

MATHEMATICAL TOOLS  
OF MANY BODY QUANTUM THEORY

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# Chapter 1

## Basic rules of Quantum Mechanics

### 1.1 Observables – finite dimension

We would like to describe basic framework of quantum mechanics – the description of **states** and **observables**. To avoid technical complications, in this section we will always assume that the Hilbert space  $\mathcal{H}$  describing a quantum system is finite dimensional, so that it can be identified with  $\mathbb{C}^N$ , for some  $N$ .

In basic courses on Quantum Mechanics we learn that quantum states are described by a **density matrix**  $\rho$  and a **yes/no experiment** by an **orthogonal projection**  $P$ . The probability of the affirmative outcome of such an experiment equals

$$\text{Tr}\rho P.$$

Two orthogonal projections  $P_1$  and  $P_2$  are simultaneously measurable iff they commute.

We say that a family of orthogonal projections  $P_1, \dots, P_n$  is an **orthogonal partition of unity on  $\mathcal{H}$**  iff

$$\sum_{i=1}^n P_i = \mathbb{1}, \quad P_i P_j = \delta_{ij} P_j, \quad i, j = 1, \dots, n.$$

Clearly, all elements of an orthogonal partition of unity commute with one another. Therefore, in principle, one can design an experiment that measures simultaneously all of them.

Note that if  $P_1, \dots, P_n$  is an orthogonal partition of unity, then setting  $\mathcal{H}_i := \text{Ran}P_i$ ,  $i = 1, \dots, n$ , we obtain an **orthogonal direct sum decomposition**  $\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i$ . Thus specifying an orthogonal partition of unity is equivalent to specifying an orthogonal direct sum decomposition.

Let  $P_1, \dots, P_n$  be a **maximal** orthogonal partition of unity, that is, an orthogonal partition of unity satisfying  $\dim P_i = 1$ ,  $i = 1, \dots, n$ . If we choose a

normalized vector  $\Psi_i$  in the range of  $P_i$  for  $i = 1, \dots, n$ , we obtain an **orthonormal basis** of  $\mathcal{H}$ .

To any **self-adjoint operator**  $A$  we can associate an orthogonal partition of unity given by the **spectral projections of  $A$  onto its eigenvalues**:

$$\mathbb{1}_{\{a\}}(A), \quad a \in \text{sp}(A). \quad (1.1)$$

By **measuring the observable  $A$**  we mean measuring the partition of unity (1.1). Clearly,  $A = \sum_{a \in \text{sp}(A)} a \mathbb{1}_a(A)$ . Hence, the average eigenvalue of  $A$  in such an experiment equals

$$\text{Tr} \rho A = \sum_{a \in \text{sp}(A)} a \text{Tr} \rho \mathbb{1}_a(A). \quad (1.2)$$

We call (1.2) the **expectation value of the observable  $A$  in the state  $\rho$** .

Let self-adjoint operators  $A_1, \dots, A_n$  commute. Then so do their spectral projections. Define the **joint spectrum** of  $A_1, \dots, A_n$  by

$$\text{sp}(A_1, \dots, A_n) := \{(a_1, \dots, a_n) \in \mathbb{R}^n : \mathbb{1}_{\{a_1\}}(A_1) \cdots \mathbb{1}_{\{a_n\}}(A_n) \neq 0\}.$$

For any subset  $\Omega \subset \text{sp}(A_1, \dots, A_n)$  we can define the **spectral projection of  $A_1, \dots, A_n$  onto  $\Omega$** :

$$\mathbb{1}_\Omega(A_1, \dots, A_n) := \sum_{(a_1, \dots, a_n) \in \Omega} \mathbb{1}_{\{a_1\}}(A_1) \cdots \mathbb{1}_{\{a_n\}}(A_n).$$

If the partition of unity

$$\mathbb{1}_{\{a_1\}}(A_1) \cdots \mathbb{1}_{\{a_n\}}(A_n) : (a_1, \dots, a_n) \in \text{sp}(A_1, \dots, A_n)$$

is maximal, then we say that the joint spectrum of  $A_1, \dots, A_n$  is **simple**. Physicists often try to find families of self-adjoint operators with a simple joint spectrum. Then they choose a normalized vector in  $\text{Ran} \mathbb{1}_{\{a_1\}}(A_1) \cdots \mathbb{1}_{\{a_n\}}(A_n)$ . We will use the traditional way to denote such a vector, which goes back to Dirac, that is  $|a_1, \dots, a_n\rangle$ . Note that a slightly different notation is used when this vector is on the left hand side of a scalar product: then it is written as  $\langle a_1, \dots, a_n|$ .

Clearly, the vectors  $|a_1, \dots, a_n\rangle$  form an o.n. basis of simultaneous eigenvectors of  $A_i$ :

$$A_i |a_1, \dots, a_n\rangle = a_i |a_1, \dots, a_n\rangle.$$

Note that if  $\Omega \subset \text{sp}(A_1, \dots, A_n)$ , then

$$\mathbb{1}_\Omega(A_1, \dots, A_n) = \sum_{(a_1, \dots, a_n) \in \Omega} |a_1, \dots, a_n\rangle \langle a_1, \dots, a_n|.$$

The setting of the following examples are an infinite dimensional Hilbert spaces but it can be easily understood with finite dimensional concepts.

**Example 1.1** Consider a function  $\mathbb{R} \ni x \mapsto V(x)$  such that  $\lim_{|x| \rightarrow \infty} V(x) = \infty$ . Consider the the *Schrödinger operator*  $H := -\partial_x^2 + V(x)$  on  $L^2(\mathbb{R})$ . One can show that it has a purely point simple spectrum.

**Example 1.2** Let  $\mathbb{R} \ni r \mapsto V(r)$  be a function such that  $\lim_{r \rightarrow \infty} V(r) = \infty$ . Consider the space  $L^2(\mathbb{R}^3)$ , the Schrödinger operator  $H := -\Delta + V(|x|)$ , the *square of angular momentum*  $L^2$  and the *zth component of the angular momentum*  $L_z$ .  $H, L^2, L_z$  is a triple of commuting self-adjoint operators with a simple joint spectrum.

So far we assumed that all orthogonal projections on  $\mathcal{H}$ , hence all self-adjoint operators on  $\mathcal{H}$ , correspond to possible experiments. We say that all self-adjoint elements of  $B(\mathcal{H})$  are *observable*.

Sometimes this is not the case. We are going to describe several situations where only a part of self-adjoint operators are observable.

It may happen that the Hilbert space  $\mathcal{H}$  has a distinguished direct sum decomposition

$$\mathcal{H} = \bigoplus_{i=1}^n \mathcal{H}_i \quad (1.3)$$

such that only self-adjoint operators that preserve each subspace  $\mathcal{H}_i$  are measurable. We say then that  $\mathcal{H}_i, i = 1, \dots, n$ , are *superselection sectors*.

Let  $Q_i$  denote the orthogonal projection onto  $\mathcal{H}_i$ . Then linear combinations of  $Q_i$  can be measured simultaneously with all other observables. We say that they are *classical observables*.

If we choose an o.n. basis of  $\mathcal{H}$  compatible with (1.3), then only block diagonal self-adjoint matrices are observable.

Note that states are also described by block diagonal matrices.

Superselection sectors arise typically when we have a strictly conserved quantity. For instance, the *total charge* of the system usually determines a superselection sector. States of an *even* and *odd number of fermions* form two superselection sectors.

Suppose that two quantum systems are described by Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2$ . Then the *composite system* is described by the tensor product  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . Observables of the first system are described by self-adjoint elements of  $B(\mathcal{H}_1) \otimes \mathbb{1}_{\mathcal{H}_2}$ , whereas observables of the second system are described by self-adjoint elements of  $\mathbb{1}_{\mathcal{H}_1} \otimes B(\mathcal{H}_2)$ . Note that they commute, so that one can simultaneously measure them. From the point of view of the first system only self-adjoint elements of  $B(\mathcal{H}_1) \otimes \mathbb{1}_{\mathcal{H}_2}$  are observable. Again, we have a situation where not all self-adjoint elements of  $B(\mathcal{H})$  are observable.

Let  $\mathcal{H}_1 = \mathbb{C}^p$  with an o.n. basis  $e_1, \dots, e_p$  and  $\mathcal{H}_2 = \mathbb{C}^q$  with an o.n. basis  $f_1, \dots, f_q$ . Then  $e_i \otimes f_j, i = 1, \dots, p, j = 1, \dots, q$  is an o.n. basis of  $\mathcal{H}_1 \otimes \mathcal{H}_2$ . Matrices in  $B(\mathbb{C}^p) \otimes \mathbb{1}_{\mathbb{C}^q}$  have the form

$$\begin{bmatrix} A & 0 & & \\ 0 & A & & \\ & & \ddots & \\ & & & A \end{bmatrix}, \quad A \in B(\mathbb{C}^p),$$

and matrices in  $\mathbb{1}_{\mathcal{H}_1} \otimes B(\mathcal{H}_2)$  have the form

$$\begin{bmatrix} b_{11}\mathbb{1} & b_{12}\mathbb{1} & & \\ b_{21}\mathbb{1} & b_{22}\mathbb{1} & & \\ & & & \\ & & & b_{qq}\mathbb{1} \end{bmatrix}, \quad [b_{ij}] \in B(\mathbb{C}^q),$$

It may happen that  $\mathcal{H} = \mathbb{C}^N$ ,  $N = \sum_{i=1}^n p_i q_i$ ,

$$\mathcal{H} = \bigoplus_{i=1}^n \mathbb{C}^{p_i} \otimes \mathbb{C}^{q_i},$$

and observables are self-adjoint elements of

$$\mathfrak{A} := \bigoplus_{i=1}^n B(\mathbb{C}^{p_i}) \otimes \mathbb{1}_{q_i}.$$

Note that  $\mathfrak{A}$  is an example of what mathematicians call a **\*-algebra**, which we recall below. We will say that  $\mathfrak{A}$  is the **\*-algebra of observables**.

Let  $\mathfrak{A}$  be a vector space over  $\mathbb{C}$ . We say that  $\mathfrak{A}$  is an **algebra** if it is equipped with an operation

$$\mathfrak{A} \times \mathfrak{A} \ni (A, B) \mapsto AB \in \mathfrak{A}$$

satisfying

$$\begin{aligned} A(B+C) &= AB+AC, & (B+C)A &= BA+CA, \\ (\alpha\beta)(AB) &= (\alpha A)(\beta B). \end{aligned}$$

If in addition

$$A(BC) = (AB)C,$$

we say that it is an **associative algebra**. (In practice by an algebra we will usually mean an associative algebra).

The **center** of an algebra  $\mathfrak{A}$  equals

$$\mathfrak{Z}(\mathfrak{A}) = \{A \in \mathfrak{A} : AB = BA, B \in \mathfrak{A}\}.$$

Let  $\mathfrak{A}, \mathfrak{B}$  be algebras. A map  $\phi : \mathfrak{A} \rightarrow \mathfrak{B}$  is called a **homomorphism** if it is linear and preserves the multiplication, ie.

1.  $\phi(\lambda A) = \lambda\phi(A)$ ;
2.  $\phi(A+B) = \phi(A) + \phi(B)$ ;
3.  $\phi(AB) = \phi(A)\phi(B)$ .

We say that an algebra  $\mathfrak{A}$  is a **\*-algebra** if it is equipped with an antilinear map  $\mathfrak{A} \ni A \mapsto A^* \in \mathfrak{A}$  such that  $(AB)^* = B^*A^*$ ,  $A^{**} = A$  and  $A \neq 0$  implies  $A^*A \neq 0$ .

If  $\mathcal{H}$  is a Hilbert space, then  $B(\mathcal{H})$  equipped with the hermitian conjugation is a \*-algebra

If  $\mathfrak{A}, \mathfrak{B}$  are \*-algebras, then a homomorphism  $\pi : \mathfrak{A} \rightarrow \mathfrak{B}$  satisfying  $\pi(A^*) = \pi(A)^*$  is called a **\*-homomorphism**.



**Theorem 1.3** 1. Every finite dimensional  $*$ -algebra  $\mathfrak{A}$  is  $*$ -isomorphic to

$$\bigoplus_{i=1}^n B(\mathbb{C}^{p_i}),$$

for some  $p_1, \dots, p_n$

2. If in addition  $\mathfrak{A}$  is a subalgebra of  $B(\mathbb{C}^N)$  and contains the identity on  $\mathbb{C}^N$ , then there exist  $q_1, \dots, q_n$  with  $N = \sum_{i=1}^n p_i q_i$ , and a basis of  $\mathbb{C}^N$  such that

$$\mathfrak{A} = \bigoplus_{i=1}^n B(\mathbb{C}^{p_i}) \otimes \mathbb{1}_{q_i}. \quad (1.4)$$

As discussed before, in the finite dimensional case, observables of a quantum system are described by the self-adjoint part of a certain  $*$ -subalgebra of  $B(\mathcal{H})$ .

If  $\mathfrak{B} \subset B(\mathcal{H})$ , then the **commutant** of  $\mathfrak{B}$  is defined as

$$\mathfrak{B}' := \{A \in B(\mathcal{H}) : AB = BA, B \in \mathfrak{B}\}.$$

It is easy to see that a commutant is always an algebra containing  $\mathbb{1}_{\mathcal{H}}$ . If  $\mathfrak{B}$  is  $*$ -invariant, then so is  $\mathfrak{B}'$ . Therefore, in such a case  $\mathfrak{B}'$  is a  $*$ -algebra.

We say that  $\mathfrak{A}$  is a **von Neumann algebra** if  $\mathfrak{A} = \mathfrak{A}''$ . Clearly, von Neumann algebras are  $*$ -algebras.

It is easy to see that all  $*$ -subalgebras of  $B(\mathbb{C}^N)$  containing  $\mathbb{1}_N$  are von Neumann algebras. Indeed, if  $\mathfrak{A}$  is given by (1.4), then  $\mathfrak{A}$  is obviously  $*$ -invariant and

$$\mathfrak{A}' = \bigoplus_{i=1}^n \mathbb{1}_{p_i} \otimes B(\mathbb{C}^{q_i}).$$

So,  $\mathfrak{A}'' = \mathfrak{A}$ .

Physically, if we know that self-adjoint operators  $A_1, \dots, A_n$  are observables, then as the observable algebra it is natural to take

$$\mathfrak{A} = \{A_1, \dots, A_n\}''.$$

We will then say that  $\mathfrak{A}$  is the **von Neumann algebra generated** by  $A_1, \dots, A_n$ .

## 1.2 Observables – infinite dimension

In infinite dimensions we have several technical complications of the formalism developed in the previous section.

The theory of  $*$ -algebras is much richer in infinite dimension. The definition of a von Neumann algebra is still valid in any dimension. But Theorem 1.3 does not extend to infinite dimension. Besides, there are other kinds of  $*$ -algebras that are interesting candidates for a description of quantum systems, such as  **$C^*$ -algebras**. We will however stick to von Neumann algebras.

Observables are often described by **unbounded self-adjoint operators**. This is not a serious problem. What is relevant for quantum measurements are spectral

projections, which are bounded. Thus by saying that an algebra  $\mathfrak{A} \subset B(\mathcal{H})$  is generated by  $A_1, \dots, A_n$  we will mean that it is generated by spectral projections of these operators (or, equivalently, by their bounded Borel function).

Another difficulty that was absent in finite dimension is the fact that self-adjoint operators may have **continuous spectrum**. This means that  $\text{sp}A$  may not coincide with the set of eigenvalues of  $A$ . We need to change the definition of simple joint spectrum of a family of commuting self-adjoint operators  $A_1, \dots, A_n$ . We will say that they have a simple joint spectrum if  $\{A_1, \dots, A_n\}''$  is a **maximal commutative** von Neumann subalgebra of  $B(\mathcal{H})$ .

**Example.** Consider the operators  $\hat{x}_i$ ,  $i = 1, 2, 3$  on  $L^2(\mathbb{R}^3)$ . They are self-adjoint and commute. They have simple joint spectrum. The von Neumann algebra generated by  $\hat{x}_i$ ,  $i = 1, 2, 3$  is equal to the operators of multiplication by functions in  $L^\infty(\mathbb{R}^3)$ .

**Example.** Consider in addition the operators  $\hat{p}_i := i^{-1}\partial_{x_i}$ ,  $i = 1, 2, 3$  on  $L^2(\mathbb{R}^3)$ . The von Neumann algebra generated by  $\hat{x}_i$ ,  $\hat{p}_i$ ,  $i = 1, 2, 3$ , coincides with  $B(L^2(\mathbb{R}^3))$ .

**Example.** Let  $r \mapsto V(r)$  be a real function such that  $\lim_{r \rightarrow \infty} V(r) = 0$ .  $H := -\Delta + V(r)$  has a continuous spectrum  $[0, \infty[$  and it may have some point spectrum.  $H, L^2, L_z$  is a triple of commuting self-adjoint operators with a simple joint spectrum.

In the case of continuous spectrum, one often tries to find **generalized eigenfunctions**. Unfortunately, there seems to be no universal theory of generalized eigenfunctions of self-adjoint operators. In concrete situations, however, they can be useful. Let us describe an example of the momentum operator, where it is clear how to define generalized eigenfunctions.

Consider  $L^2(\mathbb{R}^d)$  and the self-adjoint operators  $\hat{p}_i := i^{-1}\partial_{x_i}$ ,  $i = 1, \dots, d$ . They commute and have simple joint spectrum equal to  $\mathbb{R}^d$ . We will sometimes write  $\hat{p}$  for  $(\hat{p}_1, \dots, \hat{p}_d)$ .

Clearly,

$$i^{-1}\partial_{x_i} e^{ix \cdot k} = k_i e^{ix \cdot k}.$$

One says that

$$e^{ix \cdot k} \tag{1.5}$$

is a generalized eigenfunction of  $\hat{p}_i$  with eigenvalue  $k_i$ ,  $i = 1, \dots, d$ . We will denote it by  $|k\rangle$ . Unfortunately,  $|k\rangle$  does not belong to  $L^2(\mathbb{R}^d)$ .

One way of giving meaning to  $|k\rangle$  is as follows. Consider the space of Schwartz test functions  $\mathcal{S}(\mathbb{R}^d)$ . Its dual, that is the space of continuous functionals on  $\mathcal{S}(\mathbb{R}^d)$  is denoted  $\mathcal{S}'(\mathbb{R}^d)$  and is called the space of tempered distributions. We have

$$\mathcal{S}(\mathbb{R}^d) \subset L^2(\mathbb{R}^d) \subset \mathcal{S}'(\mathbb{R}^d).$$

$\mathcal{S}(\mathbb{R}^d)$  is dense in  $\mathcal{S}'(\mathbb{R}^d)$ . The operators  $\hat{p}_i$  preserve  $\mathcal{S}(\mathbb{R}^d)$ , and they extend uniquely to a continuous operator on  $\mathcal{S}'(\mathbb{R}^d)$ . Thus  $|k\rangle$  are true eigenfunctions of the extended  $\hat{p}_i$  in  $\mathcal{S}'(\mathbb{R}^d)$ . In particular, if  $\Psi \in \mathcal{S}(\mathbb{R}^d)$ , then  $(\Psi|k_i)$  is well defined.

One of the main applications of generalized eigenfunctions is an explicit expression for spectral projections. If  $\Omega$  is a Borel subset of  $\mathbb{R}^d = \text{sp}(\hat{p})$ , then

$$\mathbb{1}_\Omega(\hat{p}) = \int_\Omega |k\rangle\langle k| dk. \quad (1.6)$$

In other words, the integral kernel of (1.6) equals

$$\mathbb{1}_\Omega(\hat{p})(x, y) = \int_\Omega e^{ik(x-y)} dk.$$

Note that (1.6) is a priori defined as a continuous operator from  $\mathcal{S}(\mathbb{R}^d)$  to  $\mathcal{S}'(\mathbb{R}^d)$ . It extends to a bounded operator on  $L^2(\mathbb{R}^d)$ .

### 1.3 Quantum dynamics

Suppose that a quantum system described by a Hilbert space  $\mathcal{H}$  is invariant with respect to time. This is usually described by considering a strongly continuous 1-parameter unitary group on  $\mathcal{H}$ , that is, a strongly continuous function  $\mathbb{R} \ni t \mapsto U(t) \in U(\mathcal{H})$  such that

$$U(t_1)U(t_2) = U(t_1 + t_2), \quad t_1, t_2 \in \mathbb{R},$$

The **Stone Theorem** says that  $U(t) := e^{-itH}$  for a uniquely defined self-adjoint operator  $H$ , called a **Hamiltonian**.

If we prepare a state  $\rho$  at time 0 and measure an observable  $A$  at time  $t > 0$ , then the expectation value of the measurement equals

$$\text{Tr} \rho e^{itH} A e^{-itH}. \quad (1.7)$$

There are two equivalent ways of computing (1.7)

1. **The Schrödinger picture:** We let the state evolve  $\rho_t := e^{-itH} \rho e^{itH}$  and keep the observable constant. Then (1.7) equals  $\text{Tr} \rho_t A$ .
2. **The Heisenberg picture:** We let the observable evolve  $A_t := e^{itH} A e^{-itH}$  and keep the state constant. Then (1.7) equals  $\text{Tr} \rho A_t$ .

(By the Schrödinger picture one also means the unitary evolution  $\Psi_t := e^{-itH} \Psi$  on  $\mathcal{H}$ .)

Note that neither the evolution of an observable in the Heisenberg picture nor of a state in the Schrödinger picture changes if we add a real constant to the Hamiltonian.

If we consider two noninteracting quantum systems described by Hilbert spaces  $\mathcal{H}_1, \mathcal{H}_2$  and Hamiltonians  $H_1, H_2$ , then the composite system has the Hilbert space  $\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2$  and the Hamiltonian

$$H := H_1 \otimes \mathbb{1} + \mathbb{1} \otimes H_2.$$

Note that  $\text{sp}H = \text{sp}H_1 + \text{sp}H_2$ , and  $e^{itH} = e^{itH_1} \otimes e^{itH_2}$ .

The following assumptions are often satisfied for real quantum systems:

1. The Hamiltonian is **bounded from below**, that means  $E := \inf \text{sp}H > -\infty$ .
2. The Hamiltonian has a **unique ground state**, that means  $\dim \mathbb{1}_E(H) = 1$ .

A partial justification for the above two assumptions is given by the following proposition:

**Proposition 1.4** *Consider two quantum systems described by  $\mathcal{H}_i, H_i, i = 1, 2$ .*

1. *Suppose both Hamiltonians are bounded from below. Then the Hamiltonian of the composite system is also bounded from below.*
2. *If both systems have a unique ground state, then the composite system has a unique ground state.*

## 1.4 Symmetries

Let  $\mathcal{H}$  be a Hilbert space. Let us denote the group of **\*-automorphisms** of  $B(\mathcal{H})$  by  $\text{Aut}(B(\mathcal{H}))$ . One can show that every  $\sigma \in \text{Aut}(B(\mathcal{H}))$  has the form

$$\sigma(A) = UAU^*, \quad A \in B(\mathcal{H}),$$

for some  $U \in U(\mathcal{H})$ .

Consider a quantum system described by a Hilbert space  $\mathcal{H}$  with algebra of observables  $B(\mathcal{H})$ . A **symmetry** of this system should correspond to a linear map on  $B(\mathcal{H})$ , which transforms self-adjoint elements onto self-adjoint elements. Clearly, all elements of  $\text{Aut}(B(\mathcal{H}))$  have this property.

Suppose that  $G$  is a group of symmetries of a quantum system. More precisely, suppose that we have a homomorphism

$$G \ni g \mapsto \sigma_g \in \text{Aut}(B(\mathcal{H})). \quad (1.8)$$

The most obvious approach to obtain (1.8) is to consider a **unitary representation**

$$G \ni g \mapsto U(g) \in U(\mathcal{H}).$$

If  $G$  is a topological group, we will assume that its action is **strongly continuous**. In the Heisenberg picture we obtain the action on the observable algebra

$$\sigma_g(A) := U(g)AU(g)^*, \quad A \in B(\mathcal{H}), \quad g \in G.$$

Sometimes one assumes that only the operators invariant wrt  $\sigma$  are observables. In this case we often say that  $G$  is a **gauge group** of our system.

Thus the algebra of observables coincides with the **fixed point algebra**

$$\begin{aligned} B(\mathcal{H})^\sigma &:= \{A \in B(\mathcal{H}) : \sigma_g(A) = A, \quad g \in G\} \\ &= \{U(g) : g \in G\}'. \end{aligned} \quad (1.9)$$

Note that (1.9) is a von Neumann algebra.

Let  $\hat{G}$  denote the **set of equivalence classes of irreducible unitary representations**. For each equivalence class  $\pi \in \hat{G}$  we choose its representative and denote it by

$$G \ni g \mapsto \pi_i(g) \in U(\mathcal{V}_i).$$

A rather complete understanding an arbitrary unitary representation is available in the case of **compact groups**. If  $G$  is a compact group, then there exist Hilbert spaces  $\mathcal{K}_\pi$ ,  $\pi \in \hat{G}$ , such that

$$\mathcal{H} \simeq \bigoplus_{\pi_i \in \hat{G}} \mathcal{V}_i \otimes \mathcal{K}_i, \quad U(g) \simeq \bigoplus_{\pi_i \in \hat{G}} \pi_i(g) \otimes \mathbb{1}_{\mathcal{K}_i}.$$

We have then

$$B(\mathcal{H})^\sigma = \bigoplus_{\pi_i \in \hat{G}} \mathbb{1}_{\mathcal{V}_i} \otimes B(\mathcal{K}_i).$$

Thus superselection sectors are parametrized by irreducible representations of  $G$

**Example.** Consider  $G = U(1)$ . Irreducible unitary representations are 1-dimensional and are parametrized by integers called often the **charge**:

$$U(1) \ni \theta \mapsto e^{in\theta}.$$

**Example.** Consider  $G = SO(3)$ . Irreducible unitary representations are parametrized by numbers  $j = 0, 1, \dots$  called the **spin**. The representation of spin  $j$  is  $2j + 1$  dimensional.

## 1.5 Projective representations

What is physically relevant is not the action of the group on the Hilbert space, but on the algebra of observables. Having a **representation in automorphisms**

$$G \ni g \mapsto \sigma_g \in \text{Aut}(B(\mathcal{H}))$$

is equivalent to a **projective unitary representation** of  $G$  on  $\mathcal{H}$ , that is a map

$$G \ni g \mapsto U(g) \in U(\mathcal{H})$$

such that

$$U(g_1)U(g_2) = c(g_1, g_2)U(g_1g_2),$$

with  $|c(g_1, g_2)| = 1$ . Of course, we can always replace  $U(g)$  with  $b(g)U(g)$ ,  $|b(g)| = 1$ .

Not every projective representation can be modified so that it becomes a true representation.

Suppose we have several projective representations of the same group

$$G \ni g \mapsto U_i(g) \in U(\mathcal{H}_i) \tag{1.10}$$

which have different multipliers

$$G \times G \ni (g_2, g_1) \mapsto c_i(g_2, g_1) \in \mathbb{C} \quad (1.11)$$

Then as a Hilbert space of our system we can use

$$\mathcal{H} := \bigoplus_i \mathcal{H}_i. \quad (1.12)$$

and we can define a map

$$G \ni g \mapsto U(g) := \bigoplus_i U_i(g) \in U(\mathcal{H}). \quad (1.13)$$

However, (1.13) is no longer a projective representation and if we set

$$\sigma_g(A) := U(g)AU(g)^{-1} \in \text{Aut}(B(\mathcal{H})), \quad (1.14)$$

then  $G \ni g \mapsto \sigma_g$  is no longer a homomorphism. Nevertheless, if we restrict (1.14) to the block diagonal algebra

$$\mathfrak{A} := \bigoplus_i B(\mathcal{H}_i), \quad (1.15)$$

then  $G \ni g \mapsto \sigma_g$  is a homomorphism, and hence a symmetry of our system. Thus, if we insist that  $G$  is a symmetry, then we the decomposition (1.12) define strict superselection sectors of our system.

### 1.5.1 Covering group

We still assume that  $G$  is a symmetry group of our system. It is usually inconvenient to use projective representations. Instead, one tries to use true representations of a larger group  $\tilde{G}$ . This formalism we describe below. If we choose  $\tilde{G}$  appropriately, then each projective irreducible representation of  $G$  corresponds to a true irreducible representation of  $\tilde{G}$ . This construction goes back to Shur (1904 and 1907), where  $\tilde{G}$  is called a ‘‘representation group’’. After choosing  $\tilde{G}$  we can treat it as the true group of symmetries of physics.

Suppose we have another group  $\tilde{G}$  such that

$$\mathbb{1} \rightarrow N \rightarrow \tilde{G} \xrightarrow{\phi} G \rightarrow \mathbb{1}. \quad (1.16)$$

**Proposition 1.5** *Let*

$$G \xrightarrow{\psi} \tilde{G}$$

*be a right inverse of the homomorphism  $\tilde{G} \rightarrow G$ . This means that  $\phi \circ \psi = \text{id}$ . ( $\psi$  does not need to be a homomorphism itself, nor continuous—any choice is allowed). Let*

$$\tilde{G} \ni \tilde{g} \mapsto \tilde{U}(\tilde{g}) \in U(\mathcal{H}) \quad (1.17)$$

*be a representation such that  $\tilde{U}(n)$  are scalar (proportional to the identity) for  $n \in N$ . Then*

$$U(g) := \tilde{U}(\psi(g)) \quad (1.18)$$

defines a projective representation of  $G$ .

If  $\psi$  is also a representation (which means that the short exact sequence (1.16) is “split”) then (1.18) is a true representation.

**Proof.** For any  $g_1, g_2 \in G$

$$\phi(\psi(g_1)\psi(g_2)) = \phi(\psi(g_1))\phi(\psi(g_2)) = g_1g_2 = \phi(\psi(g_1g_2)). \quad (1.19)$$

Hence, there exists  $n \in N$  such that

$$\psi(g_1)\psi(g_2) = n\psi(g_1g_2). \quad (1.20)$$

Therefore,

$$U(g_1)U(g_2) = \tilde{U}(\psi(g_1))\tilde{U}(\psi(g_2)) \quad (1.21)$$

$$= \tilde{U}(\psi(g_1)\psi(g_2)) = \tilde{U}(n\psi(g_1g_2)) \quad (1.22)$$

$$= \tilde{U}(n)\tilde{U}(\psi(g_1g_2)) = \tilde{U}(n)U(g_1g_2), \quad (1.23)$$

where  $\tilde{U}(n)$  is a scalar factor depending on  $g_1, g_2$ .

If  $\psi$  is also a representation, then  $U$  is a composition of homomorphisms, hence (1.18) is a true representation.  $\square$

**Proposition 1.6** *Assume that  $N$  is contained in the center of  $\tilde{G}$ , that means  $n \in N, h \in \tilde{G}$  implies  $nh = hn$ . Let  $\tilde{\pi}$  be an irreducible representation of  $\tilde{G}$ . Then elements of  $N$  are represented by scalar operators (of the form  $c\mathbb{1}$ ).*

**Proof.** Indeed,  $\tilde{\pi}(n)$ , for  $n \in N$  commute with  $\tilde{\pi}(h)$ ,  $h \in \tilde{G}$ . Hence by Schur’s Lemma they have to be scalar.  $\square$

We can try to treat  $\tilde{G}$  as the symmetry group of our quantum system. Let  $\hat{\tilde{G}}$  be the set of representations of  $\tilde{G}$ . For any  $\tilde{\pi} \in \hat{\tilde{G}}$  we have a projective representation  $\pi$  with the multipliers  $g_1, g_2 \mapsto c_\pi(g_1, g_2)$ . We can introduce in  $\hat{\tilde{G}}$  an equivalence relation: we say that two irreducible representations belong to the same category if they have the same multipliers. We obtain a partition

$$\hat{\tilde{G}} = \bigsqcup_k S_k. \quad (1.24)$$

Let  $\tilde{U} : \tilde{G} \rightarrow U(\mathcal{H})$  be a unitary representation. Then we can write

$$\mathcal{H} \simeq \bigoplus_{\tilde{\pi} \in \hat{\tilde{G}}} \mathcal{V}_{\tilde{\pi}} \otimes \mathcal{K}_{\tilde{\pi}}, \quad \tilde{U}(g) \simeq \bigoplus_{\tilde{\pi} \in \hat{\tilde{G}}} \tilde{\pi}(g) \otimes \mathbb{1}_{\mathcal{K}_{\tilde{\pi}}}.$$

Then

$$\mathcal{H}_k := \bigoplus_{\tilde{\pi} \in S_k} \mathcal{V}_{\tilde{\pi}} \otimes \mathcal{K}_{\tilde{\pi}}, \quad (1.25)$$

form separate superselection sectors.

## 1.6 Group $SO(3)$

This problem arises in the case of the group  $SO(3)$ . One can show that to get all irreducible projective representations one needs to consider the group  $SU(2)$ , which is a two-fold covering of  $SO(3)$ :

$$\mathbb{1} \rightarrow \{\mathbb{1}, -\mathbb{1}\} \rightarrow SU(2) \rightarrow SO(3) \rightarrow \mathbb{1}.$$

In particular, the preimage of any element of  $SO(3)$  consists of a pair of the form  $\{r, -r\} \subset SU(2)$ .

Irreducible representations of  $SU(2)$  are parametrized by numbers  $j = 0, \frac{1}{2}, 1, \dots$  called the **spin**. They are

$$SU(2) \ni A \mapsto \oplus_{\mathbb{S}}^{2j} A \in U(\oplus_{\mathbb{S}}^{2j} \mathbb{C}^2) \simeq U(\mathbb{C}^{2j+1}). \quad (1.26)$$

For entire  $j$ ,  $-\mathbb{1} \in SU(2)$  is mapped on  $\mathbb{1}$ . Therefore, these representations are the compositions of a representation of  $SO(3)$  and of the homomorphism  $SU(2) \rightarrow SO(3)$ . We have  $U_j(r) = U_j(-r)$ .

For non-entire  $j$ ,  $-\mathbb{1} \in SU(2)$  is mapped on  $-\mathbb{1}$ . Therefore, these representations do not come from representations of  $SO(3)$ . We have  $U_j(r) = -U_j(-r)$ .

Therefore, if our quantum system has the symmetry  $SO(3)$ , this means that the Hilbert space is equipped with the representation  $\tilde{U} : SU(2) \rightarrow U(\mathcal{H})$ . Then  $\tilde{U}(-\mathbb{1})$  has two spectral subspaces,  $\mathcal{H}_+$  and  $\mathcal{H}_-$ .

On  $\mathcal{H}_+$  the multiplier can be chosen to be trivial. On  $\mathcal{H}_-$  the multiplier is always nontrivial. Therefore,  $\mathcal{H}_+$  and  $\mathcal{H}_-$  are superselection sectors.