HOMOGENEOUS SCHRÖDINGER OPERATORS ON HALFLINE

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1. HOMOGENEOUS SCHRÖDINGER OPERATORS
   (in collaboration with LAURENT BRUNEAU
   and VLADIMIR GEORGESCU)
2. ALMOST HOMOGENEOUS SCHRÖDINGER OPERATORS
   (in collaboration with SERGE RICHARD)
3. HOMOGENEOUS RANK ONE PERTURBATIONS

HOMOGENEOUS SCHRÖDINGER OPERATORS

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Consider a formal differential expression

$$L_\alpha = -\partial_x^2 + \big(-\frac{1}{4} + \alpha\big)\frac{1}{x^2}.$$

We would like to interpret it as a well-defined (unbounded) operator. To do this we need to specify its domain.

We will obtain operators with surprisingly rich mathematical phenomenology, which should be close to physicists' hearts: the "running coupling constant" flows under the action of the "renormalization group", there are two "phase transitions", attractive and repulsive fixed points, limit cycles, breakdown of conformal symmetry, etc.

I will discuss both the self-adjoint and non-self-adjoint cases. The latter have quite curious properties and I am looking for their physical applications.

Let U_τ be the group of dilations on $L^2[0,\infty[$, that is

$$(U_\tau f)(x) = \mathrm{e}^{\tau/2} f(e^\tau x).$$

We say that B is homogeneous of degree ν if

$$U_\tau B U_\tau^{-1} = \mathrm{e}^{\nu\tau} B.$$

Clearly, L_α is homogeneous of degree -2.

Here are two natural questions:

1. If $\alpha \in \mathbb{R}$, how to interpret L_α as a self-adjoint operator on $L^2[0, \infty[$? When is it homogeneous of degree -2?
2. If $\alpha \in \mathbb{C}$, how to interpret L_α as a closed operator on $L^2[0, \infty[$? When is it homogeneous of degree -2?

L_α, and closely related operators H_m, which we introduce shortly, are interesting for many reasons.

- They appear as the radial part of the Laplacian in all dimensions, in the decomposition of the Aharonov-Bohm Hamiltonian, in the membranes with conical singularities, in the theory of many body systems with contact interactions and in the Efimov effect.
- They have rather subtle and rich properties illustrating various concepts of the operator theory in Hilbert spaces (eg. the Friedrichs and Krein extensions, holomorphic families of closed operators).

- Essentially all basic objects related to H_m, such as their resolvents, spectral projections, wave and scattering operators, can be explicitly computed.
- A number of nontrivial identities involving special functions, especially from the Bessel family, find an appealing operator-theoretical interpretation in terms of the operators H_m. Eg. the Barnes identity leads to the formula for wave operators.

Two naive interpretations of L_α:

1. The minimal operator $L_\alpha^{\min}$: We start from L_α on $C_{\mathrm{c}}^\infty]0,\infty[$, and then we take its closure.
2. The maximal operator $L_\alpha^{\max}$: We consider the domain consisting of all $f \in L^2[0,\infty[$ such that $L_\alpha f \in L^2[0,\infty[$.

Clearly, $\mathrm{Dom}(L_{\min}) \subset \mathrm{Dom}(L_{\max})$ and

$$L_{\max}\Big|_{\mathrm{Dom}(L_{\min})} = L_{\min}.$$

In other words $L_{\min} \subset L_{\max}$.

We will see that it is often natural to write $\alpha = m^2$

Theorem 1 .

1. *For* $1 \leq \operatorname{Re} m$, $L_{m^2}^{\min} = L_{m^2}^{\max}$.
2. *For* $-1 < \operatorname{Re} m < 1$, $L_{m^2}^{\min} \subsetneq L_{m^2}^{\max}$, *and the codimension of their domains is* 2.
3. $(L_{\alpha}^{\min})^* = L_{\overline{\alpha}}^{\max}$. *Hence, for* $\alpha \in \mathbb{R}$, $L_{\alpha}^{\min}$ *is Hermitian*.
4. $L_{\alpha}^{\min}$ *and* $L_{\alpha}^{\max}$ *are homogeneous of degree* -2.

Notice that

$$Lx^{\frac{1}{2}\pm m} = 0.$$

Let ξ be a compactly supported cutoff equal 1 around 0.

Let $-1 < \operatorname{Re} m$. Note that $x^{\frac{1}{2}+m}\xi$ belongs to $\operatorname{Dom} L_{m^2}^{\max}$.

This suggests to define the operator H_m to be the restriction of $L_{m^2}^{\max}$ to

$$\operatorname{Dom} L_{m^2}^{\min} + \mathbb{C}x^{\frac{1}{2}+m}\xi.$$

Theorem 2 *.*

1. *For* $1 \leq \operatorname{Re} m$*,* $L_{m^2}^{\min} = H_m = L_{m^2}^{\max}$*.*
2. *For* $-1 < \operatorname{Re} m < 1$*,* $L_{m^2}^{\min} \subsetneq H_m \subsetneq L_{m^2}^{\max}$ *and the codimension of the domains is* 1*.*
3. $H_m^* = H_{\overline{m}}$*. Hence, for* $m \in]-1, \infty[$*,* H_m *is self-adjoint.*
4. H_m *is homogeneous of degree* -2*.*
5. $\operatorname{sp} H_m = [0, \infty[$*.*
6. $\{\operatorname{Re} m > -1\} \ni m \mapsto H_m$ *is a holomorphic family of closed operators.*

Theorem 3 *.*

1. *For $\alpha \geq 1$, $L_\alpha^{\min} = H_{\sqrt{\alpha}}$ is essentially self-adjoint on $C_c^\infty]0,\infty[$.*
2. *For $\alpha < 1$, $L_\alpha^{\min}$ is not essentially self-adjoint on $C_c^\infty]0,\infty[$.*
3. *For $0 \leq \alpha < 1$, the operator $H_{\sqrt{\alpha}}$ is the Friedrichs extension and $H_{-\sqrt{\alpha}}$ is the Krein extension of $L_\alpha^{\min}$.*
4. *$H_{\frac{1}{2}}$ is the Dirichlet Laplacian and $H_{-\frac{1}{2}}$ is the Neumann Laplacian on halfline.*
5. *For $\alpha < 0$, $L_\alpha^{\min}$ has no homogeneous selfadjoint extensions.*

It is easy to see that

$$x^{-\frac{1}{2}}\Big(-\partial_x^2+(-\frac{1}{4}+\alpha)\frac{1}{x^2}\pm 1\Big)x^{\frac{1}{2}}$$
$$=-\partial_x^2-\frac{1}{x}\partial_x+(-\frac{1}{4}+\alpha)\frac{1}{x^2}\pm 1,$$

which is the (modified) Bessel operator.

Therefore, it is not surprising that various objects related to H_m can be computed with help of functions from the Bessel family.

Theorem 4 . *If $R_m(\lambda; x, y)$ is the integral kernel of the operator $(\lambda - H_m)^{-1}$, then for $\operatorname{Re} k > 0$ we have*

$$R_m(-k^2; x, y) = \begin{cases} \sqrt{xy} I_m(kx) K_m(ky) & \text{if } x < y, \\ \sqrt{xy} I_m(ky) K_m(kx) & \text{if } x > y, \end{cases}$$

where I_m is the modified Bessel function and K_m is the MacDonald function.

Proposition 5 . *For* $0 < a < b < \infty$, *the integral kernel of* $\mathbb{1}_{[a,b]}(H_m)$ *is*

$$\mathbb{1}_{[a,b]}(H_m)(x,y) = \int_{\sqrt{a}}^{\sqrt{b}} \sqrt{xy} J_m(kx) J_m(ky) k \mathrm{d}k,$$

where J_m *is the Bessel function.*

Let $\mathcal{F}_m$ be the operator on $L^2(0,\infty)$ given by

$$\mathcal{F}_m : f(x) \mapsto \int_0^\infty J_m(kx)\sqrt{kx} f(x) \mathrm{d}x$$

$\mathcal{F}_m$ is the so-called Hankel transformation. Define also the operator $Xf(x) := xf(x)$.

Theorem 6 . *$\mathcal{F}_m$ is a bounded invertible involution on $L^2[0,\infty[$ diagonalizing H_m and anticommuting with the self-adjoint generator o dilations $A = \frac{1}{2\mathrm{i}}(x\partial_x + \partial_x x)$:*

$$\mathcal{F}^2 = \mathbb{1},$$
$$\mathcal{F}_m H_m \mathcal{F}_m^{-1} = X^2,$$
$$\mathcal{F}_m A = -A\mathcal{F}_m.$$

Theorem 7 *Set*

$$\mathcal{I}f(x) = x^{-1}f(x^{-1}), \quad \Xi_m(t) = \mathrm{e}^{\mathrm{i}\ln(2)t}\frac{\Gamma(\frac{m+1+\mathrm{i}t}{2})}{\Gamma(\frac{m+1-\mathrm{i}t}{2})}.$$

Then

$$\mathcal{F}_m = \Xi_m(A)\mathcal{I}.$$

Therefore, we have the identity

$$H_m := \Xi_m^{-1}(A)X^{-2}\Xi_m(A)$$

(Result obtained independently by Bruneau, Georgescu, D, and by Richard and Pankrashkin).

Theorem 8 . *The wave operators associated to the pair* H_m, H_k *exist and*

$$\begin{aligned} \Omega_{m,k}^{\pm} &:= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH_m}\mathrm{e}^{-\mathrm{i}tH_k} \\ &= \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\mathcal{F}_m\mathcal{F}_k \\ &= \mathrm{e}^{\pm\mathrm{i}(m-k)\pi/2}\frac{\Xi_k(A)}{\Xi_m(A)}. \end{aligned}$$

The formula

$$H_m := \Xi_m^{-1}(A) X^{-2} \Xi_m(A) \tag{1}$$

valid for $\operatorname{Re} m > -1$ can be used as an alternative definition of the family H_m also beyond this domain. It defines a family of closed operators for the parameter m that belongs to

$$\{m \mid \operatorname{Re} m \neq -1, -2, \dots\} \cup \mathbb{R}. \tag{2}$$

Its spectrum is always equal to $[0, \infty[$ and it is analytic in the interior of (2).

In fact, $\Xi_m(A)$ is a unitary operator for all real values of m. Therefore, for $m \in \mathbb{R}$, (1) is well-defined and self-adjoint.

$\Xi_m(A)$ is bounded and invertible also for all m such that $\operatorname{Re} m \neq -1, -2, \dots$. Therefore, the formula (1) defines an operator for all such m.

One can then pose various questions:

- What happens with this operator along the lines $\mathrm{Re}\, m = -1, -2, \ldots$?
- What is the meaning of the operator to the left of $\mathrm{Re} = -1$? (It is not a differential operator!)

The definition (or actually a number of equivalent definitions) of a holomorphic family of bounded operators is quite obvious and does not need to be recalled. In the case of unbounded operators the situation is more subtle.

Suppose that Θ is an open subset of $\mathbb{C}$, $\mathcal{H}$ is a Banach space, and $\Theta \ni z \mapsto H(z)$ is a function whose values are closed operators on $\mathcal{H}$. We say that this is a holomorphic family of closed operators if for each $z_0 \in \Theta$ there exists a neighborhood Θ_0 of z_0, a Banach space $\mathcal{K}$ and a holomorphic family of injective bounded operators $\Theta_0 \ni z \mapsto B(z) \in B(\mathcal{K}, \mathcal{H})$ such that $\mathrm{Ran}\, B(z) = \mathcal{D}(H(z))$ and

$$\Theta_0 \ni z \mapsto H(z)B(z) \in B(\mathcal{K}, \mathcal{H})$$

is a holomorphic family of bounded operators.

We have the following practical criterion:

Theorem 9 . *Suppose that* $\{H(z)\}_{z\in\Theta}$ *is a function whose values are closed operators on* $\mathcal{H}$*. Suppose in addition that for any* $z \in \Theta$ *the resolvent set of* $H(z)$ *is nonempty. Then* $z \mapsto H(z)$ *is a holomorphic family of closed operators if and only if for any* $z_0 \in \Theta$ *there exists* $\lambda \in \mathbb{C}$ *and a neighborhood* Θ_0 *of* z_0 *such that* λ *belongs to the resolvent set of* $H(z)$ *for* $z \in \Theta_0$ *and* $z \mapsto (H(z) - \lambda)^{-1} \in B(\mathcal{H})$ *is holomorphic on* Θ_0*.*

The above theorem indicates that it is more difficult to study holomorphic families of closed operators that for some values of the complex parameter have an empty resolvent set.

Conjecture 10 . *It is impossible to extend*

$$\{\mathrm{Re}\, m > -1\} \ni m \mapsto H_m$$

to a holomorphic family of closed operators on a larger connected open subset of $\mathbb{C}$*.*

ALMOST HOMOGENEOUS SCHRÖDINGER OPERATORS (in collaboration with SERGE RICHARD)

For any $\kappa \in \mathbb{C} \cup \{\infty\}$ let $H_{m,\kappa}$ be the restriction of $L_{m^2}^{\max}$ to the domain

$$\begin{aligned} \mathrm{Dom}(H_{m,\kappa}) = & \Big\{ f \in \mathrm{Dom}(L_{m^2}^{\max}) \mid \text{ for some } c \in \mathbb{C}, \\ & f(x) - c\big(x^{1/2-m} + \kappa x^{1/2+m}\big) \in \mathrm{Dom}(L_{m^2}^{\min}) \\ & \text{around } 0 \Big\}, \qquad \kappa \neq \infty; \\ \mathrm{Dom}(H_{m,\infty}) = & \Big\{ f \in \mathrm{Dom}(L_{m^2}^{\max}) \mid \text{ for some } c \in \mathbb{C}, \\ & f(x) - c x^{1/2+m} \in \mathrm{Dom}(L_{m^2}^{\min}) \text{ around } 0 \Big\}. \end{aligned}$$

For $\nu \in \mathbb{C} \cup \{\infty\}$, let H_0^ν be the restriction of $L_0^{\max}$ to

$$\begin{aligned}
\mathrm{Dom}(H_0^\nu) \ = \ &\Big\{ f \in \mathrm{Dom}(L_0^{\max}) \mid \text{ for some } c \in \mathbb{C}, \\
&f(x) - c\big(x^{1/2}\ln x + \nu x^{1/2}\big) \in \mathrm{Dom}(L_0^{\min}) \\
&\text{around } 0\Big\}, \qquad \nu \neq \infty; \\
\mathrm{Dom}(H_0^\infty) \ = \ &\Big\{ f \in \mathrm{Dom}(L_0^{\max}) \mid \text{ for some } c \in \mathbb{C}, \\
&f(x) - c x^{1/2} \in \mathrm{Dom}(L_0^{\min}) \text{ around } 0\Big\}.
\end{aligned}$$

Proposition 11 *.*

1. *For any* $|\mathrm{Re}\,(m)| < 1$*,* $\kappa \in \mathbb{C} \cup \{\infty\}$

$$H_{m,\kappa} = H_{-m,\kappa^{-1}}.$$

2. $H_{0,\kappa}$ *does not depend on* κ*, and these operators coincide with* H_0^{∞}*.*

Proposition 12 . *For any* m *with* $|\mathrm{Re}\,(m)| < 1$ *and any* $\kappa, \nu \in \mathbb{C} \cup \{\infty\}$, *we have*

$$U_\tau H_{m,\kappa} U_{-\tau} = \mathrm{e}^{-2\tau} H_{m,\mathrm{e}^{-2\tau m}\kappa},$$
$$U_\tau H_0^\nu U_{-\tau} = \mathrm{e}^{-2\tau} H_0^{\nu+\tau}.$$

In particular, only

$$\begin{aligned} H_{m,0} &= H_{-m}, \\ H_{m,\infty} &= H_m, \\ H_0^\infty &= H_0 \end{aligned}$$

are homogeneous.

Proposition 13 *.*

$$H_{m,\kappa}^* = H_{\overline{m},\overline{\kappa}} \qquad \textit{and} \qquad H_0^{\nu *} = H_0^{\overline{\nu}}.$$

In particular,

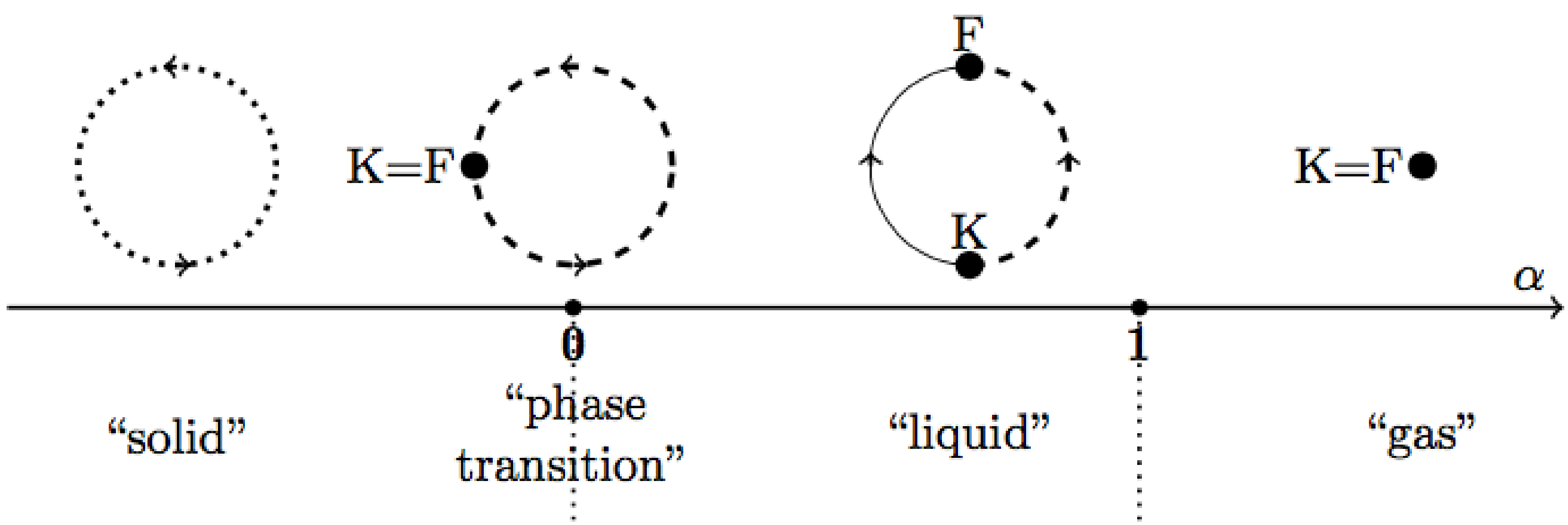
(i) $H_{m,\kappa}$ *is self-adjoint for* $m \in]-1,1[$ *and* $\kappa \in \mathbb{R} \cup \{\infty\}$*, and for* $m \in i\mathbb{R}$ *and* $|\kappa| = 1$*.*

(ii) H_0^ν *is self-adjoint for* $\nu \in \mathbb{R} \cup \{\infty\}$*.*

Self-adjoint extensions of the Hermitian operator

$$L_\alpha = -\partial_x^2 + \left(-\frac{1}{4} + \alpha\right)\frac{1}{x^2}.$$

K—Krein, F—Friedrichs, dashed line—single bound state, dotted line—infinite sequence of bound states.



The essential spectrum of $H_{m,\kappa}$ and H_0^ν is $[0,\infty[$.

Theorem 14 *.*

1. *$z \in \mathbb{C}\backslash[0,\infty[$ belongs to the point spectrum of $H_{m,\kappa}$ iff*
$$(-z)^{-m} = \kappa\frac{\Gamma(m)}{\Gamma(-m)}.$$
2. *H_0^ν possesses an eigenvalue iff $-\pi < \operatorname{Im} 2\nu < \pi$, and then it is $z = -\mathrm{e}^{-2\nu}$.*

For a given m, κ all eigenvalues form a geometric sequence that lies on a logarithmic spiral, which should be viewed as a curve on the Riemann surface of the logarithm. Only its “physical sheet” gives rise to eigenvalues.

For m which are not purely imaginary, only a finite piece of the spiral is on the “physical sheet” and therefore the number of eigenvalues is finite.

If m is purely imaginary, this spiral degenerates to a half-line starting at the origin.

If m is real, the spiral degenerates to a circle. But then the operator has at most one eigenvalue.

Theorem 15 . *Let* $m = m_\mathrm{r} + \mathrm{i}m_\mathrm{i} \in \mathbb{C}^\times$ *with* $|m_\mathrm{r}| < 1$*.*

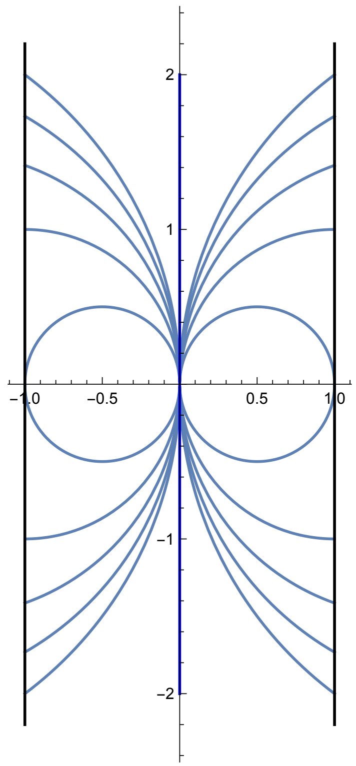
(i) Let $m_\mathrm{r} = 0$*.*

(a) If $\frac{\ln\left|\kappa\frac{\Gamma(m)}{\Gamma(-m)}\right|}{m_\mathrm{i}} \in]-\pi,\pi[$*, then* $\#\sigma_\mathrm{p}(H_{m,\lambda}) = \infty$*,*

(a) if $\frac{\ln\left|\kappa\frac{\Gamma(m)}{\Gamma(-m)}\right|}{m_\mathrm{i}} \notin]-\pi,\pi[$ *then* $\#\sigma_\mathrm{p}(H_{m,\lambda}) = 0$*.*

(ii) If $m_\mathrm{r} \neq 0$ *and if* $N \in \mathbb{N}$ *satisfies* $N < \frac{m_\mathrm{r}^2 + m_\mathrm{i}^2}{|m_\mathrm{r}|} \leq N + 1$*, then*

$$\#\sigma_\mathrm{p}(H_{m,\lambda}) \in \{N, N+1\}.$$



2
1
−1.0
−0.5
0.5
1.0
−1
−2

HOMOGENEOUS RANK ONE PERTURBATIONS

Consider the Hilbert space $\mathcal{H} = L^2(\mathbb{R}_+)$ and operator X

$$Xf(x) := xf(x).$$

Let $m \in \mathbb{C}$, $\lambda \in \mathbb{C} \cup \{\infty\}$. We consider a family of operators formally given by

$$H_{m,\lambda} := X + \lambda |x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|.$$

Note that X is homogeneous of degree 1.

$|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ is homogeneous of degree $1+m$. However strictly speaking, it is not a well defined operator, because $x^{\frac{m}{2}}$ is never square integrable.

If $-1 < \operatorname{Re} m < 0$, the perturbation $|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|$ is form bounded relatively to X and then $H_{m,\lambda}$ can be defined.

The perturbation is formally rank one. Therefore,

$$(z - H_{m,\lambda})^{-1} = (z - X)^{-1}$$

$$+ \sum_{n=0}^{\infty} (z - X)^{-1} |x^{\frac{m}{2}}\rangle (-\lambda)^{n+1} \langle x^{\frac{m}{2}}| (z - X)^{-1} |x^{\frac{m}{2}}\rangle^{n} \langle x^{\frac{m}{2}}| (z - X)^{-1}$$

$$= (z - X)^{-1}$$

$$+ \left(\lambda^{-1} - \langle x^{\frac{m}{2}}| (z - X)^{-1} |x^{\frac{m}{2}}\rangle\right)^{-1} (z - X)^{-1} |x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}| (z - X)^{-1}.$$

By straightforward complex analysis methods we obtain

$$\langle x^{\frac{m}{2}}|(z-X)^{-1}|x^{\frac{m}{2}}\rangle$$
$$=\int_0^\infty x^m(z-x)^{-1}\mathrm{d}x=(-z)^m\frac{\pi}{\sin \pi m}.$$

Therefore, the resolvent of $H_{m,\lambda}$ can be given in a closed form:

$$(z-H_{m,\lambda})^{-1}=(z-X)^{-1}$$
$$+\Big(\lambda^{-1}-(-z)^m\frac{\pi}{\sin \pi m}\Big)^{-1}(z-X)^{-1}|x^{\frac{m}{2}}\rangle\langle x^{\frac{m}{2}}|(z-X)^{-1}.$$

The above formula defines a resolvent of a closed operator for all $-1 < \operatorname{Re} m < 1$ and $\lambda \in \mathbb{C} \cup \{\infty\}$. This allows us to define a holomorphic family of closed operators $H_{m,\lambda}$.

Note that $H_{m,0} = X$.

$m = 0$ is special: $H_{0,\lambda} = X$ for all λ.

We introduce H_0^ρ for any $\rho \in \mathbb{C} \cup \{\infty\}$ by

$$(z - H_0^\rho)^{-1} = (z - X)^{-1}$$
$$-\big(\rho + \ln(-z)\big)^{-1}(z - X)^{-1}|x^0\rangle\langle x^0|(z - X)^{-1}.$$

In particular, $H_0^\infty = X$.

The group of dilations ("the renormalization group") acts on our operators in a simple way:

$$U_\tau H_{m,\lambda} U_\tau^{-1} = \mathrm{e}^\tau H_{m,\mathrm{e}^{\tau m}\lambda},$$
$$U_\tau H_0^\rho U_\tau^{-1} = \mathrm{e}^\tau H_0^{\rho+\tau}.$$

Define the unitary operator

$$(If)(x) := x^{-\frac{1}{4}} f(2\sqrt{x}).$$

Its inverse is

$$(I^{-1}f)(x) := \left(\frac{y}{2}\right)^{\frac{1}{2}} f\left(\frac{y^2}{4}\right).$$

Note that

$$\begin{aligned} I^{-1}XI &= \frac{X^2}{4}, \\ I^{-1}AI &= \frac{A}{2}. \end{aligned}$$

We change slightly notation: the almost homogeneous Schrödinger operators H_m, $H_{m,\kappa}$ and H_0^ν will be denoted $\tilde{H}_m$, $\tilde{H}_{m,\kappa}$ and $\tilde{H}_0^\nu$

Recall that we introduced the Hankel transformation $\mathcal{F}_m$, which is a bounded invertible involution satisfying

$$\begin{aligned} \mathcal{F}_m \tilde{H}_m \mathcal{F}_m^{-1} &= X^2, \\ \mathcal{F}_m A \mathcal{F}_m^{-1} &= -A. \end{aligned}$$

Theorem 16 *.*

1.

$$\mathcal{F}_m^{-1} I^{-1} H_{m,\lambda} I \mathcal{F}_m = \frac{1}{4}\tilde{H}_{m,\kappa},$$

where

$$\lambda \frac{\pi}{\sin(\pi m)} = \kappa \frac{\Gamma(m)}{\Gamma(-m)},$$

2.

$$\mathcal{F}_m^{-1} I^{-1} H_0^{\rho} I \mathcal{F}_m = \frac{1}{4}\tilde{H}_0^{\nu},$$

where $\rho = -2\nu$*.*