SCATTERING THEORY AT LOW ENERGIES

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Special class of potentials

The main topic of our work was scattering theory for a certain special class of potentials. Scattering for this class has a very interesting behavior at low energies. The exact conditions defining this class are somewhat complicated, for this talk we will adopt simpler and more restrictive conditions:

$V \in C^\infty(\mathbb{R}^d)$, $\gamma > 0$, $0 < \mu < 2$, $\epsilon > 0$ and

$$\left|\partial_x^\alpha (V(x) + \gamma |x|^{-\mu})\right| \leq C_\alpha |x|^{-|\alpha|-\mu-\epsilon}, \quad |x| > 1.$$

Thus $V(x)$ is a small perturbation of $-\gamma|x|^{-\mu}$. We do not assume the spherical symmetry of V.

Standard class of potentials

We will compare properties of potentials from the special class with properties of more general potentials used in scattering theory. For simplicity, we will restrict ourselves to the class given by the following condition:
$V \in C^\infty(\mathbb{R}^d)$, $0 < \mu$,

$$|\partial_x^\alpha V(x)| \leq C_\alpha |x|^{-|\alpha|-\mu}, \quad |x| > 1.$$

Using the jargon of PDE, we will say that *the potential is a symbol.*

Hamiltonians

We will consider first the classical Hamiltonian on the phase space $\mathbb{R}^d \times \mathbb{R}^d$

$$H(x, \xi) = \frac{1}{2}\xi^2 + V(x),$$

and then the quantum Hamiltonian on the Hilbert space $L^2(\mathbb{R}^d)$

$$H = \frac{1}{2}D^2 + V(x),$$

where $D = \frac{1}{\mathrm{i}}\nabla$ is the momentum operator.

Classical scattering in the short-range case

Let V be a symbol, $\mu > 1$. Then for any $\xi, x \in \mathbb{R}^d$, $\xi \neq 0$ from an appropriate incoming/outgoing region there exists a unique solution of

$$\begin{aligned} \ddot{y}^{\pm}(t) &= -\nabla V(y^{\pm}(t)), \\ \lim_{t\to\pm\infty} (y^{\pm}(t) - t\xi) &= x. \end{aligned}$$

Classical scattering in the long-range case

Let V be a symbol, $\mu > 0$. Then for any $\xi, x \in \mathbb{R}^d$, $\xi \neq 0$ from an appropriate incoming/outgoing region, there exists a unique solution of

$$\begin{aligned} \ddot{y}^{\pm}(t) &= -\nabla V(y^{\pm}(t)), \\ y^{\pm}(0) &= x, \\ \lim_{t \to \pm\infty} \dot{y}^{\pm}(t) &= \xi. \end{aligned}$$

All unbounded orbits of positive energy have this form. (Clearly, the energy $\frac{1}{2}\dot{y}^2(t) + V(y(t)) = \frac{1}{2}\xi^2$ is a constant of motion).

Eikonal equation

One obtains a family $y^{\pm}(t,x,\xi)$ of solutions smoothly depending on parameters. Using these solutions, in an appropriate incoming/outgoing region one can construct a solution $\phi^{\pm}(x,\xi)$ to the eikonal equation

$$\frac{1}{2}\left(\nabla_x \phi^{\pm}(x,\xi)\right)^2 + V(x) = \frac{1}{2}\xi^2$$

satisfying $\nabla_x \phi^{\pm}(x,\xi) = \dot{y}^{\pm}(0,x,\xi)$.

Exactly solvable potential

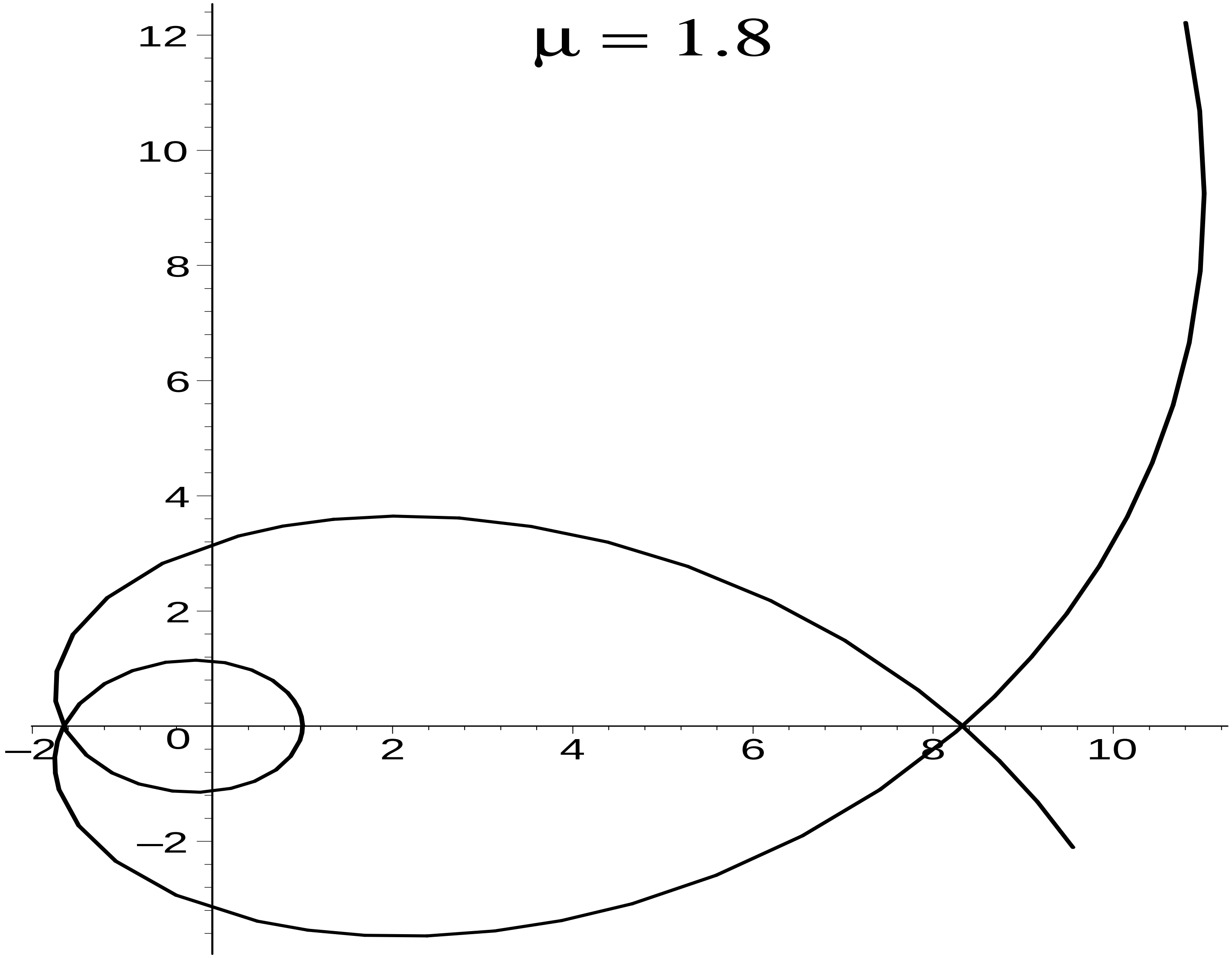
We can find exactly the orbits for the potential $-\gamma|x|^{-\mu}$ at the energy 0. The motion is clearly restricted to 2 dimensions. We can use polar coordinates (r, θ). Collision orbits are

$$r(t) = ct^{\frac{2}{2+\mu}}, \quad \theta = \text{const},$$

and the non-collision orbits satisfy

$$\sin(1 - \frac{\mu}{2})\theta(t) = \Big(\frac{r(t)}{r_{\text{tp}}}\Big)^{-1+\frac{\mu}{2}}.$$

Example of an orbit



Blowing up zero energy

Suppose that the potential belongs to the special class. We blow up the zero energy by replacing the variable $\xi \in \mathbb{R}^d$ with $\lambda \in [0, \infty[$, $\omega \in S^{d-1}$ such that $\xi = \sqrt{2\lambda}\omega$. For any $\xi \neq 0$, there exists a unique solution of

$$\begin{aligned}
\ddot{y}^{\pm}(t) &= -\nabla V(y^{\pm}(t)), \\
\frac{1}{2}\dot{y}^{\pm}(t)^2 + V(y^{\pm}(t)) &= \lambda, \\
y^{\pm}(0) &= x, \\
\lim_{t \to \pm\infty} y^{\pm}(t)/|y^{\pm}(t)| &= \omega.
\end{aligned}$$

All unbounded orbits have this form.

Eikonal equation

One obtains a family $y^{\pm}(t, x, \omega, \lambda)$ of solutions smoothly depending on parameters. Using these solutions, in an appropriate incoming/outgoing region one can construct a solution $\phi^{\pm}(x, \omega, \lambda)$ to the eikonal equation

$$\frac{1}{2}\left(\nabla_x \phi^{\pm}(x, \omega, \lambda)\right)^2 + V(x) = \lambda,$$

satisfying $\nabla_x \phi^{\pm}(x, \omega, \lambda) = \dot{y}^{\pm}(0, x, \omega, \lambda)$.

Quantum scattering in the short-range case

Let V be a symbol, $\mu > 1$. Then there exists

$$W_{\mathrm{sr}}^{\pm} := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}tH_0}\,.$$

$W_{\mathrm{sr}}^{\pm}$ is isometric, $W_{\mathrm{sr}}^{\pm}H_0 = HW_{\mathrm{sr}}^{\pm}$. The scattering operator $S_{\mathrm{sr}} := W_{\mathrm{sr}}^{+*}W_{\mathrm{sr}}^{-}$ is unitary and $S_{\mathrm{sr}}H_0 = H_0S_{\mathrm{sr}}$.

Quantum scattering in the long-range case I

Let V be a symbol, $\mu > 0$. Following Isozaki-Kitada, one introduces modifiers $J^{\pm}$ with the integral kernel

$$J^{\pm}(x,y) := (2\pi)^{-d} \int \mathrm{e}^{\mathrm{i}\phi^{\pm}(x,\xi) - \mathrm{i}\xi\cdot y}\, a^{\pm}(x,\xi)\mathrm{d}\xi.$$

Here $a^{\pm}(x,\xi)$ is an appropriate cut-off supported in the domain of the definition of $\phi^{\pm}(x,\xi)$, equal to one in the incoming/outgoing region.

Quantum scattering in the long-range case II

Then one constructs modified wave operators

$$W^{\pm} := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} J^{\pm} \mathrm{e}^{-\mathrm{i}tH_0} .$$

$W^{\pm}$ are isometric, $W^{\pm}H_0 = HW^{\pm}$. The scattering operator $S := W^{+*}W^{-}$ is unitary and $SH_0 = H_0S$.

Thus in the long-range case modified wave and scattering operators enjoy the same properties as in the short-range case, except that their definition is non-canonical.

Abstract definition of wave operators I

There exists the asymptotic velocity $D^{\pm}$ defined by

$$g(D^{\pm}) := \mathrm{s}-\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\, g(D)\, \mathrm{e}^{-\mathrm{i}tH}\, 1_{\mathrm{c}}(H).$$

We say that $\breve{W}^{\pm}$ is an outgoing/incoming wave operator associated with H if it is isometric and satisfies

$$\breve{W}^{\pm} D = D^{\pm} \breve{W}^{\pm}.$$

Operators of the form $\breve{S} = \breve{W}^{+*} \breve{W}^{-}$ are then called scattering operators.

Abstract definition of wave operators II

If $\breve{W}_1^{\pm}$ and $\breve{W}_2^{\pm}$ are two wave operators for a given H, then there exist functions $\psi^{\pm}$ such that

$$\breve{W}_1^{\pm} = \breve{W}_2^{\pm}\,\mathrm{e}^{\mathrm{i}\psi^{\pm}(D)}\,.$$

Thus

$$\breve{S}_1 = \mathrm{e}^{-\mathrm{i}\psi^{+}(D)}\,\breve{S}_2\,\mathrm{e}^{\mathrm{i}\psi^{-}(D)}\,.$$

Therefore, scattering cross-sections

$$|\breve{S}(\lambda;\omega,\omega')|^2,$$

which are usually considered to be the only measurable quantities in scattering theory, are insensitive to the choice of a wave operator.

Abstract definition of wave operators in the radial case

If the potential is radial we can restrict ourselves to spherically symmetric wave and scattering operators. The arbitrariness of the phase is then significantly reduced:

$$\begin{aligned} \breve{W}_1 &= \breve{W}_2\, \mathrm{e}^{\mathrm{i}\chi^{\pm}(H_0)}, \\ \breve{S}_1 &= \breve{S}_2\, \mathrm{e}^{\mathrm{i}\chi(H_0)}\,. \end{aligned}$$

Scattering matrices

$$\mathcal{F}(\lambda)f(\omega) = (2\lambda)^{(d-2)/4}\hat{f}(\sqrt{2\lambda}\omega), \quad f \in L^2(\mathbb{R}^d).$$

identifies $L^2(\mathbb{R}^d)$ with $\int_0^\infty \oplus L^2(S^{d-1})\mathrm{d}\lambda$ and diagonalizes H_0.

$[S, H_0] = 0$ implies the existence of a decomposition

$$S = \int_0^\infty \oplus S(\lambda)\mathrm{d}\lambda,$$

where $S(\lambda) \in U(L^2(S^{d-1}))$ is defined for almost all λ.

One can show that one can choose this decomposition so that $]0, \infty[\ni \lambda \mapsto S(\lambda)$ is continuous.

Wave matrices

Let $s > \frac{1}{2}$. One can show that there exists a unique strongly continuous function

$$]0, \infty[\ni \lambda \mapsto W^{\pm}(\lambda) \in B(L^2(S^{d-1}), \langle x \rangle^s L^2(\mathbb{R}^d))$$

such that for $f \in \langle x \rangle^{-s} L^2(\mathbb{R}^d)$

$$W^{\pm} f = \int_0^{\infty} W^{\pm}(\lambda) \mathcal{F}(\lambda) f \mathrm{d}\lambda.$$

Wave matrices at zero energy

Suppose that the potential belongs to the special class.

Theorem. There exists the wave matrices at zero energy:

$$W^{\pm}(0) = \lim_{\lambda \searrow 0} W^{\pm}(\lambda)$$

in the sense of operators in $\mathcal{B}(L^2(S^{d-1}), \langle x \rangle^s L^2(\mathbb{R}^d))$, where $s > \frac{1}{2} + \frac{\mu}{4}$. (Note that we had to change the weight.)

Wave matrices at zero energy

There also exists the scattering matrix at zero energy

$$S(0) = \mathrm{s-}\lim_{\lambda \searrow 0} S(\lambda).$$

in the sense of operators in $B(L^2(S^{d-1}))$. $S(0)$ is unitary.

Low energy asymptotics of short-range wave matrices

Note that in general $W_{\rm sr}^{\pm}(\lambda)$ do not have a limit at zero energy. One can say that $W^{\pm}(\lambda)$ are better behaved than $W_{\rm sr}^{\pm}(\lambda)$, even if not canonical. They can be used to give the asymptotics of $W_{\rm sr}^{\pm}(\lambda)$:

$$W_{\rm sr}^{\pm}(\lambda) \ = W^{\pm}(\lambda) \exp\left(\mathrm{i}O(\lambda^{\frac{1}{2}-\frac{1}{\mu}})\right), \ 1 < \mu < 2.$$

This asymptotics was first obtained in the 1-dimensional case by D. Yafaev.

Wave equation on the sphere

Let Λ be the operator on $L^2(S^{d-1})$ such that $\Lambda Y_l = (l + d/2 - 1)Y_l$, where Y_l is a spherical harmonic of order l. Alternatively, it can be introduced as follows:

$$\Lambda := \sqrt{L^2 + (d/2 - 1)^2},$$

where

$$L^2 = \sum_{1 \le i < j \le d} L_{ij}^2, \quad \mathrm{i} L_{ij} = x_i \partial_{x_j} - x_j \partial_{x_i}.$$

The natural wave equation on S^{d-1} is

$$(\partial_t^2 - \Lambda^2) f(t, \omega) = 0.$$

Evolution operator of the wave equation on the sphere

Theorem $\mathrm{e}^{\mathrm{i}\theta\Lambda}$ belongs to the class of Fourier integral operators of order 0 in the sense of Hörmander.
Moreover, let $k \in \mathbb{Z}$.

1. If $\theta = \pi 2k$, then $\mathrm{e}^{\mathrm{i}\theta\Lambda} = (-1)^{kd}$.
2. If $\theta = \pi(2k+1)$, then $\mathrm{e}^{\mathrm{i}\theta\Lambda} = (-1)^{(2k+1)(d/2-1)}P$, where P is the parity.
3. If $\theta \in]\pi 2k, \pi(2k+1)[$, then
$$\mathrm{e}^{\mathrm{i}\theta\Lambda}(\omega,\omega') = (2\pi)^{-d/2}\sin\theta\,\Gamma(d/2)\,\mathrm{e}^{-\mathrm{i}\pi/2}(-\omega\cdot\omega'+\cos\theta-\mathrm{i}0)^{-d/2}.$$
4. If $\theta \in]\pi(2k-1), \pi 2k[$, then
$$\mathrm{e}^{\mathrm{i}\theta\Lambda}(\omega,\omega') = (2\pi)^{-d/2}\sin\theta\,\Gamma(d/2)\,\mathrm{e}^{-\mathrm{i}\pi/2}(-\omega\cdot\omega'+\cos\theta+\mathrm{i}0)^{-d/2}.$$

Type of singularity of the scattering matrix

Assume that the potential belongs to the special class.

Theorem

$$S(0) = \mathrm{e}^{\mathrm{i}c}\, \mathrm{e}^{-\mathrm{i}\frac{\mu\pi}{2-\mu}\Lambda} + K,$$

where K is compact.