# EXTENDED WEAK COUPLING LIMIT

**Jan Dereziński**

**Wojciech De Roeck**

# PAULI FIERZ OPERATORS

## Bosonic Fock spaces.

1-particle Hilbert space: $\mathcal{H}_{\mathrm{R}}$.

Fock space: $\Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}}) := \bigoplus_{n=0}^{\infty} \otimes_{\mathrm{s}}^{n}\mathcal{H}_{\mathrm{R}}$.

Vacuum vector: $\Omega = 1 \in \otimes_{\mathrm{s}}^{0}\mathcal{H}_{\mathrm{R}} = \mathbb{C}$.

If $z \in \mathcal{H}_{\mathrm{R}}$, then

$$a(z)\Psi := \sqrt{n}(z|\otimes 1^{(n-1)\otimes}\Psi \in \otimes_{\mathrm{s}}^{n-1}\mathcal{H}_{\mathrm{R}}, \quad \Psi \in \otimes_{\mathrm{s}}^{n}\mathcal{H}_{\mathrm{R}}$$

is called the annihilation operator of $z$ and $a^*(z) := a(z)^*$ the corresponding creation operator.

# Second quantization

For an operator $q$ on $\mathcal{H}_\mathrm{R}$ we define the operator $\Gamma(q)$ on $\Gamma_\mathrm{s}(\mathcal{H}_\mathrm{R})$ by

$$\Gamma(q)\Big|_{\otimes_\mathrm{s}^n \mathcal{H}_\mathrm{R}} = q \otimes \cdots \otimes q.$$

For an operator $h$ on $\mathcal{H}_\mathrm{R}$ we define the operator $\mathrm{d}\Gamma(h)$ on $\Gamma_\mathrm{s}(\mathcal{H}_\mathrm{R})$ by

$$\mathrm{d}\Gamma(h)\Big|_{\otimes_\mathrm{s}^n \mathcal{H}_\mathrm{R}} = h \otimes 1^{(n-1)\otimes} + \cdots 1^{(n-1)\otimes} \otimes h.$$

Note the identity $\Gamma(\mathrm{e}^{\mathrm{i}th}) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}$.

# Creation/annihilation operators in coupled spaces

If $\mathcal{K}$ is a Hilbert space and $V \in B(\mathcal{K}, \mathcal{K} \otimes \mathcal{H}_{\mathrm{R}})$, then for $\Psi \in \mathcal{K} \otimes \otimes_{\mathrm{s}}^{n} \mathcal{H}_{\mathrm{R}}$ we set

$$a(V)\Psi := \sqrt{n} V^{*} \otimes 1^{(n-1)\otimes} \Psi \in \mathcal{K} \otimes \otimes_{\mathrm{s}}^{n-1} \mathcal{H}_{\mathrm{R}}.$$

$a(V)$ is called the annihilation operator of $V$ and $a^{*}(V) := a(V)^{*}$ the corresponding creation operator. They are closable operators on $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$. In particular, if $V = \nu \otimes |b)$, then

$$a^{*}(V) = \nu \otimes a^{*}(b), \quad a(V) = \nu^{*} \otimes a(b).$$

## Pauli-Fierz operators

Consider a Hilbert space $\mathcal{H} := \mathcal{K} \otimes \Gamma_{\rm s}(\mathcal{H}_{\rm R})$, where $\mathcal{H}_{\rm R}$ is the 1-particle space of the reservoir and $\Gamma_{\rm s}(\mathcal{H}_{\rm R})$ is the corresponding bosonic Fock space. The composite system is described by the self-adjoint operator

$$\begin{aligned} H_\lambda &= K \otimes 1 + 1 \otimes {\rm d}\Gamma(H_{\rm R}) \\ &\quad + \lambda(a^*(V) + a(V)) \end{aligned}$$

Here $K$ describes the Hamiltonian of the small system, ${\rm d}\Gamma(H_{\rm R})$ describes the dynamics of the reservoir expressed by the second quantization of $H_{\rm R}$, and $a^*(V)/a(V)$ are the creation/annihilation operators of an operator $V \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{H}_{\rm R})$.

## Alternative notation

Identify $\mathcal{H}_{\mathrm{R}}$ with $L^2(\Xi, \mathrm{d}\xi)$, for some measure space $(\Xi, \mathrm{d}\xi)$, so that one can introduce $a_\xi^*/a_\xi$ – the usual creation/annihilation operators. Let $h$ be the multiplication operator by $x(\xi)$. $V$ can be identified with a function $\Xi \ni \xi \mapsto v(\xi) \in B(\mathcal{K})$.

$$\begin{aligned}
\mathrm{d}\Gamma(H_{\mathrm{R}}) &= \int x(\xi) a_\xi^* a_\xi \mathrm{d}\xi, \\
a^*(V) &= \int v(\xi) a_\xi^* \mathrm{d}\xi, \quad a(V) = \int v^*(k) a_\xi \mathrm{d}\xi, \\
H &= K + \int x(\xi) a_\xi^* a_\xi \mathrm{d}\xi + \lambda \int \left( v(\xi) a_\xi^* + v^*(\xi) a_\xi \right) \mathrm{d}\xi.
\end{aligned}$$

# QUANTUM LANGEVIN DYNAMICS

## C.p.u.p. semigroups

Let $\mathcal{K}$ be a finite dimensional Hilbert space. Suppose that we are given $M$, the generator of a completely positive unity preserving semigroup on $B(\mathcal{K})$. Then there exists an operator $\Upsilon$, an auxiliary Hilbert space $\mathfrak{h}$ and an operator $\nu$ from $\mathcal{K}$ to $\mathcal{K} \otimes \mathfrak{h}$ such that

$$-\mathrm{i}\Upsilon + \mathrm{i}\Upsilon^* = -\nu^*\nu$$

and $M$ can be written in the Lindblad form

$$M(A) = -\mathrm{i}(\Upsilon A - A\Upsilon^*) + \nu^* \; A\otimes 1 \; \nu, \qquad A \in B(\mathcal{K}).$$

# Quantum Langevin dynamics I

Let $(1|$ denote the (unbounded) linear form on $L^2(\mathbb{R})$:

$$(1|f := \int f(x)\mathrm{d}x.$$

$|1)$ will denote the adjoint form. We define the 1-particle space $\mathcal{Z}_{\mathrm{R}} := \mathfrak{h} \otimes L^2(\mathbb{R})$. The full Hilbert space is $\mathcal{Z} := \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{R}})$. $Z_{\mathrm{R}}$ is the operator of multiplication by the variable $x$ on $L^2(\mathbb{R})$.

# Quantum Langevin dynamics II

We choose a basis $(b_j)$ in $\mathfrak{h}$ and write

$$\nu = \sum \nu_j \otimes |b_j).$$

Set

$$\begin{aligned} \nu_j^+ &= \nu_j, \\ \nu_j^- &= \nu_j^*. \end{aligned}$$

We will denote by $I_{\mathcal{K}}$ the embedding of $\mathcal{K} \simeq \mathcal{K} \otimes \Omega$ in $\mathcal{Z}$.

# Quantum Langevin dynamics III

For $t \geq 0$ we define the quadratic form

$$
\begin{aligned}
U_t \;:=\; & \mathrm{e}^{-\mathrm{i}\mathrm{d}\Gamma(Z_{\mathrm{R}})} \sum_{n=0}^{\infty} \int\limits_{t\geq t_n\geq\cdots\geq t_1\geq 0} \mathrm{d}t_n \cdots \mathrm{d}t_1 \\
& \times (2\pi)^{-\frac{n}{2}} \sum_{j_1,\ldots,j_n} \sum_{\epsilon_1,\ldots,\epsilon_n\in\{+,-\}} \\
& \times (-\mathrm{i})^n \, \mathrm{e}^{-\mathrm{i}(t-t_n)\Upsilon} \, \nu_{j_n}^{\epsilon_n} \, \mathrm{e}^{-\mathrm{i}(t_n-t_{n-1})\Upsilon} \cdots \nu_{j_1}^{\epsilon_1} \, \mathrm{e}^{-\mathrm{i}(t_1-0)\Upsilon} \\
& \times \prod_{k=1,\ldots,n:\quad \epsilon_k=+} a^*(\mathrm{e}^{\mathrm{i}t_k Z_R}|1)\otimes b_{j_k}) \\
& \times \prod_{k'=1,\ldots,n:\quad \epsilon_{k'}=-} a(\mathrm{e}^{\mathrm{i}t_{k'} Z_R}|1)\otimes b_{j_{k'}}).
\end{aligned}
$$

# Quantum Langevin dynamics IV

For $t < 0$ we set $U_{-t} := U_t^*$.

Theorem. $U_t$ is a strongly continuous unitary group on $\mathcal{Z}$, and hence can be written as $U_t = \mathrm{e}^{-\mathrm{i}tZ}$ for some self-adjoint operator $Z$. For $t \geq 0$ we have

$$\begin{aligned} I_{\mathcal{K}}^* \, \mathrm{e}^{-\mathrm{i}tZ} \, I_{\mathcal{K}} &= \mathrm{e}^{-\mathrm{i}t\Upsilon}, \\ I_{\mathcal{K}}^* \, \mathrm{e}^{\mathrm{i}tZ} \; A \otimes 1 \; \mathrm{e}^{-\mathrm{i}tZ} \, I_{\mathcal{K}} &= \mathrm{e}^{tM}(A). \end{aligned}$$

## Quantum Langevin dynamics V

Formally (and also rigorously with an appropriate regularization)

$$
\begin{aligned}
Z &= \frac{1}{2}(\Upsilon + \Upsilon^*) + \mathrm{d}\Gamma(Z_{\mathrm{R}}) \\
&\quad + (2\pi)^{-\frac{1}{2}} a^* \left(\nu \otimes |1)\right) + (2\pi)^{-\frac{1}{2}} a \left(\nu \otimes |1)\right)
\end{aligned}
$$

## Quantum Langevin equation I

(Hudson - Parthasaraty)

The cocycle $W_t := \mathrm{e}^{\mathrm{i}tZ_0}\,\mathrm{e}^{-\mathrm{i}tZ}$, for $Z_0 := \mathrm{d}\Gamma(Z_\mathrm{R})$ solves

$$
\begin{aligned}
& \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}W_t \\
= \; & \Big(\frac{1}{2}(\Upsilon + \Upsilon^*) \\
& +(2\pi)^{-\frac{1}{2}}a^*\left(\nu\otimes|\,\mathrm{e}^{-\mathrm{i}tZ_\mathrm{R}}\,1)\right) + (2\pi)^{-\frac{1}{2}}a\left(\nu\otimes|\,\mathrm{e}^{-\mathrm{i}tZ_\mathrm{R}}\,1)\right)\Big)W_t,
\end{aligned}
$$

# Quantum Langevin equation II

Apply the Fourier transformation on $L^2(\mathbb{R})$, so that $(2\pi)^{-\frac{1}{2}}|1)$ will correspond to $|\delta_0)$. Writing $\hat{W}_t$ for $W_t$ after this transformation, we obtain the quantum Langevin equation in a more familiar form:

$$\begin{aligned} & \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{W}_t \\ = \; & \left(\frac{1}{2}(\Upsilon + \Upsilon^*) + a^*\left(\nu \otimes |\delta_t)\right) + a\left(\nu \otimes |\delta_t)\right)\right)\hat{W}_t. \end{aligned}$$

## Stochastic Schrödinger equation

Let $\mathcal{D}_0 := \mathfrak{h} \otimes (C(\mathbb{R}) \cap L^2(\mathbb{R}))$. Let $\overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D}_0)$, denote the corresponding algebraic Fock space and $\mathcal{D} := \mathcal{K} \otimes \overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D}_0)$. In the sense of quadratic forms on $\mathcal{D}$ the cocycle $\hat{W}(t)$ solves

$$
\begin{aligned}
\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\hat{W}(t) &= (\Upsilon + a^*(\nu \otimes |\delta_t))\hat{W}_t \\
&\quad + \sum_j \nu_j^* \hat{W}_t a(b_j \otimes |\delta_t))
\end{aligned}
$$

## The "age" of observables

For any Borel set $I \subset \mathbb{R}$, the space $L^2(I)$ can be treated as a subspace of $L^2(\mathbb{R})$. Therefore, we have the decomposition

$$\Gamma_{\mathrm{s}}(\mathfrak{h} \otimes L^2(I)) \otimes \Gamma_{\mathrm{s}}(\mathfrak{h} \otimes L^2(\mathbb{R} \setminus I)).$$

Therefore,

$$\begin{aligned} \mathfrak{M}_{\mathrm{R}}(I) &:= 1_{\mathcal{K}} \otimes B\left(\Gamma_{\mathrm{s}}(\mathfrak{h} \otimes L^2(I))\right), \\ \mathfrak{M}(I) &:= B\left(\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathfrak{h} \otimes L^2(I))\right), \end{aligned}$$

are well defined as von Neumann subalgebras of $B(\mathcal{Z})$.

# Quantum Langevin dynamics and the observables

A quantum Langevin dynamics makes the bosons "older". At the time $t = 0$ they may become entangled with the small system.

**Theorem.** If $t > 0$ and $I \subset \mathbb{R} \setminus ]-t, 0[$, then

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}tZ}\, \mathfrak{M}_{\mathrm{R}}(I)\, \mathrm{e}^{-\mathrm{i}tZ} &= \mathfrak{M}_{\mathrm{R}}(I + t), \\
\mathrm{e}^{\mathrm{i}tZ}\, \mathfrak{M}([-t, 0])\, \mathrm{e}^{-\mathrm{i}tZ} &= \mathfrak{M}([0, t]).
\end{aligned}
$$

# WEAK COUPLING LIMIT FOR PAULI-FIERZ OPERATORS

We consider a Pauli-Fierz operator on the Hilbert space $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$

$$H_\lambda \;=\; K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(H_{\mathrm{R}}) + \lambda(a^*(V) + a(V)).$$

## Reduced weak coupling limit

We assume that $\mathcal{K}$ is finite dimensional and for any $A \in B(\mathcal{K})$ we have $\int \|V^* A \otimes 1 \ \mathrm{e}^{-\mathrm{i}tH_0} V\| \mathrm{d}t < \infty$.

**Theorem.** (E.B.Davies) There exists a c.p.u.p. semigroup $\mathrm{e}^{tM}$ such that

$$\lim_{\lambda \searrow 0} \mathrm{e}^{-\mathrm{i}tK/\lambda^2} I_{\mathcal{K}}^* \mathrm{e}^{\mathrm{i}tH_\lambda/\lambda^2} \ A \otimes 1 \ \mathrm{e}^{-\mathrm{i}tH_\lambda/\lambda^2} I_{\mathcal{K}} \mathrm{e}^{\mathrm{i}tK/\lambda^2} = \mathrm{e}^{tM}(A),$$

and a contractive semigroup $\mathrm{e}^{-\mathrm{i}t\Upsilon}$ such that

$$\lim_{\lambda \searrow 0} \mathrm{e}^{\mathrm{i}tK/\lambda^2} I_{\mathcal{K}}^* \mathrm{e}^{-\mathrm{i}tH_\lambda/\lambda^2} I_{\mathcal{K}} = \mathrm{e}^{-\mathrm{i}t\Upsilon}.$$

# Assumptions on the continuity of spectrum

Assumption. Suppose that for any $\omega \in \mathrm{sp}K - \mathrm{sp}K$ there exists open $I_\omega \subset \mathbb{R}$ such that $\omega \in I_\omega$ and

$$\mathrm{Ran}1_{I_\omega}(H_\mathrm{R}) \ \simeq \ \mathfrak{h}_\omega \otimes L^2(I_\omega, \mathrm{d}x),$$

$1_{I_\omega}(H_\mathrm{R})H_\mathrm{R}$ is the multiplication operator by the variable $x \in I_\omega$ and

$$1_{I_\omega}(H_\mathrm{R})V \ \simeq \ \int_{I_\omega}^{\oplus} v(x)\mathrm{d}x.$$

We assume that $I_\omega$ are disjoint for distinct $\omega$ and $x \mapsto v(x) \in B(\mathcal{K}, \mathcal{K}\otimes\mathfrak{h}_\omega)$ is continuous at $\omega$.

## Formula for the Davies generator I

Let $\mathfrak{h} := \bigoplus_{\omega} \mathfrak{h}_\omega$. We define $\nu_\omega : \mathcal{K} \to \mathcal{K} \otimes \mathfrak{h}_\omega$

$$\nu_\omega := (2\pi)^{\frac{1}{2}} \sum_{\omega = k - k'} 1_k(K) v(\omega) 1_{k'}(K),$$

and $\nu : \mathcal{K} \to \mathcal{K} \otimes \mathfrak{h}$

$$\nu := \sum_{\omega} \nu_\omega.$$

# Formula for the Davies generator II

The operator $\Upsilon : \mathcal{K} \to \mathcal{K}$ is

$$\Upsilon \quad := \quad -\mathrm{i} \sum_{\omega} \sum_{k-k'=\omega} \int_0^\infty 1_k(K) V^* 1_{k'}(K)\, \mathrm{e}^{-\mathrm{i}t(H_\mathrm{R}-\omega)}\, V\, 1_k(K) \mathrm{d}t.$$

Note that

$$\begin{aligned}
\mathrm{i}\Upsilon - \mathrm{i}\Upsilon^* \quad &= \quad \sum_{\omega} \sum_{k-k'=\omega} \int_{-\infty}^\infty 1_k(K) V^* 1_{k'}(K)\, \mathrm{e}^{-\mathrm{i}t(H_\mathrm{R}-\omega)}\, V\, 1_k(K) \mathrm{d}t \\
&= \quad \sum_{\omega} \sum_{k-k'=\omega} 1_k(K) v^*(\omega) 1_{k'}(K) v(\omega)\, 1_k(K) \\
&= \quad \nu^* \nu.
\end{aligned}$$

## Formula for the Davies generator III

The generator of a c.p.u.p. semigroup that arises in the reduced weak coupling limit, called sometimes the Davies generator, is

$$M(A) = -\mathrm{i}(\Upsilon A - A\Upsilon^*) + \nu^* A{\otimes}1\nu, \qquad A \in B(\mathcal{K}).$$

# Asymptotic space and dynamics

Recall that given $(\Upsilon, \nu, \mathfrak{h})$ we can define the space $\mathcal{Z}_{\mathrm{R}}$ and the Langevin dynamics $\mathrm{e}^{-\mathrm{i}tZ}$ on the space $\mathcal{Z} := \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{R}})$. Recall that

$$\mathcal{Z}_{\mathrm{R}} = \bigoplus_{\omega} \mathfrak{h}_{\omega} \otimes L^2(\mathbb{R}).$$

## Scaling

For $\lambda > 0$, we define the family of partial isometries $J_{\lambda,\omega} : \mathfrak{h}_\omega \otimes L^2(\mathbb{R}) \to \mathfrak{h}_\omega \otimes L^2(I_\omega)$:

$$(J_{\lambda,\omega} g_\omega)(y) = \begin{cases} \frac{1}{\lambda} g_\omega(\frac{y-\omega}{\lambda^2}), & \text{if } y \in I_\omega; \\ 0, & \text{if } y \in \mathbb{R}\backslash I_\omega. \end{cases}$$

We set $J_\lambda : \mathcal{Z}_{\mathrm{R}} \to \mathcal{H}_{\mathrm{R}}$, defined for $g = (g_\omega)$ by

$$J_\lambda g := \sum_\omega J_{\lambda,\omega} g_\omega.$$

Note that $J_\lambda$ are partial isometries and

$$\mathrm{s-}\lim_{\lambda \searrow 0} J_\lambda^* J_\lambda = 1.$$

# Extended weak coupling limit

(Inspired by Accardi-Frigerio-Lu).

**Theorem.** D., De Roeck.

$$\mathrm{s}^* - \lim_{\lambda \searrow 0} \Gamma(J_\lambda^*)\, \mathrm{e}^{\mathrm{i}\lambda^{-2} t H_0}\, \mathrm{e}^{-\mathrm{i}\lambda^{-2}(t-t_0) H_\lambda}\, \mathrm{e}^{\mathrm{i}\lambda^{-2} t_0 H_0}\, \Gamma(J_\lambda)$$

$$= \quad \mathrm{e}^{\mathrm{i}t Z_0}\, \mathrm{e}^{-\mathrm{i}(t-t_0) Z}\, \mathrm{e}^{-\mathrm{i}t_0 Z_0}\,.$$

Thus the physical dynamics converges to a quantum Langevin dynamics (both in the interaction picture).

# Asymptotics of correlation functions

Corrolary Let $A_\ell, \dots, A_1 \in B(\mathcal{Z})$ and $t, t_\ell, \dots, t_1, t_0 \in \mathbb{R}$. Then

$$
\begin{aligned}
& \mathrm{s}^* - \lim_{\lambda \searrow 0} I_{\mathcal{K}}^* \, \mathrm{e}^{\mathrm{i}\lambda^{-2} t H_0} \, \mathrm{e}^{-\mathrm{i}\lambda^{-2}(t-t_\ell)H_\lambda} \, \mathrm{e}^{-\mathrm{i}\lambda^{-2} t_\ell H_0} \\
& \times \Gamma(J_\lambda) A_\ell \Gamma(J_\lambda^*) \cdots \Gamma(J_\lambda) A_1 \Gamma(J_\lambda^*) \\
& \mathrm{e}^{\mathrm{i}\lambda^{-2} t_1 H_0} \, \mathrm{e}^{-\mathrm{i}\lambda^{-2}(t_1-t_0)H_\lambda} \, \mathrm{e}^{-\mathrm{i}\lambda^{-2} t_0 H_0} \, I_{\mathcal{K}} \\
= \quad & I_{\mathcal{K}}^* \, \mathrm{e}^{\mathrm{i}t Z_0} \, \mathrm{e}^{-\mathrm{i}(t-t_\ell)Z} \, \mathrm{e}^{-\mathrm{i}t_\ell Z_0} \, A_\ell \\
& \cdots A_1 \, \mathrm{e}^{\mathrm{i}t_1 Z_0} \, \mathrm{e}^{-\mathrm{i}(t_1-t_0)Z} \, \mathrm{e}^{-\mathrm{i}t_0 Z_0} \, I_{\mathcal{K}}.
\end{aligned}
$$