# $C^*$-algebras

Jan Dereziński

Department of Mathematical Methods in Physics
Warsaw University
Hoża 74, 00-682, Warszawa, Poland

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## Contents

# 1 $C^*$-algebras

## 1.1 $*$-algebras

Let $\mathfrak{A}$ be an algebra. A mapping $\mathfrak{A} \ni A \mapsto A^* \in \mathfrak{A}$ is an antilinear involution iff

$$A^{**} = A, \quad (AB)^* = B^*A^*, \quad (\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*.$$

An algebra with an involution is called a $*$-algebra.

Let $\mathfrak{A}$ be a $*$-algebra. $A \in \mathfrak{A}$ is invertible iff $A^*$ is, and $(A^{-1})^* = (A^*)^{-1}$.

A subset $\mathfrak{B}$ of $\mathfrak{A}$ is called self-adjoint iff $B \in \mathfrak{B} \Rightarrow B^* \in \mathfrak{B}$.

Let $\mathfrak{A}$, $\mathfrak{B}$ be $*$-algebras. A homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ is called a $*$-homomorphism iff $\pi(A^*) = \pi(A)^*$.

## 1.2 $C^*$-algebras

$\mathfrak{A}$ is a $C^*$-algebra if it is a Banach algebra equipped with an involution $*$ satisfying

$$\|A^*A\| = \|A\|^2. \tag{1.1}$$

$$\|A^*\| = \|A\|. \tag{1.2}$$

We can weaken the conditions (1.1) and (1.2) in the definition of a $C^*$ -algebra as follows:

**Theorem 1.1** *If $\mathfrak{A}$ is a Banach algebra with an involution $*$ satisfying*

$$\|A\|^2 \leq \|A^*A\|, \tag{1.3}$$

*then it is a $C^*$-algebra.*

**Proof.** Clearly,

$$\|A^*A\| \leq \|A^*\|\|A\|. \tag{1.4}$$

Hence, by (1.3), $\|A\| \leq \|A^*\|$. Using $A^{**} = A$, this gives $\|A^*\| \geq \|A\|$. Hence (1.2) is true.

(1.2) and (1.4) give $\|A^*A\| \leq \|A\|^2$. This and (1.3) imply (1.1). □

Let $\mathfrak{A}$ be a fixed $C^*$-algebra. A subset of $\mathfrak{A}$ is a $C^*$-algebra iff it is a self-adjoint closed algebra. If $\mathfrak{B} \subset \mathfrak{A}$, then $C^*(\mathfrak{B})$ will denote the smallest $C^*$-subalgebra in $\mathfrak{A}$ containing $\mathfrak{B}$.

Let $\mathcal{H}$ be a Hilbert space. Then $B(\mathcal{H})$ is a $C^*$-algebra. A $C^*$-algebra inside $B(\mathcal{H})$ is called a concrete $C^*$-algebra.

A concrete $C^*$-algebra is called nondegenerate if for $\Phi \in \mathcal{H}$, $A\Phi = 0$ for all $A \in \mathfrak{A}$ implies $\Phi = 0$.

If $\mathfrak{A}$ is not necessarily non-degenerate, and $\mathcal{H}_1 := \{\Phi \in \mathcal{H} \ : \ Ax = 0, \ A \in \mathfrak{A}\}$, then $\mathfrak{A}$ restricted to $\mathcal{H}_1^\perp$ is nondegenerate.

**Theorem 1.2** *If $1 \in \mathfrak{A}$, then $\|1\| = 1$.*

**Proof.** By the uniqueness of the identity, we have $1 = 1^*$. Hence $\|1\|^2 = \|1^*1\| = \|1\|$. □

## 1.3 Special elements of a $*$-algebra

$A \in \mathfrak{A}$ is called normal if $AA^* = A^*A$. It is called self-adjoint if $A^* = A$. $\mathfrak{A}_{\mathrm{sa}}$ denotes the set of self-adjoint elements of $\mathfrak{A}$

$P \in \mathfrak{A}$ is called a projection if it is a self-adjoint idempotent. $P(\mathfrak{A})$ denotes the set of projections of $\mathfrak{A}$.

**Theorem 1.3** *Let $P^* = P$ and $P^2 = P^3$. Then $P$ is a projection.*

$U \in \mathfrak{A}$ is called a partial isometry iff $U^*U$ is a projection. If this is the case, then $UU^*$ is also a projection. $U^*U$ is called the right support of $U$ and $UU^*$ is called the left support of $U$.

$U$ is called an isometry if $U^*U = 1$.

$U$ is called a unitary if $U^*U = UU^* = 1$. $U(\mathfrak{A})$ denotes the set of unitary elements of $\mathfrak{A}$.

$U$ is called a partial isometry iff $U^*U$ and $UU^*$ are projections.

We can actually weaken the above condition:

**Theorem 1.4** *Let either $U^*U$ or $UU^*$ be a projection. Then $U$ is a partial isometry.*

## 1.4 Spectrum of elements of $C^*$-algebras

**Theorem 1.5** *Let $A \in \mathfrak{A}$ be normal. Then*

$$\mathrm{sr}(A) = \|A\|.$$

**Proof.**

$$\|A^2\|^2 = \|A^{2*}A^2\| = \|(A^*A)^2\| = \|A^*A\|^2 = \|A\|^4.$$

Thus $\|A^{2^n}\| = \|A\|^{2^n}$. Hence, using the formula for the spectral radius of $A$ we get $\|A^{2^n}\|^{2^{-n}} = \|A\|$. □

**Theorem 1.6** (1) *Let $V \in \mathfrak{A}$ be isometric. Then* $\mathrm{sp}(V) \subset \{|z| \leq 1\}$.

(2) *$U \in \mathfrak{A}$ is unitary $\Rightarrow$ $U$ is normal and* $\mathrm{sp}(U) \subset \{z \ : \ |z| = 1\}$.

(3) *$A \in \mathfrak{A}$ is self-adjoint $\Rightarrow$ $A$ is normal and* $\mathrm{sp}(A) \subset \mathbb{R}$.

**Proof.** (1) We have $\|V\|^2 = \|V^*V\| = \|1\| = 1$. Hence, $\mathrm{sp}(V) \subset \{|z| \leq 1\}$.

(2) Clearly, $U$ is normal.

$U$ is an isometry, hence $\mathrm{sp}(U) \subset \{|z| \leq 1\}$.

$U^{-1}$ is also an isometry, hence $\mathrm{sp}(U^{-1}) \subset \{|z| \leq 1\}$. This implies $\mathrm{sp}(U) \subset \{|z| \geq 1\}$.

(3) For $|\lambda^{-1}| > \|A\|$, $1 + \mathrm{i}\lambda A$ is invertible. We check that $U := (1 - \mathrm{i}\lambda A)(1 + \mathrm{i}\lambda A)^{-1}$ is unitary. Hence, by (2).$\Rightarrow$, $\mathrm{sp}(U) \subset \{|z| = 1\}$. By the spectral mapping theorem, $\mathrm{sp}(A) \subset \mathbb{R}$. □

Note that in (2) and (3) we can actually replace $\Rightarrow$ $\Leftrightarrow$, which will be proven later.

## 1.5 Dependence of spectrum on the Banach algebra

**Theorem 1.7** *Let $\mathfrak{B}$ be a closed subalgebra of a Banach algebra $\mathfrak{A}$ and $1, A \in \mathfrak{B}$.*

(1) $\mathrm{rs}_{\mathfrak{B}}(A)$ *is an open and closed subset of* $\mathrm{rs}_{\mathfrak{A}}(A)$ *containing a neighborhood of* $\infty$.

(2) *The connected components of* $\mathrm{rs}_{\mathfrak{A}}(A)$ *and of* $\mathrm{rs}_{\mathfrak{B}}(A)$ *containing a neighborhood of infinity coincide.*

(3) *If* $\mathrm{rs}_{\mathfrak{A}}(A)$ *is connected, then* $\mathrm{rs}_{\mathfrak{A}}(A) = \mathrm{rs}_{\mathfrak{B}}(A)$.

**Proof.** $\mathrm{rs}_{\mathfrak{B}}(A)$ is open in $\mathbb{C}$. Hence also in $\mathrm{rs}_{\mathfrak{A}}(A)$.

Let $z_0 \in \mathrm{rs}_{\mathfrak{A}}(A)$ and $z_n \in \mathrm{rs}_{\mathfrak{B}}(A)$, $z_n \to z_0$. Then $(z_n - A)^{-1} \to (z_0 - A)^{-1}$ in $\mathfrak{A}$, hence also in $\mathfrak{B}$. Therefore, $z_0 \in \mathrm{rs}_{\mathfrak{B}}(A)$. Hence $\mathrm{rs}_{\mathfrak{B}}(A)$ is closed in $\mathrm{rs}_{\mathfrak{A}}(A)$. This proves 1.

(2) and (3) follow immediately from (1). □

If $A \in \mathfrak{A}$, define $\widetilde{\mathrm{Ban}}(A)$ to be the closed algebra generated by $\widetilde{\mathrm{Alg}}(A)$.

**Lemma 1.8** *Let $U \subset \mathbb{C}$ be open. Then there exists a countable family of open connected sets $\{U_i \; : \; i \in I\}$ such that $U = \cup_{i\in I} U_i$ and $U_i^{\mathrm{cl}} \cap U_j = \emptyset$, $i \neq j$. Besides, $U_i$ are isolated in $U$.*

**Proof.** For $z_1, z_2 \in U$ we will write $z_1 \sim z_2$ iff there exists a continuous path in $U$ connecting $z_1$ and $z_2$. This is an equivalence relation. Let $\{U_i \; : \; i \in I\}$ be the family of equivalence classes. Clearly, $U_i$ are open (and hence also open in the relative topology of $U$).

Suppose that $z_0 \in U_i^{\mathrm{cl}} \cap U_j$. Then there esists $\epsilon > 0$ with $K(z_0, \epsilon) \subset U_j$. There exists $z_1 \in U_i \cap K(z_0, \epsilon)$. Clearly, $z_1 \sim z_0$. Hence $U_i = U_j$.

Thus $U_i^{\mathrm{cl}} \cap U = U_i$. Thus it is closed in the relative topology of $U$. □

The sets $U_i$ described in the above lemma will be called connected components of $U$. Clearly, if $\mathbb{C}\backslash U$ is compact, then one of them is a neighborhood of infinity.

**Theorem 1.9** (1) *If $\mathrm{rs}_{\mathfrak{A}}(A)$ is connected, then $\widetilde{\mathrm{Ban}}(A) = \mathrm{Ban}(A, 1)$.*

(2) *If $\mathrm{rs}_{\mathfrak{A}}(A)$ is disconnected, choose one point $\lambda_1, \lambda_2, \ldots$ in every connected component of $\mathrm{rs}_{\mathfrak{A}}(A)$ that does not contain a neighborhood of infinity. Then $\widetilde{\mathrm{Ban}}(A) = \mathrm{Ban}(A, (\lambda_1 - A)^{-1}, (\lambda_2 - A)^{-1}, \ldots)$.*

**Example 1.10** *Let $U \in L^2(\mathbb{N})$, $Ue_n := e_{n+1}$. Consider the algebras $\mathfrak{B} := \mathrm{Ban}(1, U)$, $\mathfrak{A} = \mathrm{Ban}(1, U, U^*)$. Then*

$$\mathrm{sp}_{\mathfrak{B}} U = \{|z| \leq 1\}, \quad \mathrm{sp}_{\mathfrak{A}} U = \{|z| = 1\},$$

*because*

$$(z - U)^{-1} = -\sum_{n=0}^{\infty} z^n U^{*(n+1)}.$$

## 1.6 Invariance of spectrum in $C^*$-algebras

**Lemma 1.11** *Let $A$ be invertible in $\mathfrak{A}$. Then $A^{-1}$ belongs to $C^*(1, A)$.*

**Proof.** First assume that $A$ is self-adjoint. Then $\mathrm{sp}_{\mathfrak{A}}(A) \subset \mathbb{R}$. Hence $\mathrm{rs}_{\mathfrak{A}}(A)$ is connected. But $\mathfrak{C} := C^*(1, A) = \mathrm{Ban}(1, A)$. Hence, by Theorem 1.7,

$$\mathrm{rs}_{\mathfrak{C}}(A) = \mathrm{rs}_{\mathfrak{A}}(A) \tag{1.5}$$

$A$ is invertible iff $0 \in \mathrm{rs}_{\mathfrak{A}}(A)$. By (1.5), this means that $0 \in \mathrm{rs}_{\mathfrak{C}}(A)$ and hence $A^{-1} \in \mathfrak{C}$.

Next assume that $A$ be an arbitrary invertible element of $\mathfrak{A}$. Clearly, $A^*$ is invertible in $\mathfrak{A}$ and $(A^*)^{-1} = (A^{-1})^*$. Likewise, $A^*A$ is invertible in $\mathfrak{A}$ and $(A^*A)^{-1} = (A^*)^{-1}A^{-1}$. But $A^*A$ is self-adjoint and hence $(A^*A)^{-1} \in C^*(1, A^*A) \subset C^*(1, A)$. Next we check that $A^{-1} = (A^*A)^{-1}A^*$. □

**Theorem 1.12** *Let $\mathfrak{B} \subset \mathfrak{A}$ be $C^*$-algebras and $A, 1 \in \mathfrak{B}$. Then $\mathrm{sp}_{\mathfrak{B}}(A) = \mathrm{sp}_{\mathfrak{A}}(A)$.*

**Proof.** By Lemma 1.11, $\mathrm{sp}_{\mathfrak{A}}(A) = \mathrm{sp}_{\mathfrak{C}}(A)$, where $\mathfrak{C} := C^*(1, A)$. But $\mathfrak{C} \subset \mathfrak{B} \subset \mathfrak{A}$. □

Motivated by the above theorem, when speaking about $C^*$-algebras, we will write $\mathrm{sp}(A)$ instead of $\mathrm{sp}_{\mathfrak{A}}(A)$.

### 1.7 Holomorphic spectral theorem for normal operators

If $K$ is a compact subset of $\mathbb{C}$ let $C_{\rm hol}(K)$ be the completion of $\mathrm{Hol}(K)$ in $C(K)$.

**Theorem 1.13** *Let $\mathfrak{A}$ be unital and $A \in \mathfrak{A}$ be normal. Then there exists a unique continuous isomorphism*

$$C_{\rm hol}(\mathrm{sp}(A)) \ni f \mapsto f(A) \in C^*(1, A) \subset \mathfrak{A},$$

*such that*

(1) $\mathrm{id}(A) = A$ *if* $\mathrm{id}(z) = z$.
*Moreover, we have*

(2) *If* $f \in \mathrm{Hol}(\mathrm{sp}(A))$*, then* $f(A)$ *coincides with* $f(A)$ *defined in (**??**).*

(3) $\mathrm{sp}(f(A)) = f(\mathrm{sp}(A))$.

(4) $g \in C_{\rm hol}(f(\mathrm{sp}(A))) \Rightarrow g \circ f(A) = g(f(A))$.

(5) $\|f(A)\| = \sup |f|$.

**Remark 1.14** *The previous theorem will be improved in next section so that the functional calculus will be defined on the whole* $C(\mathrm{sp}A)$.

*In the case $A$ is self-adjoint or unitary,* $C(\mathrm{sp}A) = C_{\rm hol}(\mathrm{sp}A)$*, so in this case we do not need the Gelfand theory.*

### 1.8 Fuglede's theorem

**Theorem 1.15** *Let $A, B \in \mathfrak{A}$ and let $B$ be normal. Then $AB = BA$ implies $AB^* = B^*A$.*

**Proof.** For $\lambda \in \mathbb{C}$, the operator $U(\lambda) := \mathrm{e}^{\lambda B^* - \bar{\lambda} B} = \mathrm{e}^{-\bar{\lambda} B}\mathrm{e}^{\lambda B^*}$ is unitary. Moreover, $A = \mathrm{e}^{\bar{\lambda} B} A \mathrm{e}^{-\bar{\lambda} B}$. Hence

$$\mathrm{e}^{-\lambda B^*} A \mathrm{e}^{\lambda B^*} = U(-\lambda) A U(\lambda) \tag{1.6}$$

is a uniformly bounded analytic function. Hence is constant. Differentiating it wrt $\lambda$ we get $[A, B^*] = 0$.
□

## 2 Adjoining a unit

### 2.1 Adjoining a unit in an algebra

Let $\mathfrak{A}$ be an algebra. Introduce the algebra $\mathfrak{A}_{\rm un}$ equal as a vector space to $\mathbb{C} \oplus \mathfrak{A}$ with the product

$$(\lambda, A)(\mu, B) := (\lambda\mu, \lambda B + \mu A + AB).$$

Then $\mathfrak{A}_{\rm un}$ is a unital algebra and $\mathfrak{A}$ is an ideal of $\mathfrak{A}_{\rm un}$ of codimension 1.

If $\mathfrak{A}$ is non-unital, $\mathfrak{B}$ is unital and $\pi : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then there exists a unique extension $\pi_{\rm un} : \mathfrak{A}_{\rm un} \to \mathfrak{B}$ such that $\pi(1) = 1$.

## 2.2 Unit in a Banach algebra

Let $\mathfrak{A}$ be a unital Banach algebra. Then $\|1\| \geq 1$. Besides, if $\lambda$ is the regular representation, then

$$\|A\| \leq \|\lambda(A)\|\|1\| \leq \|1\|\|A\|.$$

Thus the norms $\|A\|$ an $\|\lambda(A)\|$ are equivalent. Note also that $\|\lambda(1)\| = 1$.

This means, that if $\mathfrak{A}$ is a unital Banach algebra, then by replacing the initial norm with the equivalent norm $\lambda(A)$ we can always assume that $\|1\| = 1$. We will make always this assumption.

**Theorem 2.1** *If $\mathfrak{A}$ is a unital Banach algebra such that $\|1\| = 1$, then the regular representation $\mathfrak{A} \ni A \mapsto \lambda(A) \in L(\mathfrak{A})$ is isometric.*

**Theorem 2.2** *Let $\mathfrak{A}$ be a Banach algebra. Equip $\mathfrak{A}_{\mathrm{un}}$ with the norm*

$$\|\lambda + A\| := \|A\| + |\lambda|.$$

*Then $\mathfrak{A}_{\mathrm{un}}$ is a Banach algebra and $\mathfrak{A} \to \mathfrak{A}_{\mathrm{un}}$ is an isometry.*

Note, however, that there may be other (even more natural) norms on $\mathfrak{A}_{\mathrm{un}}$ extending the norm on $\mathfrak{A}$.

**Theorem 2.3** *Let $\mathfrak{A}$ be a unital Banach algebra. Then $1$ is an extreme point of the unit ball $(\mathfrak{A})_1$.*

**Proof.** $\mathfrak{A}$ can be isometrically embedded in $B(\mathcal{V})$, where $\mathcal{V}$ is a Banach space. Hence the theorem follows from the fact that if $\mathcal{V}$ is a Banach space, then 1 is an extreme point in $(B(\mathcal{V}))_1$. □

## 2.3 Approximate units

Let $\mathfrak{A}$ be a normed algebra. If $\mathfrak{A}$ does not have a unit, then we can use the so-called approximate unit.

We say that a net $(E_\alpha) \subset (\mathfrak{A})_1$ is a left appproximate unit in $\mathfrak{A}$, if for any $A \in \mathfrak{A}$, $\|E_\alpha A - A\| \to 0$.

Let $\mathfrak{A}$ be a Banach algebra without a unit. In $\mathfrak{A}_{\mathrm{un}}$ we define

$$\|\lambda + A\|_{\mathrm{un}} := \sup_{\|B\| \leq 1} \|\lambda B + AB\|.$$

Clearly, $\|\cdot\|_{\mathrm{un}}$ is a seminorm satisfying

$$\|(\lambda + A)(\mu + B)\|_{\mathrm{un}} \leq \|(\lambda + A)\|_{\mathrm{un}}\|(\mu + B)\|_{\mathrm{un}}.$$

**Theorem 2.4** *If $\mathfrak{A}$ possesses an approximate unit, then $\|\cdot\|_{\mathrm{un}}$ is a norm and $\mathfrak{A} \to \mathfrak{A}_{\mathrm{un}}$ is an isometry. Moreover, the regular representation $\mathfrak{A} \ni A \mapsto \lambda(A) \in L(\mathfrak{A})$ is isometric.*

Recall that if $\mathfrak{I}$ is a closed ideal in a Banach algebra $\mathfrak{A}$, then

$$\|A + \mathfrak{I}\| := \inf\{||A + I\| \ : \ I \in \mathfrak{I}\}.$$

**Theorem 2.5** *Suppose that $\mathfrak{I}$ is aclosed ideal in a Banach algebra $\mathfrak{A}$ and $\mathfrak{I}$ possesses a left approximate unit $(E_\alpha)$ such that*

$$\|1 - E_\alpha\| \leq 1. \tag{2.7}$$

*Then the norm in $\mathfrak{A}/\mathfrak{I}$ is given by*

$$\|A + \mathfrak{I}\| = \lim_{\alpha} \|(1 - E_\alpha)A\|.$$

**Proof.** Let $A \in \mathfrak{A}$ and $I \in \mathfrak{J}$. Using first $\|E_\alpha I - I\| \to 0$ and then $\|1 - E_\alpha\| \leq 1$, we get

$$\limsup_\alpha \|(1-E_\alpha)A\| = \limsup_\alpha \|(1-E_\alpha)(A+I)\| \leq \|A+I\|.$$

Hence

$$\limsup_\alpha \|(1-E_\alpha)A\| \leq \inf\{\|A+I\| \ : \ I \in \mathfrak{J}\}.$$

Moreover,

$$\begin{aligned}\liminf_\alpha \|(1-E_\alpha)A\| &\geq \inf \|(1-E_\alpha)A\| \\ &\geq \inf\{\|A+I\| \ : \ I \in \mathfrak{J}\}.\end{aligned}$$

□

## 2.4 Adjoining a unit to a $C^*$-algebra

**Theorem 2.6** *Let $\mathfrak{A}$ be an $C^*$-algebra. Then the algebra $\mathfrak{A}_{\rm un}$ with the norm given by*

$$\|\lambda + A\|_{\rm un} := \sup_{B\neq 0} \frac{\|\lambda B + AB\|}{\|B\|},$$

*and the involution $(\lambda + A)^* := (\overline{\lambda} + A^*)$ is a $C^*$-algebra.*

**Proof.** **Step 1.** Recall from the theory of Banach algebras that $\|\cdot\|_{\rm un}$ is a seminorm on $\mathfrak{A}_{\rm un}$ that satisfies

$$\|(\lambda + A)(\mu + B)\|_{\rm un} \leq \|\lambda + A\|_{\rm un}\|\mu + B\|_{\rm un}.$$

**Step 2.** We show that $\|\cdot\|_{\rm un}$ coincides on $\mathfrak{A}$ with $\|\cdot\|$. In fact, $\|A\|_{\rm un} \leq \|A\|$ is obvious for any Banach algebra. The converse inequality follows by

$$\|A\|_{\rm un} \geq \frac{\|AA^*\|}{\|A^*\|} = \|A\|.$$

**Step 3.** For any $\mu < 1$ there exists $B$ such that $\|B\| = 1$ and $\mu\|\lambda + A\|_{\rm un} \leq \|\lambda B + AB\|$. Then

$$\mu^2\|\lambda + A\|_{\rm un}^2 \leq \|\lambda B + AB\|^2 = \|B^*(\lambda + A)^*(\lambda + A)B\| \leq \|(\lambda + A)^*(\lambda + A)\|.$$

This proves

$$\|\lambda + A\|_{\rm un}^2 \leq \|(\lambda + A)^*(\lambda + A)\|.$$

□

# 3 Gelfand theory

## 3.1 Characters and maximal ideals in an algebra

Let $\mathfrak{A}$ be an algebra.

A nonzero homomorphism of $\mathfrak{A}$ into $\mathbb{C}$ is called a character. We define $\mathrm{Char}(\mathfrak{A})$ to be the set of characters of $\mathfrak{A}$. For any $A \in \mathfrak{A}$ let $\hat{A}$ be the function

$$\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \hat{A}(\phi) := \phi(A) \in \mathbb{C}. \tag{3.8}$$

$\mathrm{Char}(\mathfrak{A})$ is endowed with the weakest topology such that (3.8) is continuous for any $A \in \mathfrak{A}$. Note that thus $\mathrm{Char}(\mathfrak{A})$ becomes a Tikhonov space and a net $(\phi_\alpha)$ in $\mathrm{Char}(\mathfrak{A})$ converges to $\phi \in \mathrm{Char}(\mathfrak{A})$ iff for any $A \in \mathfrak{A}$, $\phi_\alpha(A) \to \phi(A)$.

**Theorem 3.1**
$$\mathfrak{A} \ni A \mapsto \hat{A} \in C(\mathrm{Char}(\mathfrak{A})) \tag{3.9}$$
*is a homomorphism. Moreover, the range of (3.9) separates points and does not vanish on every element of* $\mathrm{Char}(\mathfrak{A})$.

**Proof.** Let $A, B \in \mathfrak{A}$, $\phi \in \mathrm{Char}(\mathfrak{A})$. Then

$$\hat{A}(\phi)\hat{B}(\phi) = \phi(A)\phi(B) = \phi(AB) = \widehat{AB}(\phi).$$

If $\phi, \psi \in \mathrm{Char}(\mathfrak{A})$. If $\phi \neq \psi$, then there exists $A \in \mathfrak{A}$ such that $\phi(A) \neq \psi(A)$. Hence the range of (3.9) separates points. □

$\mathfrak{I}$ is a maximal ideal if it is a proper ideal such that if $\mathfrak{K}$ is a proper ideal containing $\mathfrak{I}$, then $\mathfrak{I} = \mathfrak{K}$.

Let $\mathrm{I}(\mathfrak{A})$, $\mathrm{MI}(\mathfrak{A})$ and $\mathrm{MI}_1(\mathfrak{A})$ denote the set of ideals, maximal ideals and ideals of codimesion 1 in $\mathfrak{A}$. Clearly,
$$\mathrm{MI}_1(\mathfrak{A}) \subset \mathrm{MI}(\mathfrak{A}) \subset \mathrm{I}(\mathfrak{A}).$$

**Theorem 3.2** (1) *If* $\phi \in \mathrm{Char}(\mathfrak{A})$, *then* $\mathrm{Ker}\phi$ *is an ideal of codimension 1.*

*In what follows we assume that* $\mathfrak{A}$ *is unital.*

(2) *Let* $\phi \in \mathrm{Char}(\mathfrak{A})$. *Then* $\phi(1) = 1$.

(3) *If* $\mathfrak{I}$ *is an ideal of codimension* 1, *then there exists a unique character* $\phi$ *such that* $\mathfrak{I} = \mathrm{Ker}\phi$.

**Proof.** (1) $\mathrm{Ker}\phi$ is an ideal, because $\phi$ is a homomorphism. It is of codimension 1 because $\phi$ is a nonzero linear functional onto $\mathbb{C}$. (3) If $A \in \mathfrak{I}$ and $\lambda \in \mathbb{C}$ we set $\phi(A + \lambda) := \lambda$. □

**Theorem 3.3** (1) *For any* $\phi \in \mathrm{Char}(\mathfrak{A})$ *there exists a unique extension of* $\phi$ *to a character* $\phi_{\mathrm{un}}$ *on* $\mathfrak{A}_{\mathrm{un}}$. *It is given by* $\phi_{\mathrm{un}}(\lambda + A) = \lambda + \phi(A)$.

(2) *There exists a unique* $\phi_\infty \in \mathrm{Char}(\mathfrak{A}_{\mathrm{un}})$ *such that* $\mathrm{Ker}\phi_\infty = \mathfrak{A}$.

(3) *The map*
$$\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \phi_{\mathrm{un}} \in \mathrm{Char}(\mathfrak{A}_{\mathrm{un}})\backslash\{\phi_\infty\}$$
*is a homeomorphism.*

**Theorem 3.4** *If* $\mathfrak{A}$ *is unital and* $\mathfrak{I} \subset \mathfrak{A}$ *is a proper ideal, then there exists a maximal ideal containing* $\mathfrak{I}$.

**Proof.** We use the Kuratowski-Zorn lemma. □

**Theorem 3.5** *Let* $A \in \mathfrak{A}$. *Then*

(1) $\mathrm{sp}_{\mathfrak{A}}(A) \supset \{\phi(A) \ : \ \phi \in \mathrm{Char}(\mathfrak{A})\}$.

(2) $\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \phi(A) \in \mathrm{sp}_{\mathfrak{A}}(A)$ *is a continuous map.*

**Proof.** If $\mathfrak{A}$ is non-unital, then we adjoin the identity and extend all the characters to $\mathfrak{A}_{\mathrm{un}}$.

Let $\phi \in \mathrm{Char}(\mathfrak{A})$ and $\phi(A) = \lambda$. Then $\phi(A - \lambda) = 0$. Hence $A - \lambda$ belongs to a proper ideal. Hence it is not invertible. Hence $\lambda \in \mathrm{sp}(A)$. □

**Theorem 3.6** *Let* $\mathfrak{A}$ *be a commutative unital algebra. Let* $A \in \mathfrak{A}$ *be non-invertible. Then*

(1) $\mathfrak{I} := \{AB \ : \ B \in \mathfrak{A}\}$ *is a proper ideal;*

(2) *There exists a maximal ideal containing* $A$*;*

(3) *There exists* $\phi \in \mathrm{Char}(\mathfrak{A})$ *with* $\phi(A) = 0$.

**Proof.** Clearly, $\mathfrak{I}$ is an ideal such that $1 \notin \mathfrak{I}$. This shows (1). (2) follows from Theorem 3.4. □

**Theorem 3.7** (1) *Let* $\pi : \mathfrak{A} \to \mathfrak{B}$ *be a homomorphism. Then*

$$\mathrm{Char}(\mathfrak{B}) \ni \phi \mapsto \pi^{\#}(\phi) \in \mathrm{Char}(\mathfrak{A}), \tag{3.10}$$

*defined for* $\psi \in \mathrm{Char}(\mathfrak{B})$ *by* $(\pi^{\#}\psi)(A) := \psi(\pi(A))$, *is continuous.*

(2) *If* $\mathfrak{I}$ *is an ideal in* $\mathfrak{B}$, *then* $\pi^{-1}(\mathfrak{I})$ *is an ideal in* $\mathfrak{A}$ *containing* $\mathrm{Ker}\pi$. *Thus we obtain a map*

$$\mathrm{I}(\mathfrak{B}) \ni \mathfrak{I} \mapsto \pi^{-1}(\mathfrak{I}) \in \mathrm{I}(\mathfrak{A}) \tag{3.11}$$

(3) *(3.11) maps* $\mathrm{MI}(\mathfrak{B})$ *into* $\mathrm{MI}(\mathfrak{A})$.

(4) *(3.11) maps* $\mathrm{MI}_1(\mathfrak{B})$ *into* $\mathrm{MI}_1(\mathfrak{A})$.

(5) *If* $\pi$ *is surjective, then (3.11) maps* $\mathrm{I}(\mathfrak{B})$ *bijectively onto* $\{\mathfrak{I} \in \mathrm{I}(\mathfrak{A}) \ : \ \mathrm{Ker}\pi \subset \mathfrak{I}\}$.

**Proof.** (1) Let $(\psi_i)$ be a net in $\mathrm{Char}(\mathfrak{B})$ converging to $\psi \in \mathrm{Char}(\mathfrak{B})$. Let $A \in \mathfrak{A}$. Then

$$\pi^{\#}(\psi_i)(A) = \psi_i(\pi(A)) \to \psi(\pi(A)) = \pi^{\#}(\psi)(A).$$

Hence $\pi^{\#}(\psi_i) \to \pi^{\#}(\psi)$. □

We say that an algebra is simple if it has no nontrivial ideals.

**Theorem 3.8** *Let* $\mathfrak{A}$ *be an algebra with a maximal ideal* $\mathfrak{I}$. *Then* $\mathfrak{A}/\mathfrak{I}$ *is simple.*

## 3.2 Characters and maximal ideals in a Banach algebra

**Theorem 3.9** *Let* $\mathfrak{A}$ *be a unital Banach algebra.*

(1) *Let* $\mathfrak{I}$ *be a maximal ideal in* $\mathfrak{A}$. *Then* $\mathfrak{I}$ *is closed.*

(2) *Let* $\phi$ *be a character on* $\mathfrak{A}$. *Then it is continuous and* $\|\phi\| = 1$.

(3) $\mathrm{Char}(\mathfrak{A})$ *is a compact Hausdorff space.*

(4) *The Gelfand transform*

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C(\mathrm{Char}(A))$$

*is a norm decreasing unital homomorphism of Banach algebras.*

**Proof.** (1) Invertible elements do not belong to any proper ideal. But a neighborhood of 1 consists of invertible elements. Hence the closure of any proper ideal does not contain 1.

By the continuity of operations, the closure of an ideal is an ideal. Hence if $\mathfrak{I}$ is any proper ideal, then $\mathfrak{I}^{\mathrm{cl}}$ is also a proper ideal.

(2) $\mathrm{Ker}\phi$ is a maximal ideal. Hence it is closed. Hence $\phi$ is continuous.

Suppose that $\|\phi\| > 1$. Then for some $A \in \mathfrak{A}$, $\|A\| < 1$ we have $|\phi(A)| > 1$. Now $A^n \to 0$ and $|\phi(A^n)| = |\phi(A)|^n \to \infty$, which means that $\phi$ is not continuous.

(3) and (4) follow easily from (2). □

**Theorem 3.10** *Let* $\mathfrak{A}$ *be a Banach algebra.*

(1) *Let $\phi$ be a character on $\mathfrak{A}$. Then it is continuous and $\|\phi\| \leq 1$.*

(2) $\mathrm{Char}(\mathfrak{A})$ *is a locally compact Hausdorff space.*

(3) *The Gelfand transform*

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C_\infty(\mathrm{Char}(A))$$

*is a norm decreasing homomorphism of Banach algebras.*

**Theorem 3.11 (Gelfand-Mazur)** *Let $\mathfrak{A}$ be a unital Banach algebra such that all non-zero elements are invertible. Then $\mathfrak{A} = \mathbb{C}$.*

**Proof.** Let $A \in \mathfrak{A}$. We know that $\mathrm{sp}(A) \neq \emptyset$. Hence, there exists $\lambda \in \mathrm{sp}(A)$. Thus $\lambda - A$ is not invertible. Hence $\lambda - A = 0$. Hence $A = \lambda$. □

## 3.3 Gelfand theory for commutative Banach algebras

**Theorem 3.12** *Let $\mathfrak{A}$ be a commutative unital Banach algebra. Every maximal ideal in $\mathfrak{A}$ has codimension 1. Hence the map*

$$\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \mathrm{Ker}\phi \in \mathrm{MI}(\mathfrak{A})$$

*is a bijection.*

**Proof.** Let $\phi$ be a character. Then we know that $\ker \phi$ has codimension 1 and hence is a maximal ideal by Theorem 3.4.

Conversely, let $\mathfrak{I}$ be a maximal ideal. If it has a codimension 1, then it is the kernel of a character by Theorem 3.2. Thus it is sufficient to show that every maximal ideal has the codimension 1.

Let $\mathfrak{I}$ be a ideal of $\mathfrak{A}$. Then $\mathfrak{A}/\mathfrak{I}$ is a commutative Banach algebra and $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$ is a homomorphism. Assume that the codimension of $\mathfrak{I}$ is not 1. This means that $\mathfrak{A}/\mathfrak{I}$ is not $\mathbb{C}$. By the Gelfand-Mazur theorem, $\mathfrak{A}/\mathfrak{I}$ contains non-invertible elements. Every such an element is contained in a proper ideal $\mathfrak{K}$. By theorem 3.7, $\pi^{-1}(\mathfrak{K})$ is a proper ideal of $\mathfrak{A}$ containing $\mathfrak{I}$. Hence $\mathfrak{I}$ is not maximal. □

**Theorem 3.13** *Let $\mathfrak{A}$ be a commutative unital Banach algebra. For each $A \in \mathfrak{A}$,*

$$\mathrm{sp}(A) = \{\phi(A) \ : \ \phi \in \mathrm{Char}(\mathfrak{A})\}.$$

*Hence*

$$\mathrm{sr}(A) = \sup\{|\hat{A}(\phi)| \ : \ \phi \in \mathrm{Char}(\mathfrak{A})\} = \|\hat{A}\|.$$

**Proof.** The inclusion $\supset$ was proven in Theorem 3.5.

Let $z \in \mathrm{sp}(A)$. Then $(z - A)$ is not invertible. Hence there exists a maximal ideal containing $z - A$. Therefore, exists $\phi \in \mathrm{Char}(\mathfrak{A})$ such that $\phi(z - A) = 0$. Hence $z = \phi(A) = \hat{A}(\phi)$. □

**Theorem 3.14** *Let $\mathfrak{A}$ be a commutative unital Banach algebra. Let $A \in \mathfrak{A}$. The following conditions are equivalent:*

(1) *A belongs to the intersection of all maximal ideals;*

(2) *For all $\phi \in \mathrm{Char}(\mathfrak{A})$ we have $\phi(A) = 0$*

(3) $\hat{A} = 0$*;*

(4) $\mathrm{sr}(A) = 0$*;*

(5) $\limsup \|A^n\|^{1/n} = 0$.

The set of $A \in \mathfrak{A}$ satisfying the conditions of Theorem 3.14 is called the radical of $\mathfrak{A}$. It is a closed ideal of $\mathfrak{A}$.

**Theorem 3.15** *Let $\mathfrak{A}$ be a unital Banach algebra. Let $A \in \mathfrak{A}$. Set $\mathfrak{C} := \widetilde{\mathrm{Ban}_{\mathfrak{A}}}(A)$, (which is a unital commutative Banach algebra). Recall that $\mathrm{sp}_{\mathfrak{C}}(A) = \mathrm{sp}_{\mathfrak{A}}(A)$. Recall also that we have the homomorphisms*

$$\mathrm{Hol}(\mathrm{sp}_{\mathfrak{A}}(A)) \ni f \mapsto f(A) \in \mathfrak{C}, \tag{3.12}$$

$$\mathfrak{C} \ni B \mapsto \hat{B} \in C(\mathrm{sp}(A)), \tag{3.13}$$

*where $\hat{C}$ is the Gelfand transform of $C$ with respect to the algebra $\mathfrak{C}$. Then the following holds:*

(1) *For $f \in \mathrm{Hol}(\mathrm{sp}_{\mathfrak{A}}(A))$, $\phi \in \mathrm{Char}(\mathfrak{C})$,*

$$\widehat{f(A)}(\phi) = \phi(f(A)) = f(\phi(A)).$$

*In other words, if we apply (3.12) and then (3.13), we obtain the identity.*

(2) *The following map is a homeomorphism:*

$$\mathrm{Char}(\mathfrak{C}) \ni \phi \mapsto \phi(A) \in \mathrm{sp}_{\mathfrak{C}}(A) = \mathrm{sp}_{\mathfrak{A}}(A). \tag{3.14}$$

*Thus $\mathrm{sp}_{\mathfrak{A}}(A)$ can be identified with $\mathrm{Char}(\mathfrak{C})$.*

**Proof.** (1) Let $\phi \in \mathrm{Char}(\mathfrak{C})$ and $z \in \mathbb{C}$. Then $\phi(z-A) = z-\phi(A)$. Hence, $\phi\left((z-A)^{-1}\right) = (z-\phi(A))^{-1}$, for $z \in \mathrm{rs}A$. But the span of $(z-A)^{-1}$ is dense in $\mathfrak{C}$ and $\phi$ is continuous.

(2) If $\phi_1, \phi_2 \in \mathrm{Char}(\mathfrak{C})$ and $\phi_1(A) = \phi_2(A)$, then, by (1), $\phi_1 = \phi_2$ on the range of (3.12), which is dense in $\mathfrak{C}$. Hence $\phi_1 = \phi_2$ on $\mathfrak{C}$. Therefore, (3.14) is injective. We already know that it is continuous and surjective. A continuous bijection on a compact Hausdorff space is always a homeomorphism. □

Recall that if $X$ is a compact Hausdorff space, $\mathfrak{A}$ is Banach algebra and $\gamma : C(X) \to \mathfrak{A}$ a homomorphism, we defined the spectrum of $\gamma$ as

$$\mathrm{sp}\gamma := \bigcap_{F \in \mathrm{Ker}\gamma} F^{-1}(0).$$

The following theorem gives the relationship between the above definition and the Gelfand theory.

**Theorem 3.16** *Let $X$, $\mathfrak{A}$ and $\gamma$ be as above. Identify $\mathrm{Char}(C(X))$ with $X$. Let $\tilde{\gamma}$ be $\gamma$, where we restrict the target to $\gamma(C(X))$. Let $\tilde{\gamma}^{\#}$ be defined as in (3.10). Then*

$$\mathrm{sp}\gamma = \tilde{\gamma}^{\#}(\mathrm{Char}(\gamma(C(X))))$$

## 3.4 Gelfand theory for commutative $C^*$-algebras

**Theorem 3.17** *Let $\mathfrak{A}$ be a $C^*$-algebra and $\phi$ a character on $\mathfrak{A}$. Then $\phi$ is a $*$-homomorphism and $\|\phi\| = 1$.*

**Proof.** Adjoin the unit if needed. Let $A = A^*$. Let $\tilde{\phi} := \phi\Big|_{C^*(1,A)}$. Then $\tilde{\phi}$ is a character on the commutative $C^*$-algebra $C^*(1, A)$. Hence $\tilde{\phi}(A) \in \mathrm{sp}A \subset \mathbb{R}$. Thus $\phi(A) \in \mathbb{R}$.

Let $A \in \mathfrak{A}$ be arbitrary. Then $\mathrm{Re}A := \frac{1}{2}(A+A^*)$ and $\mathrm{Im}A := \frac{1}{2\mathrm{i}}(A^*_A)$ are self-adjoint. Hence, $\phi(\mathrm{Re}A)$, $\phi(\mathrm{Im}A) \in \mathbb{R}$. By linearity, this implies

$$\phi(A^*) = \overline{\phi(A)}. \tag{3.15}$$

□

**Theorem 3.18** *Let $\mathfrak{A}$ be a unital commutative $C^*$-algebra. Then the Gelfand transform*

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C(\mathrm{Char}(\mathfrak{A}))$$

*is a $*$-isomorphism.*

**Proof. Step 1** We already know that it is a norm-decreasing homomorphism by Theorem 3.9.

**Step 2** Using (3.15) we see that the Gelfand transform is a $*$-homomorphism.

**Step 3** Every $A \in \mathfrak{A}$ is normal. Hence $\|A\| = \mathrm{sr}(A)$ by Theorem 1.5. But we know that $\|\hat{A}\| = \mathrm{sr}(A)$. This show that the Gelfand transform is isometric.

**Step 4** We know that the image of the Gelfand transform is dense in $C(\mathrm{Char}(\mathfrak{A}))$ and $\mathfrak{A}$ is complete. We proved also that it is isometric. Hence it is bijective. □

**Theorem 3.19** (1) *$U \in \mathfrak{A}$ is unitary $\Leftrightarrow$ $U$ is normal and $\mathrm{sp}(U) \subset \{z \ : \ |z| = 1\}$.*

(2) *$A \in \mathfrak{A}$ is self-adjoint $\Leftrightarrow$ $A$ is normal and $\mathrm{sp}(A) \subset \mathbb{R}$.*

**Proof.** $\Rightarrow$ was proven before.

(1)$\Leftarrow$. Consider the algebra $\mathfrak{C} := C^*(1, U)$. By the normality of $U$, it is commutative. Let $\phi \in \mathrm{Char}(\mathfrak{C})$. Then $\phi(U^*)\phi(U) = \overline{\phi(U)}\phi(U) = 1$. Hence $\mathrm{sp}(U) \subset \{|z| = 1\}$. Hence $U^*U = 1$.

(2)$\Leftarrow$. Consider the algebra $\mathfrak{C} := C^*(1, A)$. By the normality of $A$, it is commutative. Let $\phi \in \mathrm{Char}(\mathfrak{C})$, Then $\phi(A) \in \mathrm{sp}(A) \subset \mathbb{R}$. Hence $\phi(A^*) = \phi(A)$. Hence $A^* = A$. □

**Theorem 3.20** *Let $\mathfrak{A}$ be a commutative $C^*$-algebra. Then the Gelfand transform*

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C_\infty(\mathrm{Char}(\mathfrak{A}))$$

*is a $*$-isomorphism.*

## 3.5 Functional calculus for normal operators

**Theorem 3.21** *Let $\mathfrak{A}$ be a unital $C^*$-algebra. Let $A \in \mathfrak{A}$ be normal. Then there exists a unique continuous unital $*$-isomorphism*

$$C(\mathrm{sp}(A)) \ni f \mapsto f(A) \in C^*(1, A) \subset \mathfrak{A},$$

*such that*

(1) $\mathrm{id}(A) = A$ *if* $\mathrm{id}(z) = z$.

*Moreover, we have*

(2) *If $f \in \mathrm{Hol}(\mathrm{sp}(A))$, then $f(A)$ coincides with $f(A)$ defined by the holomorphic functional calculus.*

(3) $\mathrm{sp}(f(A)) = f(\mathrm{sp}(A))$.

(4) $g \in C(f(\mathrm{sp}(A))) \Rightarrow g \circ f(A) = g(f(A)$.

(5) $\|f(A)\| = \sup |f|$.

**Proof.** If $f$ is a polynomial, that is $f(z) = \sum a_{nm} z^n \overline{z}^m$, we set

$$f(A) := \sum a_{nm} A^n A^{*m}.$$

$C^*(1, A)$ is a commutative algebra. Let $\phi$ be a character on $C^*(1, A)$. Then we easily check that $\phi(f(A)) = f(\phi(A))$. Hence $\mathrm{sp}(f(A)) = f(\mathrm{sp}(A))$.

Clearly, $f(A)$ is normal. Hence

$$\|f(A)\| = \mathrm{sr}(f(A)) = \sup |f|.$$

Therefore, on polynomials the map $f \to f(A)$ is isometric. Since polynomials are dense in a complete metric space $C(\mathrm{sp}(A))$ and polynomials in $A$, $A^*$ are dense in a complete metric space $C^*(1, A)$, there is exactly one continuous extension of this map to the whole $C(\mathrm{sp}(A))$, which is an isometric bijection of $C(\mathrm{sp}(A))$ to $C^*(1, A)$.

Clearly, on polynomials, the map $f \mapsto f(A)$ is a $*$-homomorphism. Since the multiplication, and involution are continuous both in $C(\mathrm{sp}(A))$ and $C^*(1, A)$, this map is a homomorphism on $C(\mathrm{sp}(A))$. $\square$

If $\mathfrak{A}$ is not unital, either we can adjoin the identity and consider the algebra $\mathfrak{A}_{\mathrm{un}}$, or we can use the following version of the above theorem:

**Theorem 3.22** *Let $\mathfrak{A}$ be a $C^*$-algebra. Let $A \in \mathfrak{A}$ be normal. Then there exists a unique continuous $*$-isomorphism*

$$C_\infty(\mathrm{sp}(A)\backslash\{0\}) \ni f \mapsto f(A) \in C^*(A) \subset \mathfrak{A},$$

*such that* $\mathrm{id}(A) = A$ *if* $\mathrm{id}(z) = z$.

# 4 Positivity in $C^*$-algebras

## 4.1 Positive elements

Let $A \in \mathfrak{A}$. We say that $A$ is positive iff $A$ is self-adjoint and $\mathrm{sp}(A) \subset [0, \infty[$. $\mathfrak{A}_+$ will denote the set of positive elements in $\mathfrak{A}$. We will write $A \geq B$ iff $A - B \in \mathfrak{A}_+$. We will write $A > B$ iff $A \geq B$ and $A \neq B$.

**Lemma 4.1** *Let $A$ be self-adjoint. Then $\|\lambda - A\| \leq \lambda$ iff $A \geq 0$ and $\|A\| \leq 2\lambda$.*

**Theorem 4.2** (1) *$A \in \mathfrak{A}_+$ and $\lambda \geq 0$ implies $\lambda A \in \mathfrak{A}_+$.*

(2) *$A, B \in \mathfrak{A}_+$ implies $A + B \in \mathfrak{A}_+$.*

(3) *$A, -A \in \mathfrak{A}_+$ implies $A = 0$.*

(4) *$\mathfrak{A}_+$ is closed.*

*In other words, $\mathfrak{A}_+$ is a closed pointed cone.*

**Proof.** (2)

$$\big\| \|A\| + \|B\| - A - B \big\| \leq \big\| \|A\| - A \big\| + \big\| \|B\| - B \big\| \leq \|A\| + \|B\|.$$

Hence $A + B \geq 0$.

(3) $\mathrm{sp}(A), \mathrm{sp}(-A) \subset [0, \infty[$ implies $\mathrm{sp}(A) = \{0\}$. But $A$ is self-adjoint. Hence $A = 0$.

(4) Let $A_n \to A$. Then $\|A_n\| \to \|A\|$. $A_n \in \mathfrak{A}_+$ iff $\|A_n - \|A_n\|\| \leq \|A_n\|$. By taking the limit, $\|A - \|A\|\| \leq \|A\|$. Hence $A \in \mathfrak{A}_+$. $\square$

**Theorem 4.3** *Let $A \in \mathfrak{A}_+$ and $n \in \mathbb{N}\backslash\{0\}$. Then there exists a unique $B \in \mathfrak{A}_+$ such that $B^n = A$.*

**Proof.** $[0, \infty[ \ni \lambda \mapsto \lambda^{1/n}$ is a continuous function. Hence $B := A^{1/n}$ is well defined. Clearly, $B$ satisfies the requirements of the theorem.

Let $B \in \mathfrak{A}_+$, $B^n = A$. Clearly,

$$BA = B^{n+1} = AB. \tag{4.16}$$

Let $\mathfrak{C} := C^*(1, B, A)$. By (4.16), $\mathfrak{C}$ is commutative. If $\phi \in \mathrm{Char}(\mathfrak{C})$, then $\phi(A) = \phi(B^n) = \phi(B)^n$. Moreover, $\phi(B) > 0$. Hence $\phi(B) = \phi(A)^{1/n}$. Hence $B$ is uniquely determined, and equals $A^{1/n}$. $\square$

**Theorem 4.4 (Jordan decomposition of a self-adjoint operator.)** *Let $A \in \mathfrak{A}$ be self-adjoint. Then there exist unique $A_+, A_- \in \mathfrak{A}_+$ such that $A_+A_- = A_-A_+ = 0$ and $A = A_+ - A_-$.*

**Proof.** The functions $|x|_+ := \max(x, 0)$ and $|x|_- := \max(-x, 0)$ are continuous. Hence $A_+$ and $A_-$ can be defined as $|A|_+$ and $|A|_-$ by the functional calculus.

Assume that $A_-$ and $A_+$ satisfy the conditions of the theorem. Then

$$A^2 = A_-^2 + A_+^2 = (A_+ + A_-)^2.$$

By the uniqueness of the positive square root, $|A| = A_+ + A_-$. Hence $A_+ = \frac{1}{2}(|A| + A)$ and $A_- = \frac{1}{2}(|A| - A)$. □

**Theorem 4.5** *Let $A \in \mathfrak{A}$. The following conditions are equivalent*

(1) $A \geq 0$.

(2) *There exists $B \in \mathfrak{A}$ such that $A = B^*B$.*

**Proof.** (1) ⇒ (2) is contained in Theorem 4.3.

Let us prove (1) ⇐ (2). Clearly, $B^*B$ is self-adjoint. Let $B^*B = A_+ - A_-$ be its Jordan decomposition. Clearly

$$(BA_-)^*(BA_-) = A_-(A_+ - A_-)A_- = -A_-^3 \in -\mathfrak{A}_+.$$

Let $BA_- = S + \mathrm{i}T$. Then

$$\begin{aligned}(BA_-)(BA_-)^* &= S^2 + T^2 + \mathrm{i}(TS - ST)\\ &= -(BA_-)^*(BA_-) + 2(S^2 + T^2) \in \mathfrak{A}_+,\end{aligned}$$

using the fact that $\mathfrak{A}_+$ is a convex cone.

But

$$\mathrm{sp}\big((BA_-)^*(BA_-)\big) \cup \{0\} = \mathrm{sp}\big((BA_-)(BA_-)^*\big) \cup \{0\}.$$

Hence $\mathrm{sp}\big((BA_-)^*(BA_-)\big) = \{0\}$. Consequently, $(BA_-)^*(BA_-) = 0$. Consequently, $A_-^3 = 0$. By the uniqueness of the positive third root, $A_- = 0$. □

**Theorem 4.6** (1) *Let $A$ be self-adjoint, then $-\|A\| \leq A \leq \|A\|$.*

*In what follows, let $0 \leq B \leq A$. Then*

(2) $\|B\| \leq \|A\|$,

(3) *If $D^*D \leq 1$, then $DD^* \leq 1$.*

(4) $0 \leq C^*BC \leq C^*AC$.

(5) $0 \leq (\lambda + A)^{-1} \leq (\lambda + B)^{-1}, \quad 0 < \lambda$.

(6) $B(\lambda + B)^{-1} \leq A(\lambda + A)^{-1}$.

(7) $0 \leq B^\theta \leq A^\theta, \quad 0 \leq \theta \leq 1$,

**Proof.** (1) $\mathrm{sp}(A) \subset [-\|A\|, \|A\|]$. Hence $\|A\| - A \geq 0$ and $\|A\| + A \geq 0$.

(2) By (1), $A \leq \|A\|$. Hence, $B \leq \|A\|$. Hence $\mathrm{sp}(B) \subset [0, \|A\|]$. Therefore, $\|B\| \leq \|A\|$.

(3) Clearly, $\|D^*D\| \leq 1$. Hence $\|DD^*\| \leq 1$. Hence, by (1), $DD^* \leq 1$.

(4) $C^*(A - B)C = \big((A - B)^{\frac{1}{2}}C\big)^*(A - B)^{\frac{1}{2}}C \geq 0$.

(5) Clearly, $\lambda + A \geq \lambda + B \geq \lambda$. Hence $\lambda + A$ and $\lambda + B$ are positive invertible. By (4), applied with $C = (\lambda + A)^{-\frac{1}{2}}$, for $D := (\lambda + B)^{\frac{1}{2}}(\lambda + A)^{-\frac{1}{2}}$ we have $1 \geq D^*D$. Hence $1 \geq DD^*$.

(6) follows immediately from (5).

(7). We use (6) and

$$A^{\theta} = c_{\theta} \int_0^{\infty} \lambda^{\theta-1} A(\lambda + A)^{-1} \mathrm{d}\lambda.$$

□

## 4.2 Left and right ideals

**Theorem 4.7** *Let $\mathfrak{I}$ be a right ideal and $B_i \in \mathfrak{I} \cap \mathfrak{A}_+$, $\|B_i\| < 1$, $i = 1, 2$. Then there exists $B \in \mathfrak{I} \cap \mathfrak{A}_+$, such that $\|B\| < 1$ and $B_i \leq B$, $i = 1, 2$.*

**Proof.** Set $A_i := B_i(1 - B_i)^{-1}$. Note that $A_i \in \mathfrak{I} \cap \mathfrak{A}_+$ and

$$B_i = A_i(1 + A_i)^{-1}, \tag{4.17}$$

We set

$$B := (A_1 + A_2)(1 + A_1 + A_2)^{-1}. \tag{4.18}$$

Clearly $B \in \mathfrak{A}_+$, $\|B\| < 1$ Clearly, $A_i \leq A_1 + A_2$, $i = 1, 2$. Hence, by (4.17), (4.18) and Theorem 4.6, we get $B_i \leq B$. □

Let $\mathfrak{I}$ be a right ideal of $\mathfrak{A}$. Then a positive left approximate unit of $\mathfrak{I}$ is defined to be a net $\{E_\alpha\}$ of elements of $\mathfrak{I}$ such that

(1) $0 \leq E_\alpha \leq 1$,

(2) $\alpha \leq \beta$ implies $E_\alpha \leq E_\beta$,

(3) $\lim_\alpha \|E_\alpha A - A\| = 0$ for all $A \in \mathfrak{I}$.

The following theorem implies that every ideal possesses a canonical positive approximate unit.

**Theorem 4.8** *Let $\mathfrak{I}$ be a right ideal of $\mathfrak{A}$. Then*

$$\mathfrak{E} := \{A \in \mathfrak{I}_+ \ : \ \|A\| < 1\} \tag{4.19}$$

*ordered by $\leq$ is a positive approximate unit in $\mathfrak{I}$.*

**Proof.** By Theorem 4.7, $\mathfrak{E}$ is a directed set.

Let $A \in \mathfrak{I}$. Then, for any $\lambda > 0$, set $E_\lambda := AA^*(\lambda^{-1} + AA^*)^{-1}$.

Let $E \in \mathfrak{E}$, $E_\lambda \leq E$. Then

$$\begin{aligned} \|(1 - E)A\|^2 = \|A^*(1 - E)^2 A\| &\leq \|A^*(1 - E)A\| \leq \|A^*(1 - E_\lambda)A\| \\ &= \|A^*(1 + \lambda AA^*)^{-1}A\| = \|A^*A(1 + \lambda AA^*)^{-1}\| \leq \lambda^{-1}. \end{aligned}$$

□

**Theorem 4.9** *If $\mathfrak{K}$ is a closed left ideal in a $C^*$-algebra $\mathfrak{A}$, and $S \in \mathfrak{K}$, then there exists $A \in \mathfrak{A}$ and $K \in \mathfrak{K}_+$ such that $S = AK$.*

**Proof.** Set $K := (S^*S)^{1/4}$, $A_n := S(n^{-1} + K^2)^{-1/2}$. Then we easily show that $\|A_m - A_n\| \leq \sup\limits_{t \in \mathrm{sp} K^2} |\sqrt{m^{-1} + t} - \sqrt{n^{-1} + t}|$. Thus $A_n$ is a Cauchy sequence. We set $A := \lim_{n\to\infty} A_n$. □

**Corollary 4.10** *Every closed ideal is self-adjoint.*

## 4.3 Quotient algebras

**Theorem 4.11** *Let $\mathfrak{I}$ be a closed ideal of a $C^*$-algebra $\mathfrak{A}$. Then $\mathfrak{A}/\mathfrak{I}$ is a $C^*$-algebra and we have a short exact sequence of $*$-homomorphisms:*

$$0 \to \mathfrak{I} \to \mathfrak{A} \to \mathfrak{A}/\mathfrak{I} \to 0.$$

*If $(E_\alpha)$ is a positive approximate unit, then the norm in $\mathfrak{A}/\mathfrak{I}$ is given by*

$$\|A + \mathfrak{I}\| = \lim_\alpha \|A(1 - E_\alpha)\|. \tag{4.20}$$

**Proof.** The approximate unit $\mathfrak{E}$, defined in (4.19), satisfies the condition (2.7), so (4.20) holds. Now let $A \in \mathfrak{A}$ and $I \in \mathfrak{I}$.

$$\begin{aligned} \|A + \mathfrak{I}\|^2 &= \lim_\alpha \|A(1 - E_\alpha)\|^2 \\ &= \lim_\alpha \|(1 - E_\alpha)A^*A(1 - E_\alpha)\|^2 \\ &= \lim_\alpha \|(1 - E_\alpha)(A^*A + I)(1 - E_\alpha)\|^2 \\ &\leq \|A^*A + I\|. \end{aligned}$$

Hence

$$\|A + \mathfrak{I}\|^2 \leq \|(A + \mathfrak{I})^*(A + \mathfrak{I})\|.$$

Therefore, $\mathfrak{A}/\mathfrak{I}$ is a $C^*$-algebra. □

## 4.4 Homomorphisms of $C^*$-algebras

**Theorem 4.12** *Let $\mathfrak{A}, \mathfrak{B}$ be $C^*$-algebras and $\pi : \mathfrak{A} \to \mathfrak{B}$ a $*$-homomorphism. Then*

(1) $\|\pi(A)\| \leq \|A\|$;

(2) $\pi(\mathfrak{A})$ *is a $C^*$-algebra.*

(3) *The following conditions are equivalent*

  (i) $\mathrm{Ker}\pi = \{0\}$,

  (ii) $\|\pi(A)\| = \|A\|$.

**Proof.** First we would like to argue that it is sufficient to assume that $\pi$ preserves the identity. If $\mathfrak{A}$ has a unit, then $\pi(1) = P$ is a projection in $\mathfrak{B}$. We can replace $\mathfrak{B}$ with $P\mathfrak{B}P$, and then consider $\pi : \mathfrak{A} \to P\mathfrak{B}P$.

If $\mathfrak{A}$ does not have a unit, we simply adjoin the unit to $\mathfrak{A}$, if needed also to $\mathfrak{B}$, and consider the extended $*$-homomorphism $\pi_{\mathrm{un}} : \mathfrak{A}_{\mathrm{un}} \to \mathfrak{B}_{\mathrm{un}}$ such that $\pi_{\mathrm{un}}(1) = 1$.

**Proof of (1).** Clearly, if $A \in \mathfrak{A}$, then $\mathrm{sp}(\pi(A)) \subset \mathrm{sp}(A)$. If $A$ is self-adjoint, then

$$\|\pi(A)\| = \mathrm{sr}(\pi(A)) \leq \mathrm{sr}(A) = \|A\|.$$

For an arbitrary $B \in \mathfrak{A}$,

$$\|\pi(B)\|^2 = \|\pi(B)^*\pi(B)\| = \|\pi(B^*B)\| \leq \|B^*B\| = \|B\|^2.$$

**Proof of (3.i)⇒(3.ii). Step 1.** Let $\mathfrak{A}$ be commutative. Then so is $\pi(\mathfrak{A})$. We have $\mathfrak{A} \simeq C(Y)$ and $\pi(\mathfrak{A}) \simeq C(X)$ for some compact Hausdorff spaces $Y, X$. Besides, for some continuous map $p : X \to Y$, $\pi(f)(x) = f \circ p(x)$, for $f \in \mathfrak{A}$. We know that $\pi$ is injective iff $p$ is surjective. This means that $\|\pi(f)\| = \|f\|$

**Step 2.** Let $\mathfrak{A}$ be arbitrary and $A \in \mathfrak{A}$ self-adjoint. By considering the commutative $C^*$-algebra $C^*(1, A)$, Step 1 implies that $\|\pi(A)\| = \|A\|$.

**Step 3.** Let $B \in \mathfrak{A}$ be arbitrary. Then

$$\|\pi(B)\|^2 = \|\pi(B)^*\pi(B)\| = \pi(B^*B)\| = \|B^*B = \|B\|^2.$$

(3.i)$\Leftarrow$(3.ii) is obvious.

**Proof of (2).** Clearly, $\tilde{\pi} : \mathfrak{A}/\mathrm{Ker}\pi \to \pi(\mathfrak{A})$ is a $*$-isomorphism. By (3.ii) it is also isometric. Since $\mathfrak{A}/\ker\mathfrak{A}$ is a $C^*$-algebra by Theorem 4.11, so is $\pi(\mathfrak{A})$. □

## 4.5 Linear functionals

Let $\omega$ be a linear functional on $\mathfrak{A}$. The adjoint functional $\omega^*$ is defined by

$$\omega^*(A) := \overline{\omega(A^*)}.$$

We say that $\omega$ is self-adjoint iff $\omega^* = \omega$, or equivalently, if $\omega(A) \in \mathbb{R}$ for $A$ self-adjoint.

We say that $\omega$ is positive iff

$$\omega(A) \geq 0, \qquad A \in \mathfrak{A}_+.$$

The set of continuous functionals over $\mathfrak{A}$ will be denoted $\mathfrak{A}^\#$. The set of continuous positive functionals over $\mathfrak{A}$ will be denoted $\mathfrak{A}^\#_+$.

**Theorem 4.13** *If $\omega$ is a positive functional, then it is self-adjoint and*

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B). \tag{4.21}$$

**Proof.** If $A$ is self-adjoint, then we can decompose $A$ as $A = -A_- + A_+$ with $A_-, A_+$ positive. Now $\omega(A_\pm) \geq 0$. Hence $\omega(A) = \omega(A_+) - \omega(A_-) \in \mathbb{R}$.

To prove (4.21), we note that for any $\lambda \in \mathbb{C}$,

$$\omega\big((A + \lambda B)^*(A + \lambda B)\big) \geq 0.$$

□

**Theorem 4.14** *Let $\omega$ be a linear functional on a unital $C^*$-algebra $\mathfrak{A}$. The following conditions are equivalent:*

(1) *$\omega$ is positive*

(2) *$\omega$ is continuous and $\|\omega\| = \omega(1)$*

**Proof.** (1)$\Rightarrow$(2). **Step 1** Let $A \in \mathfrak{A}_+$. Then $A \leq \|A\|$. Hence $|\omega(A)| = \omega(A) \leq \|A\|\omega(1)$.

**Step 2** Let $B \in \mathfrak{A}$. Then, by (4.21), using the positivity of $B^*B$ and Step 1, we get

$$|\omega(B)|^2 \leq \omega(1)\omega(B^*B) \leq \omega(1)^2\|B^*B\| = \omega(1)^2\|B\|^2.$$

Hence $\|\omega\|^2 \leq \omega(1)^2$.

(1)$\Leftarrow$ (2). It is enough to assume that $\|\omega\| = 1$.

**Step 1** Let $A$ be self-adjoint. Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\omega(A) = \alpha + \mathrm{i}\beta$. It is enough to assume that $\omega(1) = \|\omega\| = 1$. Clearly,

$$\|\gamma - \mathrm{i}A\| = \sqrt{\gamma^2 + \|A\|^2}, \quad \omega(\gamma - \mathrm{i}A) = \gamma + \beta - \mathrm{i}\alpha.$$

But

$$|\omega(\gamma - \mathrm{i}A)|^2 \leq \|\gamma - \mathrm{i}A\|^2.$$

Hence

$$(\gamma + \beta)^2 + \alpha^2 \leq \gamma^2 + \|A\|^2.$$

For large $|\gamma|$, this is possible only if $\beta = 0$. Hence $\omega$ is self-adjoint.
**Step 2** Let $A \in \mathfrak{A}_+$. Then $\big|\|A\| - A\big| \leq \|A\|$. Therefore,

$$\big|\|A\|\omega(1) - \omega(A)\big| \leq \|A\|.$$

But $\omega(1) = 1$, and $\omega(A)$ is real. Hence $\omega(A) \geq 0$. $\square$

**Theorem 4.15** *Let $\omega$ be a linear functional on a non-unital $C^*$-algebra. The following conditions are equivalent:*

(1) *$\omega$ is positive*

(2) *$\omega$ is continuous and for some positive approximate identity $\{E_\alpha\}$ of $\mathfrak{A}$*

$$\|\omega\| = \lim_\alpha \omega(E_\alpha^2).$$

(3) *$\omega$ is continuous and if the functional $\omega_{\mathrm{un}} : \mathfrak{A}_{\mathrm{un}} \to \mathbb{C}$ is given by $\omega_{\mathrm{un}}(\lambda + A) := \lambda\|\omega\| + \omega(A)$, then $\omega_{\mathrm{un}}$ is a positive functional on $\mathfrak{A}_{\mathrm{un}}$*

*Moreover, $\omega_{\mathrm{un}}$ is the unique functional on $\mathfrak{A}_{\mathrm{un}}$ that extends $\omega$ and satisfies $\|\omega\| = \|\omega_{\mathrm{un}}\|$.*

**Proof.** (1)⇒(2). **Step 1.** We want to show that

$$c := \sup\{\omega(A) \ : \ 0 \leq A \leq 1\}$$

is finite. Suppose that it is not true, $0 \leq A_n \leq 1$ and $\omega(A_n) \to \infty$. Then we will find $\lambda_n \geq 0$ such that $\sum \lambda_n < \infty$ and $\sum \lambda_n \omega(A_n) = \infty$. But $A := \sum \lambda_n A_n$ is convergent and, for any $n$,

$$\sum_{j=1}^{n} \lambda_j \omega(A_j) \leq \omega(A) < \infty,$$

which is a contradiction.

**Step 2.** If $A \in \mathfrak{A}$, then $A = \sum_{j=0}^{3} \mathrm{i}^j A_j$ with $A_j \in \mathfrak{A}_+$ and $\|A_j\| \leq \|A\|$. Hence

$$|\omega(A)| \leq \sum_{j=0}^{3} \omega(A_j) \leq 4c\|A\|.$$

Hence $\omega$ is continuous.

**Step 3.** Let $E_\alpha$ be a positive approximate unit. $\omega(E_\alpha)$ is an increasing bounded net, so $c := \lim_\alpha \omega(E_\alpha)$ exists. Since $\|E_\alpha\| \leq 1$, we have $c \leq \|\omega\|$.

**Step 4** Let $A \in \mathfrak{A}$. Then

$$|\omega(E_\alpha A)|^2 \leq \omega(E_\alpha^2)\omega(A^*A) \leq \omega(E_\alpha^2)\|\omega\|\|A^*A\| \leq c\|\omega\|\|A\|^2.$$

Moreover, $E_\alpha A \to A$ and $\omega$ is continuous, hence the left hand side goes to $|\omega(A)|^2$. Hence $|\omega(A)|^2 \leq c\|\omega\|\|A\|^2$. Therefore, $\|\omega\| \leq c$.

(2)⇒(3). It is obvious that $\|\omega_{\mathrm{un}}\| \geq \|\omega\|$. Let us prove the converse inequality.

Let $E_\alpha$ be a positive approximative unit. We have

$$\omega_{\rm un}(\lambda + A) = \lim_\alpha \omega(\lambda E_\alpha + E_\alpha A).$$

Hence

$$\begin{aligned} |\omega_{\rm un}(\lambda + A)| &= \lim_\alpha |\omega(\lambda E_\alpha + E_\alpha A)| \leq \lim_\alpha \|\omega\| \|\lambda E_\alpha + E_\alpha A\| \\ &\leq \|\omega\| \limsup_\alpha \|E_\alpha\| \|\lambda + A\| = \|\omega\| \|\lambda + A\|. \end{aligned}$$

Hence, $\|\omega_{\rm un}\| \leq \|\omega\|$.

Thus we proved that $\|\omega\| = \|\omega_{\rm un}\|$. Therefore, $\omega_{\rm un}(1) = \|\omega_{\rm un}\|$. Therefore, $\omega$ is positive by the previous theorem.

(3)⇒(1) is obvious. □

A positive functional over $\mathfrak{A}$ satisfying $\|\omega\| = 1$ will be called a state. For a unital algebra it is equivalent to $\omega(1) = 1$. For a non-unital algebra it is equivalent to $1 = \sup\{\omega(A) \ : \ A \leq 1\}$. The set of states on a $C^*$-algebra $\mathfrak{A}$ will be denoted $\mathbb{E}(\mathfrak{A})$.

If $\omega$ is a positive functional on $\mathfrak{A}$, then

$$\omega_{\rm un}(A + \lambda) := \omega(A) + \lambda \|\omega\| \quad A \in \mathfrak{A}, \ \lambda \in \mathbb{C},$$

defines a state on $\mathfrak{A}_{\rm un}$ extending $\omega$ with $\|\omega\| = \|\omega_{\rm un}\|$.

If $\phi$ is a positive functional on $\mathfrak{A}_{\rm un}$, then

$$\phi(A + \lambda) = \theta\omega(A) + \lambda \|\phi\|, \quad A \in \mathfrak{A}, \ \lambda \in \mathbb{C},$$

where $0 \leq \theta \leq \|\phi\|$, and $\omega$ is a state on $\mathfrak{A}$.

## 4.6 The GNS representation

Let $(\mathcal{H}, \pi)$ be a $*$-representation of $\mathfrak{A}$, $\Omega \in \mathcal{H}$ and $\omega \in \mathfrak{A}_+^\#$. We say that $\Omega$ is a vector representative of $\omega$ iff

$$\omega(A) = (\Omega | \pi(A)\Omega).$$

We say that $\Omega$ is cyclic iff $\pi(\mathfrak{A})\Omega$ is dense in $\mathcal{H}$. $(\mathcal{H}, \pi, \Omega)$ is called a cyclic $*$-representation iff $(\pi, \mathcal{H})$ is a $*$-representation and $\Omega$ is a cyclic vector.

**Theorem 4.16** *Let $\omega$ be a state on $\mathfrak{A}$. Then there exists a cyclic $*$-representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ such that $\Omega_\omega$ is a vector representative of $\omega$. Such a representation is unique up to a unitary equivalence.*

**Proof.** We adjoin the unit if needed.

For $A, B \in \mathfrak{A}$, $\omega(A^*B)$ is a pre-Hilbert scalar product on $\mathfrak{A}$. Define $\mathfrak{N}_\omega := \{A \in \mathfrak{A} \ : \ \omega(A^*A) = 0\}$. Then $\mathfrak{N}$ is a closed left ideal. The scalar product on $\mathfrak{A}/\mathfrak{N}_\omega$ is well defined. Let $\mathcal{H}_\omega$ be the completion of $\mathfrak{A}/\mathfrak{N}_\omega$.

The left regular representation

$$\mathfrak{A} \ni A \mapsto \lambda(A) \in L(\mathfrak{A}), \quad \lambda(A)B := AB,$$

preserves $\mathfrak{N}_\omega$. Hence we can define the representation $\pi_\omega$ on $\mathfrak{A}/\mathfrak{N}_\omega$ by

$$\pi_\omega(A)(B + \mathfrak{N}_\omega) := AB + \mathfrak{N}_\omega.$$

We have

$$\begin{aligned} \|\pi_\omega(A)(B + \mathfrak{N}_\omega)\|^2 &= \omega(B^*A^*AB) \leq \|A^*A\| \omega(B^*B) \\ &= \|A\|^2 \|B + \mathfrak{N}_\omega\|^2. \end{aligned}$$

Hence $\|\pi_\omega(A)\| \leq \|A\|$ and $\pi_\omega$ extends to a linear map on $\mathcal{H}_\omega$.

We set $\Omega_\omega := 1 + \mathfrak{N}_\omega$. Clearly, $\pi_\omega(A)\Omega_\omega = A + \mathfrak{N}_\omega$, hence $\Omega_\omega$ is cyclic. □

## 4.7 Existence of states and representation

**Theorem 4.17** *Let $\mathfrak{A}$ be a $C^*$-algebra and $A \in \mathfrak{A}_{\rm h}$. Then there exists a state $\omega$ on $\mathfrak{A}$ such that $|\omega(A)| = \|A\|$.*

**Proof.** We adjoin the unit if needed.

Let $\mathfrak{A}_0 = C^*(1, A) \simeq C({\rm sp}A)$. Let $\omega_0$ be the character on $\mathfrak{A}_0$ with $|\omega_0(A)| = {\rm sr}A = \|A\|$. Then $\omega_0$ is a state on $\mathfrak{A}_0$. By the Hahn-Banach Theorem we extend $\omega_0$ to a functional $\omega$ on $\mathfrak{A}$ with $\|\omega\| = 1$. But $\omega(1) = \omega_0(1) = 1$, hence by Theorem 4.14, $\omega$ is a state. □

**Theorem 4.18** *There exists an injective representation $(\mathcal{H}, \pi)$ of $\mathfrak{A}$.*

**Proof.** For any $A \in \mathfrak{A}_{\rm h}$ there exists a state $\omega_A$ such that $\omega_A(A) \neq 0$. Let $(\pi_A, \mathcal{H}_A, \Omega_A)$ be the corresponding GNS representation. Then $(\Omega_A|\pi_A(A)\Omega_A) = \omega_A(A)$. Hence $\pi_A(A) \neq 0$. Set

$$\mathcal{H} := \bigoplus_{A\in\mathfrak{A}_{\rm h}} \mathcal{H}_A, \quad \pi := \bigoplus_{A\in\mathfrak{A}_{\rm h}} \pi_A.$$

Then $\pi$ is a representation of $\mathfrak{A}$ in $\mathcal{H}$ and for any $A \in \mathfrak{A}_{\rm h}$, $\pi(A) \neq 0$. Since self-adjoint elements span $\mathfrak{A}$, $\pi$ is injective. □

**Theorem 4.19** *Let $\mathfrak{A}_0$ be a $C^*$-subalgebra of a $C^*$-algebra $\mathfrak{A}$. Let $\omega_0$ be a state on $\mathfrak{A}_0$. Then there exists a state $\omega$ on $\mathfrak{A}$ extending $\omega_0$. If $\Omega_0$ is hereditary, then $\omega$ is unique.*

**Proof.** By the Hahn-Banach Theorem, there exists a linear functional $\omega$ on $\mathfrak{A}$ extending $\omega_0$ with $\|\omega_0\| = \|\omega\|$. But $\|\omega\| \geq \omega(1) \geq \|omega_0\|$. Hence $\omega\| = \omega(1)$. Therefore, $\omega$ is a state. □

## 4.8 Jordan decomposition of a form

Let $\omega \in \mathfrak{A}^\#$. Then ${\rm Re}\omega := \frac{1}{2}(\omega + \omega^*)$, ${\rm Im}\omega := \frac{1}{2{\rm i}}(\omega - \omega^*)$ are self-adjoint. Moreover, $\omega = {\rm Re}\omega + {\rm i}{\rm Im}\omega$.

**Theorem 4.20** *Let $\mathfrak{A}$ be a $C^*$-algebra and $\phi, \psi \in \mathfrak{A}_+^\#$. Then the following conditons are equivalent:*

(1) $\|\phi - \psi\| = \|\phi\| + \|\psi\|$,

(2) *For every $\epsilon > 0$ there is a $A \in \mathfrak{A}_+$ with $\|A\| \leq 1$ such that*

$$\|\phi\|1 - \phi(A) < \epsilon, \quad \psi(A) < \epsilon.$$

**Proof.** We adjoin the unit, if needed, and consider the extended $\psi_{\rm un}, \phi_{\rm un}$.

(1)⇒(2). Since $\phi - \psi$ is self-adjoint, there exists $B \in \mathfrak{A}_{\rm h}$ with $\|B\| \leq 1$ such that

$$(\phi - \psi)(B) + \epsilon \geq \|\phi - \psi\|. \tag{4.22}$$

We set $A := \frac{1}{2}(1 + B)$. Clearly, $0 \leq A \leq 1$.

The rhs of (4.22) equals $\|\phi\| + \|\psi\| = \phi(1) + \psi(1)$. Hence $\phi(1 - A) + \psi(A) < \epsilon$. Hence $\phi(1 - A) < \epsilon$, $\psi(A) < \epsilon$.

(1).⇐(2). Clearly, $\|\phi - \psi\| \leq \|\phi\| + \|\psi\|$.

Let us prove the converse inequality. Let $\epsilon > 0$ and $A$ satisfy the conditions of (2). Then $\|2A-1\| \leq 1$, and hence

$$\|\phi\| + \|\psi\| = \phi(1) + \psi(1) \leq (\phi - \psi)(2A - 1) + 4\epsilon \leq \|\phi - \psi\| + 4\epsilon.$$

But $\epsilon > 0$ was arbitrary, hence $\|\phi\| + \|\psi\| \leq \|\phi - \psi\|$. □

If the (equivalent) conditions of the above theorem are satisfied, then we will write $\phi \perp \psi$.

**Theorem 4.21 (Jordan decomposition of a self-adjoint form.)** *Let $\mathfrak{A}$ be a $C^*$-algebra. Let $\omega \in \mathfrak{A}^\#$ be self-adjoint. Then there exist unique $\omega_+, \omega_- \in \mathfrak{A}_+^\#$ such that*

$$\omega = -\omega_- + \omega_+, \quad \omega_- \perp \omega_+.$$

**Proof. Existence. Step 1.** First note that $(\mathfrak{A}_{\rm h}^\#)_1$ (the unit ball in $\mathfrak{A}_{\rm h}^\#$) is compact in the $\sigma(\mathfrak{A}_{\rm h}^\#, \mathfrak{A})$ topology. Hence $\mathbb{E}(\mathfrak{A})$ (the set of states on $\mathfrak{A}$) is compact too, because it is a closed subset of $(\mathfrak{A}_{\rm h}^\#)_1$. Therefore,

$$\mathrm{CH}(-\mathbb{E}(\mathfrak{A}) \cup \mathbb{E}(\mathfrak{A})) \tag{4.23}$$

is also compact. Clearly, (4.23) is contained in $(\mathfrak{A}_{\rm h}^\#)_1$.

**Step 2.** Suppose that $\phi_0 \in (\mathfrak{A}_{\rm h}^\#)_1$, but does not belong to (4.23). By the 2nd Separation Theorem, there exists $A \in \mathfrak{A}$ such that

$$\phi_0(A) \quad > \sup\{\mathrm{Re}\phi(A) \ : \ \phi \in \mathrm{CH}(-\mathbb{E}(\mathfrak{A}) \cup \mathbb{E}(\mathfrak{A})\}.$$

By replacing $A$ with $\frac{1}{2}(A + A^*)$ and using the self-adjointness of $\phi$, we can assume that $A \in \mathfrak{A}_{\rm h}$. Now

$$\begin{aligned}\phi_0(A) \quad &> \sup\{\phi(A) \ : \ \phi \in \mathrm{CH}(-\mathbb{E}(\mathfrak{A}) \cup \mathbb{E}(\mathfrak{A})\} \\ &= \sup\{|\phi(A)| \ : \ \phi \in \mathbb{E}(\mathfrak{A})\} = \|A\|.\end{aligned}$$

Hence $\|\phi_0\| > 1$. Therefore,

$$\mathrm{CH}(-\mathbb{E}(\mathfrak{A}) \cup \mathbb{E}(\mathfrak{A}) = (\mathfrak{A}_{\rm h}^\#)_1. \tag{4.24}$$

**Step 3.** Now let us prove the existence part of the theorem. Let $\omega \in \mathfrak{A}_{\rm h}^\#$. It is sufficient to assume that $\|\omega\| = 1$. By (4.24), there exist $\tilde\omega_-, \tilde\omega_+ \in \mathbb{E}(\mathfrak{A})$ and $\theta \in [0, 1]$ such that $\omega = -\theta\tilde\omega_- + (1-\theta)\tilde\omega_+$. We set $\omega_- := \theta\tilde\omega_-$ and $\omega_+ := (1-\theta)\tilde\omega_+$. They clearly satisfy

$$\|\omega_-\| + \|\omega_+\| = \theta + (1-\theta) = 1.$$

**Uniqueness.** Let us prove the uniqueness part of the theorem. Suppose that

$$\omega = -\omega_- + \omega_+ = -\omega'_- + \omega'_+$$

and

$$\|\omega\| = \|\omega_-\| + \|\omega_+\| = \|\omega'_-\| + \|\omega'_+\|.$$

Let $\epsilon > 0$ and choose $C \in (\mathfrak{A}_{\rm h})_1$ such that

$$\omega(C) \geq \|\omega\| - \frac{1}{2}\epsilon^2. \tag{4.25}$$

Set $B := \frac{1}{2}(1 + C)$. Clearly, $0 \leq B \leq 1$. Adding $\frac{1}{2}$ times (4.25) and $-\omega_-(\frac{1}{2}) - \omega_+(\frac{1}{2}) = -\frac{1}{2}\|\omega\|$ we get

$$\omega_-(B) + \omega_+(1 - B) < \frac{1}{4}\epsilon^2.$$

Hence,

$$\omega_-(B) < \frac{1}{4}\epsilon^2, \quad \omega_+(1 - B) < \frac{1}{4}\epsilon^2.$$

For $A \in \mathfrak{A}$, by the Cauchy-Schwarz inequality,

$$\begin{aligned}|\omega_-(BA)|^2 &\leq \omega_-(B)\omega_-(A^*BA) \leq \tfrac{1}{4}\epsilon^2\|A\|^2, \\ |\omega_+((1-B)A)|^2 &\leq \omega_+(1-B)\omega_+(A^*(1-B)A) \leq \tfrac{1}{4}\epsilon^2\|A\|^2,\end{aligned} \tag{4.26}$$

Using $\omega_- - \omega'_- = \omega_+ - \omega'_+$, we get

$$\omega_-(A) - \omega'_-(A) = \omega_-(BA) - \omega'_-(BA) + \omega_+((1-B)A) - \omega'_+((1-B)A).$$

Hence, using (4.26) and analogous inequalities for $\omega'_-$ and $\omega'_+$, we get

$$|\omega_-(A) - \omega'_-(A)| < 2\epsilon \|A\|.$$

Since the last inequality is true for any $\epsilon > 0$, $\omega_-(A) = \omega'_-(A)$. $\Box$

**Corollary 4.22** *Let $\omega \in \mathfrak{A}$. Then there exists a $*$-representation $\pi : \mathfrak{A} \to B(\mathcal{H})$ and vectors $\Phi, \Psi$ such that*

$$\omega(A) = (\Phi|\pi(A)\Psi).$$

**Theorem 4.23** *Let $\mathfrak{A}$ be a $C^*$-algebra, $\phi \in \mathfrak{A}^\#$ and $A \in \mathfrak{A}_+$. Assume that*

$$\phi(A) = \|\phi\|\|A\|.$$

*Then $\phi$ is positive.*

**Proof.** We can assume that $\|\phi\| = 1$ and $\|A\| = 1$.
**Step 1** If $\mathfrak{A}$ does not have the identity, then we can extend $\phi$ to a functional $\phi_{\rm un}$ on $\mathfrak{A}_{\rm un}$ such that $\|\phi\| = \|\phi_{\rm un}\|$. If $\phi_{\rm un}$ is positive, then so is $\phi$ is. Therefore, in what follows it is sufficient to assume that $\mathfrak{A}$ has an identity.
**Step 2** Let $\phi(1) = \alpha + {\rm i}\beta$, $\alpha, \beta, \lambda \in \mathbb{R}$. Then

$$|\phi(1 + \lambda {\rm i} A)| = |\alpha + {\rm i}(\lambda + \beta)| \geq |\lambda + \beta|, \quad \|1 + {\rm i}\lambda A\| = (1 + \lambda^2)^{\frac{1}{2}}.$$

But

$$|\phi(1 + {\rm i}\lambda A)| \leq \|1 + {\rm i}\lambda A\|.$$

Hence

$$|\lambda + \beta|^2 \leq (1 + \lambda^2).$$

If this is true for all $\lambda$, then $\beta = 0$. Hence $\phi(1) \in \mathbb{R}$.
**Step 3** We will show that $\phi(1) = 1$.

It is clear that $\phi(1) \leq \|\phi\| = 1$. Using first the positivity of $A$, $\|A\| = 1$, and then $\|\phi\| = 1$, we get

$$1 \geq \|1 - 2A\| \geq |\phi(1 - 2A)|.$$

But $\phi(1 - 2A) = \phi(1) - 2$. Hence $\phi(1) \geq 1$.

This proves that $\phi(1) = 1$. By Theorem 4.14, this means that $\phi$ is positive. $\Box$

**Theorem 4.24** *Let $\phi$ be a state on $\mathfrak{A}$ and $A \in \mathfrak{A}$. Suppose that $\mathfrak{A} \ni B \mapsto \phi(BA)$ is hermitian. Then*

$$|\phi(AH)| \leq \|A\|\phi(H), \quad H \in \mathfrak{A}_+.$$

**Proof.** Iterating $\phi(B^*A) = \overline{\phi(BA)} = \phi(A^*B^*)$ we obtain $\rho(BA^{2n}) = \phi(A^{n*}BA^N)$. If $H \in \mathfrak{A}_+$, then

$$\begin{aligned}\phi(HA^n) = \phi(H^{1/2}H^{1/2}A^n) &\leq \phi(A^{n*}HA^n)^{1/2}\phi(H)^{1/2} \\ &= \phi(HA^{2n})^{1/2}\phi(H)^{1/2}.\end{aligned}$$

Hence,

$$\begin{aligned}\rho(HA) &\leq \phi(HA^{2^n})^{2^{-n}}\phi(H)^{2^{-1}+\ldots+2^{-n}} \\ &\leq \|H\|^{2^{-n}}\|A\|\phi(H)^{1-2^{-n}} \to \|A\|\phi(H).\end{aligned}$$

$\Box$

## 4.9 Unitary elements

Let $A \in \mathfrak{A}$. Then $\mathrm{Re}A := \frac{1}{2}(A + A^*)$, $\mathrm{Im}A := \frac{1}{2\mathrm{i}}(A - A^*)$ are self-adjoint and $A = \mathrm{Re}A + \mathrm{i}A^*$.

For $A \in \mathfrak{A}$ we set $|A| := (A^*A)^{\frac{1}{2}}$.

**Theorem 4.25** *Assume that $A$ is invertible. Then there exists a unique unitary $U$ such that $A = U|A|$.*

**Theorem 4.26** *If $\mathfrak{A}$ is unital, then the unit ball $(\mathfrak{A})_1$ is the closed convex hull of unitary elements of $\mathfrak{A}$.*

**Proof.** The theorem is easy for self-adjoint elements. If $A$ is self-adjoint and of norm less than 1, then for $U = \frac{1}{2}A + \frac{\mathrm{i}}{2}\sqrt{1 - A^*A}$, we have $A = U + U^*$.

In the general case, set

$$U(z) := (1 - AA^*)^{-1/2}(z + A)(1 + zA^*)^{-1}(1 - A^*A)^{1/2}.$$

Then $U(0) = A$, $U(z)$ is unitary for $|z| = 1$ and by the Cauchy formula

$$A = \frac{1}{2\pi}\int_0^{2\pi} U(\mathrm{e}^{\mathrm{i}\phi})\mathrm{d}\phi.$$

□

## 4.10 Extreme points of the unit ball

**Theorem 4.27** *The extreme points of $(\mathfrak{A})_1 \cap \mathfrak{A}_{\mathrm{h}}$ are precisely the self-adjoint unitary elements.*

**Theorem 4.28** *The extreme points of $(\mathfrak{A})_1 \cap \mathfrak{A}_+$ are precisely the projections.*

**Theorem 4.29** *The extreme points of the unit ball $(\mathfrak{A})_1$ are precisely the elements $A \in \mathfrak{A}$ such that*

$$(1 - AA^*)\mathfrak{A}(1 - A^*A) = \{0\}.$$

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