Measure theory

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1 Measurability

1.1 Notation

2^X denotes the family of subsets of the set X. The symmetric difference is defined as

$$A\Delta B := (A \cup B)\backslash(A \cap B).$$

Let $A_1, A_2, \dots \in X$.
We write $A_n \nearrow A$, if $A_n \subset A_{n+1}$, $n \in \mathbb{N}$ and $\cup_{n=1}^{\infty} A_n = A$.
We write $A_n \searrow A$, if $A_n \supset A_{n+1}$, $n \in \mathbb{N}$ and $\cap_{n=1}^{\infty} A_n = A$.

1.2 Rings and fields

Definition 1.1 $\mathcal{R} \subset 2^X$ *is called a ring if*

(1) $A, B \in \mathcal{R} \Rightarrow A\backslash B \in \mathcal{R}$;

(2) $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$

Proposition 1.2 *Let* $\mathcal{R}$ *be a ring. Then* $A, B \in \mathcal{R} \Rightarrow A \cap B \in \mathcal{R}$.

Proof. $A \cap B = A\backslash(A\backslash B)$. □

If $(\mathcal{R}_i)_{i\in I}$ is a family of rings in X, then so is $\cap_{i\in I}\mathcal{R}_i$. Hence for any $\mathcal{T} \subset 2^X$ there exists the smallest ring containing $\mathcal{T}$. We denote it by Ring($\mathcal{T}$).

Definition 1.3 $\mathcal{R} \subset 2^X$ *is called a field if*

(1) $\emptyset \in \mathcal{R}$;

(2) $A \in \mathcal{R} \Rightarrow X\backslash A \in \mathcal{R}$;

(3) $A, B \in \mathcal{R} \Rightarrow A \cup B \in \mathcal{R}$.

Equivalently, a field is a ring containing X. (Field is a ring, because $A\backslash B = X\backslash((X\backslash A) \cup B)$).
For $\mathcal{T} \subset 2^X$, Field($\mathcal{T}$) denotes the smallest field of sets containing $\mathcal{T}$.

1.3 Ordered spaces

Suppose that $(X, \leq)$ is an ordered set. Let U be a nonempty subset of X.

We say that u_0 is a largest minorant of U if

(1) $u \in U$ implies $u_0 \leq u$

(2) $u_1 \leq u$ for all $u \in U$ implies $u_1 \leq u$

If U possesses a largest minorant, then it is uniquely defined. The largest minorant of a set $\{x_1, x_2\}$ is often denoted $x_1 \wedge x_2$ and of a set U is denoted $\underset{x \in U}{\wedge} x$.

Analogously we define the smallest majorant of U. The smallest majorant of a set $\{x_1, x_2\}$ is often denoted $x_1 \vee x_2$ and of a set U is denoted $\underset{x \in U}{\vee} x$.

We say that $(X, \leq)$ is a lattice if every two-element (hence every finite) set of elements of X possess the smallest majorant and the largest minorant. It is a countably complete lattice if every countable subset that has a majorant and a minorant has the smallest majorant and the largest minorant. It is a complete lattice if every countable subset that has a majorant and a minorant has the smallest majorant and the largest minorant.

Let $\mathcal{X}$ be a vector space. $(\mathcal{X}, \leq)$ is an ordered vector space iff

(1) $x, y, z \in \mathcal{X}$, $x \leq y \Rightarrow x + z \leq y + z$;

(2) $x \in \mathcal{X}$, $x \geq 0$, $\lambda \in \mathbb{R}$, $\lambda \geq 0 \Rightarrow \lambda x \geq 0$.

$\mathcal{X}_+ := \{x \in \mathcal{X} \ : \ x \geq 0\}$ is a cone called the positive cone.

We say that an ordered vector space $(\mathcal{X}, \leq)$ is a Riesz space if it is a lattice. It is enough to check that it has $\vee$ of two elements, since

$$x \wedge y := -(-x) \vee (-y).$$

1.4 Elementary functions

Definition 1.4 *Let $(X, \mathcal{R})$ be a space with a ring. $u : X \to \mathbb{R}$ is called an elementary function if $u(X)$ is a finite set and $u^{-1}(\alpha) \in \mathcal{R}$, $\alpha \in \mathbb{R}\backslash\{0\}$. The set of elementary functions is denoted by $\mathcal{E}(X, \mathcal{R})$ or $\mathcal{E}(X)$. Positive elementary functions will be denoted $\mathcal{E}_+(X)$.*

Lemma 1.5 (1) *Let $u, v \in \mathcal{E}(X)$ and $\alpha \in \mathbb{R}$. Then*

$$\alpha u, u + v, uv, \max(u, v), \min(u, v) \in \mathcal{E}(X).$$

In particular, $\mathcal{E}(X)$ is an algebra and a lattice.

(2) *$1 \in \mathcal{E}(X)$ iff $\mathcal{R}$ is a field.*

1.5 σ-rings and σ-fields

Definition 1.6 *$\mathcal{F} \subset 2^X$ is called a σ-ring if*

(1) *$A, B \in \mathcal{F} \Rightarrow A\backslash B \in \mathcal{F}$;*

(2) *$A_1, A_2, \dots \in \mathcal{F} \Rightarrow \bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$*

Clearly, every σ-ring is a ring.

Proposition 1.7 *Let $\mathcal{F}$ be a σ-ring. Then*

(1) *$A_1, A_2, \dots \in \mathcal{F} \Rightarrow \cap_{j=1}^{\infty} A_j \in \mathcal{F}$,*

(2) *$A_1, A_2, \dots \in \mathcal{F}$, $A_n \searrow A \Rightarrow A \in \mathcal{F}$,*

(3) $A_1, A_2, \dots \in \mathcal{F},\ A_n \nearrow A \Rightarrow A \in \mathcal{F}$,

Proof. Let us prove (1). Clearly, $A := \bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{F}$. Now by the de Morgan's law

$$\bigcap_{i=1}^{\infty} A_i = A \backslash \bigcup_{i=1}^{\infty} (A \backslash A_i) \in \mathcal{F}.$$

□

For $\mathcal{T} \subset 2^X$, $\sigma-\mathrm{Ring}(\mathcal{T})$ denotes the smallest σ-ring of sets containing $\mathcal{T}$.

Theorem 1.8 *Let $\mathcal{T} \subset 2^X$ and $A \in \sigma-\mathrm{Ring}(\mathcal{T})$. Then there exists a countable $\mathcal{T}_0 \subset \mathcal{T}$ such that $A \in \sigma-\mathrm{Ring}(\mathcal{T}_0)$.*

Proof. Let $\mathcal{F}$ be the family of $A \subset X$ such that there exists a countable $\mathcal{T}_0 \subset \mathcal{T}$ with $A \in \sigma-\mathrm{Ring}(\mathcal{T}_0)$. Then $\mathcal{T} \subset \mathcal{F}$ and $\mathcal{F}$ is a σ-ring. Hence $\sigma-\mathrm{Ring}(\mathcal{T}) \subset \mathcal{F}$. □

Definition 1.9 *$\mathcal{F} \subset 2^X$ is called a σ-field if*

(1) $\emptyset \in \mathcal{F}$;

(2) $A \in \mathcal{F} \Rightarrow X \backslash A \in \mathcal{F}$;

(3) $A_1, A_2 \dots \in \mathcal{F} \Rightarrow \cup_{j=1}^{\infty} A_j \in \mathcal{F}$.

Equivalently, a σ-field is a σ-ring containing X. Clearly, every σ-field is a field.
For $\mathcal{T} \subset 2^X$, $\sigma-\mathrm{Field}(\mathcal{T})$ denotes the smallest σ-field of sets containing $\mathcal{T}$.

1.6 Transport of subsets

Let $F : X \to X'$ be a transformation. As usual, for $A \subset X$, $F(A)$ denotes the image of A, and for $A' \subset X'$, $F^{-1}(A')$ denotes the preimage of A'. Thus we have two maps

$$2^X \ni A \mapsto F(A) \in 2^{X'},$$

$$2^{X'} \ni A' \mapsto F^{-1}(A') \in 2^X. \tag{1.1}$$

Theorem 1.10 (1) *For $A \subset X$, $F^{-1}F(A) \supset A$ and we have the equality for all A iff F is injective.*

(2) *For $A' \in X'$, $FF^{-1}(A') \subset A'$ and we have the equality for all A' iff F is surjective.*

(3) $F^{-1}(\emptyset) = \emptyset$, $F^{-1}(X') = X$, $F^{-1}(A' \cup B') = F^{-1}(A') \cup F^{-1}(B')$, $F^{-1}(A' \backslash B') = F^{-1}(A') \backslash F^{-1}(B')$,

Let $F^* : 2^{X'} \to 2^X$ be the map given by (1.1). (We prefer not to denote it by F^{-1} to avoid ambiguous notation).

For $\mathcal{C}' \subset 2^{X'}$, we can write

$$F^*(\mathcal{C}') = \{F^{-1}(A')\ :\ A' \in \mathcal{C}'\}.$$

Let $\mathcal{C} \subset 2^X$. We will write

$$F_*(\mathcal{C}) := (F^*)^{-1}(\mathcal{C}) = \{A' \in 2^{X'}\ :\ F^{-1}(A') \in \mathcal{C}\}.$$

The following facts follow from Theorem 1.10 (1), (2) applied to F^*:

Theorem 1.11 (1) *$F_*F^*(\mathcal{C}') \supset \mathcal{C}'$;*

(2) *$F^*F_*(\mathcal{C}) \subset \mathcal{C}$.*

1.7 Transport of σ-rings

Theorem 1.12 (1) *If $\mathcal{F}'$ is a σ-ring over X', then $F^*(\mathcal{F}')$ is a σ-ring over X.*

(2) *If $\mathcal{F}$ is a σ-ring over X, then $F_*(\mathcal{F})$ is a σ-ring over X'.*

(3) *If $\mathcal{C}' \subset 2^{X'}$, then*

$$F^*(\sigma\text{−Ring}(\mathcal{C}')) = \sigma\text{−Ring}(F^*(\mathcal{C}')).$$

Proof. To see (1) and (2), we use Theorem 1.10 (3), which says that F^* is a homomorphism for set-theoretical operations.

Let us prove (3). By (1), $F^*(\sigma\text{−Ring}(\mathcal{C}'))$ is a σ-ring. It contains $F^*(\mathcal{C}')$. Hence

$$F^*(\sigma\text{−Ring}(\mathcal{C}')) \supset \sigma\text{−Ring}(F^*(\mathcal{C}')).$$

By (2), $F_*(\sigma\text{−Ring}(F^*(\mathcal{C}')))$ is a σ-ring. Clearly

$$F_*(\sigma\text{−Ring}(F^*(\mathcal{C}'))) \supset F_*(F^*(\mathcal{C}')) \supset \mathcal{C}'.$$

Hence $F_*(\sigma\text{−Ring}(F^*(\mathcal{C}'))) \supset \sigma\text{−Ring}(\mathcal{C}')$. Hence,

$$\sigma\text{−Ring}(F^*(\mathcal{C}')) \supset F^*F_*(\sigma\text{−Ring}(F^*(\mathcal{C}'))) \supset F^*(\sigma\text{−Ring}(\mathcal{C}')).$$

□

For $A \in 2^X$ and $\mathcal{C} \subset 2^X$, we set

$$\mathcal{C}\Big|_A := \{A \cap C \ : \ C \in \mathcal{C}\}.$$

Theorem 1.13 *If $\mathcal{T} \subset 2^X$ and $A \subset X$, then*

$$\sigma\text{−Ring}(\mathcal{T})\Big|_A = \sigma\text{−Ring}\Big(\mathcal{T}\Big|_A\Big).$$

Proof. Consider the inclusion map $J : A \to X$. If $C \in 2^X$, then $J^{-1}(C) = C \cap A$. Hence if $\mathcal{C} \subset 2^X$, then $J^*(\mathcal{C}) = \mathcal{C}\Big|_A$. Thus it is sufficient to apply Theorem 1.12 (3). □

1.8 Measurable transformations

Definition 1.14 *Let $(X, \mathcal{F})$, $(X', \mathcal{F}')$ be spaces with σ-rings and $F : X \to X'$. Then F is called a $\mathcal{F} - \mathcal{F}'$-measurable transformation if*

$$F^*(\mathcal{F}') \subset \mathcal{F}.$$

Proposition 1.15 *The composition of measurable transformations is measurable.*

Theorem 1.16 *Let $\mathcal{C}' \subset 2^{X'}$. If $\mathcal{F}' = \sigma - \text{Ring}(\mathcal{C}')$, then $F : X \to X'$ is $\mathcal{F} - \mathcal{F}'$-measurable iff $F^*(\mathcal{C}') \subset \mathcal{F}$.*

Proof.

$$F^*(\mathcal{F}') = F^*(\sigma\text{−Ring}(\mathcal{C}')) = \sigma\text{−Ring}(F^*(\mathcal{C}')) \subset \sigma\text{−Ring}(\mathcal{F}) = \mathcal{F},$$

where we used Theorem 1.12 (3) in the second equality. □

1.9 Measurable real functions

$\mathbb{R}\cup\{-\infty,\infty\} =: [-\infty,\infty]$ is a topological space in the obvious way. We can extend the addition to $[-\infty,\infty]$ except that $\infty-\infty$ is undefined. We extend the multiplication to $[-\infty,\infty]$, adopting the convention $0(\pm\infty)=0$. Let Borel($[-\infty,\infty]$) denote the σ-field of Borel subsets of $[-\infty,\infty]$, that is the σ-field generated by open subsets of $[-\infty,\infty]$. If $Y\subset\mathbb{R}$, then Borel(Y) will denote the σ-field in X generated by open subsets in X. Note in particular that Borel($[-\infty,0[\cup]0,\infty]$) is generated by the sets $[-\infty,-\alpha[$ and $]\alpha,\infty[$ for $0\leq\alpha$.

Let $(X,\mathcal{F})$ be a space with a σ-ring. We say that

$$f : X\to[-\infty,\infty]$$

is a $\mathcal{F}$-measurable function iff for any $A\in$ Borel($[-\infty,0[\cup]0,\infty]$), $f^{-1}(A)\in\mathcal{F}$. The set of such functions will be denoted $\mathcal{M}(X,\mathcal{F})$, or for shortness, $\mathcal{M}(X)$. The set of measurable functions with values in $[0,\infty]$ will be called $\mathcal{M}_+(X)$.

Let $A\subset X$. Its characteristic function is denoted by

$$1_A(x)=\begin{cases}1, & x\in A\\ 0, & x\in X\backslash A,\end{cases}$$

1_A is $\mathcal{F}$-measurable iff $A\in\mathcal{F}$.

If f,g are real functions on X, we will write

$$\{f\geq g\} := \{x\in X\ :\ f(x)\geq g(x)\}.$$

Similarly, we define $\{f>g\}$, etc.

Lemma 1.17 $f : X\to[-\infty,\infty]$ *is* $\mathcal{F}$*-measurable if*

$$\{\pm f>\pm\alpha\}\in\mathcal{F},\quad 0\leq\alpha<\infty.$$

Lemma 1.18 *Let* $f,g\in\mathcal{M}(X)$. *Then*

(1) $\alpha f\in\mathcal{M}(X)$

(2) $f+g\in\mathcal{M}(X)$ *(if defined);*

(3) $fg\in\mathcal{M}(X)$

(4) $1\in\mathcal{M}(X)$ *iff* $\mathcal{F}$ *is a* σ*-field.*

Proof. (2) For simplicity, we assume in addition that $\mathcal{F}$ is a σ-field. Using the countability of $\mathbb{Q}$ we see that

$$\{f+g>\alpha\}=\bigcup_{\beta\in\mathbb{Q}}\{f>\alpha+\beta\}\cap\{g>-\beta\}.$$

(3) First we show that $f\in\mathcal{M}(X)$ implies $f^2\in\mathcal{M}(X)$.

By (1) and (2), $f-g$ is measurable.

Finally

$$fg=\frac{1}{4}(f+g)^2-\frac{1}{4}(f-g)^2.$$

implies that fg is measurable. □

Proposition 1.19 *Let $f_1, f_2, \dots \in \mathcal{M}(X)$. Then*

$$\sup_n f_n,\ \inf_n f_n,\ \limsup_{n\to\infty} f_n,\ \liminf_{n\to\infty} f_n$$

are measurable. If there exists the pointwise limit of f_n, then also $\lim_{n\to\infty} f_n$ is measurable.

Proof. Let $f := \sup f_n$. Then

$$\{f \le \alpha\} = \cap_{n=1}^{\infty}\{f_n \le \alpha\} \in \mathcal{F}.$$

Hence f is measurable. $\inf f_n$ is treated similarly.

Then we use

$$\limsup_{n\to\infty} f_n = \inf_{n\in\mathbb{N}} \sup_{m\ge n} f_m, \quad \liminf_{n\to\infty} f_n = \sup_{n\in\mathbb{N}} \inf_{m\ge n} f_m$$

Finally,

$$\lim_{n\to\infty} f_n = \limsup_{n\to\infty} f_n = \liminf_{n\to\infty} f_n.$$

□

Theorem 1.20 *Let $f : X \to \mathbb{R}$. Then $f \in \mathcal{M}_+(X)$ iff there exists an increasing sequence $u_n \in \mathcal{E}_+(X)$ such that*

$$f = \sup_{n\in\mathbb{N}} u_n$$

Proof. $\Leftarrow$ is obvious. Let us prove the converse statement.

Let $f \in \mathcal{M}_+(X)$. The sets

$$A_{in} := \begin{cases} \{\frac{i}{2^n} \le f < \frac{i+1}{2^n}\} & i = 0, 1, \dots, n2^n - 1, \\ \{n \le f\} & i = n2^n \end{cases}$$

are disjoint and measurable. Hence

$$u_n := \sum_{j=0}^{n2^n} \frac{j}{2^n} 1_{A_{jn}} \in \mathcal{E}_+(X).$$

The sequence u_n is increasing and $\sup_{n\in\mathbb{N}} u_n = f$. □

1.10 Spaces L^∞

Assume that $\mathcal{I} \subset \mathcal{F} \subset 2^X$ are rings. We say that say that $\mathcal{I}$ is an ideal in $\mathcal{F}$ if

$$A \in \mathcal{I},\ B \in \mathcal{F} \ \Rightarrow\ A \cap B \in \mathcal{I}.$$

In what follows let $\mathcal{I} \subset \mathcal{F} \subset 2^X$ be σ-rings and $\mathcal{I}$ be an ideal in $\mathcal{F}$. Then

Proposition 1.21 (1) $\mathcal{M}(X, \mathcal{I}) \subset \mathcal{M}(X, \mathcal{F})$

(2) $\mathcal{M}(X, \mathcal{I}) := \{f \in \mathcal{M}(X, \mathcal{F})\ :\ $ *there exists $N \in \mathcal{I}$ such that $f = 0$ on $X\backslash N$*$\}$

(3) $f \in \mathcal{M}(X, \mathcal{F})$, $g \in \mathcal{M}(X, \mathcal{I})$ *implies* $fg \in \mathcal{M}(X, \mathcal{I})$.

For $f \in \mathcal{M}(X, \mathcal{F})$ we set

$$\|f\|_\infty := \inf\{\sup\{|f(x)|\ :\ x \in X\backslash N\}\ :\ N \in \mathcal{I}\}.$$

Theorem 1.22 (1) *Given $f \in \mathcal{M}(X, \mathcal{F})$, we can always find $N \in \mathcal{I}$ such that* $\sup |f|\Big|_{X\setminus N} = \|f\|_\infty$*;*

(2) $\|f+g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$*;*

(3) $\|\alpha f\| = |\alpha| \|f\|_\infty$.

Proof. (2) We can find $N, M \in \mathcal{I}$ such that

$$\sup |f|_{X\setminus N} = \|f\|_\infty,$$

$$\sup |f|_{X\setminus M} = \|g\|_\infty.$$

Then

$$\begin{aligned}\|f+g\|_\infty &\leq \sup |f+g|\Big|_{X\setminus(N\cup M)} \\ &\leq \sup(|f|+|g|)\Big|_{X\setminus(N\cup M)} \\ &\leq \sup |f|\Big|_{X\setminus(N\cup M)} + \sup |g|\Big|_{X\setminus(N\cup M)} \\ &\leq \sup |f|\Big|_{X\setminus N} + \sup |g|\Big|_{X\setminus M} = \|f\|_\infty + \|g\|_\infty.\end{aligned}$$

Let

$$\mathcal{L}^\infty(X, \mathcal{F}, \mathcal{I}) := \{f \in \mathcal{M}(X, \mathcal{F}) \ : \ \|f\|_\infty < \infty\},$$

Theorem 1.23 (Riesz-Fischer) *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^\infty(\mu)$ satisfying the Cauchy condition, that is for any $\epsilon > 0$ there exists N such that for $n, m \geq N$*

$$\|f_n - f_m\|_\infty \leq \epsilon.$$

Then there exists $f \in \mathcal{L}^\infty(\mu)$ such that

$$\|f - f_n\|_\infty \to 0$$

We can also find a subsequence of $(f_n)_{n\in\mathbb{N}}$ pointwise convergent μ-a.e. to f.

Proof. There exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_\infty \leq 2^{-k}$, for any k. We set

$$g_k := f_{n_{k+1}} - f_{n_k}, \ \ g := \sum_{k=1}^{\infty} |g_k|.$$

Then

$$\|g\|_\infty \leq \sum_{k=1}^{\infty} \|g_k\|_\infty \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Hence $g \in \mathcal{L}^\infty$ and therefore g is finite outside of a set N in $\mathcal{I}$. Hence the series $\sum_{k=1}^{\infty} g_k$ is convergent outside of N. This means that the sequence $(f_{n_k})_{k\in\mathbb{N}}$ is convergent to a function f outside of N. Inside N we set $f := 0$. We check that $f \in \mathcal{L}^\infty$ and $\|f - f_n\|_\infty \to 0$. □

Theorem 1.24 (1) $\mathcal{M}(X, \mathcal{I}) = \{f \in \mathcal{L}^\infty(X, \mathcal{F}, \mathcal{I}) \ : \ \|f\|_\infty = 0\}$

(2) *Therefore, $\|f + \mathcal{M}(X, \mathcal{I})\|_\infty := \|f\|_\infty$ defines a norm in $L^\infty(X, \mathcal{F}, \mathcal{I}) := \mathcal{L}^\infty(X, \mathcal{F}, \mathcal{I})/\mathcal{M}(X, \mathcal{I})$.*

(3) *$L^\infty(X, \mathcal{F}, \mathcal{I})$ is a Banach space.*

(4) *Elementary functions are dense in $L^\infty(X, \mathcal{F}, \mathcal{I})$.*

(5) *$f, g \in L^\infty(X, \mathcal{F}, \mathcal{I})$, $0 \leq f \leq g$ a.e. $\Rightarrow \|f\|_\infty \leq \|g\|_\infty$.*

(6) *$L^\infty(X, \mathcal{F}, \mathcal{I})$ is a countably complete lattice.*

2 Measure and integral

2.1 Contents

Let $\mathcal{R}$ be a ring. $\nu : \mathcal{R} \to [0, \infty]$ is a content if
(1) $\nu(\emptyset) = 0$;
(2) $A_1, A_2 \in \mathcal{R}$, $A_1 \cap A_2 = \emptyset$, $\Rightarrow \nu(A_1 \cup A_2) = \nu(A_1) + \nu(A_2)$.

Theorem 2.1 *Let $(X, \mathcal{R}, \nu)$ be a content on a ring. Then if $A_1, A_2, \cdots \in \mathcal{R}$ are disjoint and $A = \bigcup_{i=1}^{\infty} A_i \in \mathcal{R}$, then*

$$\nu(A) \geq \sum_{i=1}^{\infty} \nu(A_i).$$

Proof. For any n,

$$\nu(A) \geq \nu(\bigcup_{j=1}^{n} A_j) = \sum_{j=1}^{n} \nu(A_j).$$

Passing to the limit $n \to \infty$, we obtain the inequality. □

2.2 Measures

Let $(X, \mathcal{F})$ be a space with a σ-ring. A function $\mu : \mathcal{F} \to [0, \infty]$ is called a measure if
(1) $\mu(\emptyset) = 0$,
(2) $A_1, A_2 \cdots \in \mathcal{F}$, $A_i \cap A_j = \emptyset$ for $i \neq j \Rightarrow \mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.
The triple $(X, \mathcal{F}, \mu)$ is called a space with a measure.

Proposition 2.2 (1) *If $A \subset B$, then $\mu(A) \leq \mu(B)$.*
(2) $A_1, A_2, \cdots \in \mathcal{F} \Rightarrow \mu((\cup_{n=1}^{\infty} A_n) \leq \sum_{n=1}^{\infty} \mu(A_n)$.
(3) $A_1, A_2, \cdots \in \mathcal{F}$, $A_n \nearrow A \Rightarrow \mu(A_n) \nearrow \mu(A)$.
(4) *If $A_1, A_2, \cdots \in \mathcal{F}$, $A_n \searrow A$ and for some n, $\mu(A_n) < \infty$, then $\mu(A_n) \searrow \mu(A)$.*

Definition 2.3 *Let $(X, \mathcal{F}, \mu)$ be a space with a measure. and $P(x)$ be a property defined on X. We say that $P(x)$ is true μ-almost everywhere (μ-a.e.) if*

$$\mu\left(\{x \in X \ : \ P(x) \text{ is not true } \}\right) = 0.$$

2.3 $\mu - \sigma$-finite sets

Let $(X, \mathcal{F}, \mu)$ be a measure. A set $A \in \mathcal{F}$ is called μ-null if $\mu(A) = 0$. It is μ-finite if $\mu(A) < \infty$. It is called μ-σ-finite iff there exist a sequence of μ-finite sets $A_1, A_2, \cdots \in \mathcal{F}$ such that $A_n \nearrow A$. Set

$$\mathcal{F}_\mu^0 := \{A \in \mathcal{F} \ : \ \mu(A) = 0\}.$$

$$\mathcal{F}_\mu^{\mathrm{f}} := \{A \in \mathcal{F} \ : \ A \text{ is } \mu\text{-finite}\}.$$

$$\mathcal{F}_\mu^{\sigma\mathrm{f}} := \{A \in \mathcal{F} \ : \ A \text{ is } \mu - \sigma\text{-finite}\}.$$

We say that μ is σ-finite iff $\mathcal{F} = \mathcal{F}_\mu^{\sigma\mathrm{f}}$ and μ is finite iff $\mathcal{F} = \mathcal{F}_\mu^{\mathrm{f}}$. We say that μ is probabilistic iff $\mu(X) = 1$.

Theorem 2.4 (1) $\mathcal{F}^0_\mu$ *is a* σ*-ring and an ideal in* $\mathcal{F}^{\rm f}_\mu$, $\mathcal{F}^{\sigma{\rm f}}_\mu$, $\mathcal{F}$.

(2) $\mathcal{F}^{\sigma f}_\mu$ *is a* σ*-ring and an ideal in* $\mathcal{F}$

(3) $\mathcal{F}^f_\mu$ *is a ring and an ideal in* $\mathcal{F}^{\sigma{\rm f}}_\mu$, $\mathcal{F}$. *We have* $\mathcal{F}^{\sigma{\rm f}}_\mu = \sigma-{\rm Ring}(\mathcal{F}^{\rm f}_\mu)$.

(4) *If* $\mathcal{F}$ *is a* σ*-field, then* μ *is* σ*-finite iff* X *is* μ*-*σ*-finite;* μ *is finite if* X *is* μ*-finite.*

Note that if $(X, \mathcal{F}, \mu)$ is any measure, then $X, \mathcal{F}^{\mu{\rm f}}, \mu\Big|_{\mathcal{F}^{\mu{\rm f}}})$ is a σ-finite measure. We will call say that the latter measure has been obtained from the former by restricting to $\mu - \sigma$-finite sets.

2.4 Integral on elementary functions I

Let $(X, \mathcal{R}, \mu)$ be a space with a ring and a content. For $f \in \mathcal{E}_+(X)$ we set

$$\int f{\rm d}\mu := \sum_{t\in\mathbb{R}} t\mu\left(f^{-1}\{t\}\right).$$

Theorem 2.5 *The function*

$$\mathcal{E}_+(X) \ni u \mapsto \int u{\rm d}\mu \in [0, \infty]$$

satisfies

(1) $\int 1_A{\rm d}\mu = \mu(A)$

(2) $\alpha \geq 0$ *implies* $\int(\alpha u){\rm d}\mu = \alpha\int u{\rm d}\mu$;

(3) $\int(u+v){\rm d}\mu = \int u{\rm d}\mu + \int v{\rm d}\mu$.

(4) $u \leq v$ *implies* $\int u{\rm d}\mu \leq \int u{\rm d}\mu$.

2.5 Integral on elementary functions II

Assume now that $(X, \mathcal{F}, \mu)$ is a set with a σ-ring and a measure. We define the integral on elementary functions as in the previous subsection.

Lemma 2.6 *Let* $(u_n)_{n\in\mathbb{N}}$ *be an increasing sequence in* $\mathcal{E}_+(X)$ *and* $v \in \mathcal{E}_+(X)$. *Then*

$$v \leq \sup_{n\in\mathbb{N}} u_n \Rightarrow \int v{\rm d}\mu \leq \sup_{n\in\mathbb{N}} \int u_n{\rm d}\mu.$$

Proof. It is sufficient to assume that

$$\{v \neq 0\} =: A \neq \emptyset.$$

Let $\alpha := \inf v(A)$, $\beta := \sup v$, $0 < \epsilon < \alpha$. Set

$$A_n := \{u_n \geq v - \epsilon\} \cap A.$$

Then $A_n \in \mathcal{F}$ and $A_n \nearrow A$. Hence $\mu(A_n) \nearrow \mu(A)$.

Consider two cases:

1) $\mu(A) = \infty$. Then

$$(\alpha - \epsilon)1_{A_n} \leq (v - \epsilon)1_{A_n} \leq u_n.$$

Hence

$$(\alpha - \epsilon)\mu(A_n) \leq \int u_n{\rm d}\mu$$

But the lhs tends to $(\alpha-\epsilon)\infty=\infty$. Therefore,

$$\lim_{n\to\infty}\int u_n \mathrm{d}\mu=\infty.$$

2) $\mu(A)<\infty$. Set $B_n:=A\backslash A_n$. Then $B_n\in\mathcal{F}$, $\mu(B_n)<\infty$ and $B_n\searrow\emptyset$. Thus $\mu(B_n)\searrow 0$. Adding $v1_{A_n}\le u_n+\epsilon 1_{A_n}$ and $1_{B_n}v\le\beta 1_{B_n}$ we get

$$v\le\epsilon 1_{A_n}+\beta 1_{B_n}+u_n$$

Hence

$$\int v\mathrm{d}\mu\le\epsilon\mu(A_n)+\beta\mu(B_n)+\int u_n\mathrm{d}\mu.$$

After passsing to the limit we get

$$\int v\mathrm{d}\mu\le\epsilon\mu(A)+\sup_{n\in\mathbb{N}}\int u_n\mathrm{d}\mu.$$

ϵ can be taken arbitrarily close to zero, therefore,

$$\int v\mathrm{d}\mu\le\sup_{n\in\mathbb{N}}\int u_n\mathrm{d}\mu.$$

□

Lemma 2.7 *Let $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ be increasing sequences from $\mathcal{E}_+(X)$. Then*

$$\sup_{n\in\mathbb{N}}u_n=\sup_{n\in\mathbb{N}}v_n\Rightarrow\sup_{n\in\mathbb{N}}\int u_n\mathrm{d}\mu=\sup_{n\in\mathbb{N}}\int v_n\mathrm{d}\mu.$$

Proof. For any $m=1,2,\dots$ we have $v_m\le\sup u_n$. Therefore,

$$\int v_m\mathrm{d}\mu\le\sup_{n\in\mathbb{N}}\int u_n\mathrm{d}\mu.$$

Thus

$$\sup_{m\in\mathbb{N}}\int v_m\mathrm{d}\mu\le\sup_{n\in\mathbb{N}}\int u_n\mathrm{d}\mu.$$

□

2.6 Integral on positive measurable functions I

For $f\in\mathcal{M}_+(X)$ we define

$$\int f\mathrm{d}\mu:=\sup\left\{\int u\mathrm{d}\mu\ :\ u\in\mathcal{E}_+(X),\quad u\le f\right\}.$$

Theorem 2.8 *The function*

$$\mathcal{M}_+(X)\ni f\mapsto\int f\mathrm{d}\mu\in[0,\infty]$$

satisfies

(1) *If* $u_n \in \mathcal{E}_+(X)$ *is an increasing sequence such that* $f = \sup u_n$ *(which always exists), then* $\int u_n \mathrm{d}\mu \to \int f \mathrm{d}\mu$.

(2) $\int 1_A \mathrm{d}\mu = \mu(A)$;

(3) $\int \lambda f \mathrm{d}\mu = \lambda \int f \mathrm{d}\mu$, $f \in \mathcal{M}_+(X)$, $\lambda \in [0, \infty[$;

(4) $\int (f+g) \mathrm{d}\mu = \int f \mathrm{d}\mu + \int g \mathrm{d}\mu$;

(5) *on* $\mathcal{E}_+(X)$ *it coincides with the previously defined integral.*

(6) *if* $f, g \in \mathcal{M}_+(X)$, $f \le g$, *then* $\int f \mathrm{d}\mu \le \int g \mathrm{d}\mu$.

Theorem 2.9 (Beppo Levi) *Let* $(f_n)_{n\in\mathbb{N}}$ *be an increasing sequence from* $\mathcal{M}_+(X)$. *Then* $\sup_{n\in\mathbb{N}} f_n \in \mathcal{M}_+(X)$ *and*

$$\int \sup_{n\in\mathbb{N}} f_n \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int f_n \mathrm{d}\mu.$$

Proof. Set $f := \sup_{n\in\mathbb{N}} f_n$. Using $f_n \le f$, we see that

$$\int f_n \mathrm{d}\mu \le \int f \mathrm{d}\mu.$$

Hence

$$\sup \int f_n \mathrm{d}\mu \le \int f \mathrm{d}\mu.$$

Let us prove the converse inequality.

We can find $u_{mn} \in \mathcal{E}_+(X)$ such that the sequences $(u_{mn})_{m\in\mathbb{N}}$ are increasing and $\sup_{m\in\mathbb{N}} u_{mn} = f_n$. Set

$$v_m := \sup\{u_{m1}, \dots, u_{mm}\} = \sup\{u_{ij} \ : \ i, j \le m\}$$

Then $v_m \in \mathcal{E}_+(X)$, $(v_m)_{m\in\mathbb{N}}$ is increasing and $\sup v_m = f$. Hence

$$\int f \mathrm{d}\mu = \sup \int v_n \mathrm{d}\mu.$$

Using $v_n \le f_n$, we obtain

$$\int v_n \mathrm{d}\mu \le \int f_n \mathrm{d}\mu.$$

Hence

$$\int f \mathrm{d}\mu \le \sup_n \int f_n \mathrm{d}\mu.$$

□

2.7 Integral on positive measurable functions II

Theorem 2.10 (The Fatou lemma) *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathcal{M}_+(X)$. *Then*

$$\int \liminf_{n\to\infty} f_n \mathrm{d}\mu \le \liminf_{n\to\infty} \int f_n \mathrm{d}\mu.$$

Proof. Set $f := \liminf_{n\to\infty} f_n$, $g_n := \inf_{m\ge n} f_m$. We have $f, g_n \in \mathcal{M}_+(X)$ and $g_n \nearrow f$. Hence

$$\int f \mathrm{d}\mu = \sup_{n\in\mathbb{N}} \int g_n \mathrm{d}\mu.$$

$g_n \le f_m$ for $m \ge n$, therefore

$$\int g_n \mathrm{d}\mu \le \inf_{m \ge n} \int f_m \mathrm{d}\mu.$$

□

Proposition 2.11 *For any $f \in \mathcal{M}_+(X)$*

$$\int f \mathrm{d}\mu = 0 \Leftrightarrow f = 0 \ \mu\text{-almost everywhere.}$$

Proof. Set

$$M := \{f \neq 0\}.$$

$\Rightarrow$ Let

$$M_n := \{f \ge n^{-1}\}.$$

Then $M_n \in \mathcal{F}$ and

$$n^{-1}\mu(M_n) \le \int f \mathrm{d}\mu = 0.$$

Thus $\mu(M_n) = 0$. But $M_n \nearrow M$, so $\mu(M_n) \nearrow \mu(M)$. Hence $\mu(M) = 0$.

$\Leftarrow$ Set

$$f_n := \inf\{f, n\}$$

Then $f_n \in \mathcal{M}_+(\mathcal{F})$ and $f_n \nearrow f$. So $\int f_n \mathrm{d}\mu \nearrow \int f \mathrm{d}\mu$. But

$$f_n \le n 1_M,$$

therefore

$$\int f_n \mathrm{d}\mu \le n\mu(M) = 0.$$

Hence $\int f \mathrm{d}\mu = 0$. □

Theorem 2.12 *Let $f \in \mathcal{M}_+(X)$ and*

$$\int f \mathrm{d}\mu < \infty.$$

Then $f < \infty$ μ-a.e. and $\{f \neq 0\} \in \mathcal{F}_\mu^{\sigma \mathrm{f}}$.

Proof. Let $A := \{f = \infty\}$. Then $0 \le \infty 1_A \le f$. Hence $\infty\mu(A) \le \int f \mathrm{d}\mu$. □

2.8 Integral of functions with a varying sign

For $f : X \to [-\infty, \infty]$ we set

$$f_+ := \sup(f, 0), \ f_- := -\inf(f, 0).$$

Thus

$$f = f_+ - f_-, \quad |f| = f_+ + f_-.$$

Clearly, $f \in \mathcal{M}(X)$ iff $f_+, f_- \in \mathcal{M}_+(X)$.

Definition 2.13 *Let $f \in \mathcal{M}(X)$. Assume that one of the numbers $\int f_+ \mathrm{d}\mu$, $\int f_- \mathrm{d}\mu$ is finite. Then we say that the integral of f is well defined and*

$$\int f \mathrm{d}\mu := \int f_+ \mathrm{d}\mu - \int f_- \mathrm{d}\mu.$$

Theorem 2.14 *Let $f, g \in \mathcal{M}_+(X)$ and $f = g$ μ-a.e. Then*

$$\int f \mathrm{d}\mu = \int g \mathrm{d}\mu.$$

2.9 Transport of a measure—the change of variables in an integral

If $(X,\mathcal{F})$, $(X',\mathcal{F}')$ are spaces with σ-rings, μ is a measure on $(X,\mathcal{F})$ and $T : X \to X'$ is a measurable transformation, then $T_*\mu$ is the measure on $(X',\mathcal{F}')$ defined as

$$T_*\mu(A') := \mu(T^{-1}(A')),\ A' \in \mathcal{F}'.$$

Clearly, we then have the formula for $f' \in \mathcal{M}_+(X')$:

$$\int f'\mathrm{d}T_*\mu = \int f' \circ T\mathrm{d}\mu.$$

If T is injective, and μ' is a measure on $(X',\mathcal{F}')$, then we define the measure $T^*\mu'$ on $(X,\mathcal{F})$ by

$$T^*\mu'(A) := \mu'(T(A)),\ A \in \mathcal{F}.$$

and for $f \in \mathcal{M}_+(X)$:

$$\int f\mathrm{d}\mu' = \int f \circ T\mathrm{d}T^*\mu'.$$

2.10 Integrability

Definition 2.15 *Let $f \in \mathcal{M}(X)$. If*

$$\int f_+\mathrm{d}\mu < \infty,\ \int f_-\mathrm{d}\mu < \infty,$$

or equvalently, if $\int |f|\mathrm{d}\mu < \infty$, then we say that $f \in \mathcal{L}^1$, or integrable (in the sense of $\mathcal{L}^1$) and we write

$$f \in \mathcal{L}^1(\mu), \quad \int f\mathrm{d}\mu := \int f_+\mathrm{d}\mu - \int f_-\mathrm{d}\mu.$$

Proposition 2.16 (1) *If $f \in \mathcal{L}^1(\mu)$, $g \in \mathcal{M}(X)$ and $f = g$ μ-a.e., then $g \in \mathcal{L}^1(\mu)$.*

(2) *$f \in \mathcal{L}^1(\mu)$, then $|f| < \infty$ μ-a.e. and $\{f \neq 0\} \in \mathcal{F}_\mu^{\sigma\mathrm{f}}$.*

(3) *If $f \in \mathcal{M}(X)$, $g \in \mathcal{L}^1(\mu)$ and $|f| \leq g$ μ-a.e., then $f \in \mathcal{L}^1(\mu)$.*

Lemma 2.17 *If $u, v \in \mathcal{L}^1(\mu)$, $u, v \geq 0$ and*

$$f = u - v,$$

then $f \in \mathcal{L}^1(\mu)$.

Proof. We have

$$f \leq u \leq u + v, \quad -f \leq v \leq u + v.$$

Therefore,

$$|f| \leq u + v \in \mathcal{L}^1(\mu).$$

□

Proposition 2.18 *Let $f, g \in \mathcal{L}^1(\mu)$, $\alpha \in \mathbb{R}$. Then*

(1) *$\alpha f \in \mathcal{L}^1(\mu)$;*

(2) *$f + g \in \mathcal{L}^1(\mu)$;*

(3) $\sup(f,g),\ \inf(f,g) \in \mathcal{L}^1(\mu)$;

(4) $f \le g \ \Rightarrow\ \int f \mathrm{d}\mu \le \int g \mathrm{d}\mu$

(5) $|\int f\mathrm{d}\mu| \le \int |f|\mathrm{d}\mu$.

Proof. (1) We have,

$$(\alpha f)_+ = \alpha f_+,\ (\alpha f)_- = \alpha f_-, \qquad \alpha > 0$$
$$(\alpha f)_+ = |\alpha| f_-,\ (\alpha f)_- = |\alpha| f_+, \quad \alpha < 0.$$

Hence $\alpha f \in \mathcal{L}^1(\mu)$.

(2) Next we write

$$f + g = f_+ - f_- + g_+ - g_-, \tag{2.2}$$

put

$$u := f_+ + g_+ \in \mathcal{L}^1(\mu),\ v := f_- + g_- \in \mathcal{L}^1(\mu),$$

and use Lemma 2.17, which shows that (2.2) belongs to $L^1(\mu)$.

(3) The estimates

$$|\sup(f,g)| \le |f| + |g|,\ |\inf(f,g)| \le |f| + |g|$$

and $|f| + |g| \in \mathcal{L}^1(\mu)$ show that $\sup(f,g), \inf(f,g) \in \mathcal{L}^1(\mu)$.

(4) $f \le g$ implies that $f_+ \le g_+$ and $f_- \ge g_-$. Hence

$$\int f\mathrm{d}\mu \le \int g\mathrm{d}\mu. \tag{2.3}$$

(5) We have $f \le |f|$ and $-f \le |f|$. Therefore, if we put in (2.3) $g = |f|$ we get

$$|\textstyle\int f\mathrm{d}\mu| \le \int |f|\mathrm{d}\mu.$$

□

2.11 The Hölder and Minkowski inequalities

Let $1 \le p \le \infty$.

Definition 2.19 *Let $f \in \mathcal{M}(X)$. Put*

$$\|f\|_p := (\textstyle\int |f|^p \mathrm{d}\mu)^{\frac{1}{p}},\ 1 \le p < \infty,$$
$$\|f\|_\infty := \inf\{\sup\{|f(x)|\ :\ x \in X\backslash N\}\ :\ N \in \mathcal{F}^0_\mu\}.$$

We define $\mathcal{L}^p(X,\mu)$ as the space of $f \in \mathcal{M}(X)$ such that $\|f\|_p < \infty$.

Theorem 2.20 (The Hölder inequality) *Let $1 \le p, q \le \infty$,*

$$\int |fg|\mathrm{d}\mu \le \|f\|_p \|g\|_q,\ \frac{1}{p} + \frac{1}{q} = 1.$$

Proof. Assume first that $1 < p, q < \infty$. By the convexity of e^x,

$$\frac{a}{p} + \frac{b}{q} \ge a^{\frac{1}{p}} b^{\frac{1}{q}}.$$

We substitute

$$a = \frac{|f|^p(x)}{\|f\|_p^p}, \quad b = \frac{|g|^q(x)}{\|f\|_q^q}.$$

We get

$$\frac{1}{p}\frac{|f|^p(x)}{\|f\|_p^p} + \frac{1}{q}\frac{|g|^q(x)}{\|f\|_q^q} \geq \frac{|f|(x)|g|(x)}{\|f\|_p\|g\|_q}.$$

We integrate

$$1 \geq \frac{\int |f||g|\mathrm{d}\mu}{\|f\|_p\|g\|_q}.$$

The case $p = 1$, $q = \infty$ is straightforward. □

Theorem 2.21 *Let* $1 \leq, r \leq \infty$, $0 \leq \alpha \leq 1$, $\frac{\alpha}{q} + \frac{1-\alpha}{r} = \frac{1}{p}$. *Then*

$$\|f\|_p \leq \|f\|_q^\alpha \|\|f\|_r^{1-\alpha} \leq \alpha\|f\|_q + (1-\alpha)\|f\|_r.$$

Proof.

$$\int |f|^p = \int |f|^{p\alpha}|f|^{p(1-\alpha)} \leq \||f|^{p\alpha}\|_{\frac{q}{\alpha p}} \||f|^{p(1-\alpha)}\|_{\frac{r}{(1-\alpha)p}}.$$

Theorem 2.22 (The generalized Minkowski inequality) *Let* X, Y *be spaces with measures* μ *and* ν, $1 \leq p < \infty$.

$$\left(\int \mathrm{d}\nu(y)\left|\int f(x,y)\mathrm{d}\mu(x)\right|^p\right)^{\frac{1}{p}} \leq \int \mathrm{d}\mu(x)\left(\int |f|^p(x,y)\mathrm{d}\nu(y)\right)^{\frac{1}{p}}$$

Proof. Let $\frac{1}{p} + \frac{1}{q} = 1$. It suffices to assume that $f \geq 0$. We will restrict ourselves to the case $p > 1$.

$$\begin{aligned}
& \int \mathrm{d}y \left(\int f(x,y)\mathrm{d}x\right)^p \\
= \;& \int \mathrm{d}y \left(\int f(x_1,y)\mathrm{d}x_1\right)^{p-1}\left(\int f(x_2,y)\mathrm{d}x_2\right) \\
= \;& \int \mathrm{d}x_2 \left(\int \mathrm{d}y \left(\int f(x_1,y)\mathrm{d}x_1\right)^{p-1} f(x_2,y)\right) \\
\leq \;& \int \mathrm{d}x_2 \left(\int \mathrm{d}y_1 \left(\int f(x_1,y_1)\mathrm{d}x_1\right)^{q(p-1)}\right)^{\frac{1}{q}} \left(\int f^p(x_2,y_2)\mathrm{d}y_2\right)^{\frac{1}{p}} \quad \text{(the Hölder inequality)} \\
= \;& \left(\int \mathrm{d}y_1 \left(\int f(x_1,y_1)\mathrm{d}x_1\right)^p\right)^{1-\frac{1}{p}} \left(\int \mathrm{d}x_2 \left(\int f^p(x_2,y_2)\mathrm{d}y_2\right)^{\frac{1}{p}}\right).
\end{aligned}$$

Then we divide by the first factor on the left. □

Corollary 2.23 *Setting* $X = \{1,2\}$ *with the counting measure we get*

$$\|f_1 + f_2\|_p \leq \|f_1\|_p + \|f_2\|_p.$$

2.12 Dominated Convergence Theorem

Theorem 2.24 (Lebesgue) *Assume that* $1 \leq p < \infty$, $g, f_n \in \mathcal{L}^p(\mu)$, f_n *is* μ*-a.e. pointwise convergent and*

$$|f_n| \leq g.$$

Then there exists $f \in \mathcal{L}^p(\mu)$ *such that* $f_n \to f$ μ*-a.e. and*

$$\|f - f_n\|_p \to 0.$$

Proof. We define

$$f(x) := \begin{cases} \lim_{n\to\infty} f_n(x) & \text{if } \lim_{n\to\infty} f_n(x) \text{ exists} \\ 0 & \text{if } \lim_{n\to\infty} f_n(x) \text{ does not exists} \end{cases}$$

Then $f \in \mathcal{M}(X)$ and

$$|f| \leq g \ \mu\text{-a.e.}$$

hence $f \in \mathcal{L}^p(\mu)$.

Set

$$h_n := |f - f_n|^p.$$

Then

$$0 \leq h_n \leq (|f_n| + |f|)^p \leq |2g|^p =: h.$$

Clearly, h and therefore also h_n are integrable. Besides, μ-a.e.

$$h = \lim_{n\to\infty}(h - h_n)$$

Therefore, by the Fatou Lemma applied to the sequence $h - h_n$ we get

$$\begin{aligned} \int h \mathrm{d}\mu &= \int \lim_{n\to\infty}(h - h_n)\mathrm{d}\mu \\ &\leq \liminf_{n\to\infty} \int (h - h_n)\mathrm{d}\mu \\ &= \int h\mathrm{d}\mu - \limsup_{n\to\infty} \int h_n \mathrm{d}\mu. \end{aligned}$$

Thus

$$\limsup_{n\to\infty} \int h_n \mathrm{d}\mu \leq 0$$

Using $h_n \geq 0$ we get

$$\lim_{n\to\infty} \int h_n \mathrm{d}\mu = 0.$$

□

Theorem 2.25 (Scheffe's lemma) *Let $f, f_1, f_2, \dots \in \mathcal{L}^1(\mu)$ and $f_n \to f$ a.e. Then*

$$\int |f_n - f|\mathrm{d}\mu \to 0 \ \Leftrightarrow \ \int |f_n|\mathrm{d}\mu \to \int |f|\mathrm{d}\mu.$$

2.13 L^p spaces

Theorem 2.26 (Riesz-Fischer) *Let $1 \leq p \leq \infty$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{L}^p(\mu)$ satisfying the Cauchy condition, that is for any $\epsilon > 0$ there exists N such that for $n, m \geq N$*

$$\|f_n - f_m\|_p \leq \epsilon.$$

Then there exists $f \in \mathcal{L}^p(\mu)$ such that

$$\|f - f_n\|_p \to 0$$

We will also find a subsequence of $(f_n)_{n\in\mathbb{N}}$ pointwise convergent μ-a.e. to f.

Proof. There exists a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that $\|f_{n_{k+1}} - f_{n_k}\|_p \leq 2^{-k}$, for any k. We set

$$g_k := f_{n_{k+1}} - f_{n_k},\ g := \sum_{k=1}^{\infty} |g_k|.$$

Then

$$\|g\|_p \leq \sum_{k=1}^{\infty} \|g_k\|_p \leq \sum_{k=1}^{\infty} 2^{-k} = 1.$$

Hence $g \in \mathcal{L}^p(\mu)$ and therefore g is finite μ-a.e. Hence the series $\sum_{k=1}^{\infty} g_k$ is μ-a.e. convergent. This means that the sequence $(f_{n_k})_{k\in\mathbb{N}}$ is μ-a.e. convergent.

In the case $p = \infty$ it is sufficient to take the limit and to check that it is the limit in the $\mathcal{L}_\infty$ sense.

In the case $1 \leq p < \infty$, we need to apply the Lebesgue theorem. We first check hat

$$|f_{n_k}| \leq |f_{n_1} + g_1 + \cdots + g_{k-1}| \leq |f_{n_1}| + g$$

and $g + |f_{n_1}| \in \mathcal{L}^p(\mu)$. Therefore, we will find $f \in \mathcal{L}^p(\mu)$ such that

$$\lim_{k\to\infty} \|f_{n_k} - f\|_p = 0,$$

$$\lim_{k\to\infty} f_{n_k} = f \ \mu\text{-a.e.}$$

Using the Cauchy condition we get

$$\lim_{n\to\infty} \|f_n - f\|_p = 0.$$

□

Let $\mathcal{N}(\mu)$ be the space of functions in $\mathcal{M}(X)$ equal 0 μ-a.e. (That is, $\mathcal{N}(\mu) = \mathcal{M}(X, \mathcal{F}_\mu^0)$).

Theorem 2.27 *Let* $1 \leq p \leq \infty$

(1) $\mathcal{N}(\mu)$ *is a vector subspace of* $\mathcal{L}^p(\mu)$ *such that* $\|f\|_p = 0,\ f \in \mathcal{L}^p(\mu)$ *iff* $f \in \mathcal{N}(\mu)$.

(2) *Therefore,*

$$\|f + \mathcal{N}(\mu)\|_p := \|f\|_p$$

defines a norm in

$$L^p(\mu) := \mathcal{L}^p(\mu)/\mathcal{N}(\mu).$$

(3) $L^p(\mu)$ *is a Banach space.*

(4) *Elementary functions are dense in* $L^p(\mu)$.

(5) $f, g \in L^p(\mu)$, $0 \leq f \leq g$ *a.e.* $\Rightarrow \|f\|_p \leq \|g\|_p$.

(6) *For* $1 \leq p < \infty$, *if we restrict the measure to* $\mu - \sigma$*-finite sets, we obtain the same* $L^p(\mu)$ *space.*

(7) *For* $1 \leq p < \infty$, $L^p(\mu)$ *is a complete lattice.* $L^\infty(\mu)$ *is a countably complete lattice. (Later on, we will show that under some additional conditions it is also a complete lattice).*

Proof. (4) Let $f \in \mathcal{L}^p(\mu)$. Then $f_+, f_- \in \mathcal{L}^p(\mu)$. We know that there exist sequences $u_{\pm,n} \in \mathcal{E}_+(\mu)$ with $u_{\pm,n} \nearrow f_\pm$. By the Lebesgue theorem, $\|f_\pm - u_{\pm,n}\|_p \to 0$. □

2.14 Egorov theorem

Theorem 2.28 *Let $f, f_1, f_2, \dots \in \mathcal{M}(X)$. Consider the following statements:*

(1) *$f_n(x) \to f(x)$ for a.a. $x \in X$.*

(2) *For all $\epsilon > 0$*

$$\mu\Big(\bigcap_{n=1}^{\infty} \bigcup_{j=n}^{\infty} |f_j - f| \geq \epsilon\Big) = 0.$$

(3) *For all $\epsilon > 0$*

$$\lim_{n\to\infty} \mu\Big(\bigcup_{j=n}^{\infty} |f_j - f| \geq \epsilon\Big) = 0.$$

(4) *For every $\delta > 0$, there exists $A \in \mathcal{F}$ with $\mu(A) < \delta$ and*

$$\lim_{n\to\infty} \sup_{x \in X\backslash A} |f - f_n| = 0.$$

Then (1)⇔(2)⇐(3)⇐(4). If $\mu(X) < \infty$, then (1)⇔(2)⇔(3)⇔(4). (The implication (1)⇒(4) is called the **Egorov theorem***).*

Proof. Only (3)⇒(4) is not immediate, and we are going to prove this implication. Let $\delta > 0$ and $k \in \mathbb{N}$. Then by (3) there exists n_k such that for

$$B_k := \bigcup_{j=n_k}^{\infty} \{|f - f_j| > 1/k\},$$

we have $\mu(B_k) < \delta 2^{-k}$. Set $A := \bigcup\limits_{k=1}^{\infty} B_k$. We have $\mu(A) < \delta$ and on $X\backslash A$, $|f(x) - f_j(x)| \leq 1/k$ for $j \geq n_k$. Hence on $X\backslash A$, f_n converges uniformly to f. □

3 Extension of a measure

3.1 Hereditary families

$\mathcal{T} \subset 2^X$. We say that $\mathcal{T}$ is hereditary if $A \subset B \in \mathcal{T}$ implies $A \in \mathcal{T}$. For any $\mathcal{T}$ we denote by $\mathrm{Her}(\mathcal{T})$ the smallest hereditary family containing $\mathcal{T}$.

Theorem 3.1 (1) *Let $\mathcal{R}$ be a ring. Then $\mathrm{Her}(\mathcal{R})$ is a ring.*

(2) *Let $\mathcal{T} \subset 2^X$. Then $Q \in \mathrm{Her}(\sigma{-}\mathrm{Ring}(\mathcal{T}))$ iff there exist $A_1, \dots, A_n \in \mathcal{T}$ such that $Q \subset A_1 \cup \dots \cup A_n$.*

Theorem 3.2 (1) *If $\mathcal{I}$ is a σ-ring, then $\mathrm{Her}(\mathcal{I})$ is a σ-ring.*

(2) *If $\mathcal{T} \subset 2^X$, then $Q \in \mathrm{Her}(\mathcal{T})$ iff there exist $A_1, A_2, \dots \in \mathcal{T}$ such that $Q \subset \bigcup\limits_{i=1}^{\infty} A_i$.*

Theorem 3.3 *If $\mathcal{I} \subset \mathcal{F}$ are σ-rings and $\mathcal{I}$ is an ideal in $\mathcal{F}$, then*

$$\begin{aligned}\sigma{-}\mathrm{Ring}\,(\mathcal{F} \cup \mathrm{Her}(\mathcal{I})) &= \{A \cup N \ : \ A \in \mathcal{F}_1, \quad N \in \mathrm{Her}(\mathcal{I})\} \\ &= \{A\backslash N \ : \ A \in \mathcal{F}, \quad N \in \mathrm{Her}(\mathcal{I})\}.\end{aligned}$$

3.2 Extension of a measure by null sets

Let X be a set and $(\mathcal{F}_2, \mu_2)$, $(\mathcal{F}_1, \mu_1)$ be measures on X. We say that $(\mathcal{F}_2, \mu_2)$ extends $(\mathcal{F}_1, \mu_1)$ by null sets iff

(1) $\sigma-\mathrm{Ring}(\mathcal{F}_1 \cup (\mathcal{F}_2)^0_{\mu_2}) = \mathcal{F}_2$;

(2) $\mu_2\Big|_{\mathcal{F}_1} = \mu_1$.

Theorem 3.4 *Suppose that $(\mathcal{F}_2, \mu_2)$ extends $(\mathcal{F}_1, \mu_1)$ by null sets. Then*

(1) $\mathcal{F}_2 = \{A \cup N \ : \ A \in \mathcal{F}_1, \quad N \in (\mathcal{F}_2)^0_{\mu_2})\}$.

(2) $\mathcal{N}(\mu_2) \cap \mathcal{M}(\mathcal{F}_1) = \mathcal{N}(\mu_2)$. *Hence, we can identify $\mathcal{M}(X, \mathcal{F}_1)/\mathcal{N}(\mu_1)$ with $\mathcal{M}(X, \mathcal{F}_2)/\mathcal{N}(\mu_2)$ in the obvious way.*

(3) *We can identify $L^p(\mu_1)$ with $L^p(\mu_2)$.*

3.3 Complete measures

Let $(X, \mathcal{F}, \mu)$ be a space with a measure. We say that the measure μ is complete if $\mathcal{F}^0_\mu$ is hereditary.

Let μ be a not necessarily complete measure. Set

$$\mathcal{F}^{\mathrm{cp}}_\mu := \{A \cup N \ : \ A \in \mathcal{F}, \ N \in \mathrm{Her}(\mathcal{F}^0_\mu)\} = \sigma-\mathrm{Ring}(\mathcal{F} \cup \mathrm{Her}(\mathcal{F}^0_\mu)).$$

Define $\mu^{\mathrm{cp}} : \mathcal{F}^{\mathrm{cp}}_\mu \to [0, \infty]$,

$$\mu^{\mathrm{cp}}(A \cup N) := \mu(A), \quad A \in \mathcal{F}, \quad N \in \mathrm{Her}(\mathcal{F}^0_\mu).$$

Theorem 3.5 (1) *$\mathcal{F}^{\mathrm{cp}}_\mu$ is a σ-ring and μ^{cp} is a complete measure.*

(2) *μ^{cp} is an extension of μ by null sets.*

(3) *μ^{cp} is the unique extension of μ to a content on $\mathcal{F}^{\mathrm{cp}}$.*

(4) *Every extension of $(\mathcal{F}, \mu)$ to a complete measure is an extension of $(\mathcal{F}^{\mathrm{cp}}, \mu^{\mathrm{cp}})$.*

We will call $(X, \mathcal{F}^{\mathrm{cp}}, \mu^{\mathrm{cp}})$ the completion of μ.

3.4 External measures

Definition 3.6 *A function $\mu^* : 2^X \to [0, \infty]$ is called an external measure if*

(1) $\mu^*(\emptyset) = 0$,

(2) $Q_1, Q_2 \cdots \in 2^X \ \Rightarrow \ \mu^*(\cup_{n=1}^\infty Q_n) \leq \sum_{n=1}^\infty \mu^*(Q_n)$.

(3) $Q \subset P, \ Q, P \in 2^X, \Rightarrow \mu^*(Q) \leq \mu^*(P)$.

Clearly, every measure on $(X, 2^X)$ is an external measure.

For any set X the function that assigns 0 to $\emptyset$ and 1 to a nonempty set is an external measure on X. It is not a measure if X contains more than one element.

Definition 3.7 *Let μ^* be an external measure. We say that $A \in 2^X$ is measurable wrt μ^*, if one of the following two equivalent conditions holds*

$$\mu^*(Q) \geq \mu^*(Q \cap A) + \mu^*(Q \backslash A), \ Q \in 2^X; \tag{3.4}$$

$$\mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q \backslash A), \ Q \in 2^X. \tag{3.5}$$

(The equivalence of the conditions (3.5) follows from (2) of the definition of the external measure applied to $Q \cap A$ and $Q \backslash A$.)

The family of sets measurable wrt μ^* will be denoted $\mathcal{F}^{\mathrm{ms}}$. μ^* restricted to $\mathcal{F}^{\mathrm{ms}}$ will be denoted μ^{ms}.

Theorem 3.8 *Let μ^* be an external measure on X. Let $\mathcal{F}^{\mathrm{ms}}$ and μ^{ms} be defined as above. Then*

(1) *$\mathcal{F}^{\mathrm{ms}}$ is a σ-field*

(2) *$(X, \mathcal{F}^{\mathrm{ms}}, \mu^{\mathrm{ms}})$ is a complete measure;*

(3)

$$A \in 2^X, \quad \mu^*(A) = 0 \Leftrightarrow A \in \mathcal{F}^{\mathrm{ms}}, \quad \mu^{\mathrm{ms}}(A) = 0.$$

Proof. **Step 0.** Clearly, $\emptyset, X \in \mathcal{F}^{\mathrm{ms}}$.
Step 1. $A, B \in \mathcal{F}^{\mathrm{ms}} \Rightarrow A \backslash B \in \mathcal{F}^{\mathrm{ms}}$.

Let $Q \in 2^X$.

Applying the measurability condition to $Q \cap A$ and B we get

$$\mu^*(Q \cap A) = \mu^*(Q \cap A \cap B) + \mu^*(Q \cap A \backslash B). \tag{3.6}$$

(Note that $(Q \cap B) \backslash A = Q \cap (B \backslash A)$). Then we apply it to $Q \backslash A$ and B to get

$$\mu^*(Q \backslash A) = \mu^*(Q \cap B \backslash A) + \mu^*(Q \backslash (A \cup B)). \tag{3.7}$$

Thus by (3.5)

$$\mu^*(Q) = \mu^*(Q \cap A \cap B) + \mu^*(Q \cap A \backslash B) + \mu^*(Q \cap B \backslash A) + \mu^*(Q \backslash (A \cup B)). \tag{3.8}$$

Applying the measurability condition to $Q \backslash (A \backslash B)$ and B gives

$$\mu^*(Q \backslash (A \backslash B)) = \mu^*(Q \backslash (A \cup B)) + \mu^*(Q \cap B). \tag{3.9}$$

Applying it to $Q \cap B$ and A we get

$$\mu^*(Q \cap B) = \mu^*(Q \cap B \backslash A) + \mu^*(Q \cap A \cap B). \tag{3.10}$$

Inserting (3.10) into (3.9) gives

$$\mu^*(Q \backslash (A \backslash B)) = \mu^*(Q \backslash (A \cup B)) + \mu^*(Q \cap B \backslash A) + \mu^*(Q \cap A \cap B). \tag{3.11}$$

Thus, by (3.8),

$$\mu^*(Q) = \mu^*(Q \backslash (A \backslash B)) + \mu^*(Q \cap (A \backslash B)). \tag{3.12}$$

Hence $A \backslash B \in \mathcal{F}^{\mathrm{ms}}$.
Step 2. $A, B \in \mathcal{F}^{\mathrm{ms}} \Rightarrow B \cup A \in \mathcal{F}^{\mathrm{ms}}$.

We have, applying the measurability condition to $Q \cap (A \cup B)$ and A.

$$\mu^*(Q \cap (A \cup B)) = \mu^*(Q \cap A) + \mu^*(Q \cap B \backslash A). \tag{3.13}$$

Inserting (3.6) into (3.13) we get

$$\mu^*(Q \cap (A \cup B)) = \mu^*(Q \cap A \cap B) + \mu^*(Q \cap A \backslash B) + \mu^*(Q \cap B \backslash A). \tag{3.14}$$

Hence

$$\mu^*(Q) = \mu^*(Q \backslash (A \cup B)) + \mu^*(Q \cap (A \cup B)). \tag{3.15}$$

Therefore $A \cup B \in \mathcal{F}^{\mathrm{ms}}$. Thus we proved that $\mathcal{F}^{\mathrm{ms}}$ is a field.

Step 3.

$$A_1, A_2, \dots \in \mathcal{F}^{\rm ms} \Rightarrow \cup_{j=1}^{\infty} A_j =: A \in \mathcal{F}^{\rm ms}. \tag{3.16}$$

It suffices to assume that A_j are disjoint. For any n we have

$$\begin{aligned} \mu^*(Q) &\geq \mu^*\big(\cup_{j=1}^{n}(Q \cap A_j) \cup Q\backslash A)\big) \\ &= \textstyle\sum_{j=1}^{n} \mu^*(Q \cap A_j) + \mu^*(Q\backslash A)). \end{aligned} \tag{3.17}$$

Since n was arbitrary,

$$\begin{aligned} \mu^*(Q) &\geq \textstyle\sum_{j=1}^{\infty} \mu^*(Q \cap A_j) + \mu^*(Q\backslash A)). \\ &\geq \mu^*(Q \cap A) + \mu^*(Q\backslash A)), \end{aligned} \tag{3.18}$$

Hence, by the equivalence of (3.4) and (3.5) we get

$$\mu^*(Q) = \mu^*(Q \cap A) + \mu^*(Q\backslash A)),$$

which shows $A \in \mathcal{F}^{\rm ms}$.

Step 4. As a by-product we get

$$\mu^*(Q \cap A) = \sum_{j=1}^{\infty} \mu^*(Q \cap A_j).$$

Putting $Q = A$ we see that

$$\mu^*(A) = \sum_{j=1}^{\infty} \mu^*(A_j),$$

hence μ^* restricted to $\mathcal{F}^{\rm ms}$ is a measure.

Step 5. Let $A \in 2^X$ and $\mu^*(A) = 0$. Let $Q \in 2^X$. Then $Q \cap A \subset A$, hence $\mu^*(Q \cap A) = 0$. Moreover, $Q\backslash A \subset Q$, hence $\mu^*(Q\backslash A) \leq \mu^*(Q)$. Therefore,

$$\mu^*(Q) \geq \mu^*(Q \cap A) + \mu^*(Q\backslash A).$$

Hence $A \in \mathcal{F}^{\rm ms}$ and $\mu^{\rm ms}(A) = 0$. This proves (3), which implies the completeness of the measure $\mu^{\rm ms}$. □

3.5 External measure generated by a measure

Theorem 3.9 *Let $(X, \mathcal{F}, \mu)$ be a measure. For $Q \in 2^X$ define*

$$\mu^*(Q) := \inf \{\mu(A) \ : A \in \mathcal{F},\ A \supset Q\}$$

Then

(1) *μ^* is an external measure;*

(2) *$\mu^* = \mu$ on $\mathcal{F}$.*

(3) *Let $\mathcal{F}_\mu^{\rm ms}$, $\mu^{\rm ms}$ be defined from μ^* as in the previous subsection. Then $\mathcal{F}_\mu^{\rm ms}$ is a σ-field containing $\mathcal{F}$.*

(4) *In the definition of μ^* we can replace $\mathcal{F}$ with $\mathcal{F}_\mu^{\rm f}$, $\mathcal{F}_\mu^{\sigma\rm f}$, $\mathcal{F}_\mu^{\rm cp}$, $\mathcal{F}_\mu^{\rm ms}$, etc., obtaining the same μ^*.*

Proof. (1) The properties (1) and (3) of the definition of an external measure are obvious. Let us show the property (2).

Let $Q_1, Q_2, \dots \in 2^X$. For any $\epsilon > 0$ we will find a sequence $A_1, A_2, \dots \in \mathcal{F}$ such that

$$Q_n \subset A_n,$$
$$\mu(A_n) \le \mu^*(Q_n) + 2^{-n}\epsilon.$$

Then

$$\mu^*(\cup_{n=1}^{\infty} Q_n) \le \mu(\cup_{n=1}^{\infty} A_n) \le \sum_{n=1}^{\infty} \mu(A_n) \le \sum_{n=1}^{\infty} \mu^*(Q_n) + \epsilon.$$

Hence

$$\mu^*(\cup_{n=1}^{\infty} Q_n) \le \sum_{n=1}^{\infty} \mu^*(Q_n).$$

(2) is obvious.

Let us prove (3). Let $B \in \mathcal{F}$, $Q \in 2^X$. For any $\epsilon > 0$ and suitable $A \in \mathcal{F}$ such that $Q \subset A$ we have

$$\begin{aligned} \mu^*(Q) &\ge \mu(A) - \epsilon \\ &= \mu(A \cap B) + \mu(A \backslash B) - \epsilon \ge \mu^*(Q \cap B) + \mu^*(Q \backslash B) - \epsilon. \end{aligned}$$

Therefore,

$$\mu^*(Q) \ge \mu^*(Q \cap B) + \mu^*(Q \backslash B).$$

Thus, $B \in \mathcal{F}_\mu^{\mathrm{ms}}$. □

The measure $(X, \mathcal{F}^{\mathrm{ms}}, \mu^{\mathrm{ms}})$ is called the Caratheodory completion of the measure $(X, \mathcal{F}, \mu)$.

A measure $(X, \mathcal{F}, \mu)$ is called Caratheodory complete if it coincides with its Caratheodory completion.

The Caratheodory completion of a measure is always Caratheodory complete.

Theorem 3.10 *Suppose that $(X, \mathcal{F}, \mu)$ be a set with a σ-field and a finite measure. Let μ^* be the corresponding outer measure and let $S \subset X$ satisfy $\mu^*(S) = \mu(X)$. For $A \in \mathcal{F}\Big|_S$ set*

$$\mu_S(A) := \mu^*(A).$$

Then $(S, \mathcal{F}\Big|_S, \mu_S)$ is a measure, which is isomorphic to μ modulo sets of measure zero,

3.6 Extension of a measure to localizable sets

Let $\mathcal{F} \subset 2^X$ be a σ-ring. Let

$$\mathcal{F}^{\mathrm{loc}} := \{A \in 2^X \ : \ B \in \mathcal{F} \text{ implies } A \cap B \in \mathcal{F}\}.$$

We say that $\mathcal{F}^{\mathrm{loc}}$ is the family of sets localizable in $\mathcal{F}$. It is a σ-field and $\mathcal{F}$ is its ideal.

Let $(X, \mathcal{F}, \mu)$ be a measure We can extend canonically μ to $\mathcal{F}^{\mathrm{loc}}$ by setting

$$\mu^{\mathrm{loc}}(A) := \begin{cases} \mu(A) & A \in \mathcal{F} \\ \infty & A \in \mathcal{F}^{\mathrm{loc}} \backslash \mathcal{F}. \end{cases}$$

Note that $(\mathcal{F}_{\mathrm{loc}})_{\mu_{\mathrm{loc}}}^{\sigma\mathrm{f}} = \mathcal{F}_\mu^{\sigma\mathrm{f}}$, so the L^p spaces for both measures are the same.

Theorem 3.11 *Suppose $(X, \mathcal{F}, \mu)$ and $(X, \mathcal{F}_\mu^{\mathrm{ms}}, \mu^{\mathrm{ms}})$ are as in Theorem 3.9. Then $(\mathcal{F}_\mu^{\mathrm{ms}}, \mu^{\mathrm{ms}})$ can be obtained by applying consecutively to $(\mathcal{F}, \mu)$ the following constructions:*
1. restricting to σ-finite subsets,
2. completion
3. extending to localizable sets.

3.7 Sum-finite measures

Let $(X, \mathcal{F}, \mu)$ be a measure. Define the set of locally μ-measurable sets by

$$\begin{aligned} \mathcal{F}_\mu^{\mathrm{loc}} &:= \{A \in 2^X \ : \ A \cap B \in \mathcal{F}, \ B \in \mathcal{F}_\mu^{\sigma\mathrm{f}}\} \\ &= \{A \in 2^X \ : \ A \cap B \in \mathcal{F}, \ B \in \mathcal{F}_\mu^{\mathrm{f}}\}. \end{aligned}$$

In other words, $\mathcal{F}_\mu^{\mathrm{loc}}$ is the family of all sets localizable in $\mathcal{F}_\mu^{\sigma\mathrm{f}}$ (or in $\mathcal{F}_\mu^{\mathrm{f}}$). Clearly, $\mathcal{F}_\mu^{\mathrm{loc}}$ is a σ-field containing $\mathcal{F}$ as an ideal.

A family $\{X_i \ : \ i \in I\}$ of disjoint elements of $\mathcal{F}_\mu^{\mathrm{f}}$ such that $\underset{i\in I}{\cup} X_i = X$ and

$$\mu(A) = \sum_{i \in I} \mu(A \cap X_i), \quad A \in \mathcal{F},$$

is called a localizing family for μ.

We say that the measure $(X, \mathcal{F}, \mu)$ is sum-finite if

(1) $\mathcal{F} = \mathcal{F}_\mu^{\mathrm{loc}}$;

(2) There there exists a localizing family for μ.

Clearly, every σ-finite measure is sum-finite.

Theorem 3.12 *If $\{X_i \ : \ i \in I\}$ and $\{Y_j \ : \ j \in J\}$ are localizing families for μ, then so is $\{X_i \cap Y_j \ : \ (i,j) \in I \times J\}$*

Theorem 3.13 *If $(X, \mathcal{F}, \mu)$ is a measure possessing a localizing family $\{X_i \ : \ i \in I\}$, then if for $A \in \mathcal{F}_\mu^{\mathrm{loc}}$ we set*

$$\mu^{\mathrm{loc}}(A) = \sum_{i \in I} \mu(A \cap X_i),$$

then $(X, \mathcal{F}_\mu^{\mathrm{loc}}, \mu^{\mathrm{loc}})$ is a sum-finite measure.

3.8 Boolean rings

We say that $(\mathcal{R}, \Delta, \emptyset, \cap)$ is a Boolean ring if it is an additive ring where all its elements are idempotent, that is $A \cap A = A$, $A \in \mathcal{R}$. We then set

$$A \cup B := (A \Delta B) \Delta (A \cap B), \quad A \backslash B := A \Delta (A \cap B),$$

$$A \subset B \ \Leftrightarrow \ B \supset A \ \Leftrightarrow \ A = A \cap B.$$

If there exists an identity element for $\cap$, called X, then $(\mathcal{R}, \Delta, \emptyset, \cap, X)$ is called a Boolean field.

Clearly, every ring/field in 2^X is a Boolean ring/field.

In the obvious way we introduce the notion of Boolean σ-rings, Boolean σ-fields, etc. In what follows we concentrate on σ-rings/fields.

If $\mathcal{I} \subset \mathcal{F}$ are σ-rings and $\mathcal{I}$ is an ideal in $\mathcal{F}$, then $\mathcal{F}/\mathcal{I}$ is a Boolean σ-ring.

Theorem 3.14 *Le $\mathcal{F} \subset \mathcal{F}_1$ be σ-rings. Let $\mathcal{I}_1 \subset \mathcal{F}_1$ be a σ-ring, which is an ideal in $\mathcal{F}_1$. Let $\mathcal{I} := \mathcal{I}_1 \cap \mathcal{F}$, which is clearly a σ-ring and an ideal in $\mathcal{F}$. Then the σ-rings $\mathcal{F}/\mathcal{I}$ and $\mathcal{F}_1/\mathcal{I}_1$ are in the obvious way isomorphic to one another.*

If $\mathcal{F}$ is a Boolean σ-ring, we can define the space $\mathcal{M}(\mathcal{F})$ as the set of all function

$$[-\infty, 0[\cup]0, \infty]\alpha \mapsto F(\alpha) \to \mathcal{F}$$

such that for $\alpha \neq \beta$, $F(\alpha) \cap F(\beta) = \emptyset$. If $\mathcal{F} \subset 2^X$, then we identify $f \in \mathcal{M}(X, \mathcal{F})$ with $F \in \mathcal{M}(\mathcal{F})$ where $F(\alpha) = f^{-1}(\{\alpha\})$.

If $\mathcal{F}$ is a Boolean σ-ring and $F \in \mathcal{M}(\mathcal{F})$, we set $\|F\|_\infty := \sup\{|\alpha| \ : \ F(\alpha) \neq \emptyset\}$ and define the space $L^\infty(\mathcal{F})$. If $\mathcal{I} \subset \mathcal{F} \subset 2^X$ are σ-rings and $\mathcal{I}$ is an ideal in $\mathcal{F}$, then we can identify $L^\infty(X, \mathcal{F}, \mathcal{I})$ with $L^\infty(\mathcal{F}/\mathcal{I})$.

3.9 Measures on Boolean rings

We can consider measures on Boolean σ-rings as well. Clearly, one can define L^p spaces for measures on Boolean rings.

If $(\mathcal{F}, \mu)$ is a measure on a Boolean σ-ring, we say that it is faithful if $A \in \mathcal{F}$, $\mu(A) = 0$ implies $A = \emptyset$.

If $\mathcal{F}^0_\mu$ is the family of zero sets, then we can define $\tilde{\mathcal{F}} := \mathcal{F}/\mathcal{F}^0_\mu$ and $\tilde{\mu}(A\Delta N) := \mu(A)$ for $N \in \mathcal{F}^0_\mu$. Then $(\tilde{\mathcal{F}}, \tilde{\mu})$ is a faithful measure.

Theorem 3.15 *Let X be a set and $\mathcal{F} \subset \mathcal{F}_1$ are σ-rings over X. Let $(\mathcal{F}, \mu)$ and $(\mathcal{F}_1, \mu_1)$ be measures such that*

(1) $\mu = \mu_1$ *on* $\mathcal{F}$*;*

(2) $\sigma-\text{Ring}\left(\mathcal{F} \cup (\mathcal{F}_1)^0_{\mu_1}\right) = \mathcal{F}_1$.

If $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}_1}$ are faithful measures defined as above, then they are isomorphic.

4 Construction and uniqueness of a measure

4.1 Dynkin classes

We say that $\mathcal{T} \subset 2^X$ is $\cap$-stable if $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$.

We say that $\mathcal{D}$ is a Dynkin class if

(1) $A, B \in \mathcal{D}$, $A \subset B \Rightarrow B \backslash A \in \mathcal{D}$;

(2) $A_1, A_2 \in \mathcal{D}$, $A_1 \cap A_2 = \emptyset \Rightarrow A_1 \cup A_2 \in \mathcal{D}$

Theorem 4.1 *Let $\mathcal{R} \subset 2^X$ satisfy*

(1) $A, B \in \mathcal{R}$, $A \subset B \Rightarrow B \backslash A \in \mathcal{R}$*;*

(2) $A_1, A_2 \in \mathcal{R}$, $A_1 \cap A_2 = \emptyset \Rightarrow A_1 \cup A_2 \in \mathcal{R}$

(3) $A, B \in \mathcal{R}, \Rightarrow B \cap A \in \mathcal{R}$.

Then $\mathcal{R}$ is a ring. In other words, a $\cap$-stable Dynkin class is a ring.

For $\mathcal{A} \subset 2^X$ let $\text{Dyn}(\mathcal{A})$ denote the smallest Dynkin class containing $\mathcal{A}$.

Theorem 4.2 *Let $\mathcal{C}$ be a $\cap$-stable family. Then* $\text{Dyn}(\mathcal{C}) = \text{Ring}(\mathcal{C})$.

Proof. By Theorem 4.1 every ring is a Dynkin class. Hence

$$\text{Dyn}(\mathcal{C}) \subset \text{Ring}(\mathcal{C}).$$

Let us prove the converse inclusion. For $B \in 2^X$. Set

$$\text{K}(B) := \{A \in 2^X \ : A \cap B \in \text{Dyn}(\mathcal{C})\}.$$

Note that

$$A \in \mathrm{K}(B) \ \Leftrightarrow \ B \in \mathrm{K}(A). \tag{4.19}$$

Using the fact that $\mathrm{Dyn}(\mathcal{C})$ is a Dynkin class we check that $\mathrm{K}(B)$ is a Dynkin class.
Using the fact that $\mathcal{C}$ is $\cap$-stable we see that

$$B \in \mathcal{C} \ \Rightarrow \ \mathcal{C} \subset \mathrm{K}(B) \tag{4.20}$$

Hence,

$$B \in \mathcal{C} \ \Rightarrow \ \mathrm{Dyn}(\mathcal{C}) \subset \mathrm{K}(B) \tag{4.21}$$

From (4.19) and (4.21) we get

$$A \in \mathrm{Dyn}(\mathcal{C}) \ \Rightarrow \ \mathcal{C} \subset \mathrm{K}(A) \tag{4.22}$$

Hence

$$A \in \mathrm{Dyn}(\mathcal{C}) \ \Rightarrow \ \mathrm{Dyn}(\mathcal{C}) \subset \mathrm{K}(A) \tag{4.23}$$

Therefore, $\mathrm{Dyn}(\mathcal{C})$ is $\cap$-stable. Hence, by Theorem 4.1 it is a ring. □

4.2 Semirings

Definition 4.3 $\mathcal{T} \subset 2^X$ *is called a semiring if*

(1) $A, B \in \mathcal{T} \Rightarrow A \cap B \in \mathcal{T}$;

(2) $A, B \in \mathcal{T} \Rightarrow A \backslash B = \overset{n}{\underset{i=1}{\cup}} C_i$, *where* C_i *are disjoint elements of* $\mathcal{T}$

Theorem 4.4 *Let* $\mathcal{T}$ *be a semiring. Then* $A \in \mathrm{Ring}(\mathcal{T})$ *iff* A *is a disjoint union of elements of* $\mathcal{T}$.

Proof. Let $\mathcal{R}$ be the family of finite unions of disjoint elements of $\mathcal{T}$. It is obvious that $\mathcal{R} \subset \mathrm{Ring}(\mathcal{T})$.
Let us prove the converse inclusion. To this end it is enough to prove that $\mathcal{R}$ is a ring.
Step 1. Let $A \in \mathcal{R}$, $B \in \mathcal{T}$. Then $A = \overset{n}{\underset{i=1}{\cup}} A_i$ with disjoint $A_i \in \mathcal{T}$. Now

$$A \backslash B = \overset{n}{\underset{i=1}{\cup}} (A_i \backslash B),$$

where $A_i \backslash B \in \mathcal{T}$ are disjoint and each $A_i \backslash B$ is a finite union of disjoint elements of $\mathcal{T}$. Hence $A \backslash B \in \mathcal{R}$.
Step 2. Let $A \in \mathcal{R}$, $B \in \mathcal{R}$. Then $B = \overset{n}{\underset{i=1}{\cup}} B_i$ with disjoint $B_i \in \mathcal{T}$. Now

$$A \backslash B = (\cdots (A \backslash B_1) \cdots \backslash B_n).$$

Hence, by Step 1, $A \backslash B \in \mathcal{R}$.
Step 3. Let $A \in \mathcal{R}$, $B \in \mathcal{R}$. Then $A = \overset{n}{\underset{i=1}{\cup}} A_i$ with disjoint $A_i \in \mathcal{T}$ and $B = \overset{m}{\underset{j=1}{\cup}} B_j$ with disjoint $B_i \in \mathcal{T}$. Now

$$A \cap B = \overset{n}{\underset{i=1}{\cup}} \overset{m}{\underset{j=1}{\cup}} A_i \cap B_j,$$

where $A_i \cap B_j \in \mathcal{T}$ are disjoint Hence $A \cap B \in \mathcal{T}$.
Step 4. Let $A \in \mathcal{R}$, $B \in \mathcal{R}$. Then

$$A \cup B = (A \backslash B) \cup (A \cap B) \cup (B \backslash A).$$

thus by Steps 2 and 3, the left hand side is a union of three disjoint elements of $\mathcal{R}$. Therefore, it is a union of a finite family of elements of $\mathcal{T}$. Hence, $A \cup B \in \mathcal{R}$.
Thus we proved that $\mathcal{R}$ is a ring. □

4.3 σ-Dynkin classes

We say that $\mathcal{D}$ is a σ-Dynkin class if

(1) $A, B \in \mathcal{D},\ A \subset B \Rightarrow B \backslash A \in \mathcal{D}$;

(2) $A_1, A_2, \cdots \in \mathcal{D},\ A_i \cap A_j = \emptyset,\ i \neq j, \Rightarrow \bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{D}$

Theorem 4.5 *Let $\mathcal{R} \subset 2^X$ satisfy*

(1) $A, B \in \mathcal{R},\ A \subset B \Rightarrow B \backslash A \in \mathcal{R}$;

(2) $A_1, A_2, \cdots \in \mathcal{R},\ A_i \cap A_j = \emptyset,\ i \neq j \Rightarrow \bigcup\limits_{i=1}^{\infty} A_i \in \mathcal{R}$

(3) $A, B \in \mathcal{R}, \Rightarrow B \cap A \in \mathcal{R}$.

Then $\mathcal{R}$ is a σ-ring. In other words, a $\cap$-stable σ-Dynkin class is a σ-ring.

For $\mathcal{A} \subset 2^X$ let $\sigma-\mathrm{Dyn}(\mathcal{A})$ denote the smallest σ-Dynkin class containing $\mathcal{A}$.

Theorem 4.6 *Let $\mathcal{C}$ be a $\cap$-stable family. Then $\sigma-\mathrm{Dyn}(\mathcal{C}) = \sigma-\mathrm{Ring}(\mathcal{C})$.*

4.4 Monotone classes

Let $\mathcal{M} \subset 2^X$, We say that $\mathcal{M}$ is a monotone class if

(1) $A_1, A_2, \cdots \in \mathcal{M},\ A_n \searrow A \Rightarrow A \in \mathcal{M}$;

(2) $A_1, A_2, \cdots \in \mathcal{M},\ A_n \nearrow A \Rightarrow A \in \mathcal{M}$.

Proposition 4.7 (1) *A σ-ring is a monotone class;*

(2) *A monotone ring is a σ-ring*

For $\mathcal{T} \subset 2^X$, we denote by $\mathrm{Mon}(\mathcal{T})$ the smallest monotone class containing $\mathcal{T}$.

Theorem 4.8 *Let $\mathcal{R}$ be a ring. Then*

$$\mathrm{Mon}(\mathcal{R}) = \sigma-\mathrm{Ring}(\mathcal{R}).$$

Proof. Since a σ-ring is a monotone class and since $\mathcal{R} \subset \sigma - \mathrm{Ring}(\mathcal{R})$, it follows that

$$\mathrm{Mon}(\mathcal{R}) \subset \sigma - \mathrm{Ring}(\mathcal{R}).$$

Let us prove the converse inclusion. Let $A \in 2^X$. Set

$$\mathrm{K}(A) := \{B \in 2^X \ :\ A\backslash B,\ B\backslash A,\ A \cup B \in \mathrm{Mon}(\mathcal{R})\}.$$

Note that

$$A \in \mathrm{K}(B) \Leftrightarrow B \in \mathrm{K}(A). \tag{4.24}$$

We easily check that for every $A \in 2^X$, $\mathrm{K}(A)$ is a monotone class. Clearly,

$$A \in \mathcal{R} \Rightarrow \mathcal{R} \subset \mathrm{K}(A). \tag{4.25}$$

Hence

$$A \in \mathcal{R} \Rightarrow \mathrm{Mon}(\mathcal{R}) \subset \mathrm{K}(A). \tag{4.26}$$

From (4.24) and (4.26), we get

$$A \in \mathrm{Mon}(\mathcal{R}) \Rightarrow \mathcal{R} \subset \mathrm{K}(A).$$

Hence,

$$A \in \mathrm{Mon}(\mathcal{R}) \Rightarrow \mathrm{Mon}(\mathcal{R}) \subset \mathrm{K}(A).$$

Therefore, $\mathrm{Mon}(\mathcal{R})$ is a ring. By Proposition 4.7 (2) it is a σ-ring. Hence,

$$\mathrm{Mon}(\mathcal{R}) \supset \sigma{-}\mathrm{Ring}(\mathcal{R}).$$

□

4.5 Extension and uniqueness of contents

We will need a generalization of the notion of a content to the case of $\mathcal{T} \subset 2^X$ with $\emptyset \in \mathcal{T}$. We say that $\nu : \mathcal{T} \to [0, \infty]$ is a content if

(1) $\nu(\emptyset) = 0$;

(2) $A_1, \cdots, A_n \in \mathcal{T}$, $A_i \cap A_j = \emptyset$, $i \neq j$, $A_1 \cup \cdots \cup A_n \in \mathcal{R} \Rightarrow \nu(A_1 \cup \cdots \cup A_n) = \nu(A_1) + \cdots + \nu(A_n)$.

Theorem 4.9 *Let $\mathcal{T}$ be a $\cap$-stable family containing $\emptyset$ and $\mathcal{R} = \mathrm{Ring}(\mathcal{T})$. Let ν_1, ν_2 be finite contents on $\mathcal{R}$ coinciding on $\mathcal{T}$. Then $\nu_1 = \nu_2$.*

Proof. Let $\mathcal{W} := \{A \in \mathcal{R} \ : \nu_1(A) = \nu_2(A)\}$. Then
$A, B \in \mathcal{W},\ A \subset B \ \Rightarrow \ B \backslash A \in \mathcal{W}$
$A_1, A_2 \in \mathcal{W},\ A_1 \cap A_2 = \emptyset \ \Rightarrow \ A_1 \cup A_2 \in \mathcal{W}$
Hence $\mathcal{W}$ is a Dynkin system. Hence it contains $\mathrm{Dyn}(\mathcal{T})$. But by Theorem 4.2, $\mathrm{Dyn}(\mathcal{T}) = \mathcal{R}$. Hence $\mathcal{W} = \mathcal{R}$. □

Theorem 4.10 *Suppose that $\mathcal{T}$ is a semiring and ν is a content on $\mathcal{T}$. Then there exists a unique content on $\mathrm{Ring}(\mathcal{T})$ extending ν.*

Proof. Every $A \in \mathrm{Ring}(\mathcal{T})$ can be written as $A = \bigcup_{i=1}^{n} B_i$ for some disjoint $B_i \in \mathcal{T}$. Then we set

$$\nu(A) := \sum_{i=1}^{n} \nu(B_i).$$

Suppose now that $A = \bigcup_{i=1}^{n} B_i = \bigcup_{j=1}^{m} C_j$ are decompositions of the above type. Then

$$A = \bigcup_{i=1}^{n} \bigcup_{j=1}^{m} B_i \cap C_j$$

is also a decomposition into disjoint elements of the semiring, and

$$\nu(B_i) = \sum_{j=1}^{m} \nu(B_i \cap C_j), \quad \nu(C_j) = \sum_{i=1}^{n} \nu(B_i \cap C_j).$$

Therefore,

$$\sum_{i=1}^{n} \nu(B_i) = \sum_{i=1}^{n} \sum_{j=1}^{m} \nu(B_i \cap C_j) = \sum_{j=1}^{m} \nu(C_j).$$

Hence the definition is correct.

It is easy to check that the extended ν is a content. □

4.6 Uniqueness of a measure

Theorem 4.11 *Let $(X, \mathcal{F})$ be a set with a σ-ring. Let μ_1 and μ_2 be two measures defined on $(X, \mathcal{F})$ Suppose that $\mathcal{T} \subset \mathcal{F}$ is a $\cap$-stable family such that $\mu_1 = \mu_2$ on $\mathcal{T}$ and is finite on $\mathcal{T}$. Then $\mu_1 = \mu_2$ on $\sigma-\mathrm{Ring}(\mathcal{T})$.*

Proof. Let $\mathcal{W} := \{A \in \mathcal{F} \ : \ \mu_1(A) = \mu_2(A)\}$.

Step 0. By Theorem 4.9, $\mu_1 = \mu_2$ on $\mathrm{Ring}(\mathcal{T})$. Note that $\sigma-\mathrm{Ring}(\mathcal{T}) = \sigma-\mathrm{Ring}(\mathrm{Ring}(\mathcal{T}))$. Hence in what follows it suffices to assume that $\mathcal{T}$ is a ring.

Step 1. Assume that μ_1 is finite. Clearly, $\mathcal{W}$ is a σ-Dynkin class and $\mathcal{T} \subset \mathcal{W}$. Hence σ–$\mathrm{Ring}(\mathcal{T}) \subset \mathcal{W}$ in this case.

Step 2. Assume that $A \in \mathcal{F}$. Then μ_1 restricted to $\sigma-\mathrm{Ring}\Big(\mathcal{T}\Big|_A\Big) = \sigma-\mathrm{Ring}(\mathcal{T})\Big|_A$ is finite and $\mu_1 = \mu_2$ on $\mathcal{T}\Big|_A$. Hence, by Step 1, we have $\mu_1 = \mu_2$ on $\sigma-\mathrm{Ring}(\mathcal{T})\Big|_A$.

Step 3. Let $A \in \sigma-\mathrm{Ring}(\mathcal{T})$. Then by Theorem 3.2, there exist $A_1, A_2, \cdots \in \mathcal{T}$ such that $A_n \nearrow A$ and $\mu(A_n) < \infty$. Then

$$\mu_2(A) = \lim_{n\to\infty} \mu_2(A_n) = \lim_{n\to\infty} \mu_1(A_n) = \mu_1(A).$$

□

4.7 Dense subsets in L^p spaces

Theorem 4.12 *Let $\mathcal{T} \subset \mathcal{F}$ be a semiring such that $\sigma-\mathrm{Ring}(\mathcal{T}) = \mathcal{F}$ and μ is finite on $\mathcal{T}$. Assume that there exists a localizing family $\{X_i \ : \ i \in I\}$ contained in $\mathcal{T}$. If $1 \le p < \infty$, then the span of characteristic functions of $\mathcal{T}$ is dense in $L^p(\mu)$.*

Proof. Let $\mathcal{W}$ be the family of sets whose characteristic functions can be approximated in $L^p(\mu)$ by linear combinations of characteristic functions of elements in $\mathcal{T}$.

Step 1. Assume first that μ is finite. Clearly, $\mathcal{W}$ is then a σ-Dynkin class. Hence $\mathcal{F} \subset \mathcal{W}$.

Step 2. Let μ be arbitrary. Let $A \in \mathcal{F}_\mu^{\mathrm{f}}$. Let $\{X_i \ : \ i \in I\}$ be a localizing family for μ contained in $\mathcal{T}$. Then there exists a sequence $i_1, i_2, \cdots \in I$ such that $\mu(A) = \sum_{j=1}^\infty \mu(A \cap X_{i_j})$. We apply Step 1. to μ restricted to $\mathcal{T}\Big|_{X_{i_j}}$. We conclude that $A \in \mathcal{W}$. Hence $\mathcal{F}_\mu^{\mathrm{f}} \subset \mathcal{W}$ Consequently, linear combinations of characteristic functions of elements in $\mathcal{T}$ are dense in $\mathcal{E} \cap \mathcal{L}^p(\mu)$.

Step 3. Let $f \in \mathcal{L}^p(\mu)$. There exist sequences $u_n^\pm \in \mathcal{E}_+$ such that $u_n^\pm \nearrow f_\pm$. Clearly, $u_n^\pm \in \mathcal{L}^p(\mu)$, $u_n^+ - u_n^-$ is dominated by $|f| \in \mathcal{L}^p(\mu)$, hence by the Lebesgue dominated convergence theorem $u_n^+ - u_n^- \to f$ in the $\mathcal{L}^p(\mu)$ sense. □

4.8 Premeasures

Definition 4.13 *Let $(X, \mathcal{R})$ be a set with a ring. A function $\nu : \mathcal{R} \to [0, \infty]$ is called a premeasure if*

(1) $\nu(\emptyset) = 0$,

(2) $A_1, A_2 \cdots \in \mathcal{R}$, $\cup_{n=1}^\infty A_n \in \mathcal{R}$, $A_i \cap A_j = \emptyset$ *for* $i \ne j \Rightarrow \nu\left(\cup_{n=1}^\infty A_n\right) = \sum_{n=1}^\infty \nu(A_n)$.

Clearly, every premeasure is a content.

If $(X, \mathcal{F}, \mu)$ is a measure and $\mathcal{R} \subset \mathcal{F}$ is a ring, then $(X, \mathcal{R}, \mu\Big|_{\mathcal{R}})$ is a premeasure.

Theorem 4.14 *Let* $(X,\mathcal{R},\nu)$ *be a premeasure,* $A_1,A_2,\cdots\in\mathcal{R}$ *,* $A\in\mathcal{R}$ *and* $A\subset\bigcup_{i=1}^{\infty}A_i$*. Then*

$$\nu(A)\le\sum_{i=1}^{\infty}\nu(A_i).$$

Proof. $B_n:=(A_n\setminus\bigcup_{i=1}^{n-1}A_i)\cap A$ are disjoint elements of $\mathcal{R}$, $B_i\subset A_i$ and $A=\bigcup_{i=1}^{\infty}B_i$. Hence

$$\nu(A)=\sum_{i=1}^{\infty}\nu(B_i)\le\sum_{i=1}^{\infty}\nu(A_i).$$

□

4.9 Extending a premeasure to a measure

Theorem 4.15 *Let* $(X,\mathcal{R})$ *be a set with a ring. Let* $(X,\mathcal{R},\nu)$ *be a premeasure. For any* $Q\in 2^X$ *define*

$$\mu^*(Q):=\inf\{\sum_{i=1}^{\infty}\nu(A_i)\ :\ A_i\in\mathcal{R},\ Q\subset\bigcup_{i=1}^{\infty}A_i\}.$$

Then

(1) μ^* *is an external measure;*

(2) $\nu=\mu^*$ *on* $\mathcal{R}$*;*

(3) *Let* $\mathcal{F}^{\mathrm{ms}}$ *be the* σ*-field of* μ^**-measurable sets and* μ^{ms} *the corresponding measure. Then* $(X,\mathcal{F}^{\mathrm{ms}},\mu^{\mathrm{ms}})$ *is a complete measure extending the premeasure* $(X,\mathcal{R},\nu)$*.*

(4) *Let* $\mathcal{F}:=\sigma-\mathrm{Ring}(\mathcal{R})$*. Let the restriction of* μ^* *to* $\mathcal{F}$ *be denoted* μ*. Then for* $Q\in 2^X$

$$\mu^*(Q)=\inf\{\mu(A)\ :\ Q\subset A,\ A\in\mathcal{F}\}.$$

Proof. (1) The properties (1) and (3) of the definition of an external measure are obvious. Let us show the property (2). Let $Q_1,Q_2,\cdots\in 2^X$. For any $\epsilon>0$ we will find a double sequence $(A_{nm})_{m\in\mathbb{N}}$ such that

$$Q_n\subset\cup_{m=1}^{\infty}A_{nm},$$

$$\sum_{m=1}^{\infty}\mu(A_{nm})\le\mu^*(Q_n)+2^{-n}\epsilon.$$

Then

$$\mu^*(\cup_{n=1}^{\infty}Q_n)\le\sum_{n,m=1}^{\infty}\mu(A_{nm})\le\sum_{n=1}^{\infty}\mu^*(Q_n)+\epsilon.$$

Hence

$$\mu^*(\cup_{n=1}^{\infty}Q_n)\le\sum_{n=1}^{\infty}\mu^*(Q_n).$$

(2) It is obvious that $\mu^*(A)\le\mu(A)$. The converse inequality follows by Theorem 4.14

(3) Let $A \in \mathcal{R}$, $Q \in 2^X$. For any $\epsilon > 0$ and suitable $A_1, A_2, \cdots \in \mathcal{R}$ such that $Q \subset \bigcup_{i=1}^{\infty} A_i$, we have

$$\begin{aligned} \mu^*(Q) &\geq \sum_{j=1}^{\infty} \mu(A_j) - \epsilon \\ &= \sum_{j=1}^{\infty} \mu(A_j \cap A) + \sum_{j=1}^{\infty} \mu(A_j \backslash A) - \epsilon \\ &\geq \mu^*(Q \cap A) + \mu^*(Q \backslash A) - \epsilon. \end{aligned}$$

Therefore,

$$\mu^*(Q) \geq \mu^*(Q \cap A) + \mu^*(Q \backslash A).$$

Thus, $A \in \mathcal{F}^{\mathrm{ms}}$. □

5 Tensor product of measures

5.1 Tensor product of σ-rings

Theorem 5.1 *Let $\mathcal{T}_i$ be semirings over X_i, $i = 1, 2$. Set*

$$\mathcal{T}_1 * \mathcal{T}_2 := \{A_1 \times A_2 \ : \ A_i \in \mathcal{T}_i, \ i = 1, 2\}.$$

Then $\mathcal{T}_1 * \mathcal{T}_2$ is a semiring.

Now assume that $\mathcal{F}_i$ are σ-rings. Set $\mathcal{F}_1 \otimes \mathcal{F}_2 := \sigma\mathrm{-Ring}(\mathcal{F}_1 * \mathcal{F}_2)$.

Definition 5.2 *Let $B \subset X_1 \times X_2$, $x_i \in X_i$.*

$$\pi_2^{x_1}(B) = \{x_2 \in X_2 \ : \ (x_1, x_2) \in B\},$$
$$\pi_1^{x_2}(B) = \{x_1 \in X_1 \ : \ (x_1, x_2) \in B\}.$$

Proposition 5.3 *Let $B \in \mathcal{F}_1 \otimes \mathcal{F}_2$, $x_i \in X_i$. Then $\pi_1^{x_2}(B) \in \mathcal{F}_1$ and $\pi_2^{x_1}(B) \in \mathcal{F}_2$.*

Proof. Note that

$$\pi_1^{x_2}(\emptyset) = \emptyset,$$
$$\pi_1^{x_2}(A \backslash B) = \pi_1^{x_2}(A) \backslash \pi_1^{x_2}(B),$$
$$\pi_1^{x_2}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \bigcup_{i=1}^{\infty} \pi_1^{x_2}(A_i).$$

Hence

$$\mathcal{W} := \{B \subset X_1 \times X_2 \ : \ \pi_1^{x_2}(B) \in \mathcal{F}_1\}$$

is a σ-ring. Clearly $\mathcal{W}$ contains $\mathcal{F}_1 * \mathcal{F}_2$. Hence $\mathcal{F}_1 \otimes \mathcal{F}_2 \subset \mathcal{W}$. □

5.2 Tensor product of measures

Let $(X_i, \mathcal{F}_i, \mu_i)$ be set with σ-rings and measures. Let $\mathcal{F}_{\mu_i}^{\sigma\mathrm{f}}$ be the σ-ring of μ_i-σ-finite sets.

Proposition 5.4 *Let $B \in \mathcal{F}_1 \otimes \mathcal{F}_{\mu_2}^{\sigma\mathrm{f}}$. Then the map*

$$X_1 \ni x_1 \mapsto \mu_2(\pi_2^{x_1}(B)) \text{ is } \mathcal{F}_1\text{-measurable.}$$

Proof. Set

$$s_B(x_1) := \mu_2(\pi_2^{x_1}(B)).$$

Set

$$\mathcal{W} := \{B \subset X_1 \times X_2 \ : \ s_B \ \text{ is measurable } \}.$$

Step 1. Assume that $\mu(X_2) < \infty$. Clearly, $\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma\mathrm{f}} \subset \mathcal{W}$. If $A, B \in \mathcal{W}$ with $A \subset B$, then $s_{B\backslash A} = s_B - s_A$, Hence $B\backslash A \in \mathcal{W}$. Let $B_1, B_2, \cdots \in \mathcal{W}$ be disjoint and $B = \cup_{j=1}^{\infty} B_j$. Then $s_B = \sum_{j=1}^{\infty} s_{B_j}$. Hence $\overset{\infty}{\underset{j=1}{\cup}} B_j \in \mathcal{W}$. Therefore, $\mathcal{W}$ is a σ-Dynkin class. Hence it contains $\sigma-\mathrm{Ring}(\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma\mathrm{f}})$.

Alternative version of Step 1. Assume that $\mu(X_2) < \infty$. If disjoint $B_1, B_2, \cdots \subset \mathcal{W}$ and $B = \cup_{j=1}^{\infty} B_j$, then $s_{\overset{\infty}{\underset{j=1}{\cup}} B_j} = \sum_{j=1}^{\infty} s_{B_j}$. We know that $\mathrm{Ring}(\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma\mathrm{f}})$ are disjoint unions of elements in $\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma\mathrm{f}} \subset \mathcal{W}$. Hence $\mathrm{Ring}(\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma\mathrm{f}}) \subset \mathcal{W}$.

Clearly, if $A_1, A_2, \cdots \in \mathcal{W}$ and $A_n \nearrow A$, then $s_{A_n} \nearrow s_A$. Hence, $A \in \mathcal{W}$.

Likewise, if $A_1, A_2, \cdots \in \mathcal{W}$ and $A_n \searrow A$, then $s_{A_n} \searrow s_A$. Hence, using the finiteness of X, $A \in \mathcal{W}$.

Therefore, $\mathcal{W}$ is a monotone class. Hence it contains $\sigma-\mathrm{Ring}(\mathcal{F}_1 * \mathcal{F}_{\mu_2}^{\sigma\mathrm{f}}) = \mathcal{F}_1 \otimes \mathcal{F}_{\mu_2}^{\sigma\mathrm{f}})$.

Step 2. Now drop the assumption $\mu_2(X_2) < \infty$. Let $B \in \mathcal{F}_1 \otimes \mathcal{F}_{\mu_2}^{\sigma\mathrm{f}}$. Then there exists a disjoint family $A_1, A_2, \cdots \in \mathcal{F}_2$ such that $\mu_2(A_i) < \infty$ and $B \subset X_1 \times \overset{\infty}{\underset{j=1}{\cup}} A_j$. Set $B_j := B \cap X_1 \times A_j$. Then

$$s_B = \sum_{j=1}^{\infty} s_{B_j},$$

and each s_{B_j} is measurable. Hence s_B is measurable. □

Now we assume that both measures are σ-finite. If $A \in \mathcal{F}_{\mu_1} \otimes \mathcal{F}_{\mu_2}$, we define

$$\mu_1 \otimes \mu_2(A) := \int \mu_2(\pi_2^{x_1}(A))\mathrm{d}\mu_1(x_1). \tag{5.27}$$

Theorem 5.5 (1) *$\mu_1 \otimes \mu_2$ is a measure.*

(2) *If $X_1 \times X_2 \ni (x_1, x_2) \mapsto \tau(x_1, x_2) := (x_2, x_1) \in X_2 \times X_1$ is the flip, then for $A \in \mathcal{F}_1 \otimes \mathcal{F}_2$,*

$$\mu_1 \otimes \mu_2(A) = \mu_2 \otimes \mu_1(\tau A). \tag{5.28}$$

In particular, for $A \in \mathcal{F}_{\mu_1} \otimes \mathcal{F}_{\mu_2}$ (5.28) equals

$$\int \mu_2(\pi_2^{x_1}(A))\mathrm{d}\mu_1(x_1) = \int \mu_1(\pi_1^{x_2}(A))\mathrm{d}\mu_2(x_2)$$

(3) *If ν is a measure on $\mathcal{F}_1 \otimes \mathcal{F}_2$ satisfying*

$$\nu(A_1 \times A_2) = \mu_1(A_1)\mu_2(A_2), \ A_1 \in \mathcal{F}_{\mu_1}, A_2 \in \mathcal{F}_{\mu_2},$$

then it coincides with $\mu_1 \otimes \mu_2$.

(4) $(\mu_1 \otimes \mu_2)^{\mathrm{cp}} = (\mu_1^{\mathrm{cp}} \otimes \mu_2^{\mathrm{cp}})^{\mathrm{cp}}$.

Proof. The formula (5.27) is well defined by Proposition 5.4. Then we check that it is a measure.

The uniqueness follows by Theorem 4.11, because both measures coincide on the semiring $\mathcal{F}_{\mu_1} * \mathcal{F}_{\mu_2}$. □

5.3 Multiple integrals

For any $x_2 \in X_2$, the function

$$X_1 \ni x_1 \mapsto (x_1, x_2) \in X_1 \times X_2$$

is measurable. Hence if $f \in \mathcal{M}(X_1 \times X_2)$, then

$$X_1 \ni x_1 \mapsto f(x_1, x_2) \in [-\infty, \infty]$$

belongs to $\mathcal{M}(X_1)$.

Theorem 5.6 *Let $(X_i, \mathcal{F}_i, \mu_i)$ be spaces with measures. Let $f \in \mathcal{M}_+(X_1 \times X_2)$.*

(1)

$$x_1 \mapsto \int f(x_1, x_2) \mathrm{d}\mu_2(x_2)$$

belongs to $\mathcal{M}_+(X_1)$,

(2)

$$\int f \mathrm{d}(\mu_1 \otimes \mu_2) = \int (\int f(x_1, x_2) \mathrm{d}\mu_2(x_2)) \mathrm{d}\mu_1(x_1) = \int (\int f(x_1, x_2) \mathrm{d}\mu_1(x_1)) \mathrm{d}\mu_2(x_2). \tag{5.29}$$

Proof. For elementary functions the theorem is obvious. For an arbitrary function from $\mathcal{M}_+(X_1 \times X_2)$ we use the monotone convergence. □

Theorem 5.7 (Fubini) *Let $(X_i, \mathcal{F}_i, \mu_i)$ be spaces with measures and*

$$f \in \mathcal{L}^1(\mu_1 \otimes \mu_2).$$

The map

$$x_2 \mapsto f(x_1, x_2)$$

for μ_1-almost all x_1 belongs to $\mathcal{L}^1(\mu_2)$. Let N_1 be the set of x_1 for which this is not true. Define

$$f_1(x_1) := \begin{cases} \int f(x_1, x_2) \mathrm{d}\mu_2(x_2) & x_1 \in X_1 \backslash N_1 \\ 0 & x_1 \in N_1. \end{cases}$$

Then f_1 belongs to $\mathcal{L}^1(\mu_1)$ and

$$\int f \mathrm{d}(\mu_1 \otimes \mu_2) = \int f_1 \mathrm{d}\mu_1.$$

Proof. We have $f_\pm \in \mathcal{L}^1(\mu_1 \otimes \mu_2)$. Hence

$$\infty > \int f_+ \mathrm{d}\mu_1 \otimes \mu_2 = \int \Big(\int f_+(x_1, x_2) \mathrm{d}\mu_2(x_2) \Big) \mathrm{d}\mu_1(x_1). \tag{5.30}$$

Thus $x_1 \mapsto \int f_+(x_1, x_2) \mathrm{d}\mu_2(x_2)$ belongs to $L^1(\mu_1)$. Hence, by Theorem 2.12, for μ_1-a.a. x_1,

$$\int f_+(x_1, x_2) \mathrm{d}\mu_2(x_2) < \infty.$$

In other words, for μ_1-a.a., x_1 $f_+(x_1, \cdot) \in \mathcal{L}^1(\mu_2)$.

Of course, the same is true for f_-. □

Loosely speaking, the above theorem says that for $f \in \mathcal{L}^1(\mu_1 \otimes \mu_2)$

$$\int f \mathrm{d}(\mu_1 \otimes \mu_2) = \int \Big(\int f(x_1, x_2) \mathrm{d}\mu_2(x_2) \Big) \mathrm{d}\mu_1(x_1) = \int \Big(\int f(x_1, x_2) \mathrm{d}\mu_1(x_1) \Big) \mathrm{d}\mu_2(x_2).$$

Theorem 5.8 *If $\otimes$ denotes the tensor product in the sense of Hilbert spaces, then $L^2(\mu_1) \otimes L^2(\mu_2) = L^2(\mu_1 \otimes \mu_2)$.*

5.4 Layer-cake representation

Let $(X, \mathcal{F}, \mu)$ be a measure. Let ν be a Borel measure on $[0, \infty[$ and $f \in \mathcal{M}_+(X)$. For $a, t \geq 0$, set

$$\phi(t) := \int_0^t \mathrm{d}\nu(s), \quad u(a) := \mu\{f > a\}.$$

Proposition 5.9

$$\int \phi(f(x))\mathrm{d}\mu(x) = \int_0^\infty u(a)\mathrm{d}\nu(a).$$

Proof.

$$\begin{aligned} \int_0^\infty u(a)\mathrm{d}\nu(a) &= \int \Big(\int 1_{\{f>a\}}(x)\mathrm{d}\mu(x)\Big)\mathrm{d}\nu(a) \\ &= \int \Big(\int 1_{\{f>a\}}(x)\mathrm{d}\nu(a)\Big)\mathrm{d}\mu(x) = \int \Big(\int_0^{f(x)} \mathrm{d}\nu(a)\Big)\mathrm{d}\mu(x) = \int \phi(f(x))\mathrm{d}\mu(x). \end{aligned}$$

□

Corollary 5.10 (1) $\int |f(x)|^p \mathrm{d}\mu(x) = p\int_0^\infty u(a)a^{p-1}\mathrm{d}a$.
(2) $f(x) = \int 1_{\{f>a\}}(x)\mathrm{d}a$.

Proof. For (1) we set $\phi(t) = t^p$, $\mathrm{d}\nu(t) = pt^{p-1}\mathrm{d}t$.
For (2) we put $\phi(t) := t$, $\mathrm{d}\nu(t) = \mathrm{d}t$, and μ is the Dirac delta at x. □

6 Measures in $\mathbb{R}^n$

6.1 Regular contents

Suppose that X is a topological space. Let $\mathcal{R}$ be a ring over X and ν a content on $\mathcal{R}$. We say that ν is regular iff for $F \in \mathcal{R}$ the following two conditions hold:

$$\begin{aligned} \nu(F) &= \sup\{\nu(G) \ : \ G^{\mathrm{cl}} \subset F, \ G^{\mathrm{cl}} \in \mathrm{Compact}(X)\} \\ &= \inf\{\nu(H) \ : \ F \subset H^o\}. \end{aligned}$$

Theorem 6.1 *Every regular content is a premeasure.*

Let $F_1, F_2, \dots \in \mathcal{R}$ be disjoint and $F := \bigcup_{j=1}^\infty F_j \in \mathcal{R}$. We know by Theorem 2.1 that

$$\nu(F) \geq \sum_{j=1}^\infty \nu(F_j).$$

Let us show the converse inequality. Let $\epsilon > 0$. For any $j = 1, \dots, n$ we can find $H_j \in \mathcal{R}$ such that $F_j \subset H_j^o$ and $\nu(H_j \backslash F_j) < \epsilon 2^{-j-1}$. Likewise, we can find $G \in \mathcal{R}$ such that $G^{\mathrm{cl}} \subset F$, G^{cl} is compact and $\nu(F\backslash G) < \epsilon/2$. Thus $\{H_j^o \ : \ j = 1, 2, \dots\}$ is an open cover of the compact set G^{cl}. We can choose a finite subcover $\{H_{j_k}^o \ : \ k = 1, \dots, m\}$, so that

$$G^{\mathrm{cl}} \subset \bigcup_{k=1}^m H_{j_k}^o.$$

Consequently,

$$G \subset \bigcup_{k=1}^{m} H_{j_k}.$$

Thus

$$\nu(F) \le \nu(G) + \epsilon/2 \le \sum_{k=1}^{m} \nu(H_{j_k}) + \epsilon/2 \le \sum_{j=1}^{\infty} \nu(F_j) - \epsilon.$$

□

6.2 Borel sets in $\mathbb{R}$

Let $\mathcal{T} := \{]a,b] \ : \ a,b \in \mathbb{R}, \ a \le b\}$.

Theorem 6.2 *$\mathcal{T}$ is a semiring. Moreover, let $A_1, \dots, A_n \in \mathcal{T}$ be disjoint, $A \in \mathcal{T}$ and $\bigcup_{i=1}^{n} A_i = A$. Then, after a possible renumbering of A_i, $A_i =]a_{i-1}, a_i]$, where $a_0 \le a_1 \le \dots \le a_n$. $\sigma-\mathrm{Field}(\mathcal{T})$ equals the σ-field of Borel subsets of $\mathbb{R}$ and will be denoted by* Borel($\mathbb{R}$).

6.3 Borel premeasures on $\mathbb{R}$

Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function. Define $\nu : \mathcal{T} \to [0, \infty[$ as

$$\nu(]a,b]) := f(b) - f(a).$$

Theorem 6.3 *ν is a content on $\mathcal{T}$. Hence it extends uniquely to a content on* Ring($\mathcal{T}$).

Assume now in addition that $f : \mathbb{R} \to \mathbb{R}$ is continuous from the right, that means

$$\lim_{t \downarrow t_0} f(t) = f(t_0), \quad t_0 \in \mathbb{R}.$$

Lemma 6.4 *Let $F \in \mathrm{Ring}(\mathcal{T})$ and $\epsilon > 0$. Then there exist $H, G \in \mathrm{Ring}(\mathcal{T})$ such that*

$$G^{\mathrm{cl}} \subset F \subset H^{o}, \quad \nu(F \backslash G) < \epsilon, \quad \nu(H \backslash F) < \epsilon.$$

In other words, ν is a regular content.

Proof. We can assume that $F = \bigcup_{i=1}^{n}]a_{2i-1}, a_{2i}]$ and $0 < \delta < \min\{a_{j+1} - a_j \ : \ j = 0, \dots, 2n-1\}$. By decreasing δ we can demand in addition that $f(a_i + \delta) - f(a_i) < \epsilon/n$. Then we can set

$$G := \bigcup_{j=1}^{n}]a_{2j-1} + \delta, a_{2j}], \quad H := \bigcup_{j=1}^{n}]a_{2j-1}, a_{2j} + \delta].$$

□

By Theorem 6.1, we get:

Theorem 6.5 *ν is a premeasure on* Ring($\mathcal{T}$).

6.4 Borel measures on $\mathbb{R}$

Theorem 6.6 (1) *Let $f : \mathbb{R} \to \mathbb{R}$ be an increasing function continuous from the right. Then there exists a unique measure μ_f on $(\mathbb{R}, \mathrm{Borel}(\mathbb{R}))$ such that*

$$\mu_f(]a, b]) = f(b) - f(a).$$

This measure is σ-finite.

(2) *Let $(\mathbb{R}, \mathrm{Borel}(\mathbb{R}), \mu)$ be a measure such that $\mu(A) < \infty$ for compact $A \subset \mathbb{R}$. Set*

$$f(x) := \begin{cases} -\mu(]x, 0], & x < 0 \\ \mu(]0, x]), & x \geq 0. \end{cases}$$

Then $\mu = \mu_f$.

Proof. The premeasure ν_f can be extended by the Caratheodory construction to a σ-field containing $\mathrm{Borel}(\mathbb{R})$. □

Definition 6.7 *The measure on* $\mathrm{Borel}(\mathbb{R})$ *with the distribution function $f(x) = x$ is called the Borel-Lebesgue measure, and denoted λ. Its complete extension is called the Lebesgue measure. In integrals, if the generic variable in $\mathbb{R}$ will be denoted by x, then instead of* $\mathrm{d}\lambda(x)$ *we will usually write* $\mathrm{d}x$.

Theorem 6.8 *The Borel-Lebesgue measure is the only measure on* $\mathrm{Borel}(\mathbb{R})$ *invariant wrt translations such that $\mu(]0, 1]) = 1$.*

Proof. Let $(\mathbb{R}, \mathrm{Borel}(\mathbb{R}), \mu)$ be translation invariant. This means $\mu(]a, b]) = \mu(]a + x, b + x])$. Using this and $\mu(]0, 1])$ we get $\mu(]k/n, (k + 1)/n]) = 1/n$. This easily implies $\mu(]a, b]) = b - a$ for any $a \leq b$. □

Theorem 6.9 (1) *Let μ be a measure on $2^{\mathbb{R}}$ invariant wrt translations (that means if $\tau_t(x) = x - t$, then $\tau_{t*}\mu = \mu$). Suppose that μ is finite on compact sets. Then $\mu = 0$.*

(2) *Let $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ denote the 1-dimensional torus. Let μ be a finite measure on $2^{\mathbb{T}}$ invariant wrt translations. Then $\mu = 0$.*

Proof. (2) Introduce in $\mathbb{T}$ the equivalence relation

$$x \sim y \Leftrightarrow x - y \in \mathbb{Q}.$$

Let $\mathcal{I}$ be the set of equivalence classes. Choose from every class a representant x_i. We have

$$[0, 1[= \bigcup_{y \in \mathbb{Q}} A_y$$

for

$$A_y := \{x_i + y \ : \ i \in \mathcal{I}\}.$$

But $\tau_{y_2 - y_1} A_{y_2} = A_{y_1}$. Hence

$$\mu(\mathbb{T}) = \sum_{y \in \mathbb{Q}} \mu(A_y) = \sum_{y \in \mathbb{Q}} \mu(A_0) = \infty \mu(A_0).$$

Thus $\mu(\mathbb{T}) = 0$ or $\mu(\mathbb{T}) = \infty$.

(1) Let μ be a translation invariant measure on $(\mathbb{R}, 2^{\mathbb{R}})$. Consider the canonical projection $\mathbb{R} \to \mathbb{T}$ restricted to $[0, 1[$ it is a bijection. Denote it by $\pi : [0, 1[\to \mathbb{T}$. Define $\tilde{\mu}(A) := \mu(\pi^{-1}(A))$, $A \subset \mathbb{T}$.

Let us check that $\tilde{\mu}$ is translation invariant and $\tilde{\mu}(\mathbb{T}) = 1$. By (2) it is zero. □

6.5 The Cantor set and devil's staircase

Definition 6.10 *Let $q \in \mathbb{N}$. To every sequence of numbers $(p_j)_{j\in\mathbb{N}}$ with values in $\{0, 1, \ldots, q-1\}$ we assign a number from the interval $[0, 1]$:*

$$0.p_1p_2\cdots := \sum_{j=1}^{\infty} \frac{p_j}{q^j} = x.$$

We say that $0.p_1p_2\ldots$ denotes the number x in the system based on q

Note that every $x \in [0, 1]$ has such a representation. It is ambiguous only if for some n we have $q^n x \in \mathbb{N}$. Then

$$0.p_1\ldots p_{n-1}p_n00\cdots = 0.p_1p_2\ldots p_{n-1}(p_n-1)(q-1)(q-1)\ldots.$$

Definition 6.11 *The Cantor set C, is the subset of $[0, 1]$ consisting of the numbers that in the trinary system have only 0 and 2. It can be defined also as follows: $C_0 = [0, 1]$, $C_1 = C_0 \backslash]\frac{1}{3}, \frac{2}{3}[$, $C_2 = C_1 \backslash]\frac{1}{9}, \frac{2}{9}[\cup]\frac{7}{9}, \frac{8}{9}[$, etc. We set $C = \bigcap_{n=1}^{\infty} C_n$.*

It is a closed set with an empty interior, uncountable and has zero Lebesgue measure (because $\lambda(C_n) = \frac{2^n}{3^n}$).

Definition 6.12 *Define the transformation, called devil's staircase, $F : [0, 1] \to [0, 1]$ as follows. If $x = 0.p_1p_2\cdots \in C$ in the trinary system, where $p_i \in \{0, 2\}$, then $F(x) = 0.\frac{p_1}{2}\frac{p_2}{2}\ldots$ in the binary system. If $x \in [0, 1]\backslash C$, then $x \in]x_-, x_+[$ where $x_- = 0.p_1\ldots p_n022\ldots$ and $x_+ = 0.p_1\ldots p_n200\ldots$. We see that $F(x_-) = F(x_+)$ and we set $F(x) = F(x_-) = F(x_+)$*

The function F is increasing, continuous, locally constant beyond C, and $F(1) - F(0) = 1$. It defines a Borel measure μ, which is continuous and singular wrt the Lebesgue measure, since $\mu([0, 1]\backslash C) = 0$.

6.6 Transport of the Lebesgue measure in $\mathbb{R}$

Let $g : [a, b] \to \mathbb{R}$ be an increasing function. Then there exists a unique increasing function $f : [g(a), g(b)] \to [a, b]$, which is continuous from the right and $g \circ f(x) = x$, $x \in [g(a), g(b)]$. It is easy to see that $g^*\lambda = \mu_f$. n fact,

$$g^*\lambda(]\alpha, \beta]) = \lambda(g^{-1}(]\alpha, \beta]) = \lambda(]f(\alpha), f(\beta)]) = f(\beta) - f(\alpha).$$

6.7 The Lebesgue measure in $\mathbb{R}^n$

Let Borel($\mathbb{R}^n$) denote the σ-field of Borel sets in $\mathbb{R}^n$.

In $\mathbb{R}^n$ we can define the n-dimensional Borel-Lebesgue measure as the measure on Borel($\mathbb{R}^n$) equal $\lambda^n := \lambda \otimes \cdots \otimes \lambda$.

Theorem 6.13 (1) *The n-dimensional Borel-Lebesgue measure λ^n is the unique measure on* Borel($\mathbb{R}^n$) *such that*

$$\lambda^n\Big(\mathop{\times}_{i=1}^{n}]a_i, b_i]\Big) = \prod_{i=1}^{n} |b_i - a_i|.$$

(2) *It is also the unique translation invariant measure on* Borel($\mathbb{R}^n$) *such that*

$$\lambda^n\Big(\mathop{\times}_{i=1}^{n}]0, 1]\Big) = 1.$$

We can also consider its completion, called the n-dimensional Lebesgue measure. There are several equivalent ways to construct the Lebesgue measure, described in the following theorem.

Theorem 6.14 *The following measures coincide:*

(1) $(\lambda^n)^{\mathrm{cp}}$ *(the completion of the n-dimensional Borel-Lebesgue measure).*

(2) $(\lambda^{\mathrm{cp}} \otimes \cdots \otimes \lambda^{\mathrm{cp}})^{\mathrm{cp}}$.

(3) *Let $\mathcal{T}^n$ be the semiring of sets $\underset{i=1}{\overset{n}{\times}}]a_i, b_i]$. Set*

$$\nu^n\Big(\underset{i=1}{\overset{n}{\times}}]a_i, b_i]\Big) = \prod_{i=1}^{n} |b_i - a_i|.$$

Then ν^n is a premeasure. We consider the measure obtained by the Caratheodory construction.

6.8 Transport of the Lebesgue measure in $\mathbb{R}^n$

Theorem 6.15 *Let U be an open subset of $\mathbb{R}^n$ and $\phi : U \to \mathbb{R}^n$ a C^1 bijection with $\det \phi'(x) \neq 0$, $x \in U$. Let λ be the Lebesgue measure. Then*

$$\frac{\mathrm{d}\phi^*\lambda}{\mathrm{d}\lambda} = |\det \phi'|. \tag{6.31}$$

Thus if $f \in \mathcal{M}_+(\phi(U))$, then

$$\int_{\phi(U)} f \mathrm{d}\lambda = \int_U f \circ \phi |\det \phi'| \mathrm{d}\lambda.$$

We will also write the transformation as

$$(y^1, \dots, y^n) \overset{\phi}{\mapsto} (x^1, \dots, x^n).$$

Then we can write

$$\int f(x)\mathrm{d}x = \int f(x(y)) \Big|\frac{\partial x(y)}{\partial y}\Big| \mathrm{d}y.$$

Proof. We will say that the transformation ϕ satisfies the change of variables formula iff it satisfies (6.31).

Step 1. If the transformations ϕ, ψ satisfy the change of variables formula, then the transformation $\phi \circ \psi$ satisfies it as well.

Step 2. Transformations of the form $(y^1, \dots, y^n) \mapsto (y^{\pi(1)}, \dots, y^{\pi(n)})$, where π is a permutation, satisfy the change of variables formula.

Step 3. If a transformation ϕ has the form

$$(y^1, \dots, y^n) \mapsto (x^1, \dots, x^n) = (f(y^1, \dots, y^n), y^2, \dots, y^n),$$

then it satisfies the change of variables formula. In fact,

$$\begin{aligned}
\int F(x^1, \dots, x^n)\mathrm{d}x^1 \cdots \mathrm{d}x^n &= \int \mathrm{d}x^n \cdots \int \mathrm{d}x^1 F(x^1, \dots, x^n) \\
&= \int \mathrm{d}x^n \cdots \int \mathrm{d}x^2 \int \mathrm{d}y^1 \Big|\tfrac{\partial f(y^1, x^2, \dots, x^n)}{\partial y^1}\Big| F(f(y^1, x^2, \dots, x^n), x^2, \dots, x^n) \\
&= \int \mathrm{d}y^n \cdots \int \mathrm{d}y^2 \int \mathrm{d}y^1 \Big|\tfrac{\partial f(y^1, y^2, \dots, y^n)}{\partial y^1}\Big| F(f(y^1, y^2, \dots, x^n), y^2, \dots, y^n) \\
&= \int F(x(y)) \Big|\det \tfrac{\partial x(y)}{\partial y}\Big| \mathrm{d}y.
\end{aligned}$$

Step 4. We proceed by induction wrt n. We assume that the theorem is true for $n-1$.

If $F \in \mathcal{M}_+(U)$, then we can find a sequence of functions $F_n \in \mathcal{M}_+(U)$ of compact support with $F_n \nearrow F$.

Therefore, it is sufficient to assume that the support of F is compact.

Let $a = (a^1, \dots, a^n) \in \mathrm{supp} F$. There exist i, j such that $\frac{\partial x^i}{\partial y^j}(a) \neq 0$. We can find $\delta > 0$ such that for $|y^i - a^i| < \delta$, $i = 1, \dots, n$, $\frac{\partial x^i(y^1,\dots,y^n)}{\partial y^j} > 0$, or $\frac{\partial x^i(y^1,\dots,y^n)}{\partial y^j} < 0$. Set $W_a := [a^1 - \delta, a^1 + \delta] \times \cdots \times [a^n - \delta, a^n + \delta]$. Clearly, we can find a finite family of $a_1, \dots, a_n \in \mathbb{R}^n$ such that $W_{a_1}, \dots, W_{a_n}$ covers $\mathrm{supp} F$. Then we can write $F = \sum_{i=1}^n F_i$ with $\mathrm{supp} F_i \subset W_i$.

In what follows we assume that on the support of F, $\frac{\partial x^i(y^1,\dots,y^n)}{\partial y^j} > 0$. By Step 2, we can assume that $i = j = 1$. Define

$$(y^1, \dots, y^n) \overset{\psi}{\mapsto} (z^1, \dots, z^n),$$

where

$$z^1(y^1, \dots, y^n) = x^1(y^1, \dots, y^n),\ z^2 = y^2, \dots, z^n = y^n.$$

The map ψ is injective. Define $\rho := \phi\psi^{-1}$, that is

$$(z^1, \dots, z^n) \overset{\rho}{\mapsto} (x^1, \dots, x^n).$$

Note that $x^1 = z^1$.

The map ψ is of the type considered in Step 3. Hence it satisfies the change of variables formula. We have $\phi = \rho\psi$. By Step 1, it is thus sufficient to prove that ρ satisfies the change of variables formula.

We have

$$\rho' = \left[\frac{\partial x}{\partial z}\right] = \begin{bmatrix} 1 & 0 & \dots & 0 \\ \frac{\partial x^2}{\partial z^1} & \frac{\partial x^2}{\partial z^2} & \dots & \frac{\partial x^2}{\partial z^n} \\ & & \dots & \\ \frac{\partial x^n}{\partial z^1} & \frac{\partial x^n}{\partial z^2} & \dots & \frac{\partial x^n}{\partial z^n} \end{bmatrix},$$

and hence

$$\det \frac{\partial x}{\partial z} = \det \begin{bmatrix} \frac{\partial x^2}{\partial z^2} & \dots & \frac{\partial x^2}{\partial z^n} \\ & \dots & \\ \frac{\partial x^n}{\partial z^2} & \dots & \frac{\partial x^n}{\partial z^n} \end{bmatrix}$$

Thus

$$\begin{aligned}
&\int \mathrm{d}x^1 \cdots \int \mathrm{d}x^n F(x^1, \dots, x^n) \\
&= \int \mathrm{d}x^1 \int \mathrm{d}x^2 \cdots \int \mathrm{d}x^n F(x^1, x^2 \dots, x^n) \\
&= \int \mathrm{d}x^1 \int \mathrm{d}z^2 \cdots \int \mathrm{d}z^n F(x^1, x^2(x^1, z^2, \dots, z^n), \dots, x^n(x^1, z^2, \dots, z^n)) \left| \det \begin{bmatrix} \frac{\partial x^2}{\partial z^2} & \dots & \frac{\partial x^2}{\partial z^n} \\ & \dots & \\ \frac{\partial x^n}{\partial z^2} & \dots & \frac{\partial x^n}{\partial z^n} \end{bmatrix} \right| \\
&= \int \mathrm{d}z^1 \int \mathrm{d}z^2 \cdots \int \mathrm{d}z^n F(x^1(z^1, \dots, z^n), \dots, x^n(z^1, \dots, z^n)) \left| \det \begin{bmatrix} \frac{\partial x^2}{\partial z^2} & \dots & \frac{\partial x^2}{\partial z^n} \\ & \dots & \\ \frac{\partial x^n}{\partial z^2} & \dots & \frac{\partial x^n}{\partial z^n} \end{bmatrix} \right| \\
&= \int F(x(z)) \left|\det \frac{\partial x(z)}{\partial z}\right| \mathrm{d}z.
\end{aligned}$$

□

7 Charges and the Radon-Nikodym theorem

7.1 Extension of a measure from a σ-ring

Let $\mathcal{F}, \mathcal{I}$ be σ-rings over X and let $\mathcal{I}$ be an ideal in $\mathcal{F}$. Let $(X, \mathcal{I}, \mu)$ be a space with a measure. We can then extend the measure μ from $\mathcal{I}$ to $\mathcal{F}$. We can do this in many ways.

Theorem 7.1 (1) *Define* $\mu^{\max} : \mathcal{F} \to [0, \infty]$ *by*

$$\mu^{\max}(A) := \inf\{\mu(B) \ : \ A \subset B, \ B \in \mathcal{I}\}.$$

Then $\mu^{\max}$ *is a measure. We have* $\mu^{\max}(A) = \mu(A)$, $A \in \mathcal{I}$, *and* $\mu^{\max}(A) = \infty$, $A \in \mathcal{F} \backslash \mathcal{I}$. $\mu^{\max}$ *is the largest measure on* $\mathcal{F}$ *extending* μ *onto* $\mathcal{F}$. σ*-finite and null sets coincide for* μ *and* $\mu_{\max}$.

(2) *Define* $\mu^{\min} : \mathcal{F} \to [0, \infty]$ *by*

$$\mu^{\min}(A) := \sup\{\mu(B) \ : \ B \subset A, \ B \in \mathcal{I}\}.$$

Then $\mu^{\min}$ *is a measure. We have* $\mu^{\min}(A) = \mu(A)$, $A \in \mathcal{I}$. $\mu^{\min}$ *is the smallest measure extending* μ *onto* $\mathcal{F}$.

Proof. (1) is obvious.

(2) Let us prove that $\mu^{\min}$ is σ-additive. Let $A_1, A_2, \dots \in \mathcal{F}$ be disjoint, $A = \bigcup\limits_{j=1}^{\infty} A_j$.

Let $B \in \mathcal{I}$, $B \subset A$. Then using $A_j \cap B \in \mathcal{I}$ we get

$$\mu(B) = \sum_{j=1}^{\infty} \mu(A_j \cap B) \le \sum_{j=1}^{\infty} \mu^{\min}(A_j).$$

Hence

$$\mu^{\min}(A) \le \sum_{j=1}^{\infty} \mu^{\min}(A_j).$$

Let $B_j \subset A_j$, $B_j \in \mathcal{I}$. Then $B_1, B_2, \dots$ are disjoint and $\cup_{j=1}^{\infty} B_j \subset A$, hence

$$\sum_{j=1}^{\infty} \mu(B_j) = \mu(\bigcup_{j=1}^{\infty} B_j) \le \mu^{\min}(A).$$

Thus

$$\sum_{j=1}^{\infty} \mu^{\min}(A_j) \le \mu^{\min}(A).$$

□

7.2 Measures singular and continuous wrt an ideal

Let $\mathcal{F}$ be a σ-ring over X and let $\mathcal{I}$ be a σ-ring—an ideal in $\mathcal{F}$. Let $(X, \mathcal{F}, \nu)$ be a space with a measure.

We say that ν is $\mathcal{I}$-singular if

$$\nu(A) = \sup\{\nu(B) \ : \ B \subset A, \ B \in \mathcal{I}\}, \ A \in \mathcal{F}.$$

We say that ν is $\mathcal{I}$-continuous if

$$A \in \mathcal{I} \ \Rightarrow \ \nu(A) = 0. \tag{7.32}$$

(More generally, if ν is a charge, we say it is $\mathcal{I}$-continuous if (7.32) is true).

In particular, if $(X, \mathcal{F}, \mu)$ is also a space with a measure, then

$$\mathcal{F}_\mu^0 := \{A \in \mathcal{F} \ : \ \mu(A) = 0\}$$

is an ideal in $\mathcal{F}$. We say that ν is μ-singular if it is $\mathcal{F}_\mu^0$-singular. We say that ν is μ-continuous if it is $\mathcal{F}_\mu^0$-continuous.

Theorem 7.2 *Let $(X, \mathcal{F}, \nu)$ be a measure. Let $\mathcal{I}$ be a σ-ring, an ideal in $\mathcal{F}$.*

(1) *There exists a decomposition*

$$\nu = \nu_{\mathcal{I}\mathrm{s}} + \nu_{\mathcal{I}\mathrm{c}}, \tag{7.33}$$

where $\nu_{\mathcal{I}\mathrm{s}}$ is a $\mathcal{I}$-singular measure and $\nu_{\mathcal{I}\mathrm{c}}$ is a $\mathcal{I}$-continuous measure. The $\mathcal{I}$-singular part is uniquely given by

$$\nu_{\mathcal{I}\mathrm{s}}(A) := \sup\{\nu(B) \ : \ B \subset A, \ B \in \mathcal{I}\}.$$

The $\mathcal{I}$-continuous part does not have to be unique, but there is a canonical choice given by

$$\nu_{\mathcal{I}\mathrm{c}}(A) := \inf\{\nu(A \backslash B) \ : \ B \subset A, \ B \in \mathcal{I}\}.$$

(2) *If ν is σ-finite, then the decomposition of ν into a $\mathcal{I}$-singular and a $\mathcal{I}$-continuous measure is unique.*

(3) *If X is $\nu - \sigma$-finite, then there exists a set $N \in \mathcal{I}$ such that*

$$\nu_{\mathcal{I}\mathrm{s}}(A) = \nu(A \cap N), \quad \nu_{\mathcal{I}\mathrm{c}}(A) = \nu(A \backslash N).$$

Proof. The fact that $\nu_{\mathcal{I}\mathrm{s}}$ is a measure follows from Theorem 7.1 applied to $\nu\Big|_{\mathcal{I}}$.

We need to show that $\nu_{\mathcal{I}\mathrm{c}}$ is a measure. Let $A_1, A_2, \dots \in \mathcal{F}$ be disjoint and $A = \bigcup\limits_{j=1}^{\infty} A_j$. Let us prove that

$$\nu_{\mathcal{I}\mathrm{c}}(A) \leq \sum_{j=1}^{\infty} \nu_{\mathcal{I}\mathrm{c}}(A_j). \tag{7.34}$$

It is sufficient to assume that $\nu_{\mathcal{I}\mathrm{c}}(A_j) < \infty$, $j = 1, 2 \dots$. Let $\epsilon > 0$. We will find $B_j \in \mathcal{I}$, $B_j \subset A_j$ with $\nu_{\mathcal{I}\mathrm{c}}(A_j) > \nu(A_j \backslash B_j) - 2^{-j}\epsilon$. Then

$$\nu_{\mathcal{I}\mathrm{c}}(A) \leq \nu(A \backslash \cup_{j=1}^{\infty} B_j) = \sum_{j=1}^{\infty} \nu(A_j \backslash B_j) \leq \sum_{j=1}^{\infty} \nu_{\mathcal{I}\mathrm{c}}(A_j) + \epsilon.$$

This proves (7.34).

Let us prove that

$$\sum_{j=1}^{\infty} \nu_{\mathcal{I}\mathrm{c}}(A_j) \leq \nu_{\mathcal{I}\mathrm{c}}(A). \tag{7.35}$$

It is sufficient to assume that $\nu_{\mathcal{I}\mathrm{c}}(A) < \infty$. Let $\epsilon > 0$. We will find $B \in \mathcal{I}$, $B \subset A$ with $\nu_{\mathcal{I}\mathrm{c}}(A) > \nu(A \backslash B) - \epsilon$. Then

$$\sum_{j=1}^{\infty} \nu_{\mathcal{I}\mathrm{c}}(A_j) \leq \sum_{j=1}^{\infty} \nu(A_j \backslash B) = \nu(A \backslash B) < \nu_{\mathcal{I}\mathrm{c}}(A) + \epsilon.$$

This proves (7.35)

(3) Let us prove the existence of the set N. Assume that X is ν-finite. Let

$$\alpha := \sup\{\nu(A) \ : \ A \in \mathcal{I}\}.$$

Then $\alpha < \infty$. We can find a sequence $(N_j)_{j\in\mathbb{N}}$ in $\mathcal{I}$ such that $\lim_{j\to\infty} \nu(N_j) = \alpha$. We can assume that the sequence $N_j \nearrow N$. Then $N \in \mathcal{I}$ and $\nu(N) = \alpha$.

It is obvious that $\nu(A \cap N) \leq \nu_{\mathcal{I}\mathrm{s}}(A)$. Suppose that for some $A \in \mathcal{F}$,

$$\nu(A \cap N) < \nu_{\mathcal{I}\mathrm{s}}(A).$$

Then there exists $B \in \mathcal{I}$ with $B \subset A$ and

$$\nu(A \cap N) < \nu(B).$$

Then $B \cup N \in \mathcal{I}$ and

$$\nu(N \cup B) = \nu(N\backslash B) + \nu(B) > \nu(A \cap N) + \nu(N\backslash A) = \nu(N),$$

which is a contradiction

If X is $\nu - \sigma$-finite, then we can find a sequence $X_n \nearrow X$ such that $\nu(X_n) < \infty$. We will also find sets $N_n \subset X_n$ constructed as above. We easily check that $\nu(A \cap N) = \nu_{\mathcal{I}\mathrm{s}}(A)$.

The decomposition of ν is uniquely determined on σ-finite sets. Hence it is unique. □

7.3 Pure point and continuous measures

Definition 7.3 *Suppose that $(X, \mathcal{F}, \nu)$ is a space with measure and*

$$\{\{x\} \ : \ x \in X\} \subset \mathcal{F}. \tag{7.36}$$

We say that ν is a point (atomic) measure if

$$\nu(A) = \sum_{x\in A} \nu(\{x\}).$$

ν is continuous (diffuse) if

$$\nu(\{x\}) = 0, \ x \in X.$$

Theorem 7.4 *Let $(X, \mathcal{F}, \nu)$ be a measure. Assume (7.36).*

(1) *There exists a decomposition*

$$\nu = \nu_{\mathrm{p}} + \nu_{\mathrm{c}},$$

where ν_{p} is a point measure and ν_{c} is a continuous measure. The point part is uniquely given by

$$\nu_{\mathrm{p}}(A) := \sup\{\nu(B) \ : \ B \subset A, \ B \text{ is finite}\},$$

The $\mathcal{I}$-continuous part does not have to be unique, but there is a canonical choice given by

$$\nu_{\mathrm{c}}(A) := \inf\{\nu(A\backslash B) \ : \ B \subset A, \ B \text{ is finite}\}.$$

(2) *If ν is σ-finite, then the decomposition of ν into a point and a continuous measure is unique.*

(3) *If $\mathcal{X}$ is $\nu - \sigma$-finite, there exists a countable set $N \in \mathcal{F}$ such that*

$$\nu_{\mathrm{p}}(A) = \nu(A \cap N), \quad \nu_{\mathrm{c}}(A) = \nu(A\backslash N).$$

Corollary 7.5 *Let $(X, \mathcal{F})$ be a set with a σ-field Let*

$$\{\{x\} \ : \ x \in X\} \subset \mathcal{F}$$

Let ν, μ be measures on $(X, \mathcal{F})$ and let μ be continuous. Then there exists a decomposition

$$\nu = \nu_{\rm p} + \nu_{\rm sc} + \nu_{\rm ac}$$

such that

$$\nu_{\rm p} \text{ is pure point },$$

$$\nu_{\rm sc} \text{ is } \mu\text{-singular and continuous},$$

$$\nu_{\rm ac} \text{ is } \mu\text{-continuous}.$$

If ν is σ-finite, then the decomposition is unique.

7.4 Charges (signed measures)

Let $(X, \mathcal{F})$ be a space with a σ-ring. A function $\mu : \mathcal{F} \to]-\infty, \infty]$ is called a bounded from below charge (or signed measure) if

(1) $\mu(\emptyset) = 0$,

(2) $A_1, A_2 \dots \in \mathcal{F},\ A_i \cap A_j = \emptyset$ for $i \neq j \Rightarrow \mu\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mu(A_n)$.

Proposition 7.6 (1) *If $A \subset B$, and $\mu(B) < \infty$, then $\mu(A) < \infty$.*

(2) *$A_1, A_2, \dots \in \mathcal{F}$, $A_i \cap A_j = \emptyset$, $i \neq j$, and $\mu\left(\cup_{n=1}^{\infty} A_n\right) < \infty$, then $\sum_{n=1}^{\infty} \mu(A_n)$ is absolutely convergent.*

(3) *$A_1, A_2, \dots \in \mathcal{F}$, $A_n \nearrow A \Rightarrow \mu(A_n) \to \mu(A)$.*

(4) *If $A_1, A_2, \dots \in \mathcal{F}$, $A_n \searrow A$ and for some n, $\mu(A_n) < \infty$, then $\mu(A_n) \to \mu(A)$.*

Proof. (1) $\mu(B) = \mu(A) + \mu(B \backslash A)$ and $\mu(B \backslash A) > -\infty$. Hence $\mu(A) = \mu(B) - \mu(B \backslash A)$ with both summands less than ∞.

(2) We group the sets A_i into two subfamilies: those with a positive charge and a nonpositive charge. After renumbering we can call the former $B_1, B_2, \dots$ and the latter $C_1, C_2, \dots$. We have

$$-\sum_{n=1}^{\infty} \mu(B_n) = -\mu\left(\cup_{n=1}^{\infty} B_n\right) < \infty$$

and

$$\sum_{n=1}^{\infty} \mu(C_n) = \mu\left(\cup_{n=1}^{\infty} C_n\right) = \mu\left(\cup_{n=1}^{\infty} A_n\right) - \mu\left(\cup_{n=1}^{\infty} B_n\right) < \infty.$$

□

7.5 Hahn and Jordan decompositions of a charge

Let $(X, \mathcal{F}, \mu)$ be a space with a bounded from below charge.

We say that $A \in \mathcal{F}$ is positive iff $B \in \mathcal{F}$, $B \subset A$ implies $\mu(B) \geq 0$. We say that $A \in \mathcal{F}$ is negative iff $B \in \mathcal{F}$, $B \subset A$ implies $\mu(B) \leq 0$. Let $\mathcal{F}_\mu^\pm$ denote the family of positive/negative sets.

Theorem 7.7 *$\mathcal{F}_\mu^\pm$ are σ-rings and ideals of $\mathcal{F}$. $\pm\mu$ restricted to $\mathcal{F}_\mu^\pm$ are measures.*

For $A \in \mathcal{F}$, we set

$$\mu_+(A) := \sup\{\mu(B) \ : \ B \subset A, \ B \in \mathcal{F}_\mu^+\},$$

$$\mu_-(A) := \sup\{-\mu(B) \ : \ B \subset A, \ B \in \mathcal{F}_\mu^-\}.$$

Theorem 7.8 (1) *μ_-, μ_+ are measures.*

(2) *On $\mathcal{F}_\mu^\pm$, μ coincides with $\pm\mu_\pm$.*

(3) *μ_- is finite.*

(4) *There exists $E \in \mathcal{F}_\mu^-$ with $\mu_-(E) = \mu_-(X)$. In what follows we fix such a set E.*

(5) *If $\mu(A) < \infty$, then $\mu_+(A)$ is finite.*

(6) *$X \backslash E \in \mathcal{F}_\mu^+$.*

(7) **(Jordan decomposition)** *$\mu = \mu_+ - \mu_-$.*

(8) **(Hahn decomposition)** *For $A \in \mathcal{F}$,*

$$\mu_-(A) = -\mu(A \cap E), \quad \mu_+(A) = \mu(A \backslash E).$$

(9)

$$\mu_+(A) := \sup\{\mu(B) \ : \ B \subset A, \ B \in \mathcal{F}\},$$

$$\mu_-(A) := \sup\{-\mu(B) \ : \ B \subset A, \ B \in \mathcal{F}\}.$$

Proof. (1) and (2) follow immediately from Theorem 7.1 (2).

Let $\beta := \mu_-(X)$. Then there exist negative $E_1, E_2, \ldots$ such that $\mu_-(E_n) \to \beta$. Since negative sets form a σ-ring, $E := \bigcup_{j=1}^{n} E_j \in \mathcal{F}_\mu^-$. Clearly, $\mu_-(E) \le \mu_-(E_n)$. Hence $\beta = \mu_-(E)$. This implies (4) and (3). A similar argument yields (5).

We interrupt the proof. □

Lemma 7.9 *Suppose that $\mu(X) < \mu_+(X)$. Then there exists $B \in \mathcal{F}_\mu^-$ with $\mu(B) < 0$.*

Proof. If $\mu_+(X) = 0$, then $X \in \mathcal{F}_\mu^-$ and $\mu(X) < 0$. We can thus set $B := X$

The condition $\mu(X) < \infty$ implies that $\mu_+(X) < \infty$.

Let $\mu_+(X) > 0$. We can find $q < 1$ such that

$$\mu(X) - q\mu_+(X) < 0.$$

We can find $E \in \mathcal{F}_\mu^+$ such that $\mu(E) \ge q\mu_+(X)$. Set $X_1 := X \backslash E$. Then

$$\mu_+(X_1) = \mu_+(X) - \mu_+(E) \le (1-q)\mu_+(X),$$

$$\mu(X_1) = \mu(X) - \mu(E) \le \mu(X) - q\mu_+(X) < 0.$$

By induction, we can find a sequence of disjoint sets $E_1, \cdots \in \mathcal{F}_\mu^+$ such that for $X_n := X_1 \backslash \bigcup_{j=1}^{n-1} E_j$, then

$$\mu(E_n) \ge q\mu_+(X_n).$$

(Note that $E_j \subset X_j$). Then

$$\begin{aligned} \mu_+(X_{n+1}) &= \mu_+(X_n) - \mu_+(E_n) \\ &= \mu_+(X_n) - \mu(E_n) \le (1-q)\mu_+(X_n). \end{aligned}$$

Hence,

$$\mu_+(X_n) \leq (1-q)^n \mu_+(X_0).$$

Moreover,

$$\mu(X_{n+1}) = \mu(X_1)\backslash \sum_{j=1}^n \mu(E_j) \quad \leq \mu(X_1) < 0.$$

Set $B := \bigcap\limits_j X_j$. Then

$$\mu_+(B) = \lim_{j\to\infty} \mu_+(X_j) = 0,$$

and hence $B \in \mathcal{F}_\mu^-$, and

$$\mu(B) = \lim_{j\to\infty} \mu(X_j) \leq \mu(X_1) < 0.$$

□

Continuation of the proof of Theorem 7.8. Suppose that $X\backslash E$ is not positive. This means that there exists $X_0 \subset X\backslash E$ and $\mu(X_0) < 0$. Then X_0 satisfies the conditions of Lemma 7.9. Hence X_0 contains $B \in \mathcal{F}_\mu^-$ wih $\mu(B) < 0$. Hence $E \cup B \in \mathcal{F}_\mu^-$ with $\mu(E \cup B) < \beta$, which is a contradiction. This proves (6).

Now note that for $B \in \mathcal{F}_\mu^+$ we have $\mu(B) = \mu_+(B\backslash E)$. Hence for $A \in \mathcal{F}$,

$$\mu_+(A) \quad = \sup\{\mu_+(B\backslash E) \ : \ B \subset A, \ B \in \mathcal{F}_\mu^+\} = \mu_+(A\backslash E) = \mu(A\backslash E).$$

This proves (8) and (7).

$$\begin{aligned} \mu_+(A) \quad &\leq \sup\{\mu(B) \ : \ B \subset A, \ B \in \mathcal{F}\}, \\ &\leq \sup\{\mu_+(B) \ : \ B \subset A, \ B \in \mathcal{F}\} = \mu_+(A). \end{aligned}$$

This proves (9). □

7.6 Banach space of finite charges

Let $(X, \mathcal{F})$ be a set with a σ-field. We define $\mathrm{Ch}(X, \mathcal{F})$ (or $\mathrm{Ch}(X)$) as the ordered linear space of finite charges on $(X, \mathcal{F})$. We set

$$\|\mu\| := |\mu|(X).$$

Theorem 7.10 (1) $\mathrm{Ch}(X, \mathcal{F})$ *is a Banach space.*

(2) $\mathrm{Ch}(X, \mathcal{F})$ *is a complete lattice.*

(3) $0 \leq \mu \leq \nu$ *implies* $\|\mu\| \leq \|\nu\|$.

7.7 Measures with a density

Theorem 7.11 *Let $(X, \mathcal{F}, \mu)$ be a space with a measure. Let $f \in \mathcal{M}_+(X)$ Then*

$$\mathcal{F} \ni A \mapsto \nu(A) := \int 1_A f \mathrm{d}\mu \tag{7.37}$$

is a measure. If $f \in \mathcal{M}(X)$ and $f_- \in \mathcal{L}^1(\mu)$, then (7.37) is a bounded from below charge.

Definition 7.12 *The measure ν is called the measure with the density f wrt the measure μ and is denoted $\nu = f\mu$. We will also write $f := \frac{\mathrm{d}\nu}{\mathrm{d}\mu}$.*

Theorem 7.13 (1) *For $f, g \in \mathcal{M}_+(X)$ we have*

$$f = g\ \mu\text{-a.e.} \Rightarrow f\mu = g\mu.$$

(2) *If $f\mu$ is sum-finite, then the converse implication is also true.*

Proof. The implication $\Rightarrow$ is obvious. Let us show the converse statement.

First assume that $f\mu$ is finite, or in other words $f \in \mathcal{L}^1(\mu)$. Let $N := \{f < g\}$ and

$$h := g1_N - f1_N.$$

Clearly, $f1_N \leq f$ and $g1_N \leq g$. Hence, $f1_N \in \mathcal{L}^1(\mu)$, $g1_N \in \mathcal{L}^1(\mu)$. Therefore, $h \in \mathcal{L}^1(\mu)$. Besides, $\int h\mathrm{d}\mu = 0$ and $h \geq 0$. Thus $h = 0$ μ-a.e. But $N = \{h > 0\}$. Hence $\mu(N) = 0$.

Assume now that μ is sum-finite. Let X_i be a localizing family. Then $f\mu = g\mu$ restricted to X_i. Hence $f_i = g_i$ on X_i almost everywhere wrt the measure μ restricted to X_i. This implies that $f = g$ μ-a.e. $\square$

Recall that the charge ν is called continuous wrt μ (or μ-continuous), if

$$\mu(N) = 0 \Rightarrow \nu(N) = 0, \quad N \in \mathcal{F}.$$

Theorem 7.14 (Radon-Nikodym) *Let μ be a sum-finite measure on $(X, \mathcal{F})$ and let ν be a charge. Then the following conditions are equivalent:*

(1) *there exists $f \in \mathcal{M}(X)$ such that $\nu = f\mu$ and $f_- \in \mathcal{L}^1(\mu)$;*

(2) *ν is μ-continuous.*

Proof. The implication $\Rightarrow$ is obvious. Let us show the converse statement.

Step 0. If $\mu = 0$, then $\nu = 0$, and the theorem is obviously true.

Step 1. Assume that $0 < \mu(X) < \infty$, $\nu(X) < \infty$. Let

$$\mathcal{G} := \{g \in \mathcal{M}_+(\mathcal{F}) \ : \ g\mu \leq \nu\}.$$

Clearly, $\mathcal{G}$ is non-empty, since $0 \in \mathcal{G}$.

We have

$$g, h \in \mathcal{G} \Rightarrow \sup(g, h) \in \mathcal{G}.$$

In fact, if $A_1 := \{g < h\}$, $A_2 := \{g \geq h\}$ and $A \in \mathcal{F}$, then

$$\int_A \sup(g,h)\mathrm{d}\mu = \int_{A\cap A_1} g\mathrm{d}\mu + \int_{A\cap A_2} h\mathrm{d}\mu \leq \nu(A\cap A_1) + \nu(A \cap A_2) = \nu(A).$$

Let

$$\gamma := \sup\{\textstyle\int g\mathrm{d}\mu \ : \ g \in \mathcal{G}\}. \tag{7.38}$$

Then $\gamma \leq \nu(X) < \infty$. We can find $g_n' \in \mathcal{G}$ such that

$$\lim_{n\to\infty} \textstyle\int g_n'\mathrm{d}\mu = \gamma.$$

Let

$$f := \sup(g_n')_{n\in\mathbb{N}}.$$

We claim that

$$f \in \mathcal{G} \text{ and } \int g\mathrm{d}\mu = \gamma. \tag{7.39}$$

In fact, we have

$$g_n := \sup(g'_1, \dots, g'_n) \in \mathcal{G}, \; g_n \nearrow f,$$

which immediately implies (7.39).

Suppose now that

$$(f\mu)(X) < \nu(X). \tag{7.40}$$

Using $\mu(X) < \infty$, we can find $\beta > 0$ such that

$$\beta\mu(X) < \nu(X) - (f\mu)(X).$$

Set

$$\rho(A) := \nu(A) - (f\mu)(A) - \beta\mu(A), \quad A \in \mathcal{F}.$$

ρ is a bounded μ-continuous charge satisfying $\rho(X) > 0$. By Lemma 7.9, we can find a ρ-positive set $E \in \mathcal{F}$ such that $\rho(E) > 0$. Recall that ρ-positivity of E means that

$$A \in 2^E \cap \mathcal{F} \Rightarrow \rho(A) \geq 0.$$

Hence $f_0 = f + \beta 1_E \in \mathcal{G}$.

Note that the μ-continuity of the charge ρ and $\rho(E) > 0$ implies $\mu(E) > 0$. Hence

$$\int f_0 \mathrm{d}\mu = \int f \mathrm{d}\mu + \beta\mu(E) = \gamma + \beta\mu(E) > \gamma,$$

which is a contradiction with (7.38).

Step 2. ν is σ-finite, μ is finite. We decompose X into a disjoint union of sets of finite measure ν and use **Step 1.**

Step 3. ν is arbitrary, μ is finite. Let $\mathcal{F}_\nu^{\mathrm{f}} := \{A \in \mathcal{F} \;:\; \nu(A) < \infty\}$ and $\alpha := \sup\{\mu(A) \;:\; A \in \mathcal{F}_\nu^{\mathrm{f}}\}$. We can find $A_n \in \mathcal{F}_\nu^{\mathrm{f}}$ such that $\lim_{n\to\infty} \mu(A_n) = \alpha$. We can assume that $A_n \nearrow X_0$. Then ν on X_0 is σ-finite and on $X_1 := X \backslash X_0$ it has the property

$$\text{or } \mu(A) = \nu(A) = 0,$$

$$\text{or } \mu(A) > 0, \; \nu(A) = \infty.$$

In fact, if $A \subset X_1$, $\mu(A) > 0$ and $\nu(A) < \infty$, then $A_n \cup A \in \mathcal{F}_\nu^{\mathrm{f}}$ and $\mu(A_n \cup A) \nearrow \alpha + \mu(A)$, which means $\mu(A) = 0$. We apply **Step 2.** to X_0 and on X_1 we put $\nu = \infty\mu$.

Step 4. ν is arbitrary and μ sum-finite. We decompose X into a union of disjoint sets with a finite measure μ and use **Step 3.** □

7.8 Dual of $L^p(\mu)$

Theorem 7.15 *Let $(X, \mathcal{F}, \mu)$ be a space with a measure, $1 \leq p \leq \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. For $g \in L^q(\mu)$ set*

$$\langle v_g | f \rangle := \int g f \mathrm{d}\mu, \quad f \in L^p(\mu). \tag{7.41}$$

Then

(1) *$v_g \in (L^p(\mu))^\#$ and $\|v_g\| = \|g\|_q$. Thus*

$$L^q(\mu) \ni g \mapsto v_g \in L^p(\mu)^\# \tag{7.42}$$

is an isometry.

(2)

$$\tilde{f} := \overline{f}|f|^{p-2}/(\int |f|^p)^{1/q} \tag{7.43}$$

$v_{\tilde{f}}$ is a functional tangent to f.

(3) *Let $1 \leq p < \infty$ and let μ be sum-finite. Then for v, a bounded functional on $L^p(\mu)$, there exists a unique $g \in L^q$ such that (7.41) holds. Thus, the map (7.42) is bijective.*

Proof. (1) Using the Hölder inequality we check that if $g \in L^q(\mu)$, then $v_g \in L^p(\mu)^\#$ and $\|v_g\| \leq \|g\|_q$. Setting

$$\tilde{g} := \overline{g}|g|^{q-2}/(\int |g|^q)^{1/p}, \tag{7.44}$$

we see that$\langle v_g|\tilde{g}\rangle = \|g\|_q$ and $\|\tilde{g}\|_p = 1$. Hence $\|v_g\| = \|g\|_q$.

(3) To prove the existence first assume that the measure is finite and $v \in L^p(\mu)^\#$. Then

$$\mathcal{F} \ni A \mapsto \langle v|1_A\rangle$$

is a μ-continuous finite charge. By the Radon-Nikodym Theorem, there exists $g \in \mathcal{M}(X)$ such that

$$\langle v|1_A\rangle = \int g1_A \mathrm{d}\mu,$$

Assume that $g \notin L^q(\mu)$ Clearly, $g_+ \notin L^q(\mu)$ or $g_- \notin L^q(\mu)$. Hence it is sufficient to assume that $g \geq 0$.

We can find $g_n \in \mathcal{E}_+(X)$ such that $g_n \nearrow g$. Clearly, $\|g_n\|_q \to \infty$. Set $\tilde{g}_n := g_n^{q-1}/(\int |g_n|^q)^{1/p}$. Clearly, $\tilde{g}_n \in L^p(\mu)$, $\|\tilde{g}_n\| = 1$ and

$$\langle v|\tilde{g}_n\rangle = \int g g_n^{q-1} \mathrm{d}\mu/(\int |g_n|^q)^{1/p} \geq \|g_n\|_q \to \infty.$$

Hence v is not bounded.

Thus $g \in L^q(\mu)$. We already know that v_g is bounded and it coincides with v on $\mathcal{E}(X)$, which is dense in $L^p(\mu)$. Hence (7.41) is true for all $f \in L^p(\mu)$

The uniqueness follows from Theorem (7.13) (2). □

Theorem 7.16 *Let μ be sum-finite. Then $L^\infty(\mu)$ is a complete lattice.*

8 Measures on topological spaces

8.1 δ-open and σ-closed sets

If X is a topological space, we will write Open(X), Open$_0(X)$, Closed(X) and Compact(X) for the family of open, open pre-compact, closed and compact subsets of X.

Definition 8.1 *δ-open sets are countable intersection of open sets.*
σ-closed sets are countable unions of closed sets.

The complement of a σ-closed set is a δ-open set and vice versa.

Theorem 8.2 *Let X be a metrizable space. Then every closed set is δ-open.*

Proof. Let C be closed. Define

$$C_n := \{x \in X \ : \ \mathrm{d}(x, C) < 1/n\}.$$

Then C_n are open and

$$C = \bigcap_{n=1}^{\infty} C_n.$$

Hence C is a δ-open set. □

8.2 Baire and Borel sets of 1st kind

Theorem 8.3 (1) *Let $f \in C(X, \mathbb{R})$. Then $f^{-1}(]\alpha, \infty[) \in \mathrm{Open} \cap \sigma{-}\mathrm{Closed}(X)$.*

(2) *Let X be normal and $A \in \mathrm{Open} \cap \sigma{-}\mathrm{Closed}(X)$. Then there exists $f \in C(X, \mathbb{R})$ such that $A = f^{-1}(]\alpha, \infty[)$*

Proof. We have

$$f^{-1}(]a, \infty[) = \bigcup_{n=1}^{\infty} f^{-1}([a + 1/n, \infty[) \tag{8.45}$$

Clearly, (8.45) are σ-closed.

Let

$$A = \bigcup_{n=1}^{\infty} A_n$$

be open σ-closed and A_n let be closed. We can then find $f_n \in C(X)$ such that $0 \leq f \leq 1$, $f_n = 1$ on A_n and $f_n = 0$ outside A. We define

$$f := \sum_{n=1}^{\infty} 2^{-n} f_n.$$

Then

$$A = f^{-1}(]0, \infty[).$$

□

Definition 8.4 *Let X be a topological space. Then the $\sigma-$field of Baire sets of 1st kind, denoted $\mathrm{Baire}_1(X)$, is the smallest $\sigma-$field such that all elements of $C(X, \mathbb{R})$ are measurable.*

Theorem 8.5 *Let X be normal, $A \subset B \subset X$, A be closed and B open. Then there exist $A_0 \in \mathrm{Closed} \cap \mathrm{Baire}_1(X)$ and $B_0 \in \mathrm{Open} \cap \mathrm{Baire}_1(X)$ such that $A \subset B_0 \subset A_0 \subset B$.*

Proof. We can find $f \in C(X, \mathbb{R})$ such that $f = 0$ on A and $f = 1$ on $X \backslash B$. Then $f^{-1}(] - \infty, \frac{1}{2}[) \in \mathrm{Open} \cap \mathrm{Baire}_1(X)$ and $f^{-1}(] - \infty, \frac{1}{2}]) \in \mathrm{Closed} \cap \mathrm{Baire}_1(X)$. □

Theorem 8.6 (1) *In any topological space, $\mathrm{Baire}_1(X)$ is generated e.g. by*

$$\{f^{-1}(U) \ : \ U \in \mathrm{Open}(\mathbb{R}), f \in C(X, \mathbb{R})\},$$

$$\{f^{-1}(C) \ : \ C \in \mathrm{Closed}(\mathbb{R}), f \in C(X, \mathbb{R})\},$$

$$\{f^{-1}(]\alpha, \infty[, \quad \alpha \in \mathbb{R}, \quad f \in C(X, \mathbb{R}).$$

(2) *If X is normal, then* $\mathrm{Baire}_1(X)$ *is generated by*

$$\mathrm{Open} \cap \sigma{-}\mathrm{Closed}(X),$$

$$\mathrm{Closed} \cap \delta{-}\mathrm{Open}(X).$$

Theorem 8.7 *Let X be compact Hausdorff. Then*

(1) $\mathrm{Open} \cap \mathrm{Baire}_1(X) = \mathrm{Open} \cap \sigma{-}\mathrm{Closed}(X)$,

(2) $\mathrm{Closed} \cap \mathrm{Baire}_1(X) = \mathrm{Closed} \cap \delta{-}\mathrm{Open}(X)$.

Proof. It is sufficient to prove (2) $\subset$.

Let C be closed Baire. By Theorem 1.8, there exists a countable family $C_1, C_2, \dots$ of σ-closed sets such that $C \in \sigma{-}\mathrm{Ring}(C_1, C_2, \dots)$. We can find functions $f_n \in C(X)$ such that $\{f_n = 0\} = C_n$. Then

$$d(x, y) := \sum_{n=1}^{\infty} 2^{-n} |f_n(x) - f_n(y)|$$

is a semimetric on X.

Let $(\tilde{X}, \tilde{d})$ be the reduced metric space and $T : X \to \tilde{X}$ the corresponding reduction. Clearly, $x \in C_n$, $d(x, y) = 0$ imply $y \in C_n$. Therefore, $T^* 2^{\tilde{X}}$ contains $C_1, C_2, \dots$.

$T^* 2^{\tilde{X}}$ is a σ-ring. Hence

$$T^* 2^{\tilde{X}} \supset \sigma{-}\mathrm{Ring}(C_1, C_2, \dots).$$

Thus there exists $\tilde{C} \in 2^{\tilde{X}}$ with $C = T^{-1}\tilde{C}$. But C is compact and T continuous. Therefore, $\tilde{C}$ is compact as well. Thus $\tilde{C}$ is a closed subset of a metric space, and hence it is σ-open. Hence there exist open $\tilde{U}_1, \tilde{U}_2, \dots$ in $\tilde{X}$ with $\tilde{U}_n \searrow \tilde{C}$. Now $T^{-1}\tilde{U}_n$ are open in X and $T^{-1}\tilde{U}_n \searrow C$. □

Definition 8.8 *σ-field of Borel sets of 1st kind, denoted* $\mathrm{Borel}_1(X)$*, is the σ-field generated by open sets.*

Clearly, $\mathrm{Baire}_1(X) \subset \mathrm{Borel}_1(X)$.

Theorem 8.9 *If X is metrizable, then* $\mathrm{Baire}_1(X) = \mathrm{Borel}_1(X)$.

Proof. In a metrizable space every open set is σ-closed. □

Example 8.10 *Let I be uncountable and X_i be sets of at least two elements. Let $X = \prod_{i \in I} X_i$. Then one-element sets in X are closed (hence Borel) but not δ-open (hence not Baire). In fact, let $x \in Y \subset X$ and Y be δ-open. Then Y contains a subset of the form $\prod_{i \in I} Y_i$ with $Y_i = X_i$ for all but a countable number of $i \in I$.*

8.3 Baire and Borel sets of 2nd kind

Theorem 8.11 (1) *Let $f \in C_c(X, \mathbb{R})$ and $\alpha \geq 0$. Then $f^{-1}(]0, \infty[) \in \mathrm{Open} \cap \sigma{-}\mathrm{Compact}(X)$.*

(2) *Let X be Tikhonov and $A \in \mathrm{Open} \cap \sigma{-}\mathrm{Compact}(X)$. Then there exists $f \in C_c(X, \mathbb{R})$ such that $A = f^{-1}(]0, \infty[)$*

Definition 8.12 *Let X be a topological space. Then the σ-ring of Baire sets of 2nd kind, denoted* $\mathrm{Baire}_2(X)$*, is the smallest σ-ring such that all elements of $C_c(X)$ are measurable.*

Lemma 8.13 *Let X be Tikhonov, $A \subset B \subset X$, A be compact and B open. Then there exist $A_0 \in$* $\mathrm{Compact} \cap \mathrm{Baire}_2(X)$ *and* $B_0 \in \mathrm{Open} \cap \mathrm{Baire}_2(X)$ *such that* $A \subset B_0 \subset A_0 \subset B$.

Theorem 8.14 (1) $\mathrm{Baire}_2(X)$ *is is generated by*

$$\{f^{-1}(U) \ : \ U \in \mathrm{Open}(\mathbb{R}\backslash\{0\}), \ f \in C_{\mathrm{c}}(X)\};$$

$$\{f^{-1}(C) \ : \ K \in \mathrm{Closed}(\mathbb{R}\backslash\{0\}), \ f \in C_{\mathrm{c}}(X)\},$$

$$\{f^{-1}(]\alpha, \infty[), \ f^{-1}(] - \infty, -\alpha[), \ 0 < \alpha, f \in C_{\mathrm{c}}(X)\}.$$

(2) *The closures of all elements of* $\mathrm{Baire}_2(X)$ *are σ-compact.*

(3) *If X is Tikhonov, then* $\mathrm{Baire}_2(X)$ *is generated by*

$$\mathrm{Open} \cap \sigma{-}\mathrm{Compact}(X),$$

$$\mathrm{Compact} \cap \delta{-}\mathrm{Open}(X).$$

Theorem 8.15 *Let X be Tikhonov. Then*

(1) $\mathrm{Compact} \cap \mathrm{Baire}_2(X) = \mathrm{Compact} \cap \delta{-}\mathrm{Open}(X)$.

(2) $\mathrm{Open} \cap \mathrm{Baire}_2(X) = \mathrm{Open} \cap \sigma{-}\mathrm{Compact}(X)$

Definition 8.16 *σ-ring of Borel sets of 2nd kind, denoted* $\mathrm{Borel}_2(X)$, *is the $\sigma{-}$ring generated by compact sets.*

Clearly, $\mathrm{Baire}_2(X) \subset \mathrm{Borel}_2(X)$.

Theorem 8.17 *If X is metrizable, then* $\mathrm{Baire}_2(X) = \mathrm{Borel}_2(X)$.

Theorem 8.18 *For σ-compact spaces* $\mathrm{Baire}_1(X) = \mathrm{Baire}_2(X)$ *and* $\mathrm{Borel}_1(X) = \mathrm{Borel}_2(X)$.

In what follows, we will consider σ-rings of sets only on locally compact Hausdorff spaces. We will use the σ-rings $\mathrm{Baire}_2(X)$ and $\mathrm{Borel}_2(X)$. We will call them simply Baire and Borel σ-rings and denote by $\mathrm{Baire}(X)$ and $\mathrm{Borel}(X)$ respectively. For σ-compact spaces they are in fact σ-fields and coincide with $\mathrm{Baire}_1(X)$ and $\mathrm{Borel}_1(X)$ respectively.

8.4 Baire measures on compact spaces

Let X be a compact Hausdorff space. A finite measure on $\mathrm{Baire}(X)$ is called a Baire measure on X.

A linear functional $\lambda : C(X) \to \mathbb{R}$ is called a positive functional (or a Radon measure) if

$$f \in C(X), \ f \geq 0 \ \Rightarrow \ \lambda(f) \geq 0.$$

Theorem 8.19 *Let ν be a Baire measure. Then*

$$C(X) \ni f \mapsto \int f \mathrm{d}\nu \in \mathbb{R} \tag{8.46}$$

is a positive linear functional.

(1) *If* $C \in \mathrm{Closed} \cap \mathrm{Baire}(X)$, *then*

$$\nu(C) = \inf\{\textstyle\int f \mathrm{d}\nu \ : \ f \in C(X), \ f = 1 \ \mathit{on} \ \ C, \ \ 0 \leq f \leq 1\}.$$

(2) *If* $U \in \text{Open} \cap \text{Baire}(X)$*, then*

$$\nu(U) = \sup\{\textstyle\int f \mathrm{d}\nu \ : \ f \in C(X),\ \mathrm{supp} f \subset U, \quad 0 \leq f \leq 1\}.$$

Proof. The positivity is obvious.

Let us prove (1). The inequality $\leq$ is obvious.

There exists a sequence $U_1, U_2 \cdots \in \text{Open}(X)$ such that $U_n \searrow C$. Let $f_n \in C(X)$, $\mathrm{supp} f_n \subset U_n$, $0 \leq f_n \leq 1$, and $f_n = 1$ on C. Then $f_n \to 1_C$ pointwise, $f_n \leq 1 \in \mathcal{L}^1(\mu)$. Hence by the Lebesgue theorem

$$\lim_{n\to\infty} \int f_n \mathrm{d}\nu = \nu(C).$$

This shows the inequality $\geq$. □

Theorem 8.20 (Riesz-Markov) *Let* λ *be a positive linear functional on* $C(X)$*. Then there exists a unique Baire measure* ν *satisfying*

$$\lambda(f) = \int f \mathrm{d}\nu, \quad f \in C(X). \tag{8.47}$$

The proof of Theorem 8.20 will be split into a number of steps. Let us assume that we are given a positive functional λ.

Lemma 8.21 *A Baire measure satisfying (8.47) is uniquely determined.*

Proof. By Theorem 8.19 (1), ν is uniquely determined by λ on $\text{Closed} \cap \text{Baire}(X)$. This is a $\cap$-stable family that generates $\text{Baire}(X)$. Hence ν is uniquely determined. □

For $U \in \text{Open} \cap \text{Baire}(X)$ we set

$$\nu^*(U) := \sup\{\lambda(f) \ : \ f \in C(X),\ \mathrm{supp} f \subset U,\ 0 \leq f \leq 1.\} \tag{8.48}$$

For any $A \in 2^X$ we set

$$\nu^*(A) := \inf\{\nu^*(U) \ : \ A \subset U,\ U \in \text{Open} \cap \text{Baire}(X)\}. \tag{8.49}$$

(For $U \in \text{Open} \cap \text{Baire}(X)$, (8.48) agrees with (8.49)).

Lemma 8.22 ν^* *is an external measure.*

Proof. We need to show that

$$\nu^*\left(\bigcup_{j=1}^{\infty} A_j\right) \leq \sum_{j=1}^{\infty} \nu^*\left(A_j\right).$$

If the sum on the right is infinite, the inequality is obvious. Assume it is finite. We will find $U_j \in \text{Open} \cap \text{Baire}(X)$ with $A_j \subset U_j$ and

$$\nu^*(U_j) \leq \nu^*(A_j) + 2^{-j}\epsilon.$$

Let $f \in C(X)$ satisfy $0 \leq f \leq 1$,

$$\mathrm{supp} f \subset \bigcup_{j=1}^{\infty} A_j \ \text{ i } \ \nu^*\left(\bigcup_{j=1}^{\infty} A_j\right) < \lambda(f) + \epsilon.$$

By the compactness of $\mathrm{supp} f$, for some n,

$$\mathrm{supp} f \subset \bigcup_{j=1}^{n} U_j.$$

We will find $h_1, \dots, h_n \in C(X)$ such that $h_j \geq 0$, $\mathrm{supp} h_j \subset U_j$ and $\sum_{j=1}^{n} h_j = 1$ on $\mathrm{supp} f$. Hence,

$$\begin{aligned}
&\nu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \nu^*\left(\bigcup_{n=1}^{\infty} U_n\right) \\
&\leq \lambda(f) + \epsilon = \sum_{j=1}^{n} \lambda(f h_j) + \epsilon \\
&\leq \sum_{n=1}^{\infty} \nu^*(U_n) + \epsilon \\
&\leq \sum_{n=1}^{\infty} \nu^*(A_n) + \epsilon \sum_{n=1}^{\infty} 2^{-n} + \epsilon = \sum_{n=1}^{\infty} \nu^*(A_n) + 2\epsilon.
\end{aligned}$$

□

Lemma 8.23 *All Baire sets on X are ν^*-measurable.*

Proof. It suffices to show that open Baire sets are ν^*-measurable. Let U be open and $Q \in 2^X$.

Let $A \in \mathrm{Open} \cap \mathrm{Baire}(X)$ such that $Q \subset A$ and $\nu^*(A) \leq \nu^*(Q) + \epsilon$. Consider the set $A_1 := A \cap U \in \mathrm{Open} \cap \mathrm{Baire}(X)$. Let $f_1 \in C(A)$ such that $0 \leq f_1 \leq 1$, $\mathrm{supp} f_1 \subset A_1$ and

$$\nu^*(A_1) \leq \lambda(f_1) + \epsilon.$$

Next consider set $A_2 := A \backslash \mathrm{supp} f_1 \in \mathrm{Open}$. We will find $f_2 \in C(X)$ such that $0 \leq f_2 \leq 1$, $\mathrm{supp} f_2 \subset A_2$ and

$$\nu(A_2) \leq \lambda(f_2) + \epsilon.$$

We have $0 \leq f_1 + f_2 \leq 1$ and $\mathrm{supp}(f_1 + f_2) \subset A$. Besides, $Q \cap U \subset A_1$ and $Q \backslash U \subset A_2$. Hence

$$\begin{aligned}
&\nu^*(Q \cap U) + \nu^*(Q \backslash U) \\
&\leq \nu^*(A_1) + \nu^*(A_2) \\
&\leq \lambda(f_1) + \lambda(f_2) + 2\epsilon = \lambda(f_1 + f_2) + 2\epsilon \\
&\leq \nu^*(A) + 2\epsilon \leq \nu^*(Q) + 3\epsilon.
\end{aligned}$$

Hence,

$$\nu^*(Q \cap U) + \nu^*(Q \backslash U) \leq \nu^*(A).$$

Hence U is ν^*-measurable. □

Set ν to be equal to ν^* restricted to $\mathrm{Baire}(X)$. By Lemmas 8.22 and 8.23 it is a Baire measure.

Lemma 8.24 *Let $f \in C(X)$. Then*

$$\lambda(f) = \int f \mathrm{d}\nu.$$

Proof. We can assume that $0 \leq f \leq 1$. Set

$$U_{n,j} := \{f > j/n\}, \quad C_{n,j} := \{f \geq j/n\}.$$

Let $g_{n,j} \in C(X)$ with $\mathrm{supp} g_{n,j} \subset U_{n,j}$ and $0 \leq g_{n,j} \leq 1$. Set

$$g_n := \frac{1}{n} \sum_{j=1}^{n-1} g_{n,j}.$$

Then $g_n \leq f$. Hence

$$\lambda(f) \geq \lambda(g_n) = \frac{1}{n} \sum_{j=1}^{n-1} \lambda(g_{n,j}).$$

Thus

$$\lambda(f) \geq \frac{1}{n} \sum_{j=1}^{n-1} \nu(U_{n,j}) = \int f_n \mathrm{d}\nu,$$

where $f_n := \frac{1}{n} \sum_{j=1}^{n-1} 1_{U_{n,j}}$. But $f_n \to f$ and $0 \leq f_n \leq 1$. Hence $\int f_n \mathrm{d}\nu \to \int f \mathrm{d}\nu$. Thus

$$\lambda(f) \geq \int f \mathrm{d}\nu.$$

We will find a sequence of open sets W_n such that $W_n \searrow \mathrm{supp} f$. Choose functions $\tilde{g}_{n,j}$ such that $\mathrm{supp}\tilde{g}_{n,0} \subset W_n$, $\tilde{g}_{n,0} = 1$ on $\mathrm{supp} f$; $\mathrm{supp}\tilde{g}_{n,j} \subset U_{n,j}$, $\tilde{g}_{n,j} = 1$ on $C_{n,j+1}$. Set $\tilde{g}_n := \frac{1}{n} \sum_{j=0}^{n-1} \tilde{g}_{n,j}$. Then $f \leq \tilde{g}_n$. Thus

$$\begin{aligned} \lambda(f) &\leq \lambda(\tilde{g}_n) = \tfrac{1}{n} \textstyle\sum_{j=0}^{n} \lambda(\tilde{g}_{n,j}) \\ &\leq \tfrac{1}{n}\nu(W_n) + \tfrac{1}{n} \textstyle\sum_{j=0}^{n-1} \nu(U_{n,j}) = \tfrac{1}{n}\nu(W_n) + \int \tilde{f}_n \mathrm{d}\nu, \end{aligned}$$

where $\tilde{f}_n := \frac{1}{n} \sum_{j=1}^{n-1} 1_{U_{n,j}}$. We have $\tilde{f}_n \to f$ and $0 \leq \tilde{f}_n \leq 1$. Hence, by the Lebesgue theorem, $\int \tilde{f}_n \mathrm{d}\nu \to \int f \mathrm{d}\nu$. Hence

$$\lambda(f) \leq \int f \mathrm{d}\nu.$$

□

Proof of Theorem 8.20. We define the Baire measure ν as described before Lemma 8.24. By Lemma 8.24, $\lambda(f) = \int f \mathrm{d}\nu$. By Lemma 8.21, ν is uniquely defined. □

Theorem 8.25 *Let ν be a Baire measure on X. Then it satisfies the following regularity properties:*

(1) $\nu(A) = \inf\{\nu(U) \ : \ A \subset U,\ U \in \mathrm{Open} \cap \mathrm{Baire}(X)\}, \quad A \in \mathrm{Baire}(X)$;

(2) $\nu(A) = \sup\{\nu(C) \ : \ C \subset A,\ C \in \mathrm{Closed} \cap \mathrm{Baire}(X)\} \quad A \in \mathrm{Baire}(X)$.

Proof. We define λ by the formula (8.46). We construct the corresponding ν^*. By construction, it satisfies

$$\nu^*(A) = \inf\{\nu^*(U) \ : \ A \subset U,\ U \in \mathrm{Open} \cap \mathrm{Baire}(X)\}.$$

But on Baire(X) ν^* coincides with ν. Hence it satisfies the property (1). □

8.5 Borel measures on compact spaces

Let X be a compact Hausdorff space. A finite measure on Borel(X) is called a Borel measure on X.

Theorem 8.26 *Let μ be a Borel measure on X. The following condtions are equivalent:*

(1) $\mu(A) = \inf\{\mu(U) \ : \ A \subset U, \ U \in \mathrm{Open}(X)\}, \ A \in \mathrm{Borel}(X)$.

(2) $\mu(A) = \sup\{\mu(C) \ : \ C \subset A, \ C \in \mathrm{Closed}(X)\}, \ A \in \mathrm{Borel}(X)$..

If the above conditions are satisfied then μ is called a regular Borel (or Radon) measure on X.

Proof of Theorem 8.26 Using $\mu(X) < \infty$, we get

$$\mu(A) = \mu(X) - \mu(X \backslash A),$$

$$\inf\{\mu(U) \ : \ U \in \mathrm{Open}(X)\} = \mu(X) - \sup\{\mu(C) \ : \ C \in \mathrm{Closed}(X)\}.$$

□

Theorem 8.27 *Let ν be a Baire measure on X. Then there exists a unique regular Borel measure μ extending ν. It has the following properties:*

(1) *If $U \in \mathrm{Open}(X)$, then*

$$\mu(U) = \sup\{\nu(C) \ : \ C \subset U, \ C \in \mathrm{Baire} \cap \mathrm{Closed}(X)\};$$

(2) *If $C \in \mathrm{Closed}(X)$, then*

$$\mu(C) = \inf\{\nu(U) \ : \ C \subset U, \ U \in \mathrm{Baire} \cap \mathrm{Open}(X)\}.$$

Theorems 8.19 and8.27 imply the following version of the Riesz-Markov theorem:

Theorem 8.28 *Let λ be a positive linear functional on $C(X)$. Then there exists a unique regular Borel measure μ satisfying*

$$\lambda(f) = \int f \mathrm{d}\mu, \quad f \in C(X). \tag{8.50}$$

Proof of Theorem 8.27 Let us prove (2).

The inequality $\leq$ is obvious.

For any $U \in \mathrm{Open}(X)$ such that $C \subset U$, There exists an open Baire U_1 such that $C \subset U_1 \subset U$. Therefore,

$$\begin{aligned} \mu(C) &= \inf\{\nu(U) \ : \ C \subset U, \ U \in \mathrm{Open}(X)\} \\ &\geq \inf\{\nu(U) \ : \ C \subset U_1, \ U_1 \in \mathrm{Open} \cap \mathrm{Baire}(X)\}. \end{aligned}$$

This proves the $\geq$ inequality.

It follows from (2) that μ is uniquely determined on the family of closed sets. But this family is $\cap$-stable and generates Borel(X). Hence μ is uniquely determined.

Let us now describe the proof of the existence of μ. Define λ as in (). Then for $U \in \mathrm{Open}(X)$ we set

$$\mu^*(U) := \sup\{\lambda(f) \ : \ f \in C(X), \ \mathrm{supp} f \subset U, \ 0 \leq f \leq 1.\} \tag{8.51}$$

For any $A \in 2^X$ we set

$$\mu^*(A) := \inf\{\nu^*(U) \ : \ A \subset U, \ U \in \mathrm{Open}(X)\}. \tag{8.52}$$

(For $U \in \mathrm{Open}(X)$, (8.51) agrees with (8.52)).

Exactly as in the previous subsection, we show that μ^* is an external measure and Borel sets are μ^*-measurable. We define μ to be the restriction of μ^* to $\mathrm{Borel}(X)$. By (8.52), it is a regular Borel measure.

Clearly, if $U \in \mathrm{Open} \cap \mathrm{Baire}(X)$, then

$$\nu(U) = \nu^*(U) = \mu^*(U) = \mu(U).$$

Thus ν coincides with μ on $\mathrm{Open} \cap \mathrm{Baire}(X)$. But this is a $\cap$-closed family generating $\mathrm{Baire}(X)$. Hence ν coincides with μ on $\mathrm{Baire}(X)$. □

8.6 Baire measures on locally compact spaces

Let X be a locally compact space. A measure on $\mathrm{Baire}(X)$ finite on compact sets is called a Baire measure on X.

A linear functional $\lambda : C_\mathrm{c}(X) \to \mathbb{R}$ is called a positive functional if

$$f \in C_\mathrm{c}(X),\ f \geq 0 \ \Rightarrow\ \lambda(f) \geq 0.$$

Theorem 8.29 *Let ν be a Baire measure. Then*

$$C_\mathrm{c}(X) \ni f \mapsto \int f \mathrm{d}\nu \in \mathbb{R} \tag{8.53}$$

is a positive linear functional.

(1) *If $C \in \mathrm{Compact} \cap \mathrm{Baire}(X)$, then*

$$\nu(C) = \inf\{\textstyle\int f \mathrm{d}\nu \ :\ f \in C_\mathrm{c}(X),\ f = 1 \text{ on } C,\quad 0 \leq f \leq 1\}.$$

(2) *If $U \in \mathrm{Open}_0 \cap \mathrm{Baire}(X)$, then*

$$\nu(U) = \sup\{\textstyle\int f \mathrm{d}\nu \ :\ f \in C_\mathrm{c}(X),\ \mathrm{supp} f \subset U,\quad 0 \leq f \leq 1\}.$$

The following theorem is called the Riesz-Markov Theorem.

Theorem 8.30 *Let λ be a positive linear functional on $C_\mathrm{c}(X)$. Then there exists a unique Baire measure ν satisfying*

$$\lambda(f) = \int f \mathrm{d}\nu, \quad f \in C_\mathrm{c}(X). \tag{8.54}$$

Theorem 8.31 *Let ν be a Baire measure on X. Then it satisfies the following regularity properties:*

(1) $\nu(A) = \inf\{\nu(U) \ :\ A \subset U,\ U \in \mathrm{Open}_0 \cap \mathrm{Baire}(X)\}, \quad A \in \mathrm{Baire}(X)$*;*

(2) $\nu(A) = \sup\{\nu(C) \ :\ C \subset A,\ C \in \mathrm{Compact} \cap \mathrm{Baire}(X)\} \quad A \in \mathrm{Baire}(X)$.

8.7 Borel measures on locally compact spaces

A measure on Borel(X) finite on compact sets is called a Borel measure on X.

Theorem 8.32 *Let μ be a Borel measure on X. The following condtions are equivalent:*
(1) $\mu(A) = \inf\{\mu(U) \ : \ A \subset U, \ U \in \mathrm{Open}_0(X)\}, \ A \in \mathrm{Borel}(X)$.
(2) $\mu(A) = \sup\{\mu(C) \ : \ C \subset A, \ C \in \mathrm{Compact}(X)\}, \ A \in \mathrm{Borel}(X)$.

If the above conditions are satisfied then μ is called a regular Borel (or Radon) measure on X.

Theorem 8.33 *Let ν be a Baire measure on X. Then there exists a unique regular Borel measure μ extending ν. It has the following properties:*
(1) *If $U \in \mathrm{Open}_0(X)$, then*

$$\mu(U) = \sup\{\nu(C) \ : \ C \subset U, \ C \in \mathrm{Baire} \cap \mathrm{Compact}(X)\};$$

(2) *If $C \in \mathrm{Compact}(X)$, then*

$$\mu(C) = \inf\{\nu(U) \ : \ C \subset U, \ U \in \mathrm{Baire} \cap \mathrm{Open}(X)\}.$$

Theorems 8.29 and8.33 imply the following version of the Riesz-Markov theorem:

Theorem 8.34 *Let λ be a positive linear functional on $C_c(X)$. Then there exists a unique regular Borel measure μ satisfying*

$$\lambda(f) = \int f \mathrm{d}\mu, \quad f \in C_c(X). \tag{8.55}$$

9 Measures on infinite Cartesian products

9.1 Infinite Cartesian products

Let X_i, $i \in I$ be a family of sets. For any $K \subset J \subset I$ we can define the map

$$\pi^{KJ} : \underset{j \in J}{\times} X_j \to \underset{k \in K}{\times} X_k,$$

where for $x_J = (x_j)_{j \in J} \in \underset{j \in J}{\times} X_j$, $\pi^{KJ} x_J$ is $(x_k)_{k \in K}$. Clearly, $M \subset K \subset J$ implies

$$\pi^{MK} \pi^{KJ} = \pi^{MJ}.$$

If $(X_i, \mathcal{F}_i)$, $i \in I$ is a family of sets with σ-fields, then for $J \subset I$ we set $\underset{j \in J}{*} \mathcal{F}_j$ to be the family of subsets of $\underset{j \in J}{\times} X_j$ of the form $\underset{j \in J}{\times} A_j$ with $A_j \in \mathcal{F}_j$ and $A_j = X_j$ for all but a finite number of $j \in J$. We set $\underset{j \in J}{\otimes} \mathcal{F}_j := \sigma\!-\!\mathrm{Field}(\underset{j \in J}{*} \mathcal{F}_j)$.

Clearly, the maps π^{KJ} for $K \subset J \subset I$ are measurable.

9.2 Compatible measures

Let $(X_i, \mathcal{F}_i)$, $i \in I$ is a family of sets with σ-fields. Let $K \subset J \subset I$ and μ_J, μ_K are probabilistic measures on $(\underset{j\in J}{\times} X_j, \underset{j\in J}{\otimes} \mathcal{F}_j)$ and $(\underset{k\in K}{\times} X_k, \underset{k\in K}{\otimes} \mathcal{F}_k)$ respectively. We say that they are compatible iff $\pi_{KJ\,*}\mu_J = \mu_K$, that means

$$\int f(x_K)\mathrm{d}\mu_K(x_K) = \int f(x_K)\mathrm{d}\mu_J(x_K, x_{J\setminus K}).$$

Theorem 9.1 *If μ_I is a measure on $(\underset{i\in I}{\times} X_i, \underset{i\in I}{\otimes} \mathcal{F}_i)$, then for any $K \subset J \subset I$, the measures $\pi_{KI\,*}\mu_I$ and $\pi_{JI\,*}\mu_I$ are compatible.*

9.3 Infinite tensor product of measures

Theorem 9.2 *$(X_i, \mathcal{F}_i, \mu_i)$, $i \in I$, be a family of spaces with probabilistic measures. Then there exists a unique measure $\underset{i\in I}{\otimes} \mu_i$ on $(\underset{i\in I}{\times} X_i, \underset{i\in I}{\otimes} \mathcal{F}_i)$ such that for any $A_i \in \mathcal{F}_i$, where all but a finite number of $A_i = X_i$,*

$$\underset{i\in I}{\otimes} \mu_i(\underset{i\in I}{\times} A_i) = \prod_{i\in I} \mu_i(A_i).$$

$K \subset J \subset I$, the measures $\underset{j\in J}{\otimes} \mu_j$ and $\underset{k\in K}{\otimes} \mu_k$ are compatible.

9.4 The Kolmogorov theorem

Suppose that X_i, $i \in I$, is a family of compact sets.

Theorem 9.3 *Suppose that for any finite set $J \in 2^I$ we are given a Baire measure ν_J on $\underset{j\in J}{\times} X_j$. Assume that for any finite $K, J \in 2^I$ with $K \subset J$, ν_K is compatible with ν_J. Then there exists a unique Baire measure ν_I on $\underset{i\in I}{\times} X_i$ compatible with all ν_J for finite J.*

Proof. It is easy to see that the family of measures μ_J defines a regular content on $\mathrm{Ring}\big(\underset{i\in I}{*} \mathrm{Baire}(X_i)\big)$. By Theorem 6.1, it is a premeasure. Hence it admits a unique extension to

$$\sigma\!-\!\mathrm{Ring}\left(\mathrm{Ring}\big(\underset{i\in I}{*} \mathrm{Baire}(X_i)\big)\right) = \mathrm{Baire}(\underset{i\in I}{\times} X_i).$$

□

Theorem 9.4 *Let X_i, $i \in I$ be a family of topological spaces. Then* $\mathrm{Baire}_1(\prod_{i\in I} X_i) = \underset{i\in I}{\otimes} \mathrm{Baire}_1(X_i)$.

References

[Bau] Bauer, H.: Wahrscheinlichkeitstheorie und Grundzüge der Masstheorie, Walter de Gruyter & Co, Berlin 1968

[Bour] Bourbaki, N.: Elements de mathematique. Les structures fondamentales de l'analyse. Livre VI. Integration

[Hal] Halmos, P.: Measure theory, Van Nostrand 1950

[LL] Lieb, E., Loss, M.: Analysis, AMS 1997