CONTINUOUS AND HOLOMORPHIC FUNCTIONS WITH VALUES IN CLOSED OPERATORS

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ABSTRACT. We systematically derive general properties of continuous and holomorphic functions with values in closed operators, allowing in particular for operators with empty resolvent set. We provide criteria for a given operator-valued function to be continuous or holomorphic. This includes sufficient conditions for the sum and product of operator-valued holomorphic functions to be holomorphic.

Using graphs of operators, operator-valued functions are identified with functions with values in subspaces of a Banach space. A special role is thus played by projections onto closed subspaces of a Banach space, which depend holomorphically on a parameter.

1. INTRODUCTION

Definition of continuous and holomorphic operator-valued functions. It is obvious how to define the concept of a (norm) continuous or holomorphic function with values in bounded operators. In fact, let $\mathcal{H}_1$, $\mathcal{H}_2$ be Banach spaces. The set of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$, denoted by $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$, has a natural metric given by the norm, which can be used to define the continuity. For the holomorphy, we could use the following definition:

Definition 1.1. A function $\mathbb{C} \supset \Theta \ni z \mapsto T_z \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is holomorphic if $\lim\limits_{h\to 0} \frac{T_{z+h}-T_z}{h}$ exists for all $z \in \Theta$.

There exist other equivalent definitions. For instance, we can demand that $z \mapsto \langle y|T_z x\rangle$ is holomorphic for all bounded anti-linear functionals y on $\mathcal{H}_2$ and all $x \in \mathcal{H}_1$.

It is easy to see that holomorphic functions with values in bounded operators have good properties that generalize the corresponding properties of $\mathbb{C}$-valued functions. For instance, the product of holomorphic functions is holomorphic; we have the uniqueness of the holomorphic continuation; if $z \mapsto T_z$ is holomorphic, then so is $z \mapsto T_{\bar{z}}^*$ (the "Schwarz reflection principle").

In practice, however, especially in mathematical physics and partial differential equations, one often encounters functions with values in unbounded closed operators, for which the continuity and the holomorphy are more tricky. In our paper we collect and prove various general facts concerning continuous and, especially, holomorphic functions with values in closed operators. In particular, in the context of Hilbert spaces we give a certain necessary and sufficient criterion for the holomorphy, which appears to be new and useful, and was the original motivation for writing this article. Besides, we provide sufficient conditions for the continuity and holomorphy of the product and sum of operator-valued functions.

Our main motivation is the case of Hilbert spaces. However, the main tool that we use are non-orthogonal projections, where the natural framework is that of Banach spaces, which we use for the larger part of our paper. In particular, we give a systematic discussion of elementary properties of projections on a Banach space. Some of them we have not seen in the literature.

The continuity and holomorphy of functions with values in closed operators is closely related to the continuity and holomorphy of functions with values in closed subspaces. The family of closed subspaces of a Banach space $\mathcal{H}$ will be called its *Grassmannian* and denoted $\mathrm{Grass}(\mathcal{H})$. It possesses a natural metric topology given by the *gap function*. There exist several useful characterizations of functions continuous in the gap topology. For instance, the range of a continuous function with values in bounded left-invertible operators is continuous in the gap topology. (Note that left-invertible operators have automatically closed ranges). Such a function will be called a *continuous injective resolution* of a given function with values in the Grassmanian.

It is more tricky to define the holomorphy of a Grassmannian-valued function, than to define its continuity. The simplest definition we know says that it is holomorphic iff it locally possesses a holomorphic injective resolution.

Let $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ denote the set of closed operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. It is natural to ask what is the natural concept of continuity and holomorphy for functions with values in $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$.

By identifying a closed operator with its graph we transport the gap topology from $\mathrm{Grass}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ to $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$. This yields a natural definition of the continuity of functions with values in closed operators. There are other possibilities, but the *gap topology* seems to be the most natural generalization of the norm topology from $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ to $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$.

Again, the concept of the holomorphy for closed operators seems to be more complicated than that of the continuity. The original definition is quite old and there are subclasses of operator-valued holomorphic functions which have been studied extensively [16]. However, even the seemingly simpliest questions, such as the unique continuation or the validity of the Schwarz reflection principle, are quite tricky to prove.

To formulate a definition of holomorphic function with values in closed operators, it seems natural again to go first to the Grassmannian. Recall that if a function with values in the Grassmannian has a continuous injective resolution, then it is continuous. We will say that such a function is holomorphic if it possesses a holomorphic injective resolution. Arguably, this definition seems less satisfactory than that of the (gap) continuity — but we do not know of any better one.

We can transport the above definition from the Grassmannian of $\mathcal{H}_1 \oplus \mathcal{H}_2$ to $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$, obtaining a definition of the holomorphy for functions with values in closed operators. This definition, strictly speaking due to T. Kato [16], goes back essentially to F. Rellich [21] and can be reformulated as follows:

Definition 1.2. A function $\Theta \ni z \mapsto T_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ is holomorphic around $z_0 \in \Theta$ if there exists a neighbourhood $\Theta_0 \subset \Theta$ of z_0, a Banach space $\mathcal{K}$ and a holomorphic function $\Theta_0 \ni z \mapsto W_z \in \mathcal{B}(\mathcal{K}, \mathcal{H}_1)$, such that W_z maps bijectively $\mathcal{K}$ onto $\mathrm{Dom}(T_z)$ for all $z \in \Theta_0$ and

$$\Theta_0 \ni z \mapsto T_z W_z \in \mathcal{B}(\mathcal{K}, \mathcal{H}_2)$$

is holomorphic.

At first glance Def. 1.2 may seem somehow artificial, especially when compared with Def. 1.1, which looks as natural as possible. In particular, it involves a relatively arbitrary function $z \mapsto W_z$, which will be called a *resolution of holomorphy* of $z \mapsto T_z$. The arbitrariness of a resolution of holomorphy is rarely a practical problem, because there exists a convenient criterion which works when $\mathcal{H}_1 = \mathcal{H}_2$ and T_z has a non-empty resolvent set (which is usually the case in applications). It is then enough to check the holomorphy of the resolvent of T_z. This criterion is however useless for $z \in \Theta$ corresponding to T_z with an empty resolvent set, or simply when $\mathcal{H}_1 \neq \mathcal{H}_2$. In this case, at least in the context of Hilbert spaces, our criterion given in Prop. 4.8 could be particularly useful.

Note that the word *resolution* is used in two somewhat different contexts – that of the Grassmannian and that of closed operators. Moreover, in the case of the continuity the corresponding object gives a criterion, whereas in the case of the holomorphy it provides a definition. This could be confusing, therefore below we give a summary of the main 4 properties considered in our paper:

(1) If a function with values in the Grassmannian possesses a continuous injective resolution, then it is continuous in the gap topology. If values of the function are complemented, then we can replace the implication by the equivalence.
(2) A function with values in the Grassmannian possesses a holomorphic injective resolution iff it is holomorphic. (This is a definition).
(3) If a function with values in closed operators possesses a resolution of continuity, then it is continuous in the gap topology. If values of the function have complemented graphs, then we can replace the implication by the equivalence.
(4) A function with values in closed operators possesses a resolution of holomorphy iff it is holomorphic. (In the literature this is usually adopted as a definition).

Examples. As an illustration, let us give a number of examples of holomorphic functions with values in closed operators.

(1) Let $z \mapsto T_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ be a function. Assume that $\mathrm{Dom}(T_z)$ does not depend on z and $T_z x$ is holomorphic for each $x \in \mathrm{Dom}(T_z)$. Then such function $z \mapsto T_z$ is holomorphic and it is called a *holomorphic family of type A*. Type A families inherit many good properties from Banach space-valued holomorphic functions and provide the least pathological class of examples.
(2) Let $A \in \mathcal{C}(\mathcal{H})$ have a nonempty resolvent set. Then $z \mapsto (A - z\mathbb{1})^{-1} \in \mathcal{B}(\mathcal{H})$ is holomorphic away from the spectrum of A. However, $z \mapsto$

$(A - z\mathbb{1})^{-1} \in \mathcal{C}(\mathcal{H})$ is holomorphic away from the point spectrum of A, see Example 5.5. This shows that a nonextendible holomorphic function with values in bounded operators can have an extension when we allow unbounded values.

(3) Consider the so-called Stark Hamiltonian on $L^2(\mathbb{R})$

$$H_z := -\partial_x^2 + zx.$$

(Here x denotes the variable in $\mathbb{R}$ and z is a complex parameter). For $z \in \mathbb{R}$ one can define H_z as a self-adjoint operator with $\mathrm{sp}H_z = \mathbb{R}$. H_z is also naturally defined for $z \in \mathbb{C}\backslash\mathbb{R}$, and then has an empty spectrum [13]. Thus in particular all non-real numbers belong to the resolvent set of H_z. One can show that $z \mapsto H_z$ is holomorphic only outside of the real line. On the real line it is even not continuous.

(4) Consider the Hilbert space $L^2([0, \infty[)$. For $m > 1$ the Hermitian operator

$$H_m := -\partial_x^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{x^2} \tag{1.1}$$

is essentially self-adjoint on $C_\mathrm{c}^\infty(]0, \infty[)$. We can continue holomorphically (1.1) onto the halfplane $\{\mathrm{Re}\, m > -1\}$. For all such m the spectrum of H_m is $[0, \infty[$. In [4] it was asked whether $m \mapsto H_m$ can be extended to the left of the line $\mathrm{Re}\, m = -1$. If this is the case, then on this line the spectrum will cover the whole $\mathbb{C}$ and the point $m = -1$ will be a singularity. We still do not know what is the answer to this question. We hope that the method developed in this paper will help to solve the above problem.

Main results and structure of the paper. We start by introducing in Sec. 2 the basic definitions and facts on (not necessarily orthogonal) projections in a linear algebra context.

Sec. 3 contains the essential part of the paper. We first recall the definition of the gap topology on the Grassmanian and demonstrate how it can be characterized in terms of continuity of projections. It turns out that a significant role is played by the assumption that the subspaces are complementable. We then discuss holomorphic functions with values in the Grassmanian and explain how this notion is related to Rellich's definition of operator-valued holomorphic functions. Most importantly, we deduce a result on the validity of the Schwarz reflection principle in Thm 3.42 and we recover Bruk's result on the uniqueness of analytic continuation in Thm. 3.38.

In Sec. 4 we consider the case of operators on Hilbert spaces. We derive explicit formulae for projections on the graphs of closed operators and deduce a criterion for the holomorphy of operator-valued functions, which is also valid for operators with empty resolvent set (Prop. 4.8).

In Sec. 5 we give various sufficient conditions for the continuity and holomorphy of the product and sum of operator-valued functions. More precisely, we assume that $z \mapsto A_z, B_z$ are holomorphic and A_zB_z is closed. The simpliest cases when $z \mapsto A_zB_z$ is holomorphic are discussed in Prop. 5.2 (A_z boundedly invertible or B_z bounded) and Prop. 5.9 (A_zB_z densely defined and $B_{\bar{z}}^*A_{\bar{z}}^*$ closed). More sufficient conditions are given in Thm. 5.10 ($\mathrm{rs}(A_zB_z) \cap \mathrm{rs}(B_zA_z) \neq \emptyset$) and Thm. 5.12 ($\mathrm{Dom}(A_z) + \mathrm{Ran}\, B_z = \mathcal{H}$). A result on sums is contained in Thm. 5.16.

Bibliographical notes. The standard textbook reference for continuous and holomorphic operator-valued function is the book of T. Kato [16]. Most of the results are however restricted to either holomorphic families of type A or to operators with non-empty resolvent set. The first proof of the uniqueness of a holomorphic continuation outside of these two classes is due to V.M. Bruk [3]. The strategy adopted in our paper is to a large extent a generalization of [3].

Holomorphic families of subspaces were introduced by M. Shubin [22]; the definition was then reworked by I.C. Gohberg and J. Leiterer, see [11] and references therein. We use the definition from [11, Ch.6.6]. It is worth mentioning that the original motivation for considering families of subspaces depending holomorphically on a parameter comes from problems in bounded operator theory, such as the existence of a holomorphic right inverse of a given function with values in right-invertible bounded operators, see for instance [14, 18] for recent reviews.

The gap topology was investigated by many authors, eg. [2, 8, 10, 11], however some of the results that we obtain using non-orthogonal projections appear to be new. A special role in our analysis of functions with values in the Grassmannian of a Banach space is played by subspaces that possess a complementary subspace, a review on this subject can be found in [15].

Limitations and issues. The holomorphy of functions with values in closed operators is a nice and natural concept. We are, however, aware of some limitations of its practical value. Consider for instance the Laplacian Δ on $L^2(\mathbb{R}^d)$. As discussed in Example (2), the resolvent $z \mapsto (z\mathbb{1} + \Delta)^{-1}$ extends to an entire holomorphic function. On the other hand, for many practical applications to spectral and scattering theory of Schrödinger operators another fact is much more important. Consider, for example, odd d and $f \in C_c(\mathbb{R}^d)$. Then

$$z \mapsto f(x)(z\mathbb{1} + \Delta)^{-1} f(x)$$

extends to a multivalued holomorphic function, and to make it single valued, one needs to define it on the Riemann surface of the square root (the double covering of $\mathbb{C}\backslash\{0\}$). The extension of this function to the second sheet of this Riemann surface (the so called *non-physical sheet of the complex plane*) plays an important role in the theory of resonances (cf. eg. [23]). It is however different from what one obtains from the extension of $z \mapsto (z\mathbb{1} + \Delta)^{-1}$ in the sense of holomorphic functions with values in closed operators.

Further issues are due to the fact that typical assumptions considered, eg., in perturbation theory, do not allow for a good control of the holomorphy. For instance, we discuss situations where seemingly natural assumptions on T_z and S_z do not ensure that the product $T_z S_z$ defines a holomorphic function.

Applications and outlook. The main advantage of the holomorphy in the sense of Definition 1.2 is that it uses only the basic structure of the underlying Banach space (unlike in the procedure discussed before on the example of the resolvent of the Laplacian).

Despite various problems that can appear in the general case, we conclude from our analysis that there are classes of holomorphic functions which enjoy particularly good properties. This is for instance the case for functions whose values are Fredholm operators. We prove in particular that the product of two such functions functions is again holomorphic. In view of this result it is worth mentioning that the Fredholm analytic theorem (see e.g. [23, Thm. D.4]), formulated usually for bounded operators, extends directly to the unbounded case. It seems thus interesting to investigate further consequences of these facts.

On a separate note, we expect that in analogy to the analysis performed in [4] for the operator $-\partial_x^2 + zx^{-2}$, specific operator-valued holomorphic functions should play a significant role in the description of self-adjoint extensions of exactly solvable Schrödinger operators, listed explicitly in [7] in the one-dimensional case.

A problem not discussed here is the holomorphy of the closure of a given function with values in (non-closed) unbounded operators. Such problems often appear in the context of products of holomorphic functions with values in closed operators, and one can give many examples when the product has non-closed values, but the closure yields a holomorphic function. A better understanding of this issue could lead to useful improvements of the results of the present paper.

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2. Linear space theory

Throughout this section $\mathcal{K}, \mathcal{H}$ are linear spaces.

2.1. **Operators.**

Definition 2.1. By a *linear operator T from $\mathcal{K}$ to $\mathcal{H}$* (or simply an *operator on $\mathcal{K}$*, if $\mathcal{K} = \mathcal{H}$) we will mean a *linear function* $T : \mathrm{Dom}T \to \mathcal{H}$, where $\mathrm{Dom}T$ is a linear subspace of $\mathcal{K}$. In the usual way we define its kernel $\mathrm{Ker}T$, which is a subspace of $\mathrm{Dom}T$ and its range $\mathrm{Ran}\, T$, which is a subspace of $\mathcal{H}$. If $\mathrm{Dom}T = \mathcal{K}$, then we will say that T is *everywhere defined*.

Definition 2.2. We will write $\mathcal{L}(\mathcal{K}, \mathcal{H})$ for the space of linear everywhere defined operators from $\mathcal{K}$ to $\mathcal{H}$. We set $\mathcal{L}(\mathcal{H}) := \mathcal{L}(\mathcal{H}, \mathcal{H})$.

Definition 2.3. If $\mathcal{X}$ is a subspace of $\mathcal{H}$, let $J_{\mathcal{X}}$ denote the embedding of $\mathcal{X}$ into $\mathcal{H}$.

Definition 2.4. We will write $\tilde{T}$ for the operator T understood as an operator on $\mathcal{K}$ to $\mathrm{Ran}\, T$.

Clearly, $T = J_{\mathrm{Ran}\, T}\tilde{T}$.

If an operator T from $\mathcal{K}$ to $\mathcal{H}$ is injective we can define the operator T^{-1} on $\mathcal{H}$ to $\mathcal{K}$ with $\mathrm{Dom}T^{-1} = \mathrm{Ran}\, T$ and $\mathrm{Ran}\, T^{-1} = \mathrm{Dom}T$. Clearly,

$$T^{-1}T = 1\!\!1_{\mathrm{Dom}T}, \ \ TT^{-1} = 1\!\!1_{\mathrm{Ran}\, T}.$$

Often, instead of T^{-1} we will prefer to use $\tilde{T}^{-1} : \mathrm{Ran}\, T \to \mathcal{K}$, whose advantage is that it is everywhere defined.

Definition 2.5. If T,S are two operators, their product TS is defined in the usual way on the domain

$$\mathrm{Dom}(TS) = S^{-1}\mathrm{Dom}(T) = \{v \in \mathrm{Dom}(S) : \ Sv \in \mathrm{Dom}(T)\}.$$

(The notation '$S^{-1}\mathrm{Dom}(T)$' is understood as in the last equality above, so that S is not required to be injective). The operator $T + S$ is defined in the obvious way with $\mathrm{Dom}(T + S) = \mathrm{Dom}T \cap \mathrm{Dom}S$.

2.2. **Projections.**

Definition 2.6. Let $\mathcal{X}, \mathcal{Y}$ be two subspaces of $\mathcal{H}$ with $\mathcal{X} \cap \mathcal{Y} = \{0\}$. (We do not require that $\mathcal{X} + \mathcal{Y} = \mathcal{H}$.) We will write $P_{\mathcal{X},\mathcal{Y}} \in \mathcal{L}(\mathcal{X} + \mathcal{Y})$ for the idempotent with range $\mathcal{X}$ and kernel $\mathcal{Y}$. We will say that it is the *projection onto* $\mathcal{X}$ *along* $\mathcal{Y}$.

Definition 2.7. If $\mathcal{X} \cap \mathcal{Y} = \{0\}$ and $\mathcal{X} + \mathcal{Y} = \mathcal{H}$, we will say that $\mathcal{X}$ is *complementary* to $\mathcal{Y}$, and write $\mathcal{X} \oplus \mathcal{Y} = \mathcal{H}$. In such a case $P_{\mathcal{X},\mathcal{Y}} \in \mathcal{L}(\mathcal{H})$.

Clearly $\mathbb{1} - P_{\mathcal{X},\mathcal{Y}} = P_{\mathcal{Y},\mathcal{X}}$.

The following proposition describes a useful formula for the projection corresponding to a pair of subspaces:

Proposition 2.8. *Let $J \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ be injective. Let $I \in \mathcal{L}(\mathcal{H}, \mathcal{K})$ be surjective. Then IJ is bijective iff*

$$\mathrm{Ran}\, J \oplus \mathrm{Ker} I = \mathcal{H},$$

and then we have the following formula for the projection onto $\mathrm{Ran}\, J$ *along* $\mathrm{Ker} I$*:*

$$P_{\mathrm{Ran}\, J, \mathrm{Ker} I} = J(IJ)^{-1} I \tag{2.1}$$

Proof. $\Rightarrow$: $J(IJ)^{-1}$ is injective. Hence $\mathrm{Ker} J(IJ)^{-1} I = \mathrm{Ker} I$.

$(IJ)^{-1} I$ is surjective. Hence $\mathrm{Ran}\, J(IJ)^{-1} I = \mathrm{Ran}\, J$. Clearly,

$$(J(IJ)^{-1} I)^2 = J(IJ)^{-1} I.$$

$\Leftarrow$: Suppose that IJ is not injective. Clearly, $J\mathrm{Ker} IJ \subset \mathrm{Ker} I \cap \mathrm{Ran}\, J$. But J is injective. Hence, $\mathrm{Ker} I \cap \mathrm{Ran}\, J \neq \{0\}$.

Suppose that IJ is not surjective. Clearly, $\mathrm{Ran}\, IJ = I(\mathrm{Ran}\, J + \mathrm{Ker} I)$. Since I is surjective, $\mathrm{Ran}\, J + \mathrm{Ker} I \neq \mathcal{H}$. $\square$

Here is a special case of the above construction.

Proposition 2.9. *Let $\mathcal{X}, \mathcal{X}', \mathcal{Y}$ be subspaces of $\mathcal{H}$, with $\mathcal{X}$ complementary to $\mathcal{Y}$. Then $\tilde{P}_{\mathcal{X},\mathcal{Y}} J_{\mathcal{X}'}$ is bijective iff $\mathcal{X}'$ and $\mathcal{Y}$ are complementary, and then we have the following formula for the projection onto $\mathcal{X}'$ along $\mathcal{Y}$:*

$$P_{\mathcal{X}',\mathcal{Y}} = J_{\mathcal{X}'} (\tilde{P}_{\mathcal{X},\mathcal{Y}} J_{\mathcal{X}'})^{-1} \tilde{P}_{\mathcal{X},\mathcal{Y}}. \tag{2.2}$$

3. Banach space theory

Throughout this section $\mathcal{K}, \mathcal{H}, \mathcal{H}_1, \mathcal{H}_2$ are Banach spaces. We will use the notation $\mathcal{X}^{\mathrm{cl}}$ for the closure of a subset $\mathcal{X}$. Similarly, for a closable operator T, we use the notation T^{cl} for its closure.

$\mathcal{B}(\mathcal{K}, \mathcal{H})$ will denote the space of bounded everywhere defined operators from $\mathcal{K}$ to $\mathcal{H}$. $\mathcal{C}(\mathcal{K}, \mathcal{H})$ will denote the space of closed operators from $\mathcal{K}$ to $\mathcal{H}$. We will write $\mathcal{B}(\mathcal{H}) = \mathcal{B}(\mathcal{H}, \mathcal{H})$ and $\mathcal{C}(\mathcal{H}) = \mathcal{C}(\mathcal{H}, \mathcal{H})$.

3.1. **Closed range theorem.** Below we recall one of the most useful theorems of operator theory:

Theorem 3.1 (Closed range theorem)**.** *Let T be an injective operator from $\mathcal{K}$ to $\mathcal{H}$. Then the following are equivalent:*

(1) $\mathrm{Ran}\, T$ *is closed and* $T \in \mathcal{C}(\mathcal{K}, \mathcal{H})$*;*
(2) $\mathrm{Ran}\, T$ *is closed and* $\tilde{T}^{-1}$ *is bounded;*

(3) $\operatorname{Ran} T$ *is closed and*

$$\inf_{x\neq 0} \frac{\|Tx\|}{\|x\|} > 0. \tag{3.1}$$

Besides, the number defined in (3.1) is $\|\tilde{T}^{-1}\|^{-1}$.

Definition 3.2. Operators satisfying the conditions of Thm 3.1 will be called *left invertible*. The family of such operators will be denoted by $\mathcal{C}_{\rm linv}(\mathcal{K},\mathcal{H})$. We will write $\mathcal{B}_{\rm linv}(\mathcal{K},\mathcal{H}) = \mathcal{B}(\mathcal{K},\mathcal{H}) \cap \mathcal{C}_{\rm linv}(\mathcal{K},\mathcal{H})$.

An operator T satisfying the conditions of Thm 3.1 and such that $\operatorname{Ran} T = \mathcal{H}$ is called *invertible*. The family of such operators will be denoted by $\mathcal{C}_{\rm inv}(\mathcal{K},\mathcal{H})$. We will write $\mathcal{B}_{\rm inv}(\mathcal{K},\mathcal{H}) = \mathcal{B}(\mathcal{K},\mathcal{H}) \cap \mathcal{C}_{\rm inv}(\mathcal{K},\mathcal{H})$.

Clearly, T is left invertible iff $\tilde{T}$ is invertible. The next proposition shows that left invertibility is stable under bounded perturbations.

Proposition 3.3. *Let* $T \in \mathcal{C}_{\rm linv}(\mathcal{K},\mathcal{H})$*,* $S \in \mathcal{B}(\mathcal{K},\mathcal{H})$ *and* $\|S\| < \|\tilde{T}^{-1}\|^{-1}$*. Then* $T+S \in \mathcal{C}_{\rm linv}(\mathcal{K},\mathcal{H})$*, and*

$$\left| \|\widetilde{(T+S)}^{-1}\|^{-1} - \|\tilde{T}^{-1}\|^{-1} \right| \leq \|S\|. \tag{3.2}$$

Consequently, $\mathcal{B}_{\rm linv}(\mathcal{K},\mathcal{H}) \ni T \mapsto \|\tilde{T}^{-1}\|$ *is a continuous function and* $\mathcal{B}_{\rm linv}(\mathcal{K},\mathcal{H})$ *is an open subset of* $\mathcal{B}(\mathcal{K},\mathcal{H})$.

Proof. We have the lower bound

$$\|(T+S)x\| \geq \|Tx\| - \|Sx\| \geq (\|\tilde{T}^{-1}\|^{-1} - \|S\|)\|x\|.$$

But $\|\tilde{T}^{-1}\|^{-1} - \|S\| > 0$, therefore, $T+S$ is left invertible and

$$\|\widetilde{(T+S)}^{-1}\|^{-1} \geq \|\tilde{T}^{-1}\|^{-1} - \|S\|.$$

Then we switch the roles of T and $T+S$, and obtain

$$\|\tilde{T}^{-1}\|^{-1} \geq \|\widetilde{(T+S)}^{-1}\|^{-1} - \|S\|,$$

which proves (3.2). □

Definition 3.4. The *resolvent set* of an operator T on $\mathcal{H}$, denoted $\operatorname{rs} T$, is defined to be the set of all $\lambda \in \mathbb{C}$ such that $T - \lambda \mathbb{1} \in \mathcal{C}_{\rm inv}(\mathcal{H})$. The *spectrum* of T is by definition the set $\operatorname{sp} T := \mathbb{C} \setminus \operatorname{rs} T$.

Note that according to this definition (used for instance in [5, 9, 12]), $\operatorname{rs} T \neq \emptyset$ implies that T is a closed operator (note that this differs from the terminology used in [16]).

3.2. **Bounded projections.** Let $\mathcal{X}, \mathcal{Y}$ be subspaces of $\mathcal{H}$ with $\mathcal{X} \cap \mathcal{Y} = \{0\}$. The operator $P_{\mathcal{X},\mathcal{Y}}$ is bounded iff $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{X}+\mathcal{Y}$ are closed.

Definition 3.5. Let $\operatorname{Pr}(\mathcal{H})$ denote the set of bounded projections on $\mathcal{H}$. Let $\operatorname{Grass}(\mathcal{H})$ denote the set of closed subspaces of $\mathcal{H}$ (the *Grassmannian* of $\mathcal{H}$).

The first part of Prop 2.8 can be adapted to the Banach space setting as follows:

Proposition 3.6. *Let* $J \in \mathcal{B}_{\rm linv}(\mathcal{K},\mathcal{H})$*. Let* $I \in \mathcal{B}(\mathcal{H},\mathcal{K})$ *be surjective. Clearly,* $\operatorname{Ran} J, \operatorname{Ker} I$ *are closed. Then* IJ *is invertible iff*

$$\operatorname{Ran} J \oplus \operatorname{Ker} I = \mathcal{H}.$$

Proof. Given Prop. 2.8, it suffices to note that IJ is a bijective bounded operator on a Banach space, hence it is invertible. $\square$

Here is an adaptation of the first part of Prop. 2.2 to the Banach space setting:

Proposition 3.7. *Let $\mathcal{X}, \mathcal{X}', \mathcal{Y}$ be closed subspaces of $\mathcal{H}$, with $\mathcal{X}$ complementary to $\mathcal{Y}$. Then $\tilde{P}_{\mathcal{X},\mathcal{Y}} J_{\mathcal{X}'}$ is invertible iff $\mathcal{X}'$ and $\mathcal{Y}$ are complementary.*

3.3. **Gap topology.**

Definition 3.8. If $\mathcal{X} \in \mathrm{Grass}(\mathcal{H})$, we will introduce the following notation for the *ball* in $\mathcal{X}$:

$$B_{\mathcal{X}} := \{x \in \mathcal{X} \ : \ \|x\| \leq 1\}.$$

As usual, the *distance* of a non-empty set $K \subset \mathcal{H}$ and $x \in \mathcal{H}$ is defined as

$$\mathrm{dist}\,(x, K) := \inf\{\|x - y\| \ : \ y \in K\}.$$

Definition 3.9. For $\mathcal{X}, \mathcal{Y} \in \mathrm{Grass}(\mathcal{H})$ we define

$$\delta(\mathcal{X}, \mathcal{Y}) := \sup_{x \in B_{\mathcal{X}}} \mathrm{dist}\,(x, \mathcal{Y}).$$

The *gap* between $\mathcal{X}$ and $\mathcal{Y}$ is defined as

$$\hat{\delta}(\mathcal{X}, \mathcal{Y}) := \max\big(\delta(\mathcal{X}, \mathcal{Y}), \delta(\mathcal{Y}, \mathcal{X})\big). \tag{3.3}$$

The *gap topology* is the weakest topology on $\mathrm{Grass}(\mathcal{H})$ for which the function δ is continuous on $\mathrm{Grass}(\mathcal{H}) \times \mathrm{Grass}(\mathcal{H})$.

Note that the gap defined in (3.3) is not a metric. There exists a metric that can be used to define the gap topology, but we will not need it. We refer the reader to [16] for more discussion about the gap topology.

Proposition 3.10. *If $\mathcal{X}, \mathcal{Y}$ and $\mathcal{X}', \mathcal{Y}'$ are two pairs of complementary subspaces in* $\mathrm{Grass}(\mathcal{H})$*, then*

$$\max\big(\hat{\delta}(\mathcal{X}, \mathcal{X}'), \hat{\delta}(\mathcal{Y}, \mathcal{Y}')\big) \leq \|P_{\mathcal{X},\mathcal{Y}} - P_{\mathcal{X}',\mathcal{Y}'}\|.$$

Proof. For $x \in B_{\mathcal{X}}$, we have

$$\begin{aligned} \mathrm{dist}\,(x, \mathcal{X}') &\leq \|x - P_{\mathcal{X}',\mathcal{Y}'} x\| \\ &= \|(P_{\mathcal{X},\mathcal{Y}} - P_{\mathcal{X}',\mathcal{Y}'})x\| \ \leq \ \|P_{\mathcal{X},\mathcal{Y}} - P_{\mathcal{X}',\mathcal{Y}'}\|. \end{aligned}$$

This shows

$$\delta(\mathcal{X}, \mathcal{X}') \leq \|P_{\mathcal{X},\mathcal{Y}} - P_{\mathcal{X}',\mathcal{Y}'}\|.$$

The same argument gives

$$\delta(\mathcal{X}', \mathcal{X}) \leq \|P_{\mathcal{X},\mathcal{Y}} - P_{\mathcal{X}',\mathcal{Y}'}\|.$$

Finally, we use

$$P_{\mathcal{X},\mathcal{Y}} - P_{\mathcal{X}',\mathcal{Y}'} = P_{\mathcal{Y}',\mathcal{X}'} - P_{\mathcal{Y},\mathcal{X}}. \quad \square$$

Corollary 3.11. *If $\mathcal{X}, \mathcal{Y}$ and $\mathcal{X}', \mathcal{Y}$ are complementary, then*

$$\hat{\delta}(\mathcal{X}, \mathcal{X}') \leq \|P_{\mathcal{X},\mathcal{Y}} - P_{\mathcal{X}',\mathcal{Y}}\|.$$

Lemma 3.12.

$$\|P_{\mathcal{Y},\mathcal{X}} P_{\mathcal{X}',\mathcal{Y}'}\| \leq \|P_{\mathcal{Y},\mathcal{X}}\| \|P_{\mathcal{X}',\mathcal{Y}'}\| \delta(\mathcal{X}', \mathcal{X}).$$

Proof.

$$\begin{aligned}
\|P_{\mathcal{Y},\mathcal{X}}P_{\mathcal{X}',\mathcal{Y}'}\| &= \sup_{v\in B_{\mathcal{H}}} \|P_{\mathcal{Y},\mathcal{X}}P_{\mathcal{X}',\mathcal{Y}'}v\| \\
&\leq \|P_{\mathcal{X}',\mathcal{Y}'}\| \sup_{x'\in B_{\mathcal{X}'}} \|P_{\mathcal{Y},\mathcal{X}}x'\| \\
&= \|P_{\mathcal{X}',\mathcal{Y}'}\| \sup_{x'\in B_{\mathcal{X}'}} \inf_{x\in\mathcal{X}} \|P_{\mathcal{Y},\mathcal{X}}(x'-x)\| \\
&\leq \|P_{\mathcal{Y},\mathcal{X}}\|\|P_{\mathcal{X}',\mathcal{Y}'}\| \sup_{x'\in B_{\mathcal{X}'}} \inf_{x\in\mathcal{X}} \|x'-x\|. \quad \square
\end{aligned}$$

The following proposition is essentially taken from [11].

Proposition 3.13. *Let* $\mathcal{X},\mathcal{Y}\in \mathrm{Grass}(\mathcal{H})$ *be complementary,* $\mathcal{X}',\mathcal{Y}'\in\mathrm{Grass}(\mathcal{H})$ *and*

$$\|P_{\mathcal{Y},\mathcal{X}}\|\delta(\mathcal{X}',\mathcal{X}) + \|P_{\mathcal{X},\mathcal{Y}}\|\delta(\mathcal{Y}',\mathcal{Y}) < 1, \tag{3.4}$$

$$\|P_{\mathcal{Y},\mathcal{X}}\|\delta(\mathcal{Y},\mathcal{Y}') + \|P_{\mathcal{X},\mathcal{Y}}\|\delta(\mathcal{X},\mathcal{X}') < 1. \tag{3.5}$$

Then $\mathcal{X}',\mathcal{Y}'$ *are complementary and*

$$\|P_{\mathcal{X},\mathcal{Y}} - P_{\mathcal{X}',\mathcal{Y}'}\| \leq \frac{\|P_{\mathcal{X},\mathcal{Y}}\|\|P_{\mathcal{Y},\mathcal{X}}\|\big(\delta(\mathcal{X}',\mathcal{X})+\delta(\mathcal{Y}',\mathcal{Y})\big)}{1-\|P_{\mathcal{X},\mathcal{Y}}\|\delta(\mathcal{X}',\mathcal{X})-\|P_{\mathcal{Y},\mathcal{X}}\|\delta(\mathcal{Y}',\mathcal{Y})}. \tag{3.6}$$

Proof. Step 1. Let us show that $\mathcal{X}'\cap\mathcal{Y}'=\{0\}$. Suppose it is not true. Then there exists $v\in\mathcal{X}'\cap\mathcal{Y}'$, $\|v\|=1$.

$$\begin{aligned}
1=\|v\| &\leq \|P_{\mathcal{X},\mathcal{Y}}v\| + \|P_{\mathcal{Y},\mathcal{X}}v\| \\
&\leq \|P_{\mathcal{X},\mathcal{Y}}\|\delta(\mathcal{Y}',\mathcal{Y}) + \|P_{\mathcal{Y},\mathcal{X}}\|\delta(\mathcal{X}',\mathcal{X}),
\end{aligned}$$

which is a contradiction.

Step 2. By Step 1, $P_{\mathcal{X}',\mathcal{Y}'}$ is well defined as a map on $\mathcal{X}'+\mathcal{Y}'$. We will show that it is bounded, or equivalently that $\mathcal{X}'+\mathcal{Y}'$ is closed. We will also obtain the estimate (3.6).

For simplicity, in the following estimates we assume that $\mathcal{H}=\mathcal{X}'+\mathcal{Y}'$. If $\mathcal{X}'+\mathcal{Y}'$ is strictly smaller than $\mathcal{H}$, then we should replace $P_{\mathcal{X},\mathcal{Y}}$ by $P_{\mathcal{X},\mathcal{Y}}J_{\mathcal{X}'+\mathcal{Y}'}$.

Clearly,

$$\begin{aligned}
P_{\mathcal{X},\mathcal{Y}} - P_{\mathcal{X}',\mathcal{Y}'} &= P_{\mathcal{X},\mathcal{Y}}(P_{\mathcal{Y}',\mathcal{X}'}+P_{\mathcal{X}',\mathcal{Y}'}) - (P_{\mathcal{X},\mathcal{Y}}+P_{\mathcal{Y},\mathcal{X}})P_{\mathcal{X}',\mathcal{Y}'} \\
&= P_{\mathcal{X},\mathcal{Y}}P_{\mathcal{Y}',\mathcal{X}'} - P_{\mathcal{Y},\mathcal{X}}P_{\mathcal{X}',\mathcal{Y}'}.
\end{aligned}$$

Hence,

$$\begin{aligned}
\|P_{\mathcal{X},\mathcal{Y}} - P_{\mathcal{X}',\mathcal{Y}'}\| &\leq \|P_{\mathcal{X},\mathcal{Y}}P_{\mathcal{Y}',\mathcal{X}'}\| + \|P_{\mathcal{Y},\mathcal{X}}P_{\mathcal{X}',\mathcal{Y}'}\| \\
&\leq \|P_{\mathcal{X},\mathcal{Y}}\|\|P_{\mathcal{Y}',\mathcal{X}'}\|\delta(\mathcal{Y}',\mathcal{Y}) + \|P_{\mathcal{Y},\mathcal{X}}\|\|P_{\mathcal{X}',\mathcal{Y}'}\|\delta(\mathcal{X}',\mathcal{X}) \\
&\leq \|P_{\mathcal{X},\mathcal{Y}}\|\big(\|P_{\mathcal{Y},\mathcal{X}}\| + \|P_{\mathcal{X}'\mathcal{Y}'}-P_{\mathcal{X},\mathcal{Y}}\|\big)\delta(\mathcal{Y}',\mathcal{Y}) \\
&\quad +\|P_{\mathcal{Y},\mathcal{X}}\|\big(\|P_{\mathcal{X},\mathcal{Y}}\| + \|P_{\mathcal{X}'\mathcal{Y}'}-P_{\mathcal{X},\mathcal{Y}}\|\big)\delta(\mathcal{X}',\mathcal{X}).
\end{aligned}$$

Thus

$$\begin{aligned}
&\big(1-\|P_{\mathcal{X},\mathcal{Y}}\|\delta(\mathcal{X}',\mathcal{X})-\|P_{\mathcal{Y},\mathcal{X}}\|\delta(\mathcal{Y}',\mathcal{Y})\big)\|P_{\mathcal{X},\mathcal{Y}}-P_{\mathcal{X}',\mathcal{Y}}\| \\
\leq\ &\|P_{\mathcal{X},\mathcal{Y}}\|\|P_{\mathcal{Y},\mathcal{X}}\|\big(\delta(\mathcal{X}',\mathcal{X})+\delta(\mathcal{Y}',\mathcal{Y})\big).
\end{aligned}$$

Step 3. We show that $\mathcal{X}' + \mathcal{Y}' = \mathcal{H}$. Suppose that this is not the case. We can then find $v \in \mathcal{H}$ such that $\|v\| = 1$, $\mathrm{dist}\,(v, \mathcal{X}' + \mathcal{Y}') = 1$. Now

$$\begin{aligned} 1 = \mathrm{dist}\,(v, \mathcal{X}' + \mathcal{Y}') &\leq \mathrm{dist}\,(P_{\mathcal{X},\mathcal{Y}}v, \mathcal{X}' + \mathcal{Y}') + \mathrm{dist}\,(P_{\mathcal{Y},\mathcal{X}}v, \mathcal{X}' + \mathcal{Y}') \\ &\leq \mathrm{dist}\,(P_{\mathcal{X},\mathcal{Y}}v, \mathcal{X}') + \mathrm{dist}\,(P_{\mathcal{Y},\mathcal{X}}v, \mathcal{Y}') \\ &\leq \|P_{\mathcal{X},\mathcal{Y}}\|\delta(\mathcal{X}, \mathcal{X}') + \|P_{\mathcal{Y},\mathcal{X}}\|\delta(\mathcal{Y}, \mathcal{Y}'), \end{aligned}$$

which is a contradiction. □

Corollary 3.14. *Let $\mathcal{X}, \mathcal{Y} \in \mathrm{Grass}(\mathcal{H})$ be complementary and*

$$\|P_{\mathcal{Y},\mathcal{X}}\|\delta(\mathcal{X}', \mathcal{X}) < 1, \;\; \|P_{\mathcal{X},\mathcal{Y}}\|\delta(\mathcal{X}, \mathcal{X}') < 1. \tag{3.7}$$

Then $\mathcal{X}', \mathcal{Y}$ are complementary and

$$\|P_{\mathcal{X},\mathcal{Y}} - P_{\mathcal{X}',\mathcal{Y}}\| \;\leq\; \frac{\|P_{\mathcal{X},\mathcal{Y}}\|\|P_{\mathcal{Y},\mathcal{X}}\|\delta(\mathcal{X}', \mathcal{X})}{1 - \|P_{\mathcal{X},\mathcal{Y}}\|\delta(\mathcal{X}', \mathcal{X})}. \tag{3.8}$$

Note in passing that the proof of Prop. 3.13 shows also a somewhat more general statement (which we however will not use in the sequel):

Proposition 3.15. *If in Prop. 3.13 we drop the condition (3.5), then $\mathcal{X}' \cap \mathcal{Y}' = \{0\}$, $\mathcal{X}' + \mathcal{Y}'$ is closed and the estimate (3.6) is still true if we replace $P_{\mathcal{X},\mathcal{Y}}$ with $P_{\mathcal{X},\mathcal{Y}} J_{\mathcal{X}'+\mathcal{Y}'}$.*

Proposition 3.16. *Let $T \in \mathcal{B}_{\mathrm{linv}}(\mathcal{K}, \mathcal{H})$, $S \in \mathcal{B}(\mathcal{K}, \mathcal{H})$. Then*

$$\delta(\mathrm{Ran}\, T, \mathrm{Ran}\, S) \leq \|(T - S)\tilde{T}^{-1}\|. \tag{3.9}$$

Hence, if also $S \in \mathcal{B}_{\mathrm{linv}}(\mathcal{K}, \mathcal{H})$,

$$\hat{\delta}(\mathrm{Ran}\, T, \mathrm{Ran}\, S) \leq \max\big(\|(T - S)\tilde{T}^{-1}\|, \|(T - S)\tilde{S}^{-1}\|\big).$$

Proof. Let $x \in B_{\mathrm{Ran}\, T}$. Clearly, $x = T\tilde{T}^{-1}x$. Hence

$$\begin{aligned} \mathrm{dist}\,(x, \mathrm{Ran}\, S) &\leq \|x - S\tilde{T}^{-1}x\| \\ &= \|(T - S)\tilde{T}^{-1}x\| \leq \|(T - S)\tilde{T}^{-1}\|, \end{aligned}$$

which proves (3.9). □

Definition 3.17. A closed subspace of $\mathcal{H}$ possessing a complementary subspace will be called *complemented*. Let $\mathrm{Grass}_{\mathrm{com}}(\mathcal{H})$ stand for the family of complemented closed subspaces.

For any $\mathcal{Y} \in \mathrm{Grass}(\mathcal{H})$ let $\mathrm{Grass}_{\mathcal{Y}}(\mathcal{H})$ denote the family of closed subspaces complementary to $\mathcal{Y}$.

The following fact follows immediately from Prop. 3.13:

Proposition 3.18. $\mathrm{Grass}_{\mathcal{Y}}(\mathcal{H})$ *and* $\mathrm{Grass}_{\mathrm{com}}(\mathcal{H})$ *are open subsets of* $\mathrm{Grass}(\mathcal{H})$.

Proposition 3.19. *If $T \in \mathcal{B}_{\mathrm{inv}}(\mathcal{H})$, then*

$$\mathrm{Grass}(\mathcal{H}) \ni \mathcal{X} \mapsto T\mathcal{X} \in \mathrm{Grass}(\mathcal{H})$$

is bicontinuous. It preserves the complementarity relation, and hence it maps $\mathrm{Grass}_{\mathrm{com}}(\mathcal{H})$ *into itself.*

3.4. **Continuous families of subspaces.** In this section Θ will be a locally compact space (eg. an open subset of $\mathbb{C}$.) Consider a function

$$\Theta \ni z \mapsto \mathcal{X}_z \in \mathrm{Grass}(\mathcal{H}). \tag{3.10}$$

Proposition 3.20. *Let $\mathcal{Y} \in \mathrm{Grass}(\mathcal{H})$. If (3.10) has values in $\mathrm{Grass}_{\mathcal{Y}}(\mathcal{H})$, then it is continuous iff*

$$\Theta \ni z \mapsto P_{\mathcal{X}_z,\mathcal{Y}} \in \mathrm{Pr}(\mathcal{H})$$

is continuous.

Proof. We use Corrolaries 3.11 and 3.14. □

Definition 3.21. We say that

$$\Theta \ni z \mapsto T_z \in \mathcal{B}_{\mathrm{linv}}(\mathcal{K}, \mathcal{H}) \tag{3.11}$$

is an *injective resolution* of (3.10) if, for any $z \in \Theta$, T_z is a bijection onto $\mathcal{X}_z$.

Proposition 3.22. *Let $z_0 \in \Theta$.*

(1) *If there exists an open Θ_0 such that $z_0 \in \Theta_0 \subset \Theta$ and an injective resolution of (3.10) on Θ_0 continuous at z_0, then (3.10) is continuous at z_0.*
(2) *If (3.10) has values in $\mathrm{Grass}_{\mathrm{com}}(\mathcal{H})$, then we can put "if and only if" in 1.*

Proof. (1): Suppose that (3.11) is an injective resolution of (3.10) which is continuous at z_0. We can find an open Θ_1 such that $z_0 \in \Theta_1^{\mathrm{cl}} \subset \Theta_0$ and for $z \in \Theta_1$ we have $\|T_z - T_{z_0}\| < c\|\tilde{T}_{z_0}^{-1}\|^{-1}$ with $c < 1$. Then $\|T_z\|$ and $\|\tilde{T}_z^{-1}\|$ are uniformly bounded for such z. Therefore, by Prop. 3.16, for such z we have $\hat{\delta}(\mathcal{X}_z, \mathcal{X}_{z_0}) \le C\|T_z - T_{z_0}\|$. Thus the continuity of (3.11) implies the continuity of (3.10).

(2): Suppose that (3.10) is continuous at z_0 and $\mathcal{X}_{z_0}$ is complemented. Let $\mathcal{Y} \in \mathrm{Grass}(\mathcal{H})$ be complementary to $\mathcal{X}_{z_0}$. There exists an open Θ_0 such that $z_0 \in \Theta_0 \subset \Theta$ and $\mathcal{Y}$ is complementary to $\mathcal{X}_z$, $z \in \Theta_0$. Then, by Prop. 3.7, we see that

$$\Theta_0 \ni z \mapsto P_{\mathcal{X}_z,\mathcal{Y}} J_{\mathcal{X}_{z_0}} \in \mathcal{B}_{\mathrm{linv}}(\mathcal{X}_{z_0}, \mathcal{H})$$

is an injective resolution of (3.10) restricted to Θ_0. □

To sum up, functions with values in the Grassmannian that possess continuous injective resolutions are continuous. If all values of a functions are complemented, then the existence of a continuous injective resolution can be adopted as an alternative definition of the continuity.

3.5. **Closed operators.**

Definition 3.23. For an operator T on $\mathcal{H}_1$ to $\mathcal{H}_2$ with domain $\mathrm{Dom}T$ its graph is the subspace of $\mathcal{H}_1 \oplus \mathcal{H}_2$ given by

$$\mathrm{Gr}(T) := \{(x, Tx) \in \mathcal{H}_1 \oplus \mathcal{H}_2 \ : \ x \in \mathrm{Dom}T\}.$$

This induces a map

$$\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2) \ni T \mapsto \mathrm{Gr}(T) \in \mathrm{Grass}(\mathcal{H}_1 \oplus \mathcal{H}_2). \tag{3.12}$$

From now on, we endow $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ with the gap topology transported from $\mathrm{Grass}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ by (3.12).

Proposition 3.24. *$\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is open in $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$. On $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ the gap topology coincides with the usual norm topology.*

Operators whose graphs are complemented seem to have better properties.

Definition 3.25. We denote by $\mathcal{C}_{\mathrm{com}}(\mathcal{H}_1, \mathcal{H}_2)$ the set of closed operators with complemented graphs.

Clearly, $\mathcal{C}_{\mathrm{com}}(\mathcal{H}_1, \mathcal{H}_2)$ is an open subset of $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$.

Proposition 3.26. *Let $S \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. Then the map*

$$\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2) \ni T \mapsto T + S \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2) \tag{3.13}$$

is bicontinuous and preserves $\mathcal{C}_{\mathrm{com}}(\mathcal{H}_1, \mathcal{H}_2)$

Proof. (3.13) on the level of graphs acts by

$$\begin{bmatrix} \mathbb{1} & 0 \\ S & \mathbb{1} \end{bmatrix},$$

which is clearly in $\mathcal{B}_{\mathrm{inv}}(\mathcal{H}_1 \oplus \mathcal{H}_2)$. Thus the proposition follows by Prop. 3.19. □

Proposition 3.27. *(1) Graphs of invertible operators are complemented.*
(2) Graphs of operators whose resolvent set is nonempty are complemented.

Proof. By Prop. 3.26 applied to $-\lambda\mathbb{1}$, it is enough to show (1). We will show that if $T \in \mathcal{C}_{\mathrm{inv}}(\mathcal{H}_1, \mathcal{H}_2)$, then $\mathcal{H}_1 \oplus \{0\}$ is complementary to $\mathrm{Gr}(T)$.

Indeed,

$$\mathcal{H}_1 \oplus \{0\} \cap \mathrm{Gr}(T) = \{0, 0\}$$

is obviously true for any operator T.

Any $(v, w) \in \mathcal{H}_1 \oplus \mathcal{H}_2$ can be written as

$$\big(v - T^{-1}w, 0\big) + \big(T^{-1}w, w\big). \quad \square$$

3.6. **Continuous families of closed operators.** Consider a function

$$\Theta \ni z \mapsto T_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2). \tag{3.14}$$

Proposition 3.28. *Let $z_0 \in \Theta$. Suppose that there exists an open Θ_0 such that $z_0 \in \Theta_0 \subset \Theta$, a Banach space $\mathcal{K}$ and a function*

$$\Theta_0 \ni z \mapsto W_z \in \mathcal{B}(\mathcal{K}, \mathcal{H}_1) \tag{3.15}$$

s.t. W_z maps bijectively $\mathcal{K}$ onto $\mathrm{Dom}(T_z)$ for all $z \in \Theta_0$,

$$\Theta_0 \ni z \mapsto T_z W_z \in \mathcal{B}(\mathcal{K}, \mathcal{H}_2), \tag{3.16}$$

and both (3.15) and (3.16) are continuous at z_0. Then (3.14) is continuous at z_0.

Proof. Notice that

$$\Theta_0 \ni z \mapsto \big(W_z, T_z W_z\big) \in \mathcal{B}_{\mathrm{linv}}(\mathcal{K}, \mathcal{H}_1 \oplus \mathcal{H}_2) \tag{3.17}$$

is an injective resolution of

$$\Theta_0 \ni z \mapsto \mathrm{Gr}(T_z) \in \mathrm{Grass}(\mathcal{H}_1 \oplus \mathcal{H}_2). \tag{3.18}$$

(Actually, every injective resolution of (3.18) is of the form (3.17).) The injective resolution (3.17) is continuous at z_0, hence (3.14) is continuous at z_0 by Prop. 3.22 (1). □

A function $z \mapsto W_z$ with the properties described in Prop. 3.28 will be called a *resolution of continuity of $z \mapsto T_z$ at z_0*.

Proposition 3.29. *Suppose that $z_0 \in \Theta$, there exists an open Θ_0 such that $z_0 \in \Theta_0 \subset \Theta$, and $T_z \in \mathcal{C}_{\rm inv}(\mathcal{H}_1, \mathcal{H}_2)$, $z \in \Theta_0$. Then*

$$z \mapsto T_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$$

is continuous at z_0 iff

$$z \mapsto T_z^{-1} \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_1)$$

is continuous at z_0.

Proof. $\Rightarrow$: Let $z \mapsto W_z$ be a resolution of continuity of $z \mapsto T_z$ at z_0. Then $z \mapsto V_z := T_z W_z$ is invertible bounded and continuous at z_0. Hence, so is $z \mapsto V_z^{-1}$. Therefore, $z \mapsto T_z^{-1} = W_z V_z^{-1}$ is bounded and continuous at z_0.

$\Leftarrow$: Obviously, $z \mapsto T_z^{-1}$ is a resolution of continuity of $z \mapsto T_z$ at z_0. □

The following proposition is an immediate consequence of Prop. 3.29.

Proposition 3.30. *Let $\lambda \in \mathbb{C}$ and consider a function*

$$z \mapsto T_z \in \mathcal{C}(\mathcal{H}). \tag{3.19}$$

Suppose there exists an open Θ_0 such that $z_0 \in \Theta_0 \subset \Theta$, and $\lambda \in \mathrm{rs}(T_z)$, $z \in \Theta_0$. Then (3.19) is continuous at z_0 iff

$$z \mapsto (\lambda \mathbb{1} - T_z)^{-1} \in \mathcal{B}(\mathcal{H})$$

is continuous at z_0.

3.7. **Holomorphic families of closed subspaces.** Let Θ be an open subset of $\mathbb{C}$ and suppose we are given a function

$$\Theta \ni z \mapsto \mathcal{X}_z \in \mathrm{Grass}(\mathcal{H}). \tag{3.20}$$

Definition 3.31. We will say that the family (3.20) is *complex differentiable at z_0* if there exists an open Θ_0 with $z_0 \in \Theta_0 \subset \Theta$ and an injective resolution

$$\Theta_0 \ni z \mapsto T_z \in B_{\rm linv}(\mathcal{K}, \mathcal{H}) \tag{3.21}$$

of (3.20) complex differentiable at z_0. If (3.20) is complex differentiable on the whole Θ, we say it is *holomorphic*.

Clearly, the complex differentiability implies the continuity.

Proposition 3.32. *Suppose that (3.20) is complex differentiable at $z_0 \in \Theta$. Let Θ_0 be open with $z_0 \in \Theta_0 \subset \Theta$, and let $\mathcal{Y}$ be a subspace complementary to $\mathcal{X}_z$, $z \in \Theta_0$. Then the family $\Theta_0 \ni z \mapsto P_{\mathcal{X}_z,\mathcal{Y}}$ is complex differentiable at z_0.*

Proof. By making, if needed, Θ_0 smaller, we can assume that we have an injective resolution $\Theta_0 \ni z \mapsto T_z$ complex differentiable at z_0. For such z, by Prop. 3.6, $\tilde{P}_{\mathcal{X}_{z_0},\mathcal{Y}} T_z$ is invertible. Therefore, by Prop. 2.8,

$$P_{\mathcal{X}_z,\mathcal{Y}} = T_z (\tilde{P}_{\mathcal{X}_{z_0},\mathcal{Y}} T_z)^{-1} \tilde{P}_{\mathcal{X}_{z_0},\mathcal{Y}}. \tag{3.22}$$

(3.22) is clearly complex differentiable at z_0. □

Proposition 3.33. *Let $z_0 \in \Theta$. Suppose that $\mathcal{X}_{z_0} \in \mathrm{Grass}_{\rm com}(\mathcal{H})$. The following are equivalent:*

1. *(3.20) is complex differentiable at z_0.*
2. *There exists an open Θ_0 with $z_0 \in \Theta_0 \subset \Theta$ and a closed subspace $\mathcal{Y}$ complementary to $\mathcal{X}_z$, $z \in \Theta_0$, such that the family $\Theta_0 \ni z \mapsto P_{\mathcal{X}_z,\mathcal{Y}}$ is complex differentiable at z_0.*

Proof. (1)⇒(2): Consider the injective resolution (3.21). Let $\mathcal{Y}$ be a subspace complementary to $\mathcal{X}_{z_0} = \operatorname{Ran} T_{z_0}$. We know that $\Theta \ni z \mapsto \mathcal{X}_z$ is continuous by Prop. 3.22. Hence, by taking Θ_0 smaller, we can assume that $\mathcal{Y}$ is complementary to $\mathcal{X}_z$, $z \in \Theta_0$. By Prop. 3.32, $P_{\mathcal{X}_z,\mathcal{Y}}$ is complex differentiable at z_0.

(2)⇒(1):

$$\Theta_0 \ni z \mapsto P_{\mathcal{X}_z,\mathcal{Y}} J_{\mathcal{X}_{z_0}} \in B_{\mathrm{linv}}(\mathcal{X}_{z_0}, \mathcal{H})$$

is an injective resolution of (3.20) complex differentiable at z_0. □

Proposition 3.34 (Uniqueness of analytic continuation for subspaces)**.** *Let $\Theta \subset \mathbb{C}$ be connected and open. Let*

$$\Theta \ni z \mapsto \mathcal{X}_z, \mathcal{Y}_z \in \mathrm{Grass_{com}}(\mathcal{H})$$

be holomorphic. Consider a sequence $\{z_1, z_2, \dots\} \subset \Theta$ converging to a point $z_0 \in \Theta$ s.t. $z_n \neq z_0$ for each n. Suppose $\mathcal{X}_{z_n} = \mathcal{Y}_{z_n}$, $n = 1, 2, \dots$. Then $\mathcal{X}_z = \mathcal{Y}_z$ for all $z \in \Theta$.

Proof. For holomorphic functions with values in bounded operators the unique continuation property is straightforward. Therefore, it suffices to apply Prop. 3.33. □

Proposition 3.35. *Let (3.20) and $\Theta \ni z \mapsto T_z \in \mathcal{B}(\mathcal{H}, \mathcal{H}_1)$ be holomorphic. Suppose that for all $z \in \Theta$, T_z is injective on $\mathcal{X}_z$ and $T_z\mathcal{X}_z$ is closed. Then*

$$\Theta \ni z \mapsto T_z\mathcal{X}_z \in \mathrm{Grass}(\mathcal{H}_1) \tag{3.23}$$

is holomorphic.

Proof. Let $\Theta_0 \ni z \mapsto S_z \in \mathcal{B}_{\mathrm{linv}}(\mathcal{K}, \mathcal{H})$ be a holomorphic injective resolution of (3.20). Then $\Theta_0 \ni z \mapsto T_z S_z \in \mathcal{B}_{\mathrm{linv}}(\mathcal{K}, \mathcal{H}_1)$ is a holomorphic injective resolution of (3.23). □

3.8. **Holomorphic families of closed operators.** Consider a function

$$\Theta \ni z \mapsto T_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2). \tag{3.24}$$

Definition 3.36. Let $z_0 \in \Theta$. We say that (3.24) is complex differentiable at z_0 if

$$\Theta \ni z \mapsto \mathrm{Gr}(T_z) \in \mathrm{Grass}(\mathcal{H}_1 \oplus \mathcal{H}_2) \tag{3.25}$$

is complex differentiable at z_0.

The following proposition gives an equivalent condition, which in most of the literature is adopted as the basic definition of the complex differentiablity of functions with values in closed operators.

Proposition 3.37. *(3.24) is complex differentiable at $z_0 \in \Theta$ iff there exists an open Θ_0 such that $z_0 \in \Theta_0 \subset \Theta$, a Banach space $\mathcal{K}$ and a function*

$$\Theta_0 \ni z \mapsto W_z \in \mathcal{B}(\mathcal{K}, \mathcal{H}_1) \tag{3.26}$$

s.t. W_z maps bijectively $\mathcal{K}$ onto $\mathrm{Dom}(T_z)$ for all $z \in \Theta_0$,

$$\Theta_0 \ni z \mapsto T_z W_z \in \mathcal{B}(\mathcal{K}, \mathcal{H}_2), \tag{3.27}$$

and both (3.26) and (3.27) are are complex differentiable at z_0.

Proof. We use the fact, noted in the proof of Prop. 3.28, that (3.17) is an injective resolution of (3.18), and that every injective resolution is of this form. $\Box$

A function $z \mapsto W_z$ with the properties described in Prop. 3.37 will be called a *resolution of complex differentiability of $z \mapsto T_z$ at z_0*.

The following theorem follows immediately from Thm 3.34:

Theorem 3.38 (Uniqueness of analytic continuation for closed operators [3])**.** *Let $\Theta \subset \mathbb{C}$ be connected and open. Let*

$$\Theta \ni z \mapsto T_z, S_z \in \mathcal{C}_{\mathrm{com}}(\mathcal{H}_1, \mathcal{H}_2)$$

be holomorphic. Consider a sequence $\{z_1, z_2, \dots\} \subset \Theta$ converging to a point $z_0 \in \Theta$ s.t. $z_n \neq z_0$ for each n. Suppose $T_{z_n} = S_{z_n}$, $n = 1, 2, \dots$. Then $T_z = S_z$ for all $z \in \Theta$.

We also have the holomorphic obvious analogs of Props 3.29 and 3.30, with the word "continuous" replaced by "complex differentiable".

3.9. **Holomorphic families in the dual space.**

Definition 3.39. Let $\mathcal{H}^*$ denote the *dual space* of $\mathcal{H}$. We adopt the convention that $\mathcal{H}^*$ is the space of anti-linear continuous functionals, cf. [16]. (Sometimes $\mathcal{H}^*$ is then called the *antidual space*). If $\mathcal{X} \in \mathrm{Grass}(\mathcal{H})$, we denote by $\mathcal{X}^\perp \in \mathrm{Grass}(\mathcal{H}^*)$ its *annihilator*. If $T \in \mathcal{C}(\mathcal{K}, \mathcal{H})$ is densely defined, then $T^* \in \mathcal{C}(\mathcal{H}^*, \mathcal{K}^*)$ denotes its *adjoint*.

Let $\mathcal{X}, \mathcal{Y} \in \mathrm{Grass}(\mathcal{H})$ be two complementary subspaces. Then $\mathcal{X}^\perp$, $\mathcal{Y}^\perp$ are also complementary and

$$P^*_{\mathcal{X},\mathcal{Y}} = P_{\mathcal{Y}^\perp,\mathcal{X}^\perp}.$$

In the proof of the next theorem we will use the equivalence of various definitions of the holomorphy of functions with values in bounded operators mentioned at the beginning of the introduction [16].

Proposition 3.40 (Schwarz reflection principle for subspaces)**.** *A function*

$$z \mapsto \mathcal{X}_z \in \mathrm{Grass}_{\mathrm{com}}(\mathcal{H})$$

is complex differentiable at z_0 iff

$$z \mapsto \mathcal{X}^\perp_{\bar{z}} \in \mathrm{Grass}_{\mathrm{com}}(\mathcal{H}^*) \tag{3.28}$$

is complex differentiable at $\bar{z}_0$.

Proof. Locally, we can choose $\mathcal{Y} \in \mathrm{Grass}(\mathcal{H})$ complementary to $\mathcal{X}_z$. If (3.40) is holomorphic then $z \mapsto P_{\mathcal{X}_z,\mathcal{Y}}$ is holomorphic by Prop. 3.32. But $\mathcal{Y}^\perp$ is complementary to $\mathcal{X}^\perp_z$ and

$$P_{\mathcal{X}^\perp_z,\mathcal{Y}^\perp} = 1\!\!1 - P^*_{\mathcal{X}_z,\mathcal{Y}}. \tag{3.29}$$

So $z \mapsto P_{\mathcal{X}^\perp_{\bar{z}},\mathcal{Y}^\perp}$ is complex differentiable at $\bar{z}_0$. This means that (3.28) is complex differentiable.

Conversely, if (3.28) is complex differentiable at $\bar{z}_0$ then $z \mapsto P_{\mathcal{X}^\perp_{\bar{z}},\mathcal{Y}^\perp}$ is complex differentiable. By (3.29) this implies $z \mapsto P^*_{\mathcal{X}_{\bar{z}},\mathcal{Y}}$ is complex differentiable at $\bar{z}_0$. Therefore, $z \mapsto \langle u | P^*_{\mathcal{X}_{\bar{z}},\mathcal{Y}} v\rangle$ is complex differentiable for all $u \in \mathcal{H}^{**}$, $v \in \mathcal{H}^*$. In particular, by the embedding $\mathcal{H} \subset \mathcal{H}^{**}$, $z \mapsto \langle u | P^*_{\mathcal{X}_{\bar{z}},\mathcal{Y}} v\rangle = \overline{\langle v | P_{\mathcal{X}_{\bar{z}},\mathcal{Y}} u\rangle}$ is holomorphic for all $u \in \mathcal{H}$, $v \in \mathcal{H}^*$. This proves $P_{\mathcal{X}_z,\mathcal{Y}}$ is complex differentiable at z_0, thus (3.40) is complex differentiable as claimed. $\Box$

Remark 3.41. A direct analogue of Prop. 3.40 holds for continuity, as can be easily shown using the identity $\delta(\mathcal{X},\mathcal{Y})=\delta(\mathcal{Y}^\perp,\mathcal{X}^\perp)$ for closed subspaces $\mathcal{X},\mathcal{Y}\subset\mathcal{H}$, cf. [16].

We have an analogous property for functions with values in closed operators.

Theorem 3.42 (Schwarz reflection principle for closed operators)**.** *Let*

$$z\mapsto T_z\in\mathcal{C}_{\rm com}(\mathcal{H}_1,\mathcal{H}_2) \tag{3.30}$$

have values in densely defined operators. Then it is complex differentiable at z_0 iff

$$z\mapsto T_{\bar z}^*\in\mathcal{C}_{\rm com}(\mathcal{H}_2^*,\mathcal{H}_1^*)$$

is complex differentiable at $\bar z_0$.

Proof. It is well known that $\mathrm{Gr}(T_z)^\perp=U\mathrm{Gr}(T_z^*)$, where U is the invertible operator given by $U(x,y)=(-y,x)$ for $(x,y)\in\mathcal{H}_1^*\oplus\mathcal{H}_2^*$. Thus, the equivalence of the holomorphy of $z\mapsto\mathrm{Gr}(T_{\bar z}^*)$ and $z\mapsto\mathrm{Gr}(T_z)$ follows from Prop. 3.40 and Prop. 3.35 applied to the constant bounded invertible operator U or U^{-1}. □

We will make use of the following well-known result:

Theorem 3.43 ([16, Thm. 5.13])**.** *Let $T\in\mathcal{C}(\mathcal{H}_1,\mathcal{H}_2)$ be densely defined. Then $\mathrm{Ran}\,T$ is closed iff $\mathrm{Ran}\,T^*$ is closed in $\mathcal{H}_1^*$. In such case,*

$$(\mathrm{Ran}\,T)^\perp=\mathrm{Ker}\,T^*,\quad(\mathrm{Ker}\,T)^\perp=\mathrm{Ran}\,T^*. \tag{3.31}$$

Proposition 3.44. *Let $\Theta\ni z\mapsto S_z\in\mathcal{C}_{\rm com}(\mathcal{H}_1,\mathcal{H}_2)$ be holomorphic. Assume that $\mathrm{Dom}(S_z)$ is dense and $\mathrm{Ran}\,S_z=\mathcal{H}_2$. Then*

$$\Theta\ni z\mapsto\mathrm{Ker}S_z\in\mathrm{Grass}(\mathcal{H}_1) \tag{3.32}$$

is holomorphic.

Proof. Since $z\mapsto S_z$ is holomorphic, so is $z\mapsto S_{\bar z}^*$. Let $z\mapsto W_z$ be a resolution of holomorphy of $z\mapsto S_{\bar z}^*$. By (3.31), $\mathrm{Ker}S_{\bar z}^*=(\mathrm{Ran}\,S_{\bar z})^\perp=\{0\}$. It follows that $z\mapsto S_{\bar z}^*W_z$ is a holomorphic injective resolution of

$$z\mapsto\mathrm{Ran}\,S_{\bar z}^*\in\mathrm{Grass}(\mathcal{H}_1^*). \tag{3.33}$$

Hence (3.33) is holomorphic. But $(\mathrm{Ker}S_z)^\perp=\mathrm{Ran}\,S_{\bar z}^*$. Hence $z\mapsto\mathrm{Ker}S_z$ is holomorphic by the Schwarz reflection principle for subspaces (Prop. 3.40). □

4. Hilbert space theory

Throughout this section $\mathcal{H},\mathcal{H}_1,\mathcal{H}_2$ are Hilbert spaces. Note that $\mathcal{H}^*$ can be identified with $\mathcal{H}$ itself, the annihilator can be identified with the orthogonal complement and the adjoint with the Hermitian adjoint.

4.1. **Projectors.**

Definition 4.1. We will use the term *projector* as the synonym for *orthogonal projection*. We will write $P_\mathcal{X}$ for the projector onto $\mathcal{X}$.

Thus, $P_\mathcal{X}=P_{\mathcal{X},\mathcal{X}^\perp}$.

Let $\mathcal{X},\mathcal{Y}$ be subspaces of $\mathcal{H}$. Then $\mathcal{X}\oplus\mathcal{Y}^\perp=\mathcal{H}$ is equivalent to $\mathcal{X}^\perp\oplus\mathcal{Y}=\mathcal{H}$, which is equivalent to $\|P_\mathcal{X}-P_\mathcal{Y}\|<1$.

The gap topology on the Grassmannian of a Hilbert space simplifies considerably. In particular, the gap function is a metric and has a convenient expression in terms of the projectors:

$$\hat{\delta}(\mathcal{X}_1, \mathcal{X}_2) := \|P_{\mathcal{X}_1} - P_{\mathcal{X}_2}\|, \ \ \mathcal{X}_1, \mathcal{X}_2 \in \mathrm{Grass}(\mathcal{H}).$$

Thus a function

$$\Theta \ni z \mapsto \mathcal{X}_z \in \mathrm{Grass}(\mathcal{H}) \tag{4.1}$$

is continuous iff $z \mapsto P_{\mathcal{X}_z}$ is continuous. Unfortunately, the analogous statement is not true for the holomorphy, and we have to use the criteria discussed in the section on Banach spaces. This is however simplified by the fact that in a Hilbert space each closed subspace is complemented, so that $\mathrm{Grass}(\mathcal{H}) = \mathrm{Grass}_{\mathrm{com}}(\mathcal{H})$ and $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2) = \mathcal{C}_{\mathrm{com}}(\mathcal{H}_1, \mathcal{H}_2)$.

4.2. **Characteristic matrix.**

Definition 4.2. The *characteristic matrix* of a closed operator $T \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ is defined as the projector onto $\mathrm{Gr}(T)$ and denoted M_T.

Assume that T is densely defined. We set

$$\langle T \rangle := (\mathbb{1} + T^*T)^{\frac{1}{2}}.$$

It is easy to see that

$$J_T := \begin{bmatrix} \langle T \rangle^{-1} \\ T\langle T \rangle^{-1} \end{bmatrix} : \mathcal{H}_1 \to \mathcal{H}_1 \oplus \mathcal{H}_2$$

is a partial isometry onto $\mathrm{Gr}(T)$. Therefore, by (2.1) $M_T = J_T J_T^*$. To obtain a more explicit formula for the characteristic matrix, note the identities

$$\begin{aligned} T(\mathbb{1} + T^*T)^{-1} &= \big((\mathbb{1} + TT^*)^{-1}T\big)^{\mathrm{cl}}, \\ TT^*(\mathbb{1} + TT^*)^{-1} &= \big(T(\mathbb{1} + T^*T)^{-1}T^*\big)^{\mathrm{cl}} = \big((\mathbb{1} + TT^*)^{-1}TT^*\big)^{\mathrm{cl}}. \end{aligned}$$

Note that the above formulas involve products of unbounded operators. We use the standard definition of the product of unbounded operators recalled in Def. 2.5.

In the following formula for the characteristic matrix we are less pedantic and we omit the superscript denoting the closure:

$$M_T := \begin{bmatrix} \langle T \rangle^{-2} & \langle T \rangle^{-2}T^* \\ T\langle T \rangle^{-2} & T\langle T \rangle^{-2}T^* \end{bmatrix}, \tag{4.2}$$

Consider a function

$$\Theta \ni z \mapsto T_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_1). \tag{4.3}$$

Proposition 4.3. *Let $z_0 \in \Theta$. The function (4.3) is continuous in the gap topology at z_0 iff the functions*

$$\begin{aligned} \Theta \ni z &\mapsto \langle T_z \rangle^{-2} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_1), && (4.4)\\ \Theta \ni z &\mapsto T_z\langle T_z \rangle^{-2} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2), && (4.5)\\ \Theta \ni z &\mapsto \langle T_z^* \rangle^{-2} \in \mathcal{B}(\mathcal{H}_2, \mathcal{H}_2) && (4.6) \end{aligned}$$

are continuous at z_0.

Proof. Clearly, (3.14) is continuous in the gap topology iff $\Theta \ni z \mapsto M_{T_z}$ is. Now (4.4), (4.5) resp. (4.6) are $\left(M_{T_z}\right)_{11}$, $\left(M_{T_z}\right)_{12} = \left(M_{T_z}\right)_{21}^*$, resp. $1\!\!1 - \left(M_{T_z}\right)_{22}$. □

The gap topology is not the only topology on $\mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ that on $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ coincides with the usual norm topology. Here is one of the examples considered in the literature:

Definition 4.4. We say that (3.14) is continuous at z_0 in the *Riesz topology* [1] if

$$\Theta \ni z \mapsto T_z \langle T_z \rangle^{-1} \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$$

is continuous at z_0.

It is easy to see that the Riesz topology is strictly stronger than the gap topology [17]. Indeed, the fact that it is stronger is obvious. Its non-equivalence with the gap topology on self-adjoint operators follows from the following easy fact [20]:

Theorem 4.5. *Suppose that the values of (3.14) are self-adjoint.*

(1) The following are equivalent:

(i) $\lim\limits_{z \to z_0} T_z = T_{z_0}$ *in the gap topology;*

(ii) $\lim\limits_{z \to z_0} f(T_z) = f(T_{z_0})$ *for all bounded continuous* $f : \mathbb{R} \to \mathbb{C}$ *such that* $\lim\limits_{t \to -\infty} f(t)$ *and* $\lim\limits_{t \to +\infty} f(t)$ *exist and are equal.*

(2) The following are equivalent:

(i) $\lim\limits_{z \to z_0} T_z = T_{z_0}$ *in the Riesz topology;*

(ii) $\lim\limits_{z \to z_0} f(T_z) = f(T_{z_0})$ *for all bounded continuous* $f : \mathbb{R} \to \mathbb{C}$ *such that* $\lim\limits_{t \to -\infty} f(t)$ *and* $\lim\limits_{t \to +\infty} f(t)$ *exist.*

4.3. **Relative characteristic matrix.** Let T and S be densely defined closed operators.

Theorem 4.6. *We have*

$$\mathrm{Gr}(T) \oplus \mathrm{Gr}(S)^\perp = \mathcal{H}_1 \oplus \mathcal{H}_2$$

iff $J_S^* J_T$ *is invertible. Then the projection onto* $\mathrm{Gr}(T)$ *along* $\mathrm{Gr}(S)^\perp$ *is given by*

$$J_T (J_S^* J_T)^{-1} J_S^* \tag{4.7}$$

Proof. We have $J_T \in \mathcal{B}_{\mathrm{inv}}(\mathcal{H}_1, \mathcal{H}_1 \oplus \mathcal{H}_2)$. $J_S^* \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2, \mathcal{H}_1)$ is surjective. Besides, $\mathrm{Ran}\, J_T = \mathrm{Gr}(T)$ and $\mathrm{Ker} J_S^* = \mathrm{Gr}(S)^\perp$. Therefore, it suffices to apply first Prop. 3.6, and then Prop. 2.8. □

Definition 4.7. (4.7) will be called the *relative characteristic matrix of* T, S and will be denoted $M_{T,S}$.

Clearly, $M_T = M_{T,T}$.

We can formally write

$$J_S^* J_T = \langle S \rangle^{-1} (1\!\!1 + S^* T) \langle T \rangle^{-1}. \tag{4.8}$$

To make (4.8) rigorous we interpret $(1\!\!1 + S^* T)$ as a bounded operator from $\langle T \rangle^{-1} \mathcal{H}_1$ to $\langle S \rangle \mathcal{H}_1$. Now the inverse of (4.8) is a bounded operator, which can be formally written as

$$(J_S^* J_T)^{-1} = \langle T \rangle (1\!\!1 + S^* T)^{-1} \langle S \rangle.$$

[1]Note that this name is not used in older references.

Thus we can write

$$M_{T,S} = \begin{bmatrix} \langle T\rangle^{-1} \\ T\langle T\rangle^{-1} \end{bmatrix} \langle T\rangle(\mathbb{1}+S^*T)^{-1}\langle S\rangle \begin{bmatrix} \langle S\rangle^{-1} & \langle S\rangle^{-1}S^* \end{bmatrix}$$

$$= \begin{bmatrix} (\mathbb{1}+S^*T)^{-1} & (\mathbb{1}+S^*T)^{-1}S^* \\ T(\mathbb{1}+S^*T)^{-1} & T(\mathbb{1}+S^*T)^{-1}S^* \end{bmatrix} \tag{4.9}$$

$$= \begin{bmatrix} (\mathbb{1}+S^*T)^{-1} & S^*(\mathbb{1}+TS^*)^{-1} \\ T(\mathbb{1}+S^*T)^{-1} & TS^*(\mathbb{1}+TS^*)^{-1} \end{bmatrix}. \tag{4.10}$$

Note that even though the entries of (4.9) and (4.10) are expressed in terms of unbounded operators, all of them can be interpreted as bounded everywhere defined operators (eg. by taking the closure of the corresponding expression).

4.4. **Holomorphic families of closed operators.** In order to check the holomorphy of a function

$$\Theta \ni z \mapsto T_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2) \tag{4.11}$$

using the criterion given in Prop. 3.37 one needs to find a relatively arbitrary function $z \mapsto W_z$. In the case of a Hilbert space we have a criterion for the complex differentiability involving relative characteristic matrices. We believe that this criterion should be often more convenient, since it involves a function with values in bounded operators uniquely defined for any $z_0 \in \Theta$.

Proposition 4.8. *Let $z_0 \in \Theta$ and assume (4.11) has values in densely defined operators. Then (4.11) is complex differentiable at z_0 if there exists an open Θ_0 such that $z_0 \in \Theta_0 \subset \Theta$, and for $z \in \Theta_0$,*

$$\langle T_{z_0}\rangle^{-1}(\mathbb{1}+T_{z_0}^*T_z)\langle T_z\rangle^{-1}$$

is invertible, so that we can define

$$\begin{bmatrix} (\mathbb{1}+T_{z_0}^*T_z)^{-1} & T_{z_0}^*(\mathbb{1}+T_zT_{z_0}^*)^{-1} \\ T_z(\mathbb{1}+T_{z_0}^*T_z)^{-1} & T_zT_{z_0}^*(\mathbb{1}+T_zT_{z_0}^*)^{-1} \end{bmatrix} \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2), \tag{4.12}$$

and (4.12) is complex differentiable at z_0.

5. Products and sums of operator–valued holomorphic functions

In this section we focus on the question what conditions ensure that the product and the sum of two holomorphic families of closed operators is holomorphic. Note that analogous statements hold for families continuous in the gap topology.

Throughout the section $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$ are Banach spaces.

5.1. **Products of closed operators I.** If both A, B are closed operators, then the product AB (see Def. 2.5) does not need to be closed. We recall below standard criteria for this to be true. For a more detailed discussion and other sufficient conditions we refer the reader to [1] and references therein.

Proposition 5.1. *Let*

(1) $A \in \mathcal{C}(\mathcal{H}, \mathcal{H}_2)$ and $B \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$, or
(2) $A \in \mathcal{C}_{\rm inv}(\mathcal{H}, \mathcal{H}_2)$ and $B \in \mathcal{C}(\mathcal{H}_1, \mathcal{H})$.

Then AB is closed.

The simpliest conditions which imply holomorphy of the product are listed in the proposition below. Unconveniently, they are not quite compatible with the sufficient conditions for the closedness of the product, which has to be assumed separately.

Proposition 5.2. *Let*

(1) $\Theta \ni z \mapsto A_z \in \mathcal{B}(\mathcal{H}, \mathcal{H}_2), B_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H})$ *be holomorphic, or*
(2) $\Theta \ni z \mapsto A_z \in \mathcal{C}(\mathcal{H}, \mathcal{H}_2), B_z \in \mathcal{C}_{\rm inv}(\mathcal{H}_1, \mathcal{H})$ *be holomorphic.*

If in addition $A_z B_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ *for all* $z \in \Theta$ *then* $\Theta \ni z \mapsto A_z B_z$ *is holomorphic.*

Proof. (1) Let $z \mapsto V_z$ be a resolution of holomorphy of $z \mapsto B_z$. Then it is also a resolution of holomorphy of $z \mapsto A_z B_z$.

(2) Let $z \mapsto U_z$ be a resolution of holomorphy of $z \mapsto A_z$. By the holomorphic version of Prop. 3.29, $z \mapsto B_z^{-1} \in \mathcal{B}(\mathcal{H}, \mathcal{H}_1)$ is holomorphic. It is obviously injective. Hence $B_z^{-1} U_z$ is a resolution of holomorphy of $z \mapsto A_z B_z$. □

5.2. **Examples and counterexamples.**

Definition 5.3. The *point spectrum* of T is defined as

$$\mathrm{sp_p}(T) := \{z \in \mathbb{C} \ : \ \mathrm{Ker}(A - z\mathbb{1}) \neq \{0\}\}.$$

Example 5.4. Let $T \in \mathcal{B}(\mathcal{H})$. Then $z \mapsto (z\mathbb{1} - T)^{-1}$ is holomorphic on $\mathbb{C} \backslash \mathrm{sp_p}(T)$. Indeed, $z \mapsto W_z := z\mathbb{1} - T$ is injective, holomorphic, $\mathrm{Ran}\, W_z = \mathrm{Dom}(z\mathbb{1} - T)^{-1}$ and $(z\mathbb{1} - T)^{-1} W_z = \mathbb{1}$.

Example 5.5. The above example can be generalized. Let $T \in \mathcal{C}(\mathcal{H})$ have a nonempty resolvent set. Then $z \mapsto (z\mathbb{1} - T)^{-1}$ is holomorphic on $\mathbb{C} \backslash \mathrm{sp_p}(T)$. Indeed, let $z_0 \in \mathrm{rs}(T)$. Then $z \mapsto W_z := (z\mathbb{1} - T)(z_0\mathbb{1} - T)^{-1}$ is injective, holomorphic, $\mathrm{Ran}\, W_z = \mathrm{Dom}(z\mathbb{1} - T)^{-1}$ and $(z\mathbb{1} - T)^{-1} W_z = (z_0\mathbb{1} - T)^{-1}$.

Example 5.6. Consider $A_z := T$ with $T \in \mathcal{C}(\mathcal{H})$ unbounded and $B_z := z\mathbb{1} \in \mathcal{B}(\mathcal{H})$. Then the product $A_z B_z$ is closed for all $z \in \mathbb{C}$, but the function $z \mapsto A_z B_z$ is not complex differentiable at $z = 0$ due to the fact that that it yields a bounded operator at $z = 0$, but fails to do so in any small neighbourhood (cf. Example 2.1 in [16, Ch. VII.2]).

Therefore, it is not true that if A_z and B_z are holomorphic and $A_z B_z$ is closed for all z, then $z \mapsto A_z B_z$ is holomorphic.

The more surprising fact is that even the additional requirement that $A_z B_z$ is bounded does not guarantee the holomorphy, as shows the example below.

Example 5.7. Assume that $T \in \mathcal{C}(\mathcal{H})$ has empty spectrum. Note that this implies that $\mathrm{sp}(T^{-1}) = \{0\}$ and $\mathrm{sp_p}(T^{-1}) = \emptyset$. By Example 5.4, $A_z := T(Tz - \mathbb{1})^{-1} = (z\mathbb{1} - T^{-1})^{-1}$ is holomorphic. Obviously, so is $B_z := z\mathbb{1}$. Moreover, $z \mapsto A_z B_z = \mathbb{1} + (Tz - \mathbb{1})^{-1}$ has values in bounded operators. However, it is not differentiable at zero, because

$$\partial_z A_z B_z v \Big|_{z=0} = Tv, \quad v \in \mathrm{Dom} T,$$

and T is unbounded.

5.3. **Products of closed operators II.** In this subsection, exceptionally, $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}$ are Hilbert spaces.

We quote below a useful criterion specific to that case. The proof is not difficult and can be found for instance in [6, Prop 2.35].

Proposition 5.8. *Let $A \in \mathcal{B}(\mathcal{H}, \mathcal{H}_2)$, $B \in \mathcal{C}(\mathcal{H}_1, \mathcal{H})$. If $\mathrm{Dom}(B^*A^*)$ is dense, then AB is closable, B^*A^* is closed and $(AB)^* = B^*A^*$.*

Together with Prop. 5.2, this yields the following result.

Proposition 5.9. *Let $\Theta \ni z \mapsto A_z \in \mathcal{C}(\mathcal{H}, \mathcal{H}_2)$, $\Theta \ni z \mapsto B_z \in \mathcal{B}(\mathcal{H}_1, \mathcal{H})$ be holomorphic. If A_z, A_zB_z are densely defined and $B_z^*A_z^*$ is closed for all $z \in \Theta$, then $\Theta \ni z \mapsto A_zB_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ is holomorphic.*

Proof. Since A_z is closed and densely defined, $A_z^{**} = A_z$. By Prop. 5.1, A_zB_z is closed. Thus, we can apply Prop. 5.8 to $A := B_z^*$ and $B := A_z^*$ for all $z \in \Theta$ and conclude

$$A_zB_z = A_z^{**}B_z^{**} = (B_z^*A_z^*)^* \tag{5.1}$$

By the Schwarz reflection principle and Prop. 5.2, $z \mapsto B_{\bar{z}}^*A_{\bar{z}}^*$ is holomorphic. Therefore, $z \mapsto A_zB_z$ is holomorphic by (5.1) and the Schwarz reflection principle. □

5.4. **Non-empty resolvent set case.** Here is another sufficient condition for the holomorphy of the product of operator–valued holomorphic functions, based on a different strategy.

In the statement of our theorem below, the closedness of the products A_zB_z and B_zA_z is implicitly assumed in the non-empty resolvent set condition.

Theorem 5.10. *Let $\Theta \ni z \mapsto A_z \in \mathcal{C}(\mathcal{H}_2, \mathcal{H}_1)$ and $\Theta \ni z \mapsto B_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ be holomorphic. Assume that there exists $\lambda \in \mathbb{C}$ s.t. $\lambda^2 \in \mathrm{rs}(A_zB_z) \cap \mathrm{rs}(B_zA_z)$ for all $z \in \Theta$. Then both $\Theta \ni z \mapsto A_zB_z, B_zA_z$ are holomorphic.*

The proof is based on the helpful trick of replacing the study of the product A_zB_z by the investigation of the operator T_z on $\mathcal{H}_1 \oplus \mathcal{H}_2$ defined by

$$T_z := \begin{bmatrix} 0 & A_z \\ B_z & 0 \end{bmatrix}, \quad \mathrm{Dom}(T_z) := \mathrm{Dom}(B_z) \oplus \mathrm{Dom}(A_z). \tag{5.2}$$

Its square is directly related to A_zB_z, namely

$$T_z^2 = \begin{bmatrix} A_zB_z & 0 \\ 0 & B_zA_z \end{bmatrix}, \quad \mathrm{Dom}(T_z^2) = \mathrm{Dom}(A_zB_z) \oplus \mathrm{Dom}(B_zA_z). \tag{5.3}$$

A similar idea is used in [12], where results on the relation between $\mathrm{sp}(AB)$ and $\mathrm{sp}(BA)$ are derived. The following lemma will be of use for us.

Lemma 5.11 ([12], Lem 2.1)**.** *Let $A \in \mathcal{C}(\mathcal{H}_2, \mathcal{H}_1)$, $B \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ and let $T \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ be defined as in (5.2) (with the subscript denoting dependence on z ommited). Then*

$$\mathrm{rs}\, T = \{\lambda \in \mathbb{C} : \ \lambda^2 \in \mathrm{rs}(AB) \cap \mathrm{rs}(BA)\}.$$

Proof. '⊂': Suppose $\lambda \in \mathrm{rs}\, T \setminus \{0\}$. Then one can check that the algebraic inverse of $BA - \lambda^2 \mathbb{1}$ equals

$$\lambda^{-1} P_{\mathcal{H}_2, \mathcal{H}_1} (T - \lambda \mathbb{1})^{-1} J_{\mathcal{H}_2},$$

hence $BA - \lambda^2 \mathbb{1} \in \mathcal{C}_{\mathrm{inv}}(\mathcal{H}_2)$. Analogously we obtain $AB - \lambda^2 \mathbb{1} \in \mathcal{C}_{\mathrm{inv}}(\mathcal{H}_1)$.

Suppose now $0 \in \operatorname{rs} T$. This implies that A, B are invertible and consequently AB, BA are invertible.

'$\supset$': Suppose $\lambda^2 \in \operatorname{rs}(AB) \cap \operatorname{rs}(BA)$. Obviously, $\lambda^2 \in \operatorname{rs}(T^2)$.

Suppose that $v \in \operatorname{Ker}(T + \lambda 1\!\!1)$. Then $Tv = -\lambda v \in \operatorname{Dom} T$. Hence $v \in \operatorname{Dom}(T^2)$ and

$$(T^2 - \lambda^2 1\!\!1)v = (T - \lambda 1\!\!1)(T + \lambda 1\!\!1)v = 0,$$

which implies $v = 0$. Hence $T + \lambda 1\!\!1$ is injective.

Suppose that $w \in \mathcal{H}_1 \oplus \mathcal{H}_2$. Then there exists $v \in \operatorname{Dom} T^2$ such that $(T^2 - \lambda^2)v = w$. But

$$w = (T + \lambda 1\!\!1)(T - \lambda 1\!\!1)v.$$

Hence $w \in \operatorname{Ran}(T + \lambda 1\!\!1)$.

Thus we have shown that $T + \lambda 1\!\!1$ is invertible, or $-\lambda \in \operatorname{rs} T$. The same argument shows $\lambda \in \operatorname{rs} T$. □

Proof of Thm. 5.10. Let $z \mapsto P_z$, resp. $z \mapsto Q_z$ be resolutions of holomorphy of $z \mapsto A_z$, resp. $z \mapsto B_z$. Then

$$z \mapsto W_z := \begin{bmatrix} 0 & Q_z \\ P_z & 0 \end{bmatrix}$$

is a resolution of holomorphy of $z \mapsto T_z$ defined as in (5.2). By Lemma 5.11, $\lambda, -\lambda \in \operatorname{rs} T_z$. Define $V_z := (T_z - \lambda 1\!\!1)^{-1} W_z$. Clearly, V_z is holomorphic and has values in bounded operators, and so does

$$T_z^2 V_z = (1\!\!1 + \lambda (T_z - \lambda 1\!\!1)^{-1}) T_z W_z.$$

Moreover, V_z is injective for all $z \in \Theta_0$ and

$$\begin{aligned} \operatorname{Ran} V_z &= (T_z - \lambda 1\!\!1)^{-1} \operatorname{Ran} W_z = (T_z - \lambda 1\!\!1)^{-1} \operatorname{Dom}(T_z) \\ &= (T_z - \lambda 1\!\!1)^{-1} (T_z + \lambda 1\!\!1)^{-1} (\mathcal{H}_1 \oplus \mathcal{H}_2) = \operatorname{Dom}(T_z^2). \end{aligned}$$

Hence $z \mapsto T_z^2$ is holomorphic. Therefore, $z \mapsto (\lambda^2 1\!\!1 - T_z^2)^{-1}$ is holomorphic. This implies the holomorphy of $z \mapsto (\lambda^2 1\!\!1 - A_z B_z)^{-1}$, $z \mapsto (\lambda^2 1\!\!1 - B_z A_z)^{-1}$. □

5.5. Case $\operatorname{Dom}(A_z) + \operatorname{Ran} B_z = \mathcal{H}$.

In this section we use a different strategy. The idea is to represent a subspace closely related to $\operatorname{Gr}(A_z B_z)$ as the kernel of a bounded operator which depends in a holomorphic way on z. This allows us to treat the holomorphy of the product $A_z B_z$ under the assumption that $\operatorname{Dom}(A_z) + \operatorname{Ran} B_z = \mathcal{H}$.

Theorem 5.12. *Let $\Theta \ni z \mapsto A_z \in \mathcal{C}(\mathcal{H}, \mathcal{H}_2)$ and $\Theta \ni z \mapsto B_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H})$ be holomorphic. Suppose that $A_z B_z$ is closed and*

$$\operatorname{Dom}(A_z) + \operatorname{Ran} B_z = \mathcal{H} \tag{5.4}$$

for all $z \in \Theta$. Then $\Theta \ni z \mapsto A_z B_z$ is holomorphic.

Proof. Let U_z, V_z be resolutions of holomorphy of respectively A_z, B_z, so in particular $\operatorname{Ran} U_z = \operatorname{Dom}(A_z)$, $\operatorname{Ran} B_z V_z = \operatorname{Ran} B_z$. Let S_z be defined by

$$S_z \begin{bmatrix} x \\ y \end{bmatrix} = B_z V_z x - U_z y.$$

Clearly, S_z is bounded-holomorphic and $\operatorname{Ran} S_z = \operatorname{Dom}(A_z) + \operatorname{Ran} B_z = \mathcal{H}$ by assumption. Therefore, $z \mapsto \operatorname{Ker} S_z$ is holomorphic by Prop. 3.44.

A straightforward computation shows that

$$\mathrm{Gr}(A_zB_z) = \left\{ \begin{bmatrix} V_z x \\ A_z U_z y \end{bmatrix} :\ B_z V_z x = U_z y \right\} = \begin{bmatrix} V_z & 0 \\ 0 & A_z U_z \end{bmatrix} \mathrm{Ker}\, S_z$$
$$=: T_z\, \mathrm{Ker} S_z.$$

The function $z \mapsto T_z$ has values in injective bounded operators and is holomorphic, therefore $z \mapsto T_z \mathrm{Ker} S_z$ is holomorphic by Prop. 3.35. □

An analogous theorem for continuity is proved in [19, Thm. 2.3], using however methods which do not apply to the holomorphic case.

Remark 5.13. An example when the assumptions of Thm. 5.12 are satisfied is provided by the case when A_z and B_z are densely defined Fredholm operators. It is well-known that the product is then closed (it is in fact a Fredholm operator), whereas the propriety $\mathrm{Dom}(A_z) + \mathrm{Ran}\, B_z = \mathcal{H}$ follows from $\mathrm{codim}(\mathrm{Ran}\, B_z) < \infty$ and the density of $\mathrm{Dom}(A_z)$.

5.6. **Sums of closed operators.** Using, for instance, the arguments from Prop. 3.26), it is easy to show

Proposition 5.14. *If $z \mapsto T_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ and $z \mapsto S_z \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ are holomorphic then $z \mapsto T_z + S_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ is holomorphic.*

To prove a more general statement, we reduce the problem of the holomorphy of the sum to the holomorphy of the product of suitably chosen closed operators. To this end we will need the following easy lemma.

Lemma 5.15. *A function $z \mapsto T_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ is holomorphic iff the function*

$$z \mapsto A_z := \begin{bmatrix} \mathbb{1} & 0 \\ T_z & \mathbb{1} \end{bmatrix}, \quad \mathrm{Dom}(A_z) = \mathrm{Dom}(T_z) \oplus \mathcal{H}_1$$

is holomorphic. Moreover, $\mathrm{Ran}\, A_z = \mathrm{Dom}(T_z) \oplus \mathcal{H}_1$.

Proof. The function $z \mapsto A_z$ is holomorphic iff

$$z \mapsto A_z - \mathbb{1} = \begin{bmatrix} 0 & 0 \\ T_z & 0 \end{bmatrix} \tag{5.5}$$

is holomorphic. The claim follows by remarking that the graph of (5.5) is equal to the graph of T_z up to a part which is irrelevant for the holomorphy. □

Theorem 5.16. *Let $\Theta \ni z \mapsto T_z, S_z \in \mathcal{C}(\mathcal{H}_1, \mathcal{H}_2)$ be holomorphic. Suppose that*

$$\mathrm{Dom}(S_z) + \mathrm{Dom}(T_z) = \mathcal{H}_1. \tag{5.6}$$

and $T_z + S_z$ is closed for all $z \in \Theta$. Then $\Theta \ni z \mapsto T_z + S_z$ is holomorphic.

Proof. Let A_z, B_z, resp. C_z be defined as in Lem. 5.15 from T_z, S_z, resp. $T_z + S_z$. The holomorphy of $T_z + S_z$ is equivalent to holomorphy of C_z. An easy computation shows that $C_z = A_z B_z$. By (5.6) we have

$$\mathrm{Dom}(A_z) + \mathrm{Ran}\, B_z = (\mathrm{Dom}(T_z) \oplus \mathcal{H}_1) + (\mathrm{Dom}(S_z) \oplus \mathcal{H}_1) = \mathcal{H}_1 \oplus \mathcal{H}_1.$$

Moreover, A_z, B_z are holomorphic by Lem. 5.15, therefore $A_z B_z$ is holomorphic by Thm. 5.12. □

Remark 5.17. If T_z is unbounded then the auxiliary operator A_z introduced in Lem. 5.15 satisfies $\mathrm{sp}(A_z) = \mathbb{C}$ (cf. [12, Ex. 2.7]). The proof of Thm. 5.16 is an example of a situation where even if one is interested in the end in operators with non-empty resolvent set, it is still useful to work with operators with empty resolvent set.

REFERENCES

[1] Azizov, T.Ya., Dijksma, A.: *Closedness and adjoints of products of operators, and compressions*, Integr. Equ. Oper. Theory, vol. 74, No. 2, 259–269 (2012)

[2] Berkson, E.: *Some metrics on the subspaces of a Banach space*, Pacific J. Math. 13, 7–22 (1963)

[3] Bruk, V.M.: *A uniqueness theorem for holomorphic families of operators*, Matematicheskie Zametki, Vol. 53, No. 3, 155–156 (1991)

[4] Bruneau, L., Dereziński, J., Georgescu, V.: *Homogeneous Schrödinger operators on half-line*, Ann. Henri Poincaré, vol. 12, no. 3, 547–590 (2011)

[5] Davies, E.B.: *Spectral Theory and Differential Operators*, Cambridge University Press (1995)

[6] Dereziński, J., Gérard, C.: *Mathematics of Quantization and Quantum Fields*, Cambridge University Press (2013)

[7] Dereziński, J., Wrochna, M.: *Exactly solvable Schrödinger operators*, Ann. Henri Poincaré, vol. 12, no. 2, 397–418 (2011)

[8] Dirr, G., Rakočević, V., Wimmler, H.: *Estimates for projections in Banach spaces and existence of direct complements*, Stud. Math. 170, 211-216 (2005)

[9] Gohberg, I.C., Goldberg, S., Kaashoek, M.A.: *Classes of Linear Operators*, Vol. I, Birkhäuser–Verlag, Basel Boston Berlin (1990)

[10] Gohberg, I.C., Markus, A.S.: *Two theorems on the gap between subspaces of a Banach space*, (in Russian), Uspehi Mat. Nauk 14, no. 5 (89), 135-140 (1959)

[11] Gohberg, I., Leiterer, J.: *Holomorphic Operator Functions of One Variable and Applications*, Operator Theory: Advances and Applications, vol. 192, Birkhäuser (2009)

[12] Hardt, V., Konstantinov, A., Mennicken, R.: *On the spectrum of the product of closed operators*, Math. Nachr. 215, 91–102 (2000)

[13] Herbst, I.W.: *Dilation analyticity in constant electric field. I. The two body problem*, Comm. Math. Phys. 64, No. 3, 191–298 (1979)

[14] Kaballo, W.: *Meromorphic generalized inverses of operator functions*, Indag. Math., vol. 23, 4, 970–994 (2012)

[15] Kadets, M.I., Mityagin, B.S.: *Complemented subspaces in Banach spaces*, Russ. Math. Surv. 28 77 (1973)

[16] Kato, T.: *Perturbation Theory for Linear Operators*, Springer (1966)

[17] Kaufman, W.E.: *A stronger metric for closed operators in Hilbert space*, Proc. Amer. Math. Soc. 90, 83-87 (1984)

[18] Leiterer, J., Rodman, L.: *Smoothness of generalized inverses*, Indag. Math., vol. 23, 4, 615-649 (2012)

[19] Neubauer, G.: *Homotopy properties of semi-Fredholm operators in Banach spaces*, Math. Annalen 176, 273–301 (1968)

[20] Nicolaescu, L.: *On the space of Fredholm operators*, preprint math.DG/0005089 (2000)

[21] Rellich, F.: *Störungstheorie der Spektralzerlegung III.*, Math. Ann. 116, 550–570 (1937)

[22] Shubin, M.A.: *On holomorphic families of subspaces of a Banach space*, Integr. Equ. Oper. Theory 2, no. 3, 407–420 (1979)

[23] Zworski, M.: *Semiclassical Analysis*, Graduate Studies in Mathematics 138, American Mathematical Society (2012)