1]a,b[. \tag{6.26}$$

Moreover, $G_{u,v}$ is bounded if and only if there exists $c > 0$ such that

$$\|L_{u,v}f\| \geq c\|f\|, \quad f \in \mathcal{D}(L_{u,v}). \tag{6.27}$$

If this is the case, then $G_{u,v}$ is an L^2 Green's operator related to $L_{u,v}$ as in Def. 6.6 and Prop. 6.7. In particular, 0 belongs to the resolvent set of $L_{u,v}$, we have $G_{u,v} = L_{u,v}^{-1}$, $G_{u,v}^{\#} = G_{u,v}$ and

$$\mathcal{D}(L_{u,v}) = G_{u,v}L^2]a,b[. \tag{6.28}$$

Proof. (6.23) is immendiate. The relation (6.24) follows from Green's identity (3.13). Then it is easy to see that

$$G_{u,v}L^2_{\mathrm{c}}]a,b[\subset \mathcal{D}(L_{\mathrm{c}}) + \operatorname{Span}\{\chi u, (1-\chi)v\},$$

which implies (6.25).

Let $g \in L^2]a,b[$. For $a < x < b$ we compute:

$$\partial_x G_{u,v}g(x) = u'(x)\int_x^b v(y)g(y)\mathrm{d}y + v'(x)\int_a^x u(y)g(y)\mathrm{d}y. \tag{6.29}$$

Now, $x \mapsto u'(x), v'(x), \int_x^b v(y)g(y)\mathrm{d}y$, $\int_a^x u(y)g(y)\mathrm{d}y$ belong to $AC]a,b[$. Hence (6.29) belongs to $AC]a,b[$. Therefore, (6.26) is true. Next, let

$$f = f_{\mathrm{c}} + \alpha\chi u + \beta(1-\chi)v, \quad f_{\mathrm{c}} \in \mathcal{D}(L_{\mathrm{c}}). \tag{6.30}$$

We compute, integrating by parts,

$$G_{u,v}L_{u,v}f(x) = \int_a^b \Big(\big(-\partial_y^2 + V(y)\big)G_{u,v}(x,y)\Big)f(y)\mathrm{d}y \tag{6.31}$$

$$+ \lim_{y\to a}\big(G_{u,v}(x,y)f'(y) - \partial_y G_{u,v}(x,y)f(y)\big) \tag{6.32}$$

$$- \lim_{y\to b}\big(G_{u,v}(x,y)f'(y) - \partial_y G_{u,v}(x,y)f(y)\big) \tag{6.33}$$

$$= f(x) + v(x)W(u,f;a) - u(x)W(v,f;b) \quad = \quad f(x). \tag{6.34}$$

Moreover, functions of the form (6.30) are dense in $\mathcal{D}(L_{u,v})$. Therefore, if $G_{u,v}$ is bounded, then (6.34) extends to

$$G_{u,v}L_{u,v}f = f, \quad f \in \mathcal{D}(L_{u,v}). \tag{6.35}$$

Hence $\|f\| = \|G_{u,v}L_{u,v}f\| \leq \|G_{u,v}\|\|L_{u,v}f\|$ which gives (6.27).

Assume that $G_{u,v}$ is bounded in the sense of $L^2]a,b[$. By Prop. 6.5, $G_{u,v}$ is an L^2 Green's operator. By Prop. 6.4, it is also bounded from $L^2]a,b[$ to $\mathcal{D}(L_{\max})$. Therefore (6.25) extends then to

$$G_{u,v}L^2]a,b[\subset \mathcal{D}(L_{u,v}), \tag{6.36}$$

so that

$$L_{u,v}G_{u,v}g = g, \quad g \in L^2]a,b[. \tag{6.37}$$

By (6.35) and (6.37), $G_{u,v}$ is a (bounded) inverse of $L_{u,v}$ so that (6.27) and (6.28) are true.

Now assume that (6.27) holds. By (6.25), we then have

$$g = L_{u,v}G_{u,v}g, \quad g \in L^2_{\mathrm{c}}]a,b[. \tag{6.38}$$

Hence,

$$\|g\| = \|L_{u,v}G_{u,v}g\| \geq c\|G_{u,v}g\| \tag{6.39}$$

on $L^2_{\mathrm{c}}]a,b[$, which is dense in $L^2]a,b[$. Therefore, $G_{u,v}$ is bounded. □

6.5. **Classification of realizations possessing non-empty resolvent set.** In applications operators possessing non-empty resolvent set are by far the most useful. The following theorem describes a classification of realizations of L with this property. We will denote by $\mathrm{rs}(A)$ the resolvent set of an operator A.

Theorem 6.18. *Suppose that $L_\bullet$ is a realization of L with a non-empty resolvent set. Then exactly one of the following statements is true.*

(1) $L_\bullet = L_{\max}$.
Then also $L_{\min} = L_\bullet$, so that L possesses a unique realization. We have $\nu(L) = 0$
If $\lambda \in \mathrm{rs}(L_\bullet)$, then $\dim \mathrm{Ker}(L_{\max} - \lambda) = 0$, $\dim \mathcal{U}_a(\lambda) = \dim \mathcal{U}_b(\lambda) = 1$ and $\mathcal{U}_a(\lambda) \neq \mathcal{U}_b(\lambda)$. If $u \in \mathcal{U}_a(\lambda)$ and $v \in \mathcal{U}_b(\lambda)$ with $W(v,u) = 1$, then

$$(L_\bullet - \lambda)^{-1} = G_{u,v}.$$

$L_\bullet$ is self-transposed and has separated boundary conditions.

(2) The inclusion $\mathcal{D}(L_\bullet) \subset \mathcal{D}(L_{\max})$ is of codimension 1.
Then the inclusion $\mathcal{D}(L_{\min}) \subset \mathcal{D}(L_\bullet)$ is of codimension 1 or 3 (1 if Conjecture 5.9 holds). We have $\nu(L) = 2$ or 4 (2 if Conjecture 5.9 holds).
If $\lambda \in \mathrm{rs}(L_\bullet)$, then $\dim \mathrm{Ker}(L_{\max} - \lambda) = 1$, $\dim \mathcal{U}_a(\lambda) = 2$ and $\dim \mathcal{U}_b(\lambda) = 1$, or $\dim \mathcal{U}_a(\lambda) = 1$ and $\dim \mathcal{U}_b(\lambda) = 2$. We can find $u \in \mathcal{U}_a(\lambda)$ and $v \in \mathcal{U}_b(\lambda)$ with $W(v,u) = 1$ such that

$$(L_\bullet - \lambda)^{-1} = G_{u,v}.$$

$L_\bullet$ is self-transposed and has separated boundary conditions.

(3) The inclusion $\mathcal{D}(L_\bullet) \subset \mathcal{D}(L_{\max})$ is of codimension 2.
Then the inclusion $\mathcal{D}(L_{\min}) \subset \mathcal{D}(L_\bullet)$ is of codimension 2. We have $\nu(L) = 4$.
The spectrum of $L_\bullet$ is discrete and its resolvents are Hilbert-Schmidt. For any $\lambda \in \mathbb{C}$ we have $\dim \mathrm{Ker}(L_{\max} - \lambda) = 2$, $\dim \mathcal{U}_a(\lambda) = 2$ and $\dim \mathcal{U}_b(\lambda) = 2$.
If in addition $L_\bullet$ is separated and self-transposed, and $\lambda \in \mathrm{rs}(L_\bullet)$, then we can find $u \in \mathcal{U}_a(\lambda)$ and $v \in \mathcal{U}_b(\lambda)$ with $W(v,u) = 1$ such that

$$(L_\bullet - \lambda)^{-1} = G_{u,v}.$$

If, instead, $L_\bullet$ is separated and not self-transposed, then it has empty spectrum and one of the following possibilities hold:
(i) *$L_\bullet = L_a$ and $(L_\bullet - \lambda)^{-1}$ is given by the forward Green's operator.*
(ii) *$L_\bullet = L_b$ and $(L_\bullet - \lambda)^{-1}$ is given by the backward Green's operator.*
We have $L_a^{\#} = L_b$, and both (i) and (ii) are described in Prop. 6.14.

6.6. **Existence of realizations with non-empty resolvent set.** $\mathbb{C}\backslash\mathbb{R}$ is contained in the resolvent set of all self-adjoint operators. The following proposition gives a generalization of this fact.

Proposition 6.19. *Let V_{R} and V_{I} be the real and imaginary part of V. Let $\|V_{\mathrm{I}}\|_\infty =: \beta < \infty$. Then*

$$\{\lambda \in \mathbb{C} \mid |\operatorname{Im}\lambda| > \beta\} \tag{6.40}$$

is contained in the resolvent set of some realizations of L. All realizations of L possess only disrete spectrum in (6.40).

Proof. Let $L_{\mathrm{R}} := -\partial_x^2 + V_{\mathrm{R}}$. By Theorem 3.4, $L_{\mathrm{R,min}}$ is densely defined and $L_{\mathrm{R,min}}^{\#} = L_{\mathrm{R,max}} \supset L_{\mathrm{R,min}}$. By the reality of V_{R}, $L_{\mathrm{R,min}}^{*} = L_{\mathrm{R,min}}^{\#}$. Therefore, $L_{\mathrm{R,min}}^{*} \supset L_{\mathrm{R,min}}$. This means that $L_{\mathrm{R,min}}$ is Hermitian (symmetric). Let us now apply the well-known theory of self-adjoint extensions of Hermitian operators. Let $d_\pm := \mathrm{Ker}\,\big(L_{\mathrm{R,min}}^{*} \mp \mathrm{i}\big)$ be the deficiency indices (see the proof of Proposition 5.13). Using the fact that $L_{\mathrm{R,min}}$ is real we conclude that $d_+ = d_-$. Therefore, $L_{\mathrm{R,min}}$ possesses at least one self-adjoint extension, which we denote $L_{\mathrm{R},\bullet}$. By the self-adjointness of $L_{\mathrm{R},\bullet}$ we have $\|(L_{\mathrm{R},\bullet}-\lambda)^{-1}\| \leq |\operatorname{Im}\lambda|^{-1}$ for all $\lambda \notin \mathbb{R} > 0$. Set $L_\bullet := L_{\mathrm{R},\bullet} + \mathrm{i}V_{\mathrm{I}}$. Clearly,

$$L_{\max} \supset L_\bullet \supset L_{\min}. \tag{6.41}$$

For $|\operatorname{Im}\lambda| > \beta$, λ belongs to the resolvent set of $L_\bullet$, and its resolvent is given by

$$(L_\bullet - \lambda)^{-1} = (L_{\mathrm{R},\bullet} - \lambda)^{-1}\big(\mathbb{1} + \mathrm{i}V_{\mathrm{I}}(L_{\mathrm{R},\bullet} - \lambda)^{-1}\big)^{-1}. \qquad \square$$

Note that the above proposition can be improved to cover some singularities of V_{I}. In fact, if there are numbers α, β with $0 \leq \alpha < 1$ such that

$$\|V_{\mathrm{I}} f\|^2 \leq \alpha^2\big(\|L_{\mathrm{R},\bullet} f\|^2 + \beta^2\|f\|^2\big), \quad \forall f \in \mathcal{D}(L_{\mathrm{R},\bullet}),$$

then still

$$\|V_{\mathrm{I}}(L_{\mathrm{R},\bullet} - \lambda)^{-1}\| \leq \alpha < 1,$$

and the conclusion of Prop. 6.19 holds.

6.7. **"Pathological" spectral properties.** We construct now Sturm-Liouville operators whose realizations have an empty resolvent set. Such operators seem to be rather pathological and not very interesting for applications.

Proposition 6.20. *There is $V \in L^\infty_{\mathrm{loc}}]0,\infty[$ such that if $L = -\partial^2 + V$ then any operator $L_\bullet$ on $L^2]0,\infty[$ with $L_{\min} \subset L_\bullet \subset L_{\max}$ has empty resolvent set, hence $\sigma(L_\bullet) = \mathbb{C}$.*

Proof. Let $I_n =]n^2 - n, n^2 + n[$ with $n \geq 1$ integer. Then I_n is an open interval of length $|I_n| = 2n$ and I_{n+1} starts with the point $n^2 + n$ which is the end point of I_n. Thus $\cup_n I_n$ is a disjoint union equal to $]0,\infty[\setminus\{n^2+n \mid n \geq 1\}$. Let $\mathbb{P}$ be the set prime numbers $\mathbb{P} = \{2,3,5,\dots\}$ and for each prime p denote $J_p = \cup_{k\geq 1} I_{p^k}$. We get a family of open subsets J_p of $]0,\infty[$ which are pairwise disjoint and each of them contains intervals of length as large as we wish. Now let $p \mapsto c_p$ be a bijective map from $\mathbb{P}$ to the set of complex rational numbers and let us define a function $V :]0,\infty[\to \mathbb{C}$ by the following rules: if $x \in J_p$ for some prime p then $V(x) = c_p$ and $V(x) = 0$ if $x \notin \cup_p J_p$. Then V is a locally bounded function whose range contains all the complex rational numbers. We set $L = -\partial^2 + V(x)$ and we prove that the spectrum of any $L_\bullet$ with $L_{\min} \subset L_\bullet \subset L_{\max}$ is equal to $\mathbb{C}$. Since the spectrum is closed, it suffices to show that any complex rational number c belongs to the spectrum of any $L_\bullet$. If not, there is a number $\alpha > 0$ such that $\|(L_\bullet - c)\phi\| \geq \alpha\|\phi\|$ for any $\phi \in \mathcal{D}(L_\bullet)$. If r is a (large) positive number then there is an open interval I of length $\geq r$ such that $V(x) = c$ on I. Let $\phi \in C_{\mathrm{c}}^\infty(I)$ such that $\phi(x) = 1$ for x at distance ≥ 1 from the boundary of I and with $|\phi''| \leq \beta$ with a constant β independent of r (take $r > 3$ for example). Then $\phi \in \mathcal{D}(L_{\min})$ and $(L - c)\phi = -\phi'' + V\phi - c\phi = -\phi''$ hence $\|\phi''\| = \|(L_\bullet - c)\phi\| \geq \alpha\|\phi\|$ so $\alpha\|\phi\| \leq 2\beta$ which is impossible because the left hand side is of order $\sqrt{r}$. One may choose V of class C^∞ by a simple modification of this construction. $\square$

7. Potentials with a negative imaginary part

7.1. **Dissipative operators.** Recall that an operator A is called *dissipative* if

$$\operatorname{Im}(f|Af) \leq 0, \quad f \in \mathcal{D}(A), \tag{7.1}$$

that is, if its numerical range is contained in $\{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda \leq 0\}$. It is called *maximal dissipative* if in addition its spectrum is contained in $\{\lambda \in \mathbb{C} \mid \operatorname{Im}\lambda \leq 0\}$. The following criterion is well-known.

Proposition 7.1. *Assume A is closed, densely defined and dissipative. Then A is maximal dissipative if and only if $-A^*$ is dissipative and then $-A^*$ is maximal dissipative.*

Proof. Note first that A is dissipative if and only if $\|(A-\mathrm{i}\lambda)f\| \geq \lambda\|f\|$ $\forall f \in \mathcal{D}(A)$ and $\forall\lambda > 0$. Indeed, if A is dissipative then $\|Af - \mathrm{i}\lambda f\|^2 = \|Af\|^2 + \lambda^2\|f\|^2 - 2\lambda\,\mathrm{Im}(f|Af) \geq \lambda^2\|f\|^2$ and reciprocally, if $\|(A-\mathrm{i}\lambda)f\| \geq \lambda\|f\|$ $\forall\lambda > 0$ then $\|Af\|^2 - 2\lambda\,\mathrm{Im}(f|Af) \geq 0$ by the same computation hence by making $\lambda \to \infty$ we see that A is dissipative.

Note one more fact: if A is dissipative then by (7.1) the operator $A-\mu$ is dissipative for any real μ hence we get $\|(A-\mu-\mathrm{i}\lambda)f\| \geq \lambda\|f\|$ $\forall f \in \mathcal{D}(A)$ and $\forall\lambda > 0$ and $\forall\mu \in \mathbb{R}$. Thus if A is dissipative then $\|(A-z)f\| \geq \mathrm{Im}\, z\|f\|$ $\forall f \in \mathcal{D}(A)$ and $\forall z \in \mathbb{C}$ with $\mathrm{Im}\, z > 0$.

A dissipative A is maximal dissipative if and only if $A-z$ is surjective for some z with $\mathrm{Im}\, z > 0$. Indeed, then z belongs to the resolvent set of A and $\|R\| \leq 1/\,\mathrm{Im}\, z$ if $R = (A-z)^{-1}$. So if $\zeta \in \mathbb{C}$

$$A - \zeta = A - z + z - \zeta = \big[1 + (z-\zeta)R\big](A-z)$$

and $\|(z-\zeta)R\| \leq |\zeta - z|/\,\mathrm{Im}\, z$. Thus if $|\zeta - z| < \mathrm{Im}\, z$ the operator $A-\zeta$ will be a bijective map $\mathcal{D}(A) \to L^2$, so ζ belongs to the resolvent set of A. It is now geometrically obvious that any ζ in the open upper half plane belongs to the resolvent set of A, so A is maximal dissipative.

If A is closed and dissipative and if $\mathrm{Im}\, z > 0$ then clearly $A-z$ is injective with closed range. If A is also densely defined then A^* is a closed densely defined operator and from $(A-z)^* = A^* - \overline{z}$ and Theorem A.3 we get that $A^* - \overline{z}$ is surjective. Thus if $-A^*$ is also dissipative then it will be maximal dissipative and again Theorem A.3 implies the surjectivity of $A-z$ so the maximal dissipativity of A. □

Remark 7.2. The relation $\mathcal{D}(A) \cap \mathcal{D}(A^*) = \{0\}$ is not exceptional in our context. Indeed, consider an operator $L = -\partial^2 + V$ so that $\mathrm{Im}\, V \leq 0$ and $L_{\min} = L_{\max}$. By Remark 3.10 the operator $\overline{L} = -\partial^2 + \overline{V}$ also has the property $\overline{L}_{\min} = \overline{L}_{\max}$. Then $A = L_{\min}$ is a closed densely defined dissipative operator and $A^* = -L^*_{\min} = -\overline{L}_{\max}$ hence $-A^*$ is also dissipative so A is maximal dissipative. If $\mathrm{Im}\, V$ is not square integrable on any non empty open set then $\mathcal{D}(A) \cap \mathcal{D}(A^*) = \{0\}$ by Lemma 3.9 or Remark 3.10.

7.2. **Dissipative Sturm-Liouville operators.** Recall that the complex conjugate $\overline{B}$ of an operator B, its hermitian adjoint B^*, and its transpose $B^{\#}$, are related by $B^* = \overline{B^{\#}}$ (cf. §3.1). Thus if L is the differential expression $L = -\partial^2 + V$ then $\overline{L}$ is the differential expression $\overline{L} = -\partial^2 + \overline{V}$ and for its minimal and maximal realization $\overline{L}_{\min}$ and $\overline{L}_{\max}$ we have

$$\mathcal{D}(\overline{L}_{\min}) = \{\overline{f} \mid f \in \mathcal{D}(L_{\min})\} \quad \text{and} \quad \mathcal{D}(\overline{L}_{\max}) = \{\overline{f} \mid f \in \mathcal{D}(L_{\max})\}$$

hence $\overline{L}_{\min} = \overline{L_{\min}}$ and $\overline{L}_{\max} = \overline{L_{\max}}$. Then $L^*_{\min} = \overline{L^{\#}_{\min}} = \overline{L_{\max}} = \overline{L}_{\max}$ and $L^*_{\max} = \overline{L}_{\min}$.

Proposition 7.3. *The operator $L_{\min}$ is dissipative if and only if* $\mathrm{Im}\, V \leq 0$.

Proof. If $f \in \mathcal{D}(L_{\max})$ then

$$\begin{aligned}(f|L_{\max}f) &= \int_a^b \big(\overline{f}'f' - (\overline{f}f')' + V\overline{f}f\\ &= \lim_{\substack{a_1\to a\\ b_1\to b}} \left(\overline{f}(a_1)f'(a_1) - \overline{f}(b_1)f'(b_1) + \int_{a_1}^{b_1} \big(|f'|^2 + V|f|^2\big)\right)\end{aligned}$$

hence

$$\mathrm{Im}(f|L_{\max}f) = \lim_{\substack{a_1\to a\\ b_1\to b}} \left(\int_{a_1}^{b_1} \mathrm{Im}(V)|f|^2 + \mathrm{Im}(\overline{f}(a_1)f'(a_1)) - \mathrm{Im}(\overline{f}(b_1)f'(b_1))\right). \tag{7.2}$$

Thus $\mathrm{Im}(f|L_{\min}f) = \int_{a_1}^{b_1} \mathrm{Im}(V)|f|^2$ for $f \in \mathcal{D}(L_{\mathrm{c}})$ which clearly implies the proposition. □

Since $L^*_{\min} = \overline{L}_{\max}$ the operator $L_{\min}$ will not be maximal dissipative in general (unless $L_{\min} = L_{\max}$). In the rest of this subsection and in the next one we study the dissipativity of the realisations of L introduced in Definition 4.11.

Let us point out a certain difficulty which appears in this context. If $L_{\min} \subset L_\bullet \subset L_{\max}$ then $L^*_{\max} \subset L^*_\bullet \subset L^*_{\min}$ hence $\overline{L}_{\min} \subset L^*_\bullet \subset \overline{L}_{\max}$. But we may have $\mathcal{D}(\overline{L}_{\max}) \cap \mathcal{D}(L_{\max}) = \{0\}$

(Lemma 3.9) hence we could have $\mathcal{D}(L_\bullet)\cap\mathcal{D}(L_\bullet^*)=\{0\}$ which is annoying when trying to prove the dissipativity of a restriction of $L_{\max}$. Indeed, although $W_x(\overline{f},f)=2\mathrm{i}\,\mathrm{Im}(\overline{f}(x)f'(x))$, we cannot use in (7.2) the existence of the limits (3.11) and (3.12) because in general $\overline{f}\notin\mathcal{D}(L_{\max})$. Of course, if for example a is a regular end then f,f' extend to continuous functions on $[a,b[$ so there is no problem in taking the limit as $a_1\to a$ in each term in the right hand side of (7.2) and if b is also regular then we get the simple expression

$$\mathrm{Im}(f|L_{\max}f)=\int_a^b\mathrm{Im}(V)|f|^2+\mathrm{Im}(\overline{f}(a)f'(a))-\mathrm{Im}(\overline{f}(b)f'(b)).$$

Then the dissipativity of some realization $L_\bullet$ means

$$\mathrm{Im}(\overline{f}(a)f'(a))-\mathrm{Im}(\overline{f}(b)f'(b))\le\int_a^b\mathrm{Im}(-V)|f|^2\quad\forall f\in\mathcal{D}(L_\bullet)$$

so we clearly have $\mathrm{Im}(V)\le0$. If $L_\bullet$ is defined by separated boundary conditions then this implies

$$\mathrm{Im}(\overline{f}(a)f'(a))\le\int_a^b\mathrm{Im}(-V)|f|^2\quad\forall f\in\mathcal{D}(L_\bullet)\text{ which is zero near }b$$

from which we easily get $\mathrm{Im}(\overline{f}(a)f'(a))\le0$ for all $f\in\mathcal{D}(L_\bullet)$ and similarly at b. Thus

$$L_\bullet\text{ dissipative }\Rightarrow\mathrm{Im}(V)\le0\text{ and }\mathrm{Im}(\overline{f}(a)f'(a))\le0,\ \mathrm{Im}(\overline{f}(b)f'(b))\ge0\ \forall f\in\mathcal{D}(L_\bullet).$$

We will complete this argument in §7.3 and here we treat the general case under a certain simplifying hypothesis. Note first the following sesquilinear version of Green's identity (3.13).

Lemma 7.4. *Suppose that $f,\overline{f},g\in\mathcal{D}(L_{\max})$. Then $\mathrm{Im}(V)f\in L^2]a,b[$ and*

$$(L_{\max}f|g)-(f|L_{\max}g)=-2\mathrm{i}\int_a^b\mathrm{Im}(V)\overline{f}g+W_b(\overline{f},g)-W_a(\overline{f},g).\tag{7.3}$$

Proof. The left hand side of (7.3) is

$$\begin{aligned}\langle\overline{L_{\max}f}|g\rangle-\langle\overline{f}|L_{\max}g\rangle&=\langle\overline{L_{\max}f}|g\rangle-\langle L_{\max}\overline{f}|g\rangle &(7.4)\\ &\quad+\langle L_{\max}\overline{f}|g\rangle-\langle\overline{f}|L_{\max}g\rangle. &(7.5)\end{aligned}$$

Then we apply $\overline{L_{\max}}-L_{\max}=-2\mathrm{i}\,\mathrm{Im}(V)$ to (7.4) and Green's identity (3.13) to (7.5). □

We consider now the realizations of L introduced in Definition 4.11. Fix $\alpha\in\mathcal{B}_a$ and $\beta\in\mathcal{B}_b$ and let $L_{\alpha\beta}$ be the restriction of $L_{\max}$ to the domain $\mathcal{D}(L_{\alpha\beta})=\{f\in\mathcal{D}(L_{\max})\mid\alpha(f)=\beta(f)=0\}$. Note that if for example $\alpha\neq0$ then the relation $\alpha(f)=0$ is equivalent to: there is a complex number $c(f)$ such that $\vec{f_a}=c(f)\alpha$. We will assume that $\alpha\neq0$ if $\mathcal{B}_a\neq\{0\}$ and similarly for $\beta\neq0$ if $\mathcal{B}_b\neq\{0\}$. If for example $\mathcal{B}_b=\{0\}$ then we set $L_\alpha=L_{\alpha0}$ (no boundary condition at b).

We will first compute the hermitian adjoint of $L_{\alpha\beta}$. This result is also a consequence of Proposition 6.17, but a direct proof is easy and instructive. We need the following notion.

The *conjugate* of $\alpha\in\mathcal{B}(L)$ is the boundary functional $\overline{\alpha}\in\mathcal{B}(\overline{L})$ given by

$$\overline{\alpha}(f):=\overline{\alpha(\overline{f})}.\tag{7.6}$$

Clearly $\alpha\mapsto\overline{\alpha}$ is a bijective anti-linear map $\mathcal{B}(L)\to\mathcal{B}(\overline{L})$ which sends $\mathcal{B}_a(L)$ into $\mathcal{B}_a(\overline{L})$ and $\mathcal{B}_b(L)$ into $\mathcal{B}_b(\overline{L})$. Then if $g\in\mathcal{D}(L_{\max})$ is a representative of $\alpha\in\mathcal{B}_a$, so that

$$\alpha(f)=W_a(g,f),\quad f\in\mathcal{D}(L_{\max}),\tag{7.7}$$

then

$$\overline{\alpha}(f)=W_a(\overline{g},f),\quad f\in\mathcal{D}(\overline{L}_{\max}).\tag{7.8}$$

Recall that $\mathcal{B}_a$ and $\mathcal{B}_b$ are equipped with symplectic forms σ_a and σ_b, see (4.13).

Proposition 7.5. $L_{\alpha\beta}^*=\overline{L}_{\overline{\alpha}\,\overline{\beta}}$.

Proof. We consider only the case $\alpha \neq 0 \neq \beta$. By interchanging L and $\overline{L}$ in $L_{\min}^* = \overline{L}_{\max}$ we get $\overline{L}_{\min}^* = L_{\max}$ hence $\overline{L}_{\min} = L_{\max}^*$. Then from $L_{\min} \subset L_{\alpha\beta} \subset L_{\max}$ we get $\overline{L}_{\min} \subset L_{\alpha\beta}^* \subset \overline{L}_{\max}$ hence we have $f \in \mathcal{D}(L_{\alpha\beta}^*)$ if and only if $f \in \mathcal{D}(\overline{L}_{\max})$ and $(f|Lg) = (\overline{L}f|g)$ $\forall g \in \mathcal{D}(L_{\alpha\beta})$. This may be written $\langle \overline{f}|Lg\rangle = \langle L\overline{f}|g\rangle$ $\forall g \in \mathcal{D}(L_{\alpha\beta})$ and since $f \in \mathcal{D}(\overline{L}_{\max}) \Rightarrow \overline{f} \in \mathcal{D}(L_{\max})$ the Green's identity (3.13) gives

$$\langle L\overline{f}|g\rangle - \langle \overline{f}|Lg\rangle = W_b(\overline{f}, g) - W_a(\overline{f}, g) \quad \forall g \in \mathcal{D}(L_{\alpha\beta}).$$

Thus $f \in \mathcal{D}(L_{\alpha\beta}^*)$ if and only if $f \in \mathcal{D}(\overline{L}_{\max})$ and $W_b(\overline{f}, g) = W_a(\overline{f}, g)$ for all $g \in \mathcal{D}(L_{\alpha\beta})$ and this is clearly equivalent to $W_a(\overline{f}, g) = 0$ and $W_b(\overline{f}, g) = 0$ for all $g \in \mathcal{D}(L_{\alpha\beta})$.

Hence $\overline{f} \in \mathcal{D}(L_{\max})$ and $W_a(\overline{f}, g) = 0$ for all $g \in \mathcal{D}(L_{\max})$ such that $\alpha(g) = 0$. From Theorem 4.5 it follows that there is $g \in \mathcal{D}(L_{\max})$ such that $\vec{g}_a = \alpha$ and this implies $\alpha(g) = 0$. Then $W_a(g, \overline{f}) = 0$, which means $\alpha(\overline{f}) = 0$ or $\overline{\alpha}(f) = 0$. Similarly $\overline{\beta}(f) = 0$. □

For the rest of the argument we need the equality of the domains $\mathcal{D}(\overline{L}_{\max}) = \mathcal{D}(L_{\max})$ which, by Lemma 3.9, is equivalent to $\operatorname{Im} V \in L^2_{\mathrm{loc}}]a, b[$. Then we have $\mathcal{B}(L) = \mathcal{B}(\overline{L})$ hence $\alpha \mapsto \overline{\alpha}$ is a conjugation in $\mathcal{B}(L)$ which leaves invariant the subspaces $\mathcal{B}_a(L)$ and $\mathcal{B}_b(L)$. In particular, the number $\sigma_a(\overline{\alpha}, \alpha)$ is well defined for any $\alpha \in \mathcal{B}_a(L)$.

Lemma 7.6. *If $\operatorname{Im} V \in L^2_{\mathrm{loc}}]a, b[$ and $\alpha \in \mathcal{B}_a$ then the number $\sigma_a(\overline{\alpha}, \alpha)$ is purely imaginary and*

$$\frac{1}{2\mathrm{i}}\sigma_a(\overline{\alpha}, \alpha) \geq 0 \iff \frac{1}{2\mathrm{i}} W_a(\overline{f}, f) \geq 0 \ \forall f \in \mathcal{D}(L_{\max}) \text{ with } \alpha(f) = 0.$$

Proof. Let $g \in \mathcal{D}(L_{\max})$ be a representative of α, so that (7.7) and (7.8) are true. Then

$$\sigma_a(\overline{\alpha}, \alpha) = W_a(\overline{g}, g) = \lim_{c \searrow a} \big(\overline{g}(c)g'(c) - \overline{g}'(c)g(c)\big),$$

which proves that $\sigma_a(\overline{\alpha}, \alpha)$ is purely imaginary. Now, by the Kodaira identity

$$W_a(\overline{g}, g)W_a(\overline{f}, f) = |W_a(g, f)|^2 - |W_a(\overline{g}, f)|^2.$$

But $\alpha(f) = 0$ means $W_a(g, f) = 0$. Therefore, $\sigma_a(\overline{\alpha}, \alpha)W_a(\overline{f}, f) \leq 0$. □

Theorem 7.7. *If $\operatorname{Im} V \in L^2_{\mathrm{loc}}]a, b[$ we have*

$$L_{\alpha\beta} \text{ is dissipative} \iff \operatorname{Im} V \leq 0, \ \frac{1}{2\mathrm{i}}\sigma_a(\overline{\alpha}, \alpha) \leq 0, \text{ and } \frac{1}{2\mathrm{i}}\sigma_b(\overline{\beta}, \beta) \geq 0. \tag{7.9}$$

And then $L_{\alpha\beta}$ is maximal dissipative.

Proof. We consider only the case $\alpha \neq 0, \beta \neq 0$. Lemma 7.4 gives

$$\operatorname{Im}(f|L_{\max}f) = \int_a^b \operatorname{Im}(V)|f|^2 + \frac{1}{2\mathrm{i}}W_a(\overline{f}, f) - \frac{1}{2\mathrm{i}}W_b(\overline{f}, f) \quad \forall f \in \mathcal{D}(L_{\max}) \tag{7.10}$$

and this implies that $L_{\alpha\beta}$ is dissipative if and only if

$$\frac{1}{2\mathrm{i}}W_a(\overline{f}, f) - \frac{1}{2\mathrm{i}}W_b(\overline{f}, f) \leq \int_a^b \operatorname{Im}(-V)|f|^2 \quad \forall f \in \mathcal{D}(L_{\alpha\beta}). \tag{7.11}$$

If $L_{\alpha\beta}$ is dissipative, by taking $f \in \mathcal{D}(L_{\mathrm{c}})$ in (7.11) we get $\operatorname{Im}(-V) \geq 0$. Then by choosing $f \in \mathcal{D}(L_{\alpha\beta})$ equal to zero near b we get $\frac{1}{2\mathrm{i}}W_a(\overline{f}, f) \leq \int_a^b \operatorname{Im}(-V)|f|^2$. If we fix such an f and replace it in this estimate by $f\theta$ where $\theta \in C^\infty(\mathbb{R})$ with $0 \leq \theta \leq 1$ and $\theta(x) = 1$ on a neighborhood of a the we get $\frac{1}{2\mathrm{i}}W_a(\overline{f}, f) \leq \int_a^b \operatorname{Im}(-V)|f\theta|^2$. Since the right hand side here can be made as small as we wish by taking θ equal to zero for $x > d > a$ with d close to a, we see that we must have $\frac{1}{2\mathrm{i}}W_a(\overline{f}, f) \leq 0$ and this clearly implies the same inequality for any $f \in \mathcal{D}(L_{\alpha\beta})$. Then we get $\frac{1}{2\mathrm{i}}\sigma_a(\overline{\alpha}, \alpha) \leq 0$ by Lemma 7.6. We similarly prove $\frac{1}{2\mathrm{i}}\sigma_b(\overline{\beta}, \beta) \geq 0$.

We proved the implication $\Rightarrow$ in (7.9) and $\Leftarrow$ is clear by (7.11). It remains to show the maximal dissipativity assertion. Due to Propositions 7.1 and 7.5 it suffices to prove that the operator $-L^*_{\alpha\beta} = -\overline{L}_{\overline{\alpha}\,\overline{\beta}}$ is dissipative. Observe first that the relation $\mathcal{D}(\overline{L}_{\max}) = \mathcal{D}(L_{\max})$ implies $\mathcal{D}(\overline{L}_{\overline{\alpha}\,\overline{\beta}}) = \mathcal{D}(L_{\overline{\alpha}\,\overline{\beta}})$. Then (7.10) gives

$$\operatorname{Im}(f| - \overline{L}_{\max} f) = \int_a^b \operatorname{Im}(V)|f|^2 - \frac{1}{2\mathrm{i}} W_a(\overline{f}, f) + \frac{1}{2\mathrm{i}} W_b(\overline{f}, f) \quad \forall f \in \mathcal{D}(L_{\max}) \tag{7.12}$$

hence instead of (7.11) we get the condition

$$-\frac{1}{2\mathrm{i}} W_a(\overline{f}, f) + \frac{1}{2\mathrm{i}} W_b(\overline{f}, f) \le \int_a^b \operatorname{Im}(-V)|f|^2 \quad \forall f \in \mathcal{D}(L_{\overline{\alpha}\,\overline{\beta}}).$$

As above we get $\frac{1}{2\mathrm{i}} W_a(\overline{f}, f) \ge 0$ and $\frac{1}{2\mathrm{i}} W_b(\overline{f}, f) \le 0$ for any $f \in \mathcal{D}(L_{\overline{\alpha}\,\overline{\beta}})$. Thus, if $f \in \mathcal{D}(L_{\max})$ and $\overline{\alpha}(f) = 0$ then $\frac{1}{2\mathrm{i}} W_a(\overline{f}, f) \ge 0$ and by Lemma 7.6 this means $\frac{1}{2\mathrm{i}}\sigma_a(\alpha, \overline{\alpha}) \ge 0$ which is equivalent to $\frac{1}{2\mathrm{i}}\sigma_a(\overline{\alpha}, \alpha) \le 0$. Similarly we get $\frac{1}{2\mathrm{i}}\sigma_b(\overline{\beta}, \beta) \ge 0$ and the last two conditions are satisfied by the assumptions in the right hand side of (7.9). Hence $-L^*_{\alpha\beta}$ is dissipative. ☐

7.3. **Regular boundary conditions.** Suppose that the operator L has a regular left endpoint at a. As we noted several times, for regular boundary conditions $\mathcal{B}_a$ can be identified with $\mathbb{C}^2$. Indeed,

$$\alpha(f) = \alpha_0 f'(a) - \alpha_1 f(a),$$

is a general form of a boundary functional, with $\alpha = (\alpha_0, \alpha_1) \in \mathbb{C}^2$ and $f \in \mathcal{D}(L_{\max})$.

The space $\mathcal{B}_a$ is equipped with the symplectic form σ_a, which coincides with the usual (two-dimensional) vector product:

$$\sigma_a(\alpha, \beta) = \alpha_0\beta_1 - \alpha_1\beta_0 = \alpha \times \beta.$$

Thus, if we write $\vec{f}_a := \big(f(a), f'(a)\big)$, an alternative notation for $\alpha(f)$ is

$$\alpha(f) = \alpha \times \vec{f}_a.$$

Note that there is no guarantee that $\mathcal{D}(L_{\min})$ and $\mathcal{D}(L_{\max})$ are invariant wrt the complex conjugation. However the space $\mathcal{B}_a \simeq \mathbb{C}^2$ is equipped with the obvious complex conjugation:

$$\overline{\alpha}(f) = \overline{\alpha}_0 f'(a) - \overline{\alpha}_1 f(a) = \overline{\alpha} \times \vec{f}_a.$$

Lemma 7.8. *(1) $\alpha \times \beta = 0$ if and only if the vectors α, β are collinear.*

(2) $\overline{\alpha} \times \alpha \in \mathrm{i}\mathbb{R}$ and $\overline{\alpha} \times \alpha = 0$ if and only if α is proportional to a real vector.

(3) $(\overline{\alpha} \times \alpha)(\overline{\beta} \times \beta) = |\alpha \times \beta|^2 - |\overline{\alpha} \times \beta|^2$

Proof. (1): If $\alpha_0\beta_1 = \alpha_1\beta_0$ and $\beta \ne 0$ then $\beta_k = 0 \Rightarrow \alpha_k = 0$ and if $\beta_0 \ne 0 \ne \beta_1$ then $\alpha_0/\beta_0 = \alpha_1/\beta_1$. (2): If $\overline{\alpha} \times \alpha = 0$ we get $\overline{\alpha} = c^2\alpha$ for some complex c with $|c| = 1$ which implies $(c\alpha)^* = c\alpha$. (3) follows by the Kodaira identity. ☐

Here is a version of Lemma 7.4 for the regular case.

Lemma 7.9. *Let $V \in L^1]a, b[$. Suppose that $f, g \in \mathcal{D}(L_{\max})$.*

$$(L_{\max} f|g) - (f|L_{\max} g) = -2\mathrm{i} \int_a^b \operatorname{Im}(V)\overline{f}g + W_b(\overline{f}, g) - W_a(\overline{f}, g). \tag{7.13}$$

Next we have a version of Thm 7.7 for the regular case. Fix *nonzero vectors* α, β and define $L_{\alpha\beta}$ by imposing the boundary conditions at a and b:

$$f(a)\alpha_1 - f'(a)\alpha_0 = 0, \quad f(b)\beta_1 - f'(b)\beta_0 = 0.$$

In this context it is quite easy to prove that $L^*_{\alpha\beta} = \overline{L}_{\overline{\alpha}\,\overline{\beta}}$.

Theorem 7.10. *Suppose that a, b are finite and $V \in L^1]a, b[$. Then $L_{\alpha\beta}$ is dissipative if and only if $\operatorname{Im} V \le 0$ and $\operatorname{Im}(\overline{\alpha}_0\alpha_1) \le 0$, $\operatorname{Im}(\overline{\beta}_0\beta_1) \ge 0$. And in this case $L_{\alpha\beta}$ is maximal dissipative.*

Proof. The proof is similar to that of Theorem 7.7, but much simpler. We use Lemma 7.9 instead of Lemma 7.4 and get the same relation (7.11) as necessary and sufficient condition for dissipativity. Then we use

$$\frac{1}{2\mathrm{i}}\sigma_a(\overline{\alpha},\alpha) = \frac{1}{2\mathrm{i}}(\overline{\alpha}_0\alpha_1 - \overline{\alpha}_1\alpha_0) = \mathrm{Im}(\overline{\alpha}_0\alpha_1)$$

and a similar relation for β. Finally, when checking the dissipativity of $-L_{\alpha\beta}^*$, note that this operator is associated to the differential expression $\partial^2 - \overline{V}$, which explains a difference of sign. □

7.4. **Weyl circle in the regular case.** In this subsection we fix a regular operator L and prove Theorem 7.11, which will be needed in the next subsection §7.5. We will use an argument essentially due to H. Weyl in the real case, cf. [5, 20, 21] for example. Potentials with semi-bounded imaginary part were first treated in [23], see [3] for more recent results.

Let us denote $U = \mathrm{Im}(\lambda - V)$ and

$$(f|g)_U = \int_a^b \overline{f} g U. \tag{7.14}$$

We set $\|f\|_U^2 = (f|f)_U$ and note that if $U \geq 0$ then $(\cdot|\cdot)_U$ is a positive hermitian form and we denote $\|\cdot\|_U$ is the corresponding seminorm. Now if $f, g \in \mathcal{D}(L_{\max})$ and $Lf = \lambda f$, $Lg = \lambda g$ for some complex number λ then (7.3) can be rewritten as

$$2\mathrm{i}(f|g)_U = W_a(\overline{f}, g) - W_b(f, g). \tag{7.15}$$

Theorem 7.11. *Assume that* $\mathrm{Im}\, V \leq 0$ *and* $\mathrm{Im}\,\lambda > 0$. *Let* u, v *be solutions of the equation* $Lf = \lambda f$ *with real boundary condition at* a *and satisfying* $W(v, u) = 1$. *If* w *is a solution of* $Lf = \lambda f$ *with a real boundary condition at* b, *then there is a unique* $m \in \mathbb{C}$ *such that* $w = mu + v$; *this number is on the circle*

$$\int_a^b |mu + v|^2\, \mathrm{Im}(\lambda - V) = \mathrm{Im}\, m, \tag{7.16}$$

which has

$$\textit{center } c = \frac{\mathrm{i}/2 - (u|v)_U}{\|u\|_U^2} = \frac{W_b(\overline{u}, v)}{2\mathrm{i}\|u\|_U^2} \quad \textit{and radius } r = \frac{1}{2\|u\|_U^2}. \tag{7.17}$$

Conversely, let m *be a complex number on the circle* (7.16), *and define* w *by* $w = mu + v$. *Then* w *has a real boundary condition at* b *and* $W(w, u) = 1$.

Proof. From Lemma 7.8 (2) and the reality of the boundary conditions at a we get

$$W_a(\overline{u}, u) = 0, \quad W_a(\overline{v}, v) = 0. \tag{7.18}$$

This implies

$$\|u\|_U^2 = \frac{\mathrm{i}}{2} W_b(\overline{u}, u), \quad \|v\|_U^2 = \frac{\mathrm{i}}{2} W_b(\overline{v}, v), \tag{7.19}$$

due to (7.15). And if w is as in the first part of the theorem then the same argument gives

$$\|w\|_U^2 = \frac{1}{2\mathrm{i}} W_a(\overline{w}, w). \tag{7.20}$$

Since u, v are linearly independent solutions of $Lf = \lambda f$, if w is another solution then we have $w = mu + nv$ for uniquely determined complex numbers m, n. Since $W(v, u) = 1$ we see that $n = 1$.

Now fix $w = mu + v$. Using (7.18) and $W_a(u, v) = -1$, we get

$$W(\overline{w}, w)_a = |m|^2 W_a(\overline{u}, u) + \overline{m} W_a(\overline{u}, v) + m W_a(\overline{v}, u) + W_a(\overline{v}, v) = 2\mathrm{i}\, \mathrm{Im}\, m. \tag{7.21}$$

From (7.20) and (7.21) we get

$$\|w\|_U^2 = \mathrm{Im}\, m. \tag{7.22}$$

From this relation we get

$$\mathrm{Im}\, m = \|mu + v\|_U^2 = |m|^2 \|u\|_U^2 + 2\, \mathrm{Re}\left(\overline{m}(u|v)_U\right) + \|v\|_U^2 \tag{7.23}$$

and since $\operatorname{Im} m = 2\operatorname{Re}(\overline{m}\,\mathrm{i}/2)$ we may rewrite this as

$$|m|^2\|u\|_U^2 - 2\operatorname{Re}\big(\overline{m}(\mathrm{i}/2-(u|v)_U)\big) + \|v\|_U^2 = 0. \tag{7.24}$$

Clearly, $\|w\|_U > 0$ hence $\operatorname{Im} m > 0$ by (7.22) so (7.24) may be rewritten

$$|m|^2 - 2\operatorname{Re}\left(\overline{m}\frac{\mathrm{i}/2-(u|v)_U}{\|u\|_U^2}\right) + \frac{\|v\|_U^2}{\|u\|_U^2} = 0. \tag{7.25}$$

If $c\in\mathbb{C}$ and $d\in\mathbb{R}$ then $|m|^2-2\operatorname{Re}(\overline{m}c)+d = |m-c|^2-(|c|^2-d)$. Hence there is m such that $|m|^2-2\operatorname{Re}(\overline{m}c)+d=0$ if and only if $d\le|c|^2$, and then $|m|^2-2\operatorname{Re}(\overline{m}c)+d=0$ is the equation of a circle with center c and radius $\sqrt{|c|^2-d}$. Thus (7.25) is the equation of the circle with

$$\text{center}\quad c=\frac{\mathrm{i}/2-(u|v)_U}{\|u\|_U^2}\quad\text{and square of radius}\quad r^2=\frac{|\mathrm{i}/2-(u|v)_U|^2-\|u\|_U^2\|v\|_U^2}{\|u\|_U^4}.$$

From (7.15) we get $2\mathrm{i}(u|v)_U = W_a(\overline{u},v)-W_b(\overline{u},v) = -1-W_b(\overline{u},v)$, hence $\mathrm{i}/2-(u|v)_U = W_b(\overline{u},v)/2\mathrm{i}$. Then (7.19) implies

$$\|u\|_U^2\|v\|_U^2 = -\frac{1}{4}W_b(\overline{u},u)W_b(\overline{v},v),$$

hence

$$|\mathrm{i}/2-(u|v)_U|^2-\|u\|_U^2\|v\|_U^2 = \big(|W_b(\overline{u},v)|^2+W_b(\overline{u},u)W_b(\overline{v},v)\big)/4.$$

But by the Kodaira identity $W_b(\overline{u},u)W_b(\overline{v},v)=1-|W_b(\overline{u},v)|^2$, hence we get

$$|\mathrm{i}/2-(u|v)_U|^2-\|u\|_U^2\|v\|_U^2 = 1/4$$

so (7.25) is just the circle described by (7.17).

To prove the reciprocal part of the theorem, consider a point m on this circle and let $w=mu+v$. Clearly $Lw=\lambda w$ and $W(w,u)=1$ and the computation (7.21) gives us $W_a(\overline{w},w)=2\mathrm{i}\operatorname{Im} m$. We also have (7.23) because this just says that m is on the circle (7.17). Thus we have

$$\|w\|_U^2 = \operatorname{Im} m = W_a(\overline{w},w)/2\mathrm{i}$$

and then (7.15) implies $W_b(\overline{w},w)=0$. Therefore, by Lemma 7.15 w has a real boundary condition at b. This proves the final assertion of the theorem. □

7.5. **Limit point/circle.** In this section we assume that $V\in L^1_{\mathrm{loc}}]a,b[$ and $\operatorname{Im} V\le 0$. This class of potentials has first been considered in [23]; see [3] for more general conditions. We also assume that a is a regular endpoint for L. If not, the analysis should be done separately on intervals $[a,a_1[$ and $]b_1,b[$ and one gets similar results on each of these intervals. What follows is an immediate consequence of Theorem 7.11.

Fix a number λ with $\operatorname{Im}\lambda>0$. Let u,v be solutions of $Lf=\lambda f$ on $]a,b[$ with real boundary conditions at a and such that $W(v,u)=1$.

Definition 7.12. *Then for any $d\in]a,b[$ we define*

$$\textit{the } \text{Weyl circle}\quad \mathscr{C}_d := \Big\{m\in\mathbb{C}\mid \int_a^d |mu+v|^2\operatorname{Im}(\lambda-V)=\operatorname{Im} m\Big\},$$

$$\textit{the } \text{open Weyl disk}\quad \mathscr{C}_d^\circ := \Big\{m\in\mathbb{C}\mid \int_a^d |mu+v|^2\operatorname{Im}(\lambda-V)<\operatorname{Im} m\Big\},$$

$$\textit{the } \text{closed Weyl disk}\quad \mathscr{C}_d^\bullet := \Big\{m\in\mathbb{C}\mid \int_a^d |mu+v|^2\operatorname{Im}(\lambda-V)\le\operatorname{Im} m\Big\} = \mathscr{C}_d^\circ\cup\mathscr{C}_d.$$

Thus the Weyl circle is given by the condition (7.16) with b replaced by d. Since the left hand side of (7.16) growth like $|m|^2$ when $m\to\infty$, it follows that $\mathscr{C}_d^\circ$ is inside $\mathscr{C}_d$. If $d_1<d_2$ then

$$\int_a^{d_2}|mu+v|^2\operatorname{Im}(\lambda-V)\le\operatorname{Im} m \Rightarrow \int_a^{d_1}|mu+v|^2\operatorname{Im}(\lambda-V)<\operatorname{Im} m$$

hence $\mathscr{C}_{d_2}^\bullet\subset\mathscr{C}_{d_1}^\circ$ strictly if $d_1<d_2<b$.

Definition 7.13. *We set*

$$\mathscr{C}_b^\bullet := \bigcap_{d<b} \mathscr{C}_d^\bullet$$
$$\mathscr{C}_b := \text{the boundary of } \mathscr{C}_b^\bullet.$$

It follows that either $\mathscr{C}_b^\bullet = \mathscr{C}_b$ is a point, or $\mathscr{C}_b^\bullet$ is a disk and $\mathscr{C}_b$ is a circle of radius > 0.

Definition 7.14. *We say that b is* limit point *if $\mathscr{C}_b$ is a point. We say that b is* limit circle *if $\mathscr{C}_b$ is a circle of a positive radius.*

Lemma 7.15. *Let $m \in \mathscr{C}_b^\bullet$. Then*

$$\int_a^b |w|^2 \operatorname{Im}(\lambda - V) \le \operatorname{Im}(m). \tag{7.26}$$

If b is limit point then $\int_a^b |u|^2 \operatorname{Im}(\lambda - V) = \infty$.

Proof. For any $d \in]a, b[$, we have $\mathscr{C}_b^\bullet \subset \mathscr{C}_d^\bullet$. Therefore,

$$\int_a^d |w|^2 \operatorname{Im}(\lambda - V) \le \operatorname{Im}(m). \tag{7.27}$$

Then we take the limit $d \nearrow b$. If b is limit point then the radius of the Weyl circle $\mathscr{C}_d$ tends to zero as $d \to b$ hence $\lim_{d\to b} \int_a^d |u|^2 \operatorname{Im}(\lambda - V) = \infty$ by the last relation in (7.17). □

The above lemma implies immediately the following theorem:

Theorem 7.16. *If b is limit circle, then all solutions of $(L - \lambda)f - 0$ satisfy*

$$\int_a^b |w|^2 \operatorname{Im}(\lambda - V) \quad \textit{is bounded.} \tag{7.28}$$

If b is limit point, then there exists only one (modulo a complex factor) solution of $(L - \lambda)f = 0$ satisfying (7.28).

Note that $\operatorname{Im}(\lambda - V) \ge \operatorname{Im} \lambda > 0$. Therefore, (7.28) implies the square integrability of w.

Thus, for the potentials with a negative imaginary part instead of Weyl's dichotomy we have three possibilities (we think of solutions modulo a complex factor):

(1) limit point case, only one solution satisfies (7.28), only one solution is square integrable;
(2) limit point case, only one solution satisfies (7.28), all solutions are square integrable;
(3) limit circle, all solutions satisfy (7.28), and hence all solutions are square integrable.

We emphasize that *the limit point/circle terminology is interpreted here in the geometric sense described above* (based on Theorem 7.11). If V is real then one can say without ambiguity that L is limit point at b if for any λ there is at most one solution of $Lf = \lambda f$ which is square integrable near b: indeed, this is equivalent to the geometric meaning of the terminology. But this is not the case if V is complex.

Thus the complex case differs from the real one in an important aspect: if V is real, then the case (2) is absent and we have the usual Weyl's dichotomy.

There exist examples of (2) in the literature. *In the limit point case, it is possible that we have only one nonzero solution satisfying* (7.28)*, whereas all solutions are square integrable with respect to the Lebesgue measure.* Indeed, Sims [23, p. 257] has shown that this happens in simple examples like $V(x) = x^6 - 3ix^2/2$ on $]1, \infty[$. See also the discussion in [3].

We also note that if V is real then for any non-real λ there is at least one nonzero solution of $Lf = \lambda f$ which is square integrable near b. But it does not seem to be known whether for arbitrary complex V there is λ such that $Lf = \lambda f$ has a nonzero solution which is square integrable near b.

Appendix A. Abstract lemmas

Lemma A.1. *Let S, T be linear operators on a Hilbert space $\mathcal{H}$ such that:*
(1) $(Sf|g) = (f|Tg)$ *for all* $f \in \mathcal{D}(S)$ *and* $g \in \mathcal{D}(T)$,
(2) T *is surjective,*
(3) $(\operatorname{Ran} S)^{\perp} \subset \operatorname{Ker} T$.
Then S is densely defined.

Proof. We must show:

$$(f|h) = 0,\ \forall f \in \mathcal{D}(S) \Rightarrow h = 0.$$

Since T is surjective, there is $g \in \mathcal{D}(T)$ such that $h = Tg$ and then we get $0 = (f|h) = (f|Tg) = (Sf|g)$ by (1) for all $f \in \mathcal{D}(S)$. Thus $g \in (\operatorname{Ran} S)^{\perp}$ and (3) gives $Tg = 0$ hence $h = 0$. □

Lemma A.2. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{K}$ a closed subspace of finite codimension. If $\mathcal{Z}$ is a dense subspace of $\mathcal{H}$, then $\mathcal{Z} \cap \mathcal{K}$ is dense in $\mathcal{K}$.*

Proof. The lemma is obvious if the codimension is 1. Then we apply induction. □

We also recall the closed range theorem [27, Sect. VII.5] in the form used in §6.1.

Theorem A.3. *Let $\mathcal{H}$ be a Banach space and $\mathcal{H}'$ its topological dual. If T is a closed densely defined operator in $\mathcal{H}$ and T' is its dual operator acting in $\mathcal{H}'$* [27, Sect. VII.1]*, then the following assertions are equivalent:*
(1) $\operatorname{Ran} T$ *is closed in* X, (2) $\operatorname{Ran} T'$ *is closed in* X',
(3) $\operatorname{Ran} T = (\operatorname{Ker} T')^{\mathrm{perp}} = \{u \in \mathcal{H} \mid \langle u|u'\rangle = 0\ \forall u' \in \operatorname{Ker} T'\}$,
(4) $\operatorname{Ran} T' = (\operatorname{Ker} T)^{\mathrm{perp}} = \{u' \in \mathcal{H}' \mid \langle u|u'\rangle = 0\ \forall u \in \operatorname{Ker} T\}$.

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(Jan Dereziński) Department of Mathematical Methods in Physics, Faculty of Physics, University of Warsaw, Pasteura 5, 02-093 Warszawa, Poland

E-mail address: `jan.derezinski@fuw.edu.pl`

(Vladimir Georgescu) Département de Mathématiques, Université de Cergy-Pontoise, 95000 Cergy-Pontoise, France

E-mail address: `vladimir.georgescu@math.cnrs.fr`