Operators on $L^2(\mathbb{R}^d)$

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Lecture notes, version of Jan. 2007

January 30, 2007

Contents

These lecture notes are companions to *"Bounded operators"* and *"Unbounded operators"*. We study a number of concrete and useful examples of bounded and unbounded operators.

1 Convolutions

1.1 Introduction to convolutions

In these notes X will denote the space $\mathbb{R}^d$ equipped with the Lebesgue measure.

Let us recall two estimates, which we will often use, whose validity is not restricted to $\mathbb{R}^d$:

The Hölder inequality Let $1 \le p, q \le \infty$, $\frac{1}{p} + \frac{1}{q} = 1$:

$$\int |f(x)g(x)|\mathrm{d}x \le \|f\|_p \|g\|_q,$$

The generalized Minkowski inequality

$$\left(\int \mathrm{d}y \left|\int f(x,y)\mathrm{d}\,x\right|^p\right)^{\frac{1}{p}} \le \int \mathrm{d}x \left(\int |f|^p(x,y)\mathrm{d}y\right)^{\frac{1}{p}}$$

If g, h are functions on $\mathbb{R}^d$, then their convolution is formally defined by

$$g * h(x) := \int g(x-y)h(y)\mathrm{d}y,$$

provided this makes sense. In what follows we will give a number of conditions when the convolution is well defined.

1.2 Modulus of continuity

Lemma 1.1 *For $1 \le p < \infty$, $f \in L^p(X)$, set*

$$\omega_{p,f}(y) := \left(\int |f(x+y) - f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}};$$

and for $p = \infty$, $f \in C_\infty(\mathbb{R}^n)$

$$\omega_{\infty,f}(y) := \sup_x |f(x+y) - f(x)|.$$

Then $\omega_{p,f}(y)$ is bounded and

$$\lim_{y \to 0} \omega_{p,f}(y) = 0.$$

Proof. The boundedness follows from the Minkowski inequality. In fact, $\omega_{p,f}(y) \leq 2\|f\|_p$.

The convergence to zero is obvious for $f \in C_c(\mathbb{R}^n)$. But C_c is dense in L^p for $1 \leq p < \infty$ and in C_∞. □

1.3 The special case of the Young inequality with $\frac{1}{p} + \frac{1}{q} = 1$

Theorem 1.2 *Let $1 < p, q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, $f \in L^p$, $g \in L^q$. Then*

$$f * g \in C_\infty.$$

If $f \in L^1$, $g \in L^\infty$, then $f * g$ is uniformly continuous.

Proof. By the Hölder inequality, $f * g(x)$ is defined for all x and depends continuously on $f \in L^p(X)$ and $g \in L^q(X)$. Moreover,

$$\begin{aligned} & f * g(x_1) - f * g(x_2) \\ &= \int (f(x_1 - y) - f(x_2 - y))g(y)\mathrm{d}y \\ &\leq \left(\int |f(x_1 - y) - f(x_2 - y)|^p \mathrm{d}y\right)^{\frac{1}{p}} \|g\|_q \\ &= \omega_{p,f}(x_1 - x_2)\|g\|_q. \end{aligned}$$

Hence $f * g$ is uniformly continuous.

For $f \in C_c(X)$ obviously $f * g \in C_c(X)$. If $p, q < \infty$, then $C_c(X)$ is dense in $L^p(X)$, $L^q(X)$. Hence for such p, q, $f * g$ belongs to the closure of $C_c(X)$ in $L^\infty(X)$, which is $C_\infty(X)$. □

1.4 Convolution by an L^1 function

Theorem 1.3 *Let $g \in L^p(X)$ and $h \in L^1(X)$. Then $g * h$ is well defined almost everywhewre and*

$$\|g * h\|_p \leq \|h\|_1 \|g\|_p.$$

Proof. In the generalized Minkowski inequality set $X = Y = \mathbb{R}^n$ and $f(x, y) = h(y)g(x - y)$. □

Theorem 1.4 *Let $\phi \in L^1(\mathbb{R}^n)$ and $\int \phi(x)\mathrm{d}x = 1$. Set*

$$\phi_\epsilon(x) := \epsilon^{-n}\phi(\epsilon^{-1}x), \ \epsilon > 0.$$

Then

$$\begin{aligned} \lim_{\epsilon \to 0} \|f * \phi_\epsilon - f\|_p = 0, \quad f \in L^p(\mathbb{R}^n), \ 1 \leq p < \infty, \\ \lim_{\epsilon \to 0} \|f * \phi_\epsilon - f\|_\infty = 0, \quad f \in C_\infty(\mathbb{R}^n). \end{aligned}$$

Proof.

$$f * \phi_\epsilon(x) - f(x) = \int (f(x - y) - f(x))\phi_\epsilon(y)\mathrm{d}y.$$

$$\begin{aligned} & \|f * \phi_\epsilon(x) - f(x)\|_p \\ &\leq \int \mathrm{d}y \left(\int |f(x - y) - f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} |\phi_\epsilon(y)| \\ &= \int \omega_{p,f}(y)\phi_\epsilon(y)\mathrm{d}y = \int \omega_{p,f}(\epsilon y)\phi(y)\mathrm{d}y \to_{\epsilon \to 0} 0. \end{aligned}$$

□

1.5 The Young inequality

Theorem 1.5 *Let $1 \le p, q, r \le \infty$, $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 2$, $f, g, h \in \mathcal{M}_+(X)$ (positive, measurable functions on X). Then*

$$\int\int f(x)g(x-y)h(y)\mathrm{d}x\mathrm{d}y \le C_{p,r,n}\|f\|_p\|g\|_q\|h\|_r.$$

Proof. Let $\frac{1}{p} + \frac{1}{p'} = 1$, $\frac{1}{q} + \frac{1}{q'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Set

$$\alpha(x,y) := f(x)^{p/r'}g(x-y)^{q/r'},$$

$$\beta(x,y) := g(x-y)^{q/p'}h(y)^{r/p'},$$

$$\gamma(x,y) := f(x)^{p/q'}h(y)^{r/q'}.$$

Then

$$\begin{aligned}\int\int f(x)g(x-y)h(y)\mathrm{d}x\mathrm{d}y &= \int\int f(x)^{p(2-\frac{1}{q}-\frac{1}{r})}g(x-y)^{q(2-\frac{1}{p}-\frac{1}{r})}h(y)^{r(2-\frac{1}{p}-\frac{1}{q})}\\ &= \int\int f(x)^{p(\frac{1}{q'}+\frac{1}{r'})}g(x-y)^{q(\frac{1}{p'}+\frac{1}{r'})}h(y)^{r(\frac{1}{p'}+\frac{1}{q'})}\\ &= \int\int \alpha(x,y)\beta(x,y)\gamma(x,y)\mathrm{d}x\mathrm{d}y \le \|\alpha\|_{r'}\|\beta\|_{p'}\|\gamma\|_{q'},\end{aligned}$$

where in the last step we used the Hölder inequality noting that $\frac{1}{r'} + \frac{1}{p'} + \frac{1}{q'} = 1$. Finally,

$$\|\alpha\|_{r'} = (\int\int f(x)^p g(x-y)^q \mathrm{d}x\mathrm{d}y)^{1/r'} = \|f\|_p^{p/r'}\|g\|_q^{q/r'}.$$

□

Corollary 1.6 *If $\frac{1}{q} + \frac{1}{r} = 1 + \frac{1}{s}$, $h \in L^r(X)$, $g \in L^q(X)$, then for almost all x*

$$y \mapsto g(x-y)h(y)$$

belongs to $L^1(X)$ and

$$g * h(x) = \int g(x-y)h(y)\mathrm{d}y$$

belongs to $L^s(X)$ and

$$\|g * h\|_s \le \|g\|_q\|h\|_r. \tag{1.1}$$

Proof. We know that for $f \in L^p(X)$, $\frac{1}{p} + \frac{1}{s} = 1$ we have

$$\int |f(x)|\mathrm{d}x \int |g(x-y)h(y)|\mathrm{d}y \le \|f\|_p\|g\|_q\|h\|_r < \infty.$$

Hence for a.a x

$$|f(x)| \int |g(x-y)h(y)|\mathrm{d}y < \infty.$$

Hence for a.a. x

$$\int |g(x-y)h(y)|\mathrm{d}y < \infty.$$

From

$$|\int f(x)g * h(x)\mathrm{d}x| \le \|f\|_p\|g\|_q\|h\|_r.$$

we obtain (1.1). □

2 Fourier transformation and tempered distributions on $\mathbb{R}^d$

2.1 Fourier transformation on $L^1 \cup L^2(\mathbb{R}^d)$

For

$$f \in L^1(\mathbb{R}^d)$$

we define its Fourier transform as

$$\mathcal{F}f(\xi) = \hat{f}(\xi) := \int \mathrm{e}^{-ix\xi} f(x)\mathrm{d}x.$$

$$\check{f}(x) := f(-x)$$
$$\tau_y f(x) := f(x-y)$$
$$f_{(a)}(x) := f(ax)$$

Theorem 2.1 (1) $\|\hat{f}\|_\infty \leq \|f\|_1$;

(2) $\hat{\check{f}}(\xi) = \check{\hat{f}}(\xi) = \int \mathrm{e}^{\mathrm{i}x\xi} f(x)\mathrm{d}x$.

(3) $\overline{\hat{f}} = \check{\hat{\bar{f}}}$;

(4) $\hat{f}_{(a)}(x) = a^{-d}\hat{f}(a^{-1}x)$;

(5) $\widehat{\tau_y f}(\xi) = \mathrm{e}^{-iy\xi}\hat{f}(\xi)$;

(6) $\left(f\mathrm{e}^{i\eta\cdot}\right)\hat{}\,(\xi) = \hat{f}(\xi - \eta)$.

Example 2.2 (1) $f(x) = \mathrm{e}^{-\frac{x^2}{2}}$, $\hat{f}(\xi) = (2\pi)^{\frac{n}{2}}\mathrm{e}^{-\frac{x^2}{2}}$.

(2) $f(x) = \mathrm{e}^{-\epsilon x}x^\alpha\theta(x)$, $\hat{f}(\xi) = \frac{\Gamma(\alpha+1)}{(\epsilon+i\xi)^{\alpha+1}}$, $\mathrm{Re}\epsilon > 0$.

(3) $f(x) = \chi_{[-1,1]}(x)$, $\hat{f}(\xi) = \frac{2\sin\xi}{\xi}$.

(4) $f(x) = \mathrm{e}^{-|x|}$, $\hat{f} = \frac{1}{1+\xi^2}$.

Theorem 2.3 (The Riemann-Lebesgue Lemma) *If $f \in L^1$, then $\hat{f} \in C_\infty$.*

Proof. We know that the Fourier transformation is continuous from L^1 to L^∞. C_∞ is a closed subspace of L^∞.

Combinations of characteristic functions of intervals are dense in L^1. Their Fourier transforms, which we computed explicitly, belong to C_∞. □

Theorem 2.4 *Let $f, g \in L^1$. Then*

(1) $\int \hat{f}(\xi)g(\xi)\mathrm{d}\xi = \int f(x)\hat{g}(x)\mathrm{d}x$.

(2) $(f\hat{g})\check{} = \check{\hat{f}} * g$.

(3) $(f * g)\hat{} = \hat{f}\hat{g}$.

Proof. (2) For $f_\eta(x) = f(x)\mathrm{e}^{\mathrm{i}x\eta}$, we have $\hat{f}_\eta(\xi) = \check{\hat{f}}(\eta - \xi)$. Hence

$$\int \hat{f}_\eta(\xi) g(\xi) \mathrm{d}\xi = \check{\hat{f}} * g(\eta).$$

Besides,

$$\int f_\eta(x) \hat{g}(x) \mathrm{d}x = (h\hat{g})\check{}(\eta)$$

Therefore, it suffices to apply (1). □

Theorem 2.5 (Parseval) *Let $g, \hat{g} \in L^1$. Then*

$$\check{\hat{g}} = (2\pi)^d g.$$

Proof. Let

$$\phi_\epsilon(x) := \mathrm{e}^{-\frac{\epsilon x^2}{2}}.$$

We have

$$0 \le \phi_\epsilon \le 1, \quad \lim_{\epsilon \to 0} \phi_\epsilon = 1.$$

Using that $\hat{g} \in L^1$, by the Lebesgue Theorem we obtain

$$\phi_\epsilon \hat{g} \to \hat{g}$$

in the sense of L^1. Therefore,

$$(\phi_\epsilon \hat{g})\hat{}(x) \to \hat{\hat{g}}(x),$$

$$(\phi_\epsilon \hat{g})\check{}(x) \to \check{\hat{g}}(x),$$

in the supremum norm.

Moreover,

$$\int \phi(\xi) = (2\pi)^d,$$

$$\hat{\phi}_\epsilon(\xi) = \left(\frac{2\pi}{\epsilon}\right)^{\frac{d}{2}} \mathrm{e}^{-\frac{\xi^2}{2\epsilon}}.$$

Using that $g \in L^1$ we obtain

$$\hat{\phi}_\epsilon * g(x) \to (2\pi)^d g(x)$$

in the sense of L^1.

Finally, we use

$$\hat{\phi}_\epsilon * g = \check{\hat{\phi}}_\epsilon * g = (\phi_\epsilon \hat{g})\hat{}.$$

□

Theorem 2.6 *Let $f \in L^1$, $\hat{f} \ge 0$ and let f be continuous at 0. Then $\hat{f} \in L^1$ and we have*

$$\int \hat{f}(\xi) \mathrm{d}\xi = (2\pi)^d f(0)$$

Proof. If ϕ_ϵ is as in the proof of the Parseval Theorem, then

$$\int \phi_\epsilon(\xi)\hat{f}(\xi)\mathrm{d}\xi = \int \hat{\phi}_\epsilon(x)f(x)\mathrm{d}x.$$

The left hand side is increasing and converges to $\int \hat{f}(\xi)\mathrm{d}\xi$. The right hand side goes to $(2\pi)^d f(0)$. By the Fatou Lemma, $\hat{f}$ is integrable. □

Theorem 2.7 *Let $f \in L^1 \cap L^2$. Then*

$$\|\hat{f}\|_2 = (2\pi)^{\frac{d}{2}}\|f\|_2.$$

Proof. The function $h := \check{\overline{f}} * f$ belongs to L^1 as the convolution of functions from L^1 and continuous as the convolution of functions from L^2. Besides,

$$\hat{h} = \left(\check{\overline{f}} * f\right)\hat{} = \hat{\check{\overline{f}}}\hat{f} = \overline{\hat{f}}\hat{f} \geq 0.$$

Hence, by Theorem 2.6, $\hat{h} \in L^1$ and

$$(2\pi)^d h(0) = \int \hat{h}(\xi)\mathrm{d}\xi.$$

Finally,

$$(2\pi)^d \int |f(x)|^d \mathrm{d}x = (2\pi)^d h(0) = \int \hat{h}(\xi)\mathrm{d}\xi = \int |\hat{f}(\xi)|^2\mathrm{d}\xi.$$

□

Let $f \in L^2$. Then for any sequence $f_n \in L^1 \cap L^2$ such that

$$\lim_{n\to\infty} f_n = f$$

in L^2, there exists $\lim_{n\to\infty} \hat{f}_n = \hat{f}$. The operator

$$f \mapsto (2\pi)^{-\frac{d}{2}}\hat{f}$$

is unitary.

Theorem 2.8 *If $f \in L^1$ and $xf \in L^1$, then $\hat{f} \in C^1$ and*

$$\partial_\xi \hat{f}(\xi) = (xf)\hat{}(\xi).$$

Proof. We use the theorem about differentiation of an integral depending on a parameter. □

2.2 Tempered distributions on $\mathbb{R}^d$

Typical spaces of functions (measures) on $\mathbb{R}^d$ are

$$C_\infty(X), \quad L^p(X), \ \mathrm{Ch}(X).$$

where $\mathrm{Ch}(X)$ denotes Borel complex charges of finite variation. We have

$$C_\infty^\#(X) = \mathrm{Ch}(X), \ L^p(X)^\# = L^q(X), \quad p^1 + q^{-1} + 1, \quad 1 \leq p < \infty.$$

We have a bilinear and sesquilinear forms

$$\langle a, b\rangle = \int a(x)b(x)\mathrm{d}x, \quad (a, b) = \int \overline{a}(x)b(x)\mathrm{d}x.$$

Lemma 2.9

$$\|f\|_\infty \leq C\|(1+|x|)^{-p}f\|_1 + C\|\partial_{x_1}\dots\partial_{x_d}f\|_1,\ \ p>d$$

$$\|f\|_q \leq C\|(1+|x|)^{-k}f\|_p,\ \ \frac{1}{q}<\frac{k}{d}+\frac{1}{p}.$$

Theorem 2.10 *The following set does not depend on $1\leq p\leq\infty$:*

$$\bigcap_{\alpha,m>0}\{f\ :\ \|\partial^\alpha(1+|x|^2)^{m/2}f\|_p<\infty\}. \tag{2.2}$$

The space $\mathcal{S}(\mathbb{R}^d)$ is defined as (2.2). It is a Frechet space.
For the dual of $\mathcal{S}(\mathbb{R}^d)$ we will use the traditional notation $\mathcal{S}'(\mathbb{R}^d)$.

Example 2.11 *Elements of $\mathcal{S}'(X)$ satisfying*

$$|\langle v,\phi\rangle| \leq C\|x^m\phi\|_\infty$$

have the form

$$\langle v,\phi\rangle = \int\phi(x)\mathrm{d}\mu$$

for a certain Borel charge μ for which there exists m such that $\mu(1+|x|)^{-m}\in\mathrm{Ch}(X)$.

The operator ∂ is continuous on $\mathcal{S}(X)$. For $v\in\mathcal{S})(X)$ we define $\partial v\in\mathcal{S}'(X)$ by

$$\langle v,\partial\phi\rangle = -\langle\partial v,\phi\rangle.$$

Theorem 2.12 *Any $v\in\mathcal{S}'(X)$ has the form*

$$\sum_{\alpha<N}\partial_x^\alpha\mu_\alpha$$

for some Borel charge μ such that for some m we have $\mu(1+|x|)^{-m}\in\mathrm{Ch}(X)$.

Proof. For some α,β,

$$\langle v,\phi\rangle \leq C\sum_{|\alpha|,|\beta|\leq N}\|x^\alpha\partial_x^\beta\phi\|_\infty.$$

Introduce the locally compact space

$$\tilde{X} = \prod_{|\alpha|,|\beta|\leq N} X$$

and the map

$$\mathcal{S}(X)\ni\phi\mapsto j(\phi) = \sum_{|\alpha|,|\beta|\leq N}^{\oplus} x^\alpha\partial^\beta\phi\in C_\infty(\tilde{X})$$

Any distribution v determines a bounded functional on $j(\mathcal{S}(X))$. By the Hahn-Banach Theorem, this functional can be extended to a bounded functional $\tilde{v}$ on $C_\infty(\tilde{X})$. By the Riesz-Markov Theorem, there exists a finite Borel charge on $\tilde{X}$ Such that

$$\tilde{v}(\phi_{\alpha,\beta}) = \sum_{|\alpha|,|\beta|\leq N}\int\phi(x)\mathrm{d}\eta_{\alpha,\beta}(x).$$

□

Clearly, $\mathcal{S}(X)\subset L^1(X)$. Hence the Fourier transform is defined on $\mathcal{S}(X)$.

Theorem 2.13 *If $\phi \in \mathcal{S}(X)$, then $\hat{\phi} \in \mathcal{S}(X)$.*

Recall that for $\psi \in \mathcal{S}(X)$, $\phi \in \mathcal{S}(X)$ we have

$$\langle \psi, \hat{\phi} \rangle = \langle \hat{\psi}, \phi \rangle.$$

For $v \in \mathcal{S}'(X)$ we define

$$\langle \hat{v}, \phi \rangle := \langle v, \hat{\phi} \rangle, \ \ \phi \in \mathcal{S}(X).$$

Clearly, $L^1(X) \cup L^2 \subset \mathcal{S}'(X)$ and the Fourier transformation previously defined coincides with the presently defined on $L^1(X) \cup L^2$.

Theorem 2.14

$$\check{\hat{v}} = (2\pi)^d v, \quad v \in \mathcal{S}'(X), \tag{2.3}$$

2.3 Spaces of sequences

Below we list a couple of typical spaces of sequences indexed by $\mathbb{Z}^d$:

$$L^1(\mathbb{Z}^d) \subset L^p(\mathbb{Z}^d) \subset L^q(\mathbb{Z}^d) \subset C_\infty(\mathbb{Z}^d) \subset L^\infty(\mathbb{Z}^d), \ p \leq q$$

We have

$$C_\infty(\mathbb{Z}^d)^\# = L^1(\mathbb{Z}^d), \quad L^p(\mathbb{Z}^d)^\# = L^q(\mathbb{Z}^d), \quad p^{-1} + q^{-1} = 1, \quad 1 \leq p < \infty.$$

We have natural bilinear and sesquilinear forms:

$$\langle a|b \rangle = \sum a_n b_n, \quad (a|b) = \sum \overline{a}_n b_n.$$

Lemma 2.15

$$\|a\|_p \leq \|a\|_q, \ p \geq q,$$

$$\|a\|_q \leq \|(1+n)^{-k} a\|_p, \quad \frac{1}{q} < \frac{k}{d} + \frac{1}{p}.$$

Theorem 2.16 *The following set does not depend on $1 \leq p \leq \infty$:*

$$\bigcap_{m>0} \{a \ : \ \|(1+|n|^2)^{m/2} a\|_p < \infty\}.$$

The above space is a Frechet space, which will be denoted $\mathcal{S}(\mathbb{Z}^d)$.

Theorem 2.17 *The space dual to $\mathcal{S}(\mathbb{Z}^d)$, denoted $\mathcal{S}'(\mathbb{Z}^d)$, equals*

$$\bigcup_{m>0} \{a \ : \ \|(1+|n|^2)^{-m/2} a\|_p < \infty\}.$$

Theorem 2.18 *$\mathcal{S}(\mathbb{Z}^d)$ is dense in $\mathcal{S}'(\mathbb{Z}^d)$.*

2.4 The oscillator representation of $\mathcal{S}(X)$ and $\mathcal{S}'(X)$

For simplicity, we discuss $X = \mathbb{R}$.

Lemma 2.19

$$\lim_{n\to\infty} \left\| \mathrm{e}^{ix\xi}\mathrm{e}^{-\frac{x^2}{2}} - \sum_{j=0}^{n} \frac{(ix\xi)^j}{j!}\mathrm{e}^{-\frac{x^2}{2}} \right\| = 0$$

Proof.

$$\left| \mathrm{e}^{ix\xi}\mathrm{e}^{-\frac{x^2}{2}} - \sum_{j=0}^{n} \frac{(ix\xi)^j}{j!}\mathrm{e}^{-\frac{x^2}{2}} \right| \leq \frac{\xi^{n+1}x^{n+1}}{(n+1)!}\mathrm{e}^{-\frac{x^2}{2}}.$$

Hence the norm of the difference is estimated by

$$\int \frac{\xi^{2(n+1)}x^{2(n+1)}}{((n+1)!)^2}\mathrm{e}^{-x^2}\mathrm{d}x = \xi^{2(n+1)} \int_0^\infty \frac{s^{n+\frac{1}{2}}\mathrm{e}^{-s}\mathrm{d}s}{((n+1)!)^2} = \frac{\xi^{2(n+1)}\Gamma(n+\frac{1}{2})}{((n+1)!)^2}.$$

□

Theorem 2.20 *Linear combinations of*

$$x^n\mathrm{e}^{-\frac{x^2}{2}} \tag{2.4}$$

are dense in $L^2(\mathbb{R})$.

Proof. Let f be orthogonal to the space spanned by (2.4). Then for any ξ

$$\int f(x)\mathrm{e}^{ix\xi}\mathrm{e}^{-\frac{x^2}{2}}\mathrm{d}x = 0.$$

Hence, the Fourier transform of $f\mathrm{e}^{-\frac{x^2}{2}}$ is zero. Therefore, $f = 0$ almost everywhere. □

Let

$$A^* := \frac{1}{\sqrt{2}}\left(x - \frac{\mathrm{d}}{\mathrm{d}x}\right), \ A := \frac{1}{\sqrt{2}}\left(x + \frac{\mathrm{d}}{\mathrm{d}x}\right)$$

$$\phi_n := \pi^{-\frac{1}{4}}(n!)^{-\frac{1}{2}}(A^+)^n\mathrm{e}^{-\frac{x^2}{2}} = (2^2n!)^{-\frac{1}{2}}(-1)^n\pi^{-\frac{1}{4}}\mathrm{e}^{\frac{x^2}{2}}\partial_x^n\mathrm{e}^{-x^2}$$

$$N := A^*A + AA^* = x^2 + D^2.$$

Theorem 2.21 ϕ_n *is an orthonormal basis obtained by the Gramm-Schmidt orthonormalization of* $x^n\mathrm{e}^{-\frac{x^2}{2}}$. *They are eigenvectors of* N *and* $\mathcal{F}$*:*

$$N\phi_n = \left(n + \frac{1}{2}\right)\phi_n, \quad \mathcal{F}\phi_n = i^n(2\pi)^d\phi_n.$$

Theorem 2.22 *Suppose that for* $v \in \mathcal{S}'(\mathbb{R})$

$$v_n := \langle v, \phi_n \rangle$$

Then there exists m *such that*

$$|v_n| \leq C(1+n)^m,$$

or, in other words, $(v_n) \in \mathcal{S}'(\mathbb{N})$*. The map*

$$\mathcal{S}'(\mathbb{R}) \ni v \to (v_n) \in \mathcal{S}'(\mathbb{N})$$

is an isomorphism. $v \in \mathcal{S}(\mathbb{R})$*, iff*

$$|v_n| \leq C(1+n)^{-m}, \quad m = 0, 1, \dots$$

The map

$$\mathcal{S}(\mathbb{R}) \ni v \to (v_n) \in \mathcal{S}(\mathbb{N})$$

is an isomorphism and

$$\mathcal{S}(\mathbb{R}) = \cap_{n=0}^{\infty} \mathrm{Dom}(N^n).$$

Proof. Clearly, the seminorms $\|N^m \phi\|$ can be estimated by linear combinations of seminorms $\|\phi\|_{\alpha,\beta,2}$. Hence,

$$\mathcal{S}(\mathbb{R}) \supset \cap_{n=0}^{\infty} \mathrm{Dom}(N^n).$$

To show the inverse estimate note first that $\|\phi\|_{\alpha,\beta,2}$ can be bounded by

$$(\phi, A_1^{\natural} \dots A_n^{\natural} \phi),$$

where $A_i^{\natural} = A$ or $A_i^{\natural} = A^*$. After commuting we can estimate them by linear combinations

$$\begin{aligned} &(\phi A^k, A^{+m} \phi) \\ &\leq \tfrac{1}{2} \|A^{+k} \phi\|^2 + \tfrac{1}{2} \|A^{+m} \phi\|^2 \\ &\leq C \textstyle\sum_{j=1}^{\max\{k,m\}} \|N^j \phi\|^2. \end{aligned}$$

Hence

$$\mathcal{S}(\mathbb{R}) \subset \cap_{n=0}^{\infty} \mathrm{Dom}(N^n).$$

□

Corollary 2.23 (The Schwartz Kernel Theorem) *Every continuous bilinear functional*

$$\mathcal{S}(X_1) \times \mathcal{S}(X_2) \ni (\phi, \psi) \mapsto T(\phi, \psi)$$

has the form

$$\langle T, \phi \otimes \psi \rangle$$

for some $T \in \mathcal{S}'(X_1 \times X_2)$

Proof. We have

$$\langle T, \phi \otimes \psi \rangle = \sum t_{k,m} \phi_k \otimes \psi_m,$$

where

$$|t_{k,m}| \leq (1+|k|)^n (1+|m|)^n.$$

Hence,

$$|t_{k,m}| \leq (1+|k|+|m|)^{2n}.$$

□

2.5 Convolution of distributions

Theorem 2.24 *The following space does not depend on $1 \leq p \leq \infty$:*

$$\bigcap_\alpha \bigcup_{m_\alpha} \{f \in C^\infty(\mathbb{R}^d) \ : \ \|(1+|x|)^{-m_\alpha} D^\alpha f\|_p < \infty\}. \tag{2.5}$$

The space (2.5), which is an inductive limit of Frechet space, is denoted $\mathcal{O}(\mathbb{R}^d)$. Its dual space, for which we will use the traditional notation $\mathcal{O}'(\mathbb{R}^d)$, is called the space of rapidly decreasing distributions.

We have the inclusions

$$\mathcal{S} \subset \mathcal{O} \subset \mathcal{S}', \quad \mathcal{S} \subset \mathcal{O}' \subset \mathcal{S}'$$

Example 2.25 *If μ is a Borel charge and for any m*

$$\int (1+|x|)^m |\mathrm{d}\mu|(x) < \infty,$$

then $\mu \in \mathcal{O}'$.

Clearly, if $f \in \mathcal{O}$, then

$$\mathcal{S} \ni \phi \mapsto f\phi \in \mathcal{S} \tag{2.6}$$

is continuous. For $v \in \mathcal{S}'$ we define $fv \in \mathcal{S}'$ as the adjoint of (2.6), that is

$$\langle v, f\phi \rangle = \langle fv, \phi \rangle.$$

The operator ∂ is continuous also on $\mathcal{O}$ and $\mathcal{O}'$.

For $\phi \in \mathcal{S}$ we define

$$\check{\phi}(x) := \phi(-x).$$

Clearly,

$$\langle \psi, \check{\phi} \rangle = \langle \check{\psi}, \phi \rangle$$

For $v \in \mathcal{S}'$ we introduce

$$\langle v, \check{\phi} \rangle = \langle \check{v}, \phi \rangle$$

Note that for $\phi, \psi, \chi \in \mathcal{S}$ we have

$$\langle \chi, \psi * \phi \rangle = \langle \chi * \check{\psi}, \phi \rangle.$$

For $v \in \mathcal{S}'$, $\psi \in \mathcal{S}$ we define

$$\langle v * \psi, \phi \rangle := \langle v, \check{\psi} * \phi \rangle.$$

Theorem 2.26 *For $v \in \mathcal{S}'$, $\phi \in \mathcal{S}$ we define*

$$\phi_y(x) := \phi(x-y).$$

Then

$$v * \phi(x) := \langle v, (\check{\phi})_{-x} \rangle.$$

and

$$v * \psi \in \mathcal{O}. \tag{2.7}$$

Proof. Let us show (2.7):

$$\begin{aligned}|\partial_x^\alpha v * \phi(x)| &= |\langle v|\partial_y^\alpha \check{\phi}_{-x}\rangle| \\ &\leq C\|y^n\partial_y^{\alpha+\gamma}\phi_{-x}\|_\infty \\ &\leq C(1+|x|)^n\|y^n\partial_y^{\alpha+\gamma}\phi\|_\infty.\end{aligned}$$

□

Hence we can extend the definition of the convolution as follows. Let $w \in \mathcal{S}'$, $v \in \mathcal{O}'$. Then

$$\langle v * w, \phi\rangle := \langle v, \check{w} * \phi\rangle, \quad \phi \in \mathcal{S}.$$

Using the convolution we can easily show that $\mathcal{S}$ is dense in $\mathcal{S}'$.

Theorem 2.27 *If $v \in \mathcal{O}'$, then $\hat{v} \in \mathcal{O}$.*

Proof. Note first that

$$\partial_\xi^\beta \hat{v}(\xi) = \langle v, x^\beta \mathrm{e}^{-\mathrm{i}\xi\cdot}\rangle.$$

We know that

$$|\langle v, \phi\rangle| \leq \sum_{|\alpha|\leq N} \|(1+x^2)^{-\frac{|\beta|}{2}}\partial_x^\alpha \phi\|_\infty.$$

Hence,

$$|\partial_\xi^\beta \hat{v}(\xi)| \leq \sum_{|\alpha|\leq N} |\xi|^\alpha.$$

□

Theorem 2.28

$$(v * w)\hat{} = \hat{v}\hat{w}, \quad v \in \mathcal{S}', \; w \in \mathcal{O}' \tag{2.8}$$

Proof. First prove (2.8) for $w \in \mathcal{S}$. Let $\phi \in \mathcal{S}$. Then

$$\begin{aligned}&\langle (v * w)\hat{}, \phi\rangle \\ &= \langle v * w, \hat{\phi}\rangle \\ &= \langle v, \check{w} * \hat{\phi}\rangle \\ &= (2\pi)^{-d}\langle v, (\check{w} * \hat{\phi})\hat{\check{}}\rangle \\ &= (2\pi)^{-d}\langle \hat{\check{v}}, \hat{\check{w}}\hat{\hat{\phi}}\rangle \\ &= \langle \hat{\check{v}}, \hat{\check{w}}\check{\phi}\rangle \\ &= \langle \hat{\check{v}}\hat{\check{w}}, \check{\phi}\rangle \\ &= \langle \hat{v}\hat{w}, \phi\rangle.\end{aligned}$$

Then we assume that $v \in \mathcal{S}'$, $w \in \mathcal{O}'$ and we repeat the same reasoning. □

3 Sobolev inequalities and their consequences

3.1 The Hardy-Littlewood-Sobolev inequality

Let θ denote the Heaviside function, that is

$$\theta(t) := \begin{cases} 0 & t < 0, \\ 1 & t > 0. \end{cases}$$

Let $0 \leq \lambda \leq n$. Then

$$|x|^{-\lambda}\theta(|x|-1) \in L^p(X), \quad \infty \geq p > \frac{n}{\lambda},$$
$$|x|^{-\lambda}\theta(1-|x|) \in L^p(X), \quad 1 \leq p < \frac{n}{\lambda}.$$

Theorem 3.1 $1 < p, r < \infty$, $0 < \lambda < n$, $\frac{1}{p} + \frac{\lambda}{n} + \frac{1}{r} = 2$, $f, h \in \mathcal{M}_+(X)$. *Then*

$$\int\int f(x)|x-y|^{-\lambda}h(y)\mathrm{d}x\mathrm{d}y \leq C_{n,\lambda,r}\|f\|_p\|h\|_r.$$

Corollary 3.2 *If* $\frac{\lambda}{n} + \frac{1}{r} = 1 + \frac{1}{s}$, $h \in L^r(X)$, *then for almost all* x

$$y \mapsto |x-y|^{-\lambda}h(y)$$

belongs to $L^1(X)$ *and*

$$x \mapsto \int |x-y|^{-\lambda}h(y)\mathrm{d}y$$

belongs to $L^s(X)$ *and for* $g(x) = |x|^{-\lambda}$,

$$\|g * h\|_s \leq C_{n,\lambda,r}\|h\|_r. \tag{3.9}$$

Proof of Theorem 3.1 We will write $g(x) := |x|^{-\lambda}$. Set

$$v(a) := \int 1_{\{f>a\}}(x)\mathrm{d}x, \quad w(b) := \int 1_{\{h>b\}}(x)\mathrm{d}x, \quad u(c) := \int 1_{\{g>c\}}(x)\mathrm{d}x.$$

Note that

$$u(c) = C_n c^{-n/\lambda}, \quad u^{-1}(t) = \tilde{C}_n t^{-\lambda/n}.$$

We can assume that

$$1 = \|f\|_p^p = p\int_0^\infty a^{p-1}v(p)\mathrm{d}a, \quad 1 = \|h\|_r^r = r\int_0^\infty b^{r-1}w(b)\mathrm{d}b$$

Now

$$\begin{aligned} I &:= \int\int f(x)g(x-y)h(y)\mathrm{d}x\mathrm{d}y = \int\int\int\int\int 1_{\{f>a\}}(x)1_{\{h>b\}}(y)1_{\{g>c\}}(x-y)\mathrm{d}x\mathrm{d}y\mathrm{d}a\mathrm{d}b\mathrm{d}c \\ &= \underset{w(b)\leq v(a)}{\int\int\int} \mathrm{d}a\mathrm{d}b\mathrm{d}y 1_{\{h>b\}}(y)\int\int \mathrm{d}c\mathrm{d}x 1_{\{f>a\}}(x)1_{\{g>c\}}(x-y) \\ &+ \underset{w(b)\geq v(a)}{\int\int\int} \mathrm{d}a\mathrm{d}b\mathrm{d}x 1_{\{f>a\}}(x)\int\int \mathrm{d}c\mathrm{d}y 1_{\{h>b\}}(y)1_{\{g>c\}}(x-y). \end{aligned}$$

Now

$$\int\int \mathrm{d}c\mathrm{d}x 1_{\{f>a\}}(x)1_{\{g>c\}}(x-y) \leq \int\int_{v(a)\geq u(c)} \mathrm{d}c\mathrm{d}x 1_{\{f>a\}}(x) + \int\int_{v(a)\leq u(c)} \mathrm{d}c\mathrm{d}x 1_{\{g>c\}}(x-y)$$
$$= v(a)\int_0^{u^{-1}(v(a))} \mathrm{d}c + \int_{u^{-1}(v(a))}^{\infty} u(c)\mathrm{d}c$$
$$= v(a)u^{-1}(v(a)) + c_{n,\lambda}(u^{-1}(v(a)))^{1-n/\lambda}$$
$$= c_{n,\lambda}v(a)^{1-\lambda/n}.$$

Therefore,

$$I \leq c_{n,\lambda}\int\int_{w(b)\leq v(a)} \mathrm{d}a\mathrm{d}b w(b)v(a)^{1-\lambda/n} + c_{n,\lambda}\int\int_{w(b)\geq v(a)} \mathrm{d}a\mathrm{d}b v(a)w(b)^{1-\lambda/n}$$
$$= c_{n,\lambda}\int\int \mathrm{d}a\mathrm{d}b \min\Big(w(b)v(a)^{1-\lambda/n}, v(a)w(b)^{1-\lambda/n}\Big)$$
$$\leq c_{n,\lambda}\int_0^\infty \mathrm{d}a v(a)\int_0^{a^{p/r}} \mathrm{d}b w(b)^{1-\lambda/n} + c_{n,\lambda}\int_0^\infty \mathrm{d}b w(b)\int_0^{b^{r/p}} \mathrm{d}a v(a)^{1-\lambda/n}$$

Now setting $m := (r-1)(1-\lambda/n)$, we get

$$\int_0^{a^{p/r}} w(b)^{1-\lambda/n}\mathrm{d}b = \int_0^{a^{p/r}} w(b)^{1-\lambda/n}b^m b^{-m}\mathrm{d}b$$
$$\leq \Big(\int_0^{a^{p/r}} w(b)b^{r-1}\mathrm{d}b\Big)^{1-\lambda/n}\Big(\int_0^{a^{p/r}} b^{-mn/\lambda}\mathrm{d}b\Big)^{\lambda/n}$$
$$\leq C\Big(\int_0^\infty w(b)b^{r-1}\mathrm{d}b\Big)^{1-\lambda/n} a^{p-1}.$$

Hence

$$I \leq c_{n,\lambda,r}\int v(a)a^{p-1}\mathrm{d}a\Big(\int_0^\infty w(b)b^{r-1}\mathrm{d}b\Big)^{1-\lambda/n}$$
$$+c_{n,\lambda,r}\int_0^\infty w(b)b^{r-1}\mathrm{d}b\Big(\int v(a)a^{p-1}\mathrm{d}a\Big)^{1-\lambda/n} = 2c_{n,\lambda,r}$$

3.2 The Thomas-Fermi functional

For $\rho \in \mathcal{M}_+(\mathbb{R}^3)$ we set

$$\mathcal{E}(\rho) := \frac{3}{5}\int \rho(x)^{5/3}\mathrm{d}x - \int \frac{z}{|x|}\rho(x)\mathrm{d}x + \frac{1}{2}\int\int \frac{\rho(x)\rho(y)}{|x-y|}\mathrm{d}x\mathrm{d}y.$$

$$E_N := \inf\left\{\mathcal{E}(\rho) \ : \ \rho\in\mathcal{M}_+, \ \int \rho\mathrm{d}x = N\right\}.$$

Theorem 3.3 (1) $E_N > -\infty$
(2) *$\mathcal{E}$ is finite for $\rho \in L^1\cap L^{5/3}$.*
(3) *The function $L^1\cap L^{5/3}\rho \mapsto \mathcal{E}(\rho)$ is convex and continuous.*

Proof. (1) For any c we can split $\frac{1}{|x|} = \frac{\theta(|x|-c)}{|x|} + \frac{\theta(-|x|+c)}{|x|} = f_1 + f_2$. The first term belongs to $\bigcap_{p>3} L^p$ and the second to $\bigcap_{p<3} L^p$. Now

$$\mathcal{E}(\rho) \geq c_1\|\rho\|_{5/3}^{5/3} - z\|f_1\|_\infty\|\rho\|_1\|f_2\|_{5/2}\|\rho\|_{5/3}.$$

By choosing c we can make $\|f_2\|_{5/2}$ as small as we wish. The function $c_1 t^{5/3} - c_2 t$ is bounded from below.

To prove (2) we use

$$\int\int \frac{\rho(x)\rho(y)}{|x-y|}\mathrm{d}x\mathrm{d}y \le c\|\rho\|_{6/5} \le c\|\rho\|_1 + c\|\rho\|_{5/3}.$$

3.3 Sobolev inequalities I

Consider the operator

$$(-\Delta)^{-\frac{\alpha}{2}} = I_\alpha(D),$$

where $I_\alpha(\xi) = |\xi|^{-\alpha}$..

Lemma 3.4 *Let $0 < \alpha < n$. Then the Fourier transform of I_α equals*

$$\hat{I}_\alpha(x) = \pi^{\frac{n}{2}} 2^{n-\alpha} \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} |x|^{\alpha-n}.$$

Proof. We use the representation:

$$|\xi|^{-\alpha} = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \mathrm{e}^{-s\xi^2} s^{\frac{\alpha}{2}} \frac{\mathrm{d}s}{s}. \tag{3.10}$$

It is well known that the Fourier transform of $\mathrm{e}^{-s\xi^2}$ equals

$$\left(\frac{\pi}{s}\right)^{\frac{n}{2}} \mathrm{e}^{-\frac{\xi^2}{4s}}.$$

Hence

$$\begin{aligned} \hat{I}_\alpha(x) &= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \mathrm{e}^{-\frac{x^2}{4s}} s^{\frac{\alpha-n}{2}} \frac{\mathrm{d}s}{s} \\ &= \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \mathrm{e}^{-\frac{tx^2}{4}} t^{\frac{n-\alpha}{2}} \frac{\mathrm{d}t}{t} \\ &= \pi^{\frac{n}{2}} 2^{n-\alpha} \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right)} |x|^{\alpha-n}. \end{aligned}$$

□

Theorem 3.5 *Let $0 < \alpha < n$, $1 < p, r < \infty$, and $\frac{1}{p} + \frac{1}{r} = 1 + \frac{\alpha}{n}$. Then*

$$\left(f|(-\Delta)^{-\frac{\alpha}{2}} h\right) \le c\|f\|_p \|h\|_r.$$

Theorem 3.6 *Let $0 < \alpha < n$, $1 < q, r < \infty$, and $\frac{1}{r} = \frac{1}{q} + \frac{\alpha}{n}$. Then*

$$\|(-\Delta)^{-\frac{\alpha}{2}} h\|_q \le c\|h\|_r.$$

Corollary 3.7 *For $n = 3, 4, \ldots$, $\frac{1}{2} = \frac{1}{q} + \frac{1}{n}$,*

$$\|g\|_q^2 \le c_n \Big(g|(-\Delta) g\Big).$$

Proof. Set $g := (-\Delta)^{-\alpha/2} h$, $\alpha = 1$ and $r = 2$. □

3.4 Sobolev inequalities II

Consider the operator

$$(1-\Delta)^{-\frac{\alpha}{2}} = G_\alpha(D),$$

where $G_\alpha(\xi) = (1+|\xi|^2)^{-\alpha/2}$.

Lemma 3.8 *Let $\alpha > 0$. The Fourier transform of G_α, satisfies*

(1)

$$\hat{G}_\alpha(x) \geq 0$$

(2) *For $|x| \to 0$*

$$\hat{G}_\alpha(x) \leq \begin{cases} C(|x|^{-n+\alpha}), & 0 < \alpha < n \\ C(-\log|x| + 1), & \alpha = n \\ C, & \alpha > n. \end{cases}$$

(3)

$$\hat{G}_\alpha(x) \in L^1(\mathbb{R}^n) \begin{cases} 1 - \frac{\alpha}{n} < \frac{1}{p} \leq 1, & 0 < \alpha < n \\ 0 < \frac{1}{p} \leq 1, & \alpha = n \\ 0 \leq \frac{1}{p} \leq 1, & \alpha > n. \end{cases}$$

Proof. We use the representation

$$(1+\xi^2)^{-\frac{\alpha}{2}} = \frac{1}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \mathrm{e}^{-s(1+\xi^2)} s^{\frac{\alpha}{2}} \frac{\mathrm{d}s}{s}. \tag{3.11}$$

It implies

$$\hat{G}_\alpha(x) = \frac{\pi^{\frac{n}{2}}}{\Gamma\left(\frac{\alpha}{2}\right)} \int_0^\infty \mathrm{e}^{-s-\frac{x^2}{4s}} s^{\frac{\alpha-n}{2}} \frac{\mathrm{d}s}{s}.$$

Hence $\hat{G}_\alpha$ is positive.

If $0 < \alpha < n$, then

$$\hat{G}_\alpha(x) \leq I_\alpha(x).$$

This gives the first inequality in (2).

For $\alpha = n$

$$\begin{aligned} &\textstyle\int_0^\infty \mathrm{e}^{-s-\frac{x^2}{s}} \frac{\mathrm{d}s}{s} \\ &\leq \textstyle\int_0^{|x|} \mathrm{e}^{-\frac{x^2}{s}} \frac{\mathrm{d}s}{s} + \int_{|x|}^\infty \mathrm{e}^{-s} \frac{\mathrm{d}s}{s} \\ &= 2\textstyle\int_{|x|}^\infty \mathrm{e}^{-s} \frac{\mathrm{d}s}{s} \\ &= 2\left(\mathrm{e}^{-|x|}\log|x| + \int_{|x|}^\infty \mathrm{e}^{-s} \log s \mathrm{d}s\right). \end{aligned}$$

Finally, for $\alpha > n$ the integrand in the formula for $\hat{G}_\alpha$ is integrable uniformly in x. This ends the proof of (2).

By the inversion formula for the Fourier transformation,

$$\int \hat{G}_\alpha(x)\mathrm{d}x = (2\pi)^n (1+\xi^2)^{-\frac{\alpha}{2}}\big|_{\xi=0}.$$

Hence $\hat{G}_\alpha \in L^1$. Together with (2), this implies (3). □

Theorem 3.9 *Let $1 \leq p, r \leq \infty$. We have the inequality*

$$\left(f|(1-\Delta)^{-\frac{\alpha}{2}}h\right) \leq c\|f\|_p\|h\|_r$$

in the following cases:

(1) *Let $n < \alpha$. For $1 \leq \frac{1}{p} + \frac{1}{r}$,*

(2) *Let $\alpha = n$. For $1 \leq \frac{1}{p} + \frac{1}{r}$, except for the case $p = r = 1$,*

(3) *$0 < \alpha < n$. For $1 \leq \frac{1}{p} + \frac{1}{r} \leq 1 + \frac{\alpha}{n}$ except for $p = 1$, $r = \frac{n}{\alpha}$ and $p = \frac{n}{\alpha}$, $r = 1$.*

Theorem 3.10 *Let $1 \leq q, r \leq \infty$. We have the inequality*

$$\|(1-\Delta)^{-\frac{\alpha}{2}}h\|_q \leq c\|h\|_r$$

in the following cases:

(1) *If $n < \alpha$, for $0 \leq \frac{1}{q} \leq \frac{1}{r} \leq 1$.*

(2) *If $\alpha = n$, for $0 \leq \frac{1}{q} \leq \frac{1}{r} \leq 1$ except for the case $\frac{1}{r} = 1$, $\frac{1}{q} = 0$,*

(3) *If $0 < \alpha < n$, for $0 \leq \frac{1}{q} \leq \frac{1}{r} \leq \frac{1}{q} + \frac{\alpha}{n}$, except for $\frac{1}{q} = 0$, $\frac{1}{r} = \frac{\alpha}{n}$ and $\frac{1}{q} = 1 - \frac{\alpha}{n}$, $\frac{1}{r} = 1$.*

Theorem 3.11 *The inequality*

$$\|g\|_q^2 \leq c(g|(1-\Delta)g)$$

is valid in the following cases:

(1) *If $n = 1$, for $2 \leq q \leq \infty$.*

(2) *If $n = 2$, for $2 \leq q < \infty$*

(3) *If $n \geq 3$, for $2 \leq q \leq 2n/(n-2)$.*

3.5 Schrödinger operators

Theorem 3.12 *We have*

$$\left(g|(-\Delta + V(x))g\right) \geq -c\|g\|^2,$$

in the following cases:

(1) *If $n = 1$, for $V_- \in L^1 + L^\infty$.*

(2) *If $n = 2$, for $V_- \in L^t + L^\infty$, $1 < t$.*

(3) *If $n \geq 3$, for $V \in L^{n/2} + L^\infty$.*

Proof. Consider eg. (1). It is enough to assume that $V \leq 0$. Let $V_\infty = \max(R, V)$, $V_1 = V - V_\infty$. We have

$$(g|(-\Delta + V)g) \geq \frac{1}{c}\|g\|_\infty^2 - C\|g\|_2^2 - \|V_1\|_1\|g\|_\infty^2.$$

By choosing R big enough we can make $\|V_1\|_1$ small enough. □

Theorem 3.13

$$-\Delta + \frac{c}{|x|^2} \geq 0$$

iff $c \geq -\frac{(n-2)^2}{4}$. Otherwise it is unbounded from below.

Proof. First consider $n = 1$. Then

$$(f|Hf) = (f|(-\partial_x^2 - \tfrac{1}{4x^2})f) = \|(\partial_x - \tfrac{1}{2x})f\|^2 \geq 0.$$

This proves $\Leftarrow$.

To prove $\Rightarrow$ we first note that if $f_\lambda(x) := \lambda^{\frac{n}{2}} f(\lambda x)$, then

$$(f_\lambda|Hf_\lambda) = \lambda^2 (f|Hf).$$

Thus to prove that H is not bounded from below, it is enough to find f with $(f|Hf) < 0$, which is easy.

To get the case of the general n, we use the spherical coordinates:

$$-\Delta = -\partial_r^2 - \frac{n-1}{r}\partial_r - \frac{\Delta_\omega}{r^2},$$

where Δ_ω is the Laplace-Beltrami operator on the sphere, which is negative. Now, setting $\phi(r,\omega) = r^{(n-1)/2}\psi(r,\omega)$,

$$\begin{aligned} &\int \overline{\psi}(r,\omega) r^{n-1}(-\partial_r^2 - \tfrac{n-1}{r}\partial_r)\psi(r,\omega)\mathrm{d}r\mathrm{d}\omega \\ &= \int \overline{\phi}(r,\omega)\Big(- (\partial_r - \tfrac{n-1}{2r})^2 - \tfrac{n-1}{r}(\partial_r - \tfrac{n-1}{2r})\Big)\phi(r,\omega)\mathrm{d}r\mathrm{d}\omega \\ &= \int \overline{\phi}(r,\omega)\Big(- \partial_r^2 + \tfrac{(n-2)^2}{4r^2} - \tfrac{1}{4r^2})\Big)\phi(r,\omega)\mathrm{d}r\mathrm{d}\omega. \end{aligned}$$

4 Momentum in one dimension

4.1 Momentum on the line

The equation

$$U(t)f(x) := f(x-t), \quad f \in L^2(\mathbb{R}), \quad t \in \mathbb{R},$$

defines a unitary strongly continuous group on $L^2(\mathbb{R})$. Let the momentum operator D be defined by

$$U(t) = \mathrm{e}^{-\mathrm{i}tD}.$$

Theorem 4.1 (1) *D is a self-adjoint operator.*

(2) *The integral kernel of $(z-D)^{-1}$ equals*

$$R(z,x,y) = \begin{cases} -\mathrm{i}\theta(x-y)\mathrm{e}^{iz(x-y)}, & \mathrm{Im} z > 0, \\ +\mathrm{i}\theta(y-x)\mathrm{e}^{iz(x-y)}, & \mathrm{Im} z < 0. \end{cases},$$

where θ is the Heavyside function

(3) $\mathrm{Dom}D \subset C_\infty(\mathbb{R})$ *and* $\mathrm{Dom}D \ni f \mapsto f(x) \in \mathbb{C}$ *is a continuous functional.*

(4) $\{f \in L^2(\mathbb{R}) \cap C^1(\mathbb{R}) \ : \ f' \in L^2(\mathbb{R})\} \subset \mathrm{Dom}D$ *and for f in this space*

$$Df(x) := \frac{1}{i}\partial_x f(x). \tag{4.12}$$

(5) *If $f \in \mathrm{Dom}D$ and $Df \in C(\mathbb{R})$, then $f \in C^1(\mathbb{R})$ and (4.12) Is true.*

(6) *$C_\mathrm{c}^\infty(\mathbb{R})$ is an essential domain of D.*

(7) $\mathrm{sp}D = \mathbb{R}$.

(8) $\mathrm{sp_p}D = \emptyset$.

(9) *If* $f \in \mathrm{Dom}D$ *and* $f = 0$ *on* $]a, b[$, *then* $Df = 0$ *on* $]a, b[$.

Proof. (2) For $\mathrm{Im}z > 0$

$$(z - D)^{-1} = -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}U(t)\mathrm{d}t.$$

Hence

$$(z - D)^{-1}f(x) = -\mathrm{i}\int_0^\infty \mathrm{e}^{\mathrm{i}zt}f(x - t)\mathrm{d}t = -\mathrm{i}\int_{-\infty}^\infty \mathrm{e}^{\mathrm{i}(x-y)z}\theta(x - y)f(y)\mathrm{d}y.$$

For $\mathrm{Im}z < 0$ we can use

$$(z - D)^{-1*} = (\overline{z} - D)^{-1}.$$

(3) $\mathrm{Dom}D = \mathrm{Ran}(\mathrm{i} - D)^{-1}$. Now $(\mathrm{i} - D)^{-1}$ is the convolution with $-\mathrm{i}\theta(x)\mathrm{e}^{-|x|}$, which belongs to $L^2(\mathbb{R})$. The convolution of two $L^2(\mathbb{R})$ functions belongs to $C_\infty(\mathbb{R})$.

(4) First let $f \in C_\mathrm{c}^1(\mathbb{R})$. Then

$$t^{-1}(f(x + t) - f(x)) = t^{-1}\int_x^{x+t} f'(y)\mathrm{d}y.$$

f' is uniformy continuous. Hence we will find $t_0 > 0$ such that for $|y_1 - y_2| < t_0$, we have $|f'(y_1) - f'(y_2)| < \epsilon$. Therefore, for $|t| < t_0$

$$|t^{-1}(f(x + t) - f(x)) - f'(x)| < \epsilon.$$

Using the compactness of the support of f we obtain that $t^{-1}(f(x + t) - f(x)) - f'(x) \to 0$ in $L^2(\mathbb{R})$. Thus $C_\mathrm{c}^1(\mathbb{R}) \subset \mathrm{Dom}D$ and (4.12) is true on this subspace.

If $f \in L^2(\mathbb{R}) \cap C^1(\mathbb{R})$ and $f' \in L^2(\mathbb{R})$, then choose $j \in C_\mathrm{c}^\infty(\mathbb{R})$ such that $j = 1$ on a neighborhood of zero. Set $j_r(x) := j(x/r)$. Then $j_r f \in C_\mathrm{c}^1$, hence $Dj_r f = -\mathrm{i}\partial_x j_r f$. We easily check that $j_r f \to f$ and $Dj_r f \to -\mathrm{i}\partial_x f$ in $L^2(\mathbb{R})$. Hence, by the closedness of D we get $Df = -\mathrm{i}\partial_x f$.

(5) Let $f \in \mathrm{Dom}D$, $g \in C(\mathbb{R})$ and $Df = g$. Let $x \in \mathbb{R}$, $r > 0$. Set $h := 1_{[x,x+r]}$. Then

$$\begin{aligned} t^{-1}(h|U(t)f - f) &= t^{-1}\int_{x-t}^{x+r-t} f(y)\mathrm{d}y - t^{-1}\int_x^{x+r} f(y)\mathrm{d}y \\ &= -t^{-1}\int_{x+r-t}^{x+r} f(y)\mathrm{d}y + t^{-1}\int_{x-t}^{x} f(y)\mathrm{d}y \to -f(x + r) + f(x). \end{aligned}$$

Therefore

$$\mathrm{i}(h|g) = \mathrm{i}\int_x^{x+r} g(y)\mathrm{d}y = -f(x + r) + f(x).$$

Hence, using the continuity of g,

$$\lim_{r\to 0} \frac{f(x + r) - f(x)}{r} = -\mathrm{i}g(x).$$

(7) Let $k \in \mathbb{R}$. Consider $f_{\epsilon,k} = \sqrt{\pi\epsilon}\mathrm{e}^{-\epsilon x^2 + \mathrm{i}kx}$. Then $\|f_{\epsilon,k}\| = 1$, $f_{\epsilon,k} \in \mathrm{Dom}D$ and $(k - D)f_{\epsilon,k} \to 0$ as $\epsilon \to 0$. Hence $k \in \mathrm{sp}D$.

(8) Suppose that $f \in \mathrm{Dom}D$ and $Df = kf$. Clearly, $f \in \mathrm{Dom}D^2$. Hence, by Theorem 4.2, $f \in C^1(\mathbb{R})$ and $Df = -\mathrm{i}\partial_x f = kf$. It is well known that the only solution is $f = c\mathrm{e}^{\mathrm{i}kx}$, which does not belong to $L^2(\mathbb{R})$.

(9) is obvious for $f \in C_\mathrm{c}^1(\mathbb{R})$. It extends by density.□

4.2 Sobolev spaces in one dimension

Let $L^2_\alpha(\mathbb{R})$ be the scale of spaces associated with D. This means in particular, that $L^2_n(\mathbb{R}) = \mathrm{Dom} D^n$.

Theorem 4.2 *$L^2_{n+1}(\mathbb{R}) \subset C^n(\mathbb{R})$ and $L^2_{n+1}(\mathbb{R}) \ni f \mapsto f^{(j)}(x)$ for $j = 0, \dots, n-1$ are continuous functionals depending continuously on $x \in \mathbb{R}$.*

Proof. We use induction. The step $n = 0$ was proven in Theorem 4.1 (3).

Suppose that we know that $L^2_{n+1}(\mathbb{R}) \subset C^n(\mathbb{R})$. Let $f \in L^2_{n+2}(\mathbb{R})$. Then $(\mathrm{i} - D)f = g \in L^2_{n+1}(\mathbb{R})$. Clearly, $L^2_{n+2}(\mathbb{R}) \subset L^2_{n+1}(\mathbb{R})$ hence $f \in C^n(\mathbb{R})$. Likewise, $g \in C^n(\mathbb{R})$, by the induction assumption. Now $Df = -g + \mathrm{i}f \in C^n(\mathbb{R})$. Hence, by Theorem 4.1 (5) $f \in C^{n+1}(\mathbb{R})$. □

Define

$$L^2_{n,\min}([0,\infty[) := \{f \in L^2_n(\mathbb{R}) \ : \ f(x) = 0, \ x < 0\}.$$

Define $L^2_{n,\min}(]-\infty, 0])$ in a similar way.

Theorem 4.3 (1) *$L^2_{n,\min}([0,\infty[)$ is orthogonal to $L^2_{n,\min}(]-\infty,0])$.*

(2) *The codimension of*

$$L^2_{n,\min}(]-\infty,0]) \oplus L^2_{n,\min}([0,\infty[) \tag{4.13}$$

equals n.

(3) *(4.13) equals*

$$\{f \in L^2_n(\mathbb{R}) \ : \ f^{(j)}(0) = 0, \ j = 0, \dots, n-1\}.$$

(4) *D maps $L^2_{n,\min}([0,\infty[)$ into $L^2_{n-1,\min}([0,\infty[)$.*

(5) *$L_{n+1,\min}([0,\infty[) \subset L_{n,\min}([0,\infty[)$*

(6) *$L^2_{0,\min}([0,\infty[) = L^2([0,\infty[)$.*

We define

$$L^2_{n,\max}([0,\infty[) := L^2_n(\mathbb{R}) \ominus L^2_{n,\min}(]-\infty,0]).$$

Theorem 4.4 (1) *$L^2_{n,\min}([0,\infty[)$ is a subspace of $L^2_{n,\max}([0,\infty[)$ of codimension n.*

(2) *$L^2_{n,\min}([0,\infty[)$ equals*

$$\{f \in L^2_{n,\max}([0,\infty[) \ : \ f^{(j)}(0) = 0, \ j = 0, \dots, n-1\}.$$

(3) *D maps $L^2_{n,\max}([0,\infty[)$ into $L^2_{n-1,\max}([0,\infty[)$.*

(4) *$L_{n+1,\max}([0,\infty[) \subset L_{n,\max}([0,\infty[)$*

(5) *$L^2_{0,\max}([0,\infty[) = L^2([0,\infty[)$.*

4.3 Momentum on the half-line

Define $D_{\max}$ as an operator on $L^2([0,\infty[)$ equal to the restriction of D to $L^2_{1,\max}([0,\infty[)$. Likewise, define $D_{\min}$ as an operator on $L^2([0,\infty[)$ equal to the restriction of D to $L^2_{1,\min}([0,\infty[)$.

Theorem 4.5 (1) *$D_{\min} \subset D_{\max}$, $D^*_{\min} = D_{\max}$, $D^*_{\max} = D_{\min}$*

(2) *The operators $D_{\min}$ and $-D_{\max}$ are m-dissipative (in particular, they are closed); the operator $D_{\min}$ is hermitian.*

(3) $\mathrm{sp_p} D_{\max} = \{\mathrm{Im} z > 0\}$, $\mathrm{sp_p} D_{\min} = \emptyset$;

$$D_{\max} \mathrm{e}^{\mathrm{i}zx} = z\mathrm{e}^{\mathrm{i}zx}, \quad \mathrm{e}^{\mathrm{i}zx} \in \mathrm{Dom} D_{\max}, \quad \mathrm{Im} z > 0, \tag{4.14}$$

(4) $\mathrm{sp} D_{\max} = \{\mathrm{Im} z \geq 0\}$, $\mathrm{sp} D_{\min} = \{\mathrm{Im} z \leq 0\}$;

(5) *The integral kernels of* $(z - D_{\max})^{-1}$ *and* $(z - D_{\min})^{-1}$ *are equal*

$$R_{\max}(z, x, y) = \mathrm{i}\theta(y - x)\mathrm{e}^{\mathrm{i}z(x-y)}, \ \mathrm{Im} z < 0.$$
$$R_{\min}(z, x, y) = -\mathrm{i}\theta(x - y)\mathrm{e}^{\mathrm{i}z(x-y)}, \ \mathrm{Im} z > 0.$$

(6) *The semigroups generated by these operators:*

$$\mathrm{e}^{\mathrm{i}tD_{\max}} f(x) = f(x + t), \quad t \geq 0.$$
$$\mathrm{e}^{-\mathrm{i}tD_{\min}} f(x) = \begin{cases} f(x - t), & x \geq t \geq 0. \\ 0, & t > x, \end{cases}$$

4.4 Momentum on an interval I

We define $L^2_{n,\max}([-\pi, \pi])$ and $L^2_{n,\min}([-\pi, \pi])$ modifying in the obvious way the definitions of Subsection 4.2.

Define $D_{\max}$ as an operator on $L^2([-\pi, \pi])$ equal to the restriction of D to $L^2_{1,\max}([-\pi, \pi])$. Likewise, define $D_{\min}$ as an operator on $L^2([-\pi, \pi])$ equal to the restriction of D to $L^2_{1,\min}([-\pi, \pi])$.

Theorem 4.6 (1) $D_{\min} \subset D_{\max}$, $D^*_{\min} = D_{\max}$, $D^*_{\max} = D_{\min}$

(2) *The operators* $D_{\min}$ *and* $D_{\max}$ *are closed; the operator* $D_{\min}$ *is hermitian.*

(3) $\mathrm{sp_p} D_{\max} = \mathbb{C}$, $\mathrm{sp_p} D_{\min} = \emptyset$;

$$D_{\max} \mathrm{e}^{\mathrm{i}zx} = z\mathrm{e}^{\mathrm{i}zx}, \quad \mathrm{e}^{\mathrm{i}zx} \in \mathrm{Dom} D_{\max}, \quad z \in \mathbb{C}, \tag{4.15}$$

(4) $\mathrm{sp} D_{\max} = \mathbb{C}$, $\mathrm{sp} D_{\min} = \mathbb{C}$;

4.5 Momentum on an interval II

Let $\kappa \in \mathbb{C}$. Let the operator D_κ on $L^2([-\pi, \pi])$ be defined as the restriction of $D_{\max}$ to

$$\mathrm{Dom} D_\kappa = \{f \in L^2_{1,\max}([-\pi, \pi]) \ : \ \mathrm{e}^{\mathrm{i}2\pi\kappa} f(-\pi) = f(\pi)\}.$$

Theorem 4.7 (1) $D^*_\kappa = D_{\overline{\kappa}}$, $\quad D_\kappa = D_{\kappa+1}$.

(2) $D_{\min} \subset D_\kappa \subset D_{\max}$.

(3) *Operators* D_κ *are closed and for* $\kappa \in \mathbb{R}$ *self-adjoint.*

(4) $\mathrm{sp} D_\kappa = \mathrm{sp_p} D_\kappa = \mathbb{Z} + \kappa$,

$$D_\kappa \mathrm{e}^{\mathrm{i}(n+\kappa)x} = (n + \kappa)\mathrm{e}^{\mathrm{i}(n+\kappa)x}, \ n \in \mathbb{Z}.$$

(5) *The integral kernel of* $(z - D_\kappa)^{-1}$ *equals*

$$R_\kappa(z, x, y) = \frac{1}{2\sin\pi(z - \kappa)} \left(\mathrm{e}^{-\mathrm{i}(z-\kappa)\pi} \mathrm{e}^{\mathrm{i}z(x-y)} \theta(x - y) + \mathrm{e}^{\mathrm{i}(z-\kappa)\pi} \mathrm{e}^{\mathrm{i}z(x-y)} \theta(y - x) \right).$$

(6) *The group generated by* $\mathrm{i}D_\kappa$ *equals*

$$\mathrm{e}^{\mathrm{i}tD_\kappa} \phi(x) = \mathrm{e}^{\mathrm{i}\pi n\kappa} \phi(x + t), \quad (2n - 1)\pi < x + t < (2n + 1)\pi.$$

(7) *The operators* D_κ *are similar to one another up to an additive constant:*

$$\mathrm{Dom} D_\kappa = \mathrm{e}^{\mathrm{i}\kappa x} \mathrm{Dom} D_0, \quad D_\kappa = \mathrm{e}^{\mathrm{i}\kappa x} D_0 \mathrm{e}^{-\mathrm{i}\kappa x} + \kappa. \tag{4.16}$$

4.6 Momentum on an interval III

Let the operator $D_{\pm\mathrm{i}\infty}$ on $L^2([-\pi,\pi])$ be defined as the restriction of $D_{\max}$ to

$$\mathrm{Dom}D_{\pm\mathrm{i}\infty} = \{f \in L^2_{1,\max}([-\pi,\pi]) \ : \ f(\pm\pi) = 0\}.$$

Theorem 4.8 (1) $D^*_{\pm\mathrm{i}\infty} = D_{\mp\mathrm{i}\infty}$.

(2) $D_{\min} \subset D_{\pm\mathrm{i}\infty} \subset D_{\max}$.

(3) *The operators* $D_{\pm\mathrm{i}\infty}$ *are closed.*

(4) $\mathrm{sp}D_{\pm\mathrm{i}\infty} = \emptyset$.

(5) *The integral kernel of* $(z - D_{\pm\mathrm{i}\infty})^{-1}$ *equals*

$$R_{\pm\mathrm{i}\infty}(z,x,y) = \pm\mathrm{i}\mathrm{e}^{\mathrm{i}z(x-y\pm\pi)}\theta(\pm y \mp x), \quad z \in \mathbb{C}.$$

(6) $\pm\mathrm{i}D_{\pm\mathrm{i}\infty}$ *generate the semigroups of contractions for* $t \geq 0$*:*

$$\mathrm{e}^{\pm\mathrm{i}tD_{\mathrm{i}\infty}}f(x) = \begin{cases} f(x \pm t), & |x \pm t| \leq \pi, \\ 0 & |x \pm t| > \pi. \end{cases}$$

5 Laplacian

5.1 Laplacian on the line

The operator D^2 on $L^2(\mathbb{R})$ will be denoted $-\Delta$. Thus $\mathrm{Dom}(-\Delta) = L^2_2(\mathbb{R})$.

Theorem 5.1 (1) $-\Delta$ *is a positive self-adjoint operator.*

(2) $\mathrm{sp}(-\Delta) = [0,\infty[$.

(3) *The integral kernel of* $(k^2 - \Delta)^{-1}$*, for* $\mathrm{Re}k > 0$*, equals*

$$R(k,x,y) = \frac{1}{2k}\mathrm{e}^{-k|x-y|}.$$

(4) *The integral kernel of* $\mathrm{e}^{t\Delta}$ *equals*

$$K(t,x,y) = (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}}.$$

(5) $\mathrm{sp_p}(-\Delta) = \emptyset$.

(6) $\{f \in C^2(\mathbb{R}) \cap L^2(\mathbb{R}) \ : \ f', f'' \in L^2(\mathbb{R})\}$ *is contained in* $\mathrm{Dom}(-\Delta)$ *and on this set*

$$-\Delta f(x) = -\partial_x^2 f(x).$$

(7) $C_{\mathrm{c}}^\infty(\mathbb{R})$ *is an essential domain of* $-\Delta$.

Proof. (3) Let $\mathrm{Re}k > 0$. Then

$$(\mathrm{i}k - D)^{-1}(x,y) = -\mathrm{i}\theta(x-y)\mathrm{e}^{-k|x-y|}, \quad (-\mathrm{i}k - D)^{-1}(x,y) = \mathrm{i}\theta(y-x)\mathrm{e}^{-k|x-y|}.$$

Now

$$\begin{aligned} (k^2-\Delta)^{-1} &= (\mathrm{i}k - D)^{-1}(-\mathrm{i}k - D)^{-1} \\ &= (-2\mathrm{i}k)^{-1}\left((\mathrm{i}k - D)^{-1} - (-\mathrm{i}k - D)^{-1}\right). \end{aligned} \tag{5.17}$$

The integral kernel of (5.17) equals $(2k)^{-1}\mathrm{e}^{-k|x-y|}$.

(4) We have

$$\mathrm{e}^{t\Delta} = (2\pi\mathrm{i})^{-1}\int_\gamma (z-\Delta)^{-1}\mathrm{e}^{tz}\mathrm{d}z,$$

where γ is a contour of the form $\mathrm{e}^{-\mathrm{i}\alpha}]0,\infty[\cup\mathrm{e}^{\mathrm{i}\alpha}[0,\infty[$ bypassing 0, where $\pi/2<\alpha<\pi$. Hence

$$\mathrm{e}^{t\Delta}(x,y) = (2\pi\mathrm{i})^{-1}\int_{\tilde\gamma} \mathrm{e}^{-k|x-y|+tk^2}\mathrm{d}k$$

where $\tilde\gamma$ is a contour of the form $\mathrm{e}^{-\mathrm{i}\alpha/2}[0,\infty[\cup\mathrm{e}^{\mathrm{i}\alpha/2}[0,\infty[$. We put $k=\mathrm{i}u$ and obtain

$$\mathrm{e}^{t\Delta}(x,y) = (2\pi\mathrm{i})^{-1}\int_{-\infty}^{\infty} \mathrm{e}^{-\mathrm{i}u|x-y|-tu^2}\mathrm{i}\mathrm{d}u$$

□

5.2 Laplacian on the halfline I

Define $-\Delta_{\max}$ as an operator on $L^2([0,\infty[)$ equal to the restriction of $-\Delta$ to $L^2_{2,\max}([0,\infty[)$. Likewise, define $-\Delta_{\min}$ as an operator on $L^2([0,\infty[)$ equal to the restriction of $-\Delta$ to $L^2_{2,\max}([0,\infty])$.

Theorem 5.2 (1) $-\Delta_{\min}^* = -\Delta_{\max}$, $\quad -\Delta_{\min}\subset -\Delta_{\max}$.

(2) *The operators $-\Delta_{\min}$ and $-\Delta_{\max}$ are closed and $-\Delta_{\min}$ is hermitian.*

(3) $\mathrm{sp_p}(-\Delta_{\max}) = \mathbb{C}\backslash[0,\infty[$, $\quad \mathrm{sp_p}(-\Delta_{\min})=\emptyset$

$$-\Delta_{\max}\mathrm{e}^{ikx} = k^2\mathrm{e}^{ikx},\ \mathrm{Im}k>0,\qquad \mathrm{e}^{\mathrm{i}kx}\in\mathrm{Dom}(-\Delta_{\max}).$$

(4) $\mathrm{sp}(-\Delta_{\max})=\mathbb{C}$, $\mathrm{sp}(-\Delta_{\min})=\mathbb{C}$.

(5) $-\Delta_{\min}=D^2_{\min}$, $\quad -\Delta_{\max}=D^2_{\max}$.

5.3 Laplacian on the halfline II

Let $\mu\in\mathbb{C}\cap\{\infty\}$,

$$\mathrm{Dom}(-\Delta_\mu) = \{f\in L^2_{2,\max}([0,\infty[) \quad : \quad \mu f(0)=f'(0)\}. \tag{5.18}$$

(If $\mu=\infty$, these are the Dirichlet boundary conditions, that means $f(0)=0$, if $\mu=0$, these are the Neumann boundary conditions, that means $f'(0)=0$). Let $-\Delta_\mu$ be the restriction of $-\Delta_{\max}$ to (5.18).

Define also the form δ_μ as follows. If $\mu\in\mathbb{R}$, then $\mathrm{Dom}\delta_\mu = L^2_{1,\max}([0,\infty[)$ and

$$\delta_\mu(f,g) := \mu\overline{f(0)}g(0)+\int\overline{f'(x)}g'(x)\mathrm{d}x.$$

For $\mu=\infty$, we set $\mathrm{Dom}\delta_\infty := L^2_{1,\min}([0,\infty[)$ and

$$\delta_\infty(f,g) := \int\overline{f'(x)}g'(x)\mathrm{d}x.$$

Theorem 5.3 (1) $-\Delta_{\min}\subset -\Delta_\mu\subset -\Delta_{\max}$.

(2) $-\Delta_\mu^* = -\Delta_{\overline{\mu}}$.

(3) *The operator $-\Delta_\mu$ are generators of groups. For $\mu\in\mathbb{R}\cup\{\infty\}$ it is self-adjoint.*

(4) $\mathrm{sp_p}(-\Delta_\mu) = \begin{cases} \{-\mu^2\}, & \mathrm{Re}\mu < 0; \\ \emptyset, & \text{otherwise}; \end{cases}$
$-\Delta_\mu \mathrm{e}^{\mu x} = -\mu^2 \mathrm{e}^{\mu x},\ \mathrm{Re}\mu < 0, \qquad \mathrm{e}^{\mu x} \in \mathrm{Dom}(-\Delta_\mu)$.

(5) $\mathrm{sp}(-\Delta_\mu) = \begin{cases} \{-\mu^2\} \cup [0,\infty[, & \mathrm{Re}\mu < 0, \\ [0,\infty[, & \text{otherwise}. \end{cases}$

(6) $-\Delta_0 = D_{\max}^* D_{\max}, \quad -\Delta_\infty = D_{\min}^* D_{\min}$.

(7) *The forms* δ_μ *are closed and associated with the operator* $-\Delta_\mu$.

(8) *Let* $\mathrm{Re}k > 0$. *The integral kernel of* $(k^2 - \Delta_\mu)^{-1}$ *is equal*

$$R_\mu(k,x,y) = \frac{1}{2k}\mathrm{e}^{-k|x-y|} + \frac{1}{2k}\frac{(k-\mu)}{(k+\mu)}\mathrm{e}^{-k(x+y)},$$

in particular, for the Dirichlet boundary conditions,

$$R_\infty(z,x,y) = \frac{1}{2k}\mathrm{e}^{-k|x-y|} - \frac{1}{2k}\mathrm{e}^{-k(x+y)},$$

and for the Neumann boundary conditions

$$R_0(k,x,y) = \frac{1}{2k}\mathrm{e}^{-k|x-y|} + \frac{1}{2k}\mathrm{e}^{-k(x+y)}.$$

(9) *The semigroups* $\mathrm{e}^{t\Delta_\mu}$ *have the integral kernel*

$$K_\mu(t,x,y) = (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}} + (2\pi)^{-1}\int_{-\infty}^{\infty}\frac{\mathrm{i}u-\mu}{\mathrm{i}u+\mu}\mathrm{e}^{-\mathrm{i}u(x+y)-tu^2}\mathrm{d}u,$$

In particular, in the Dirichlet case

$$K_\infty(t,x,y) = (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}} - (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x+y)^2}{4t}},$$

and in the Neumann case

$$K_0(t,x,y) = (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x-y)^2}{4t}} + (4\pi t)^{-\frac{1}{2}}\mathrm{e}^{-\frac{(x+y)^2}{4t}}.$$

The semigroup $\mathrm{e}^{t\Delta_\mu}$ for $\mu \in \mathbb{R}$ can be used to describe the diffusion with a sink or source at the end of the halfline.

Note that for positive μ, $\mathrm{e}^{t\Delta_\mu}$ preserves the pointwise positivity. If $p_t = \mathrm{e}^{t\Delta_\mu}p_0$, $0 < a < b$, then

$$\partial_t \int_a^b p_t(x)\mathrm{d}x = p'(b) - p'(a).$$

$$\partial_t \int_0^a p_t(x)\mathrm{d}x = p'(a) - \mu p(0).$$

Thus at 0 there is a sink of p with the rate μ.

5.4 Contact perturbations of the Laplacian as examples of an Aronszajn-Donoghue Hamiltonian

5.4.1 Neumann Laplacian on a halfline

On $L^2([0,\infty[)$ we define the cosine transform

$$U_{\rm N}f(k) := \pi^{-1/2}\int \cos kx f(x){\rm d}x, \quad k\geq 0.$$

Note that $U_{\rm N}$ is unitary and $U_{\rm N}^2 = 1$.

Let $\Delta_{\rm N}$ be the Laplacian on $L^2([0,\infty[)$ with the Neumann boundary condition. Clearly,

$$-U_{\rm N}\Delta_{\rm N}U_{\rm N}^* = k^2.$$

Let $|\delta)(\delta|$ be the quadratic form given by

$$(f_1|\delta)(\delta|f_2) = \overline{f}_1(0)f_2(0),$$

Note that in the literature it is also denoted by δ (and thus is interpreted as a "potential").

Let $(1|$ denote the functional on $L^2([0,\infty[)$ given by

$$(1|g) = \int g(k){\rm d}k.$$

Using $\delta(x) = \pi^{-1}\int_0^\infty \cos kx{\rm d}x$ we deduce that

$$U_{\rm N}|\delta)(\delta|U_{\rm N}^* = \pi^{-1}|1)(1|.$$

Then

$$U_{\rm N}\left(-\Delta_{\rm N} + \lambda|\delta)(\delta|\right)U_{\rm N}^* = k^2 + \lambda\pi^{-1}|1)(1|$$

is an example of an Aronszajn-Donoghue Hamiltonian of type II.

5.4.2 Dirichlet Laplacian on a halfline

On $L^2([0,\infty[)$ we define the sine transform

$$U_{\rm D}f(k) := \pi^{-1/2}\int \sin kx f(x){\rm d}x, \quad k\geq 0.$$

Note that $U_{\rm D}$ is unitary and $U_{\rm D}^2 = 1$

Let $\Delta_{\rm D}$ be the Laplacian on $L^2([0,\infty[)$ with the Dirichlet boundary condition. Clearly,

$$-U_{\rm D}\Delta_{\rm D}U_{\rm D}^* = k^2.$$

Using $-\delta'(x) = \pi^{-1}\int_0^\infty \sin kx{\rm d}x$ we deduce that

$$U_{\rm D}|\delta')(\delta'|U_{\rm D}^* = \pi^{-1}|k)(k|.$$

Here $|\delta')(\delta'|$ is the quadratic form given by

$$(f_1|\delta')(\delta'|f_2) = \overline{f}_1'(0)f_2'(0),$$

and $(k|$ is the functional on $L^2([0,\infty[)$ given by

$$(k|g) = \int kg(k){\rm d}k.$$

Thus

$$U_{\rm D}\left(-\Delta_{\rm D} + \lambda|\delta')(\delta'|\right)U^* = k^2 + \lambda\pi^{-1}|k)(k|$$

is an example of an Aronszajn-Donoghue Hamiltonian of type III.

5.4.3 Laplacian on $L^2(\mathbb{R}^d)$ with a delta potential

On $L^2(\mathbb{R}^d)$ we consider the unitary operator $U = (2\pi)^{d/2}\mathcal{F}$, where $\mathcal{F}$ is the Fourier transformation. Note that U is unitary.

Let Δ be the usual Laplacian. Clearly,

$$-U\Delta U^* = k^2.$$

Let $|\delta)(\delta|$ be the quadratic form given by

$$(f_1|\delta)(\delta|f_2) = \overline{f}_1(0)f_2(0).$$

Note that again it can be also denoted by δ (and thus is interpreted as a "potential"). Let $(1|$ denote the functional on $L^2(\mathbb{R}^d)$ given by

$$(1|g) = \int g(k)\mathrm{d}k.$$

Using $\delta(x) = (2\pi)^{-d}\int_0^\infty \mathrm{e}^{\mathrm{i}kx}\mathrm{d}x$ we deduce that

$$U|\delta)(\delta|U^* = (2\pi)^{-d}|1)(1|.$$

Now

$$U\left(-\Delta + \lambda|\delta)(\delta|\right)U^* = k^2 + \lambda(2\pi)^{-d}|1)(1|$$

is an example of an Aronszajn-Donoghue Hamiltonian of type II, for $d = 1$ (as we have already seen). For $d = 2, 3$, on the other hand, it is an Aronszajn-Donoghue Hamiltonian of type III (so we need to renormalize λ). In dimension $d \geq 4$ we cannot use the renormalization procedure. This is reflected in the following theorem:

Theorem 5.4 *Consider* $-\Delta$ *on* $C_\mathrm{c}^\infty(\mathbb{R}^d\backslash\{0\})$

(1) *It has the defficiency index* $(2,2)$ *for* $d = 1$.

(2) *It has the defficiency index* $(1,1)$ *for* $d = 2, 3$.

(3) *It is essentially self-adjoint for* $d \geq 4$.

(4) *Its Friedrichs extension equals* Δ_D *for* $d = 1$.

(5) *Its Friedrichs extension equals* Δ *for* $d \geq 2$.

The Laplacian in d dimensions written in spherical coordinates equals

$$\Delta = \partial_r^2 + \frac{d-1}{r}\partial_r + \frac{\Delta_{\mathrm{LB}}}{r^2},$$

where Δ_{LB} is the Laplace-Beltrami operator on the sphere. For $d \geq 2$, the eigenvalues of Δ_{LB} are $-l(l+d-2)$, for $l = 0, 1, \ldots$. For $D = 1$ instead of the Laplace-Beltrami operator we consider the parity operator with the eigenvalues ± 1. We will write $l = 0$ for parity $+1$ and $l = 1$ for parity -1. Hence the radial part of the operator is

$$\partial_r^2 + \frac{d-1}{r}\partial_r - \frac{l(l+d-2)}{r^2}.$$

The indicial equation of this operator reads

$$\lambda(\lambda + d - 2) - l(l + d - 2) = 0.$$

It has the solutions $\lambda = l$ and $\lambda = 2 - l - d$.

For $l \geq 2$ only the solutions behaving as r^l around zero are locally square integrable, the solutions behaving as r^{2-1-d} have to be discarded. For $l = 0, 1$ we have the following possible square integrable behaviors around zero:

	$l = 0$	$l = 1$
$d = 1$	r^0, r^1	r^0, r^1
$d = 2$	$r^0, r^0 \ln r$	r^1
$d = 3$	r^0, r^{-1}	r^1
$d \geq 4$	r^0	r^1

In particular, in dimension $d = 2$, apart fro the usual Laplacian we have a family of self-adjoint extensions of the operator considered in (5.4) with the behavior of elements in the domain in zero given by $c \ln(r/a)$. In dimension $d = 3$, apart from the usual Laplacian we have an analogous family with $c(1 - \frac{a}{r})$. The parameter a is called the *scattering length*.

6 Operators on a lattice

6.1 Schrödinger operator on a lattice

Fix a real function $\mathbb{Z} \ni n \mapsto V_n$ and define the operator H on $l^2(\mathbb{Z})$

$$(Hf)_n = f_{n-1} + f_{n+1} + V_n f_n.$$

H is called a discrete Schrödinger operator.

Assume in addition that $V_{n+q} = V_n$. Then we can partly diagonalize H by applying the Fourier transformation. More precisely, define

$$\mathcal{F} : l^2(\mathbb{Z}) \to L^2\left(\mathbb{Z}_q \times [0, (2\pi)/q[\right)$$

by setting

$$(\mathcal{F}f)_k(\theta) := \sqrt{q/(2\pi)} \sum_{t=-\infty}^{\infty} f_{k+q+t} \mathrm{e}^{-\mathrm{i}\theta(qt+k)}.$$

Clearly, the inverse transformation equals

$$(\mathcal{F}^* f)_{k+qt} = \sqrt{q/(2\pi)} \int_0^{(2\pi)/q} f_k(\theta) \mathrm{e}^{\mathrm{i}\theta(qt+k)} \mathrm{d}\theta.$$

For $\theta \in [0, \frac{2\pi}{q}[$, introduce the operator H_θ on $L^2(\mathbb{Z}_q)$ by

$$(H_\theta f)_k := \mathrm{e}^{-\mathrm{i}\theta} f_{k-1} + \mathrm{e}^{\mathrm{i}\theta} f_{k+1} + V_k f_k.$$

Theorem 6.1 *We have $\left(\mathcal{F}H\mathcal{F}^* f\right)(\theta) = H_\theta f(\theta)$ for almost all θ. and hence*

$$\mathrm{sp} H = \bigcup_{\theta \in [0, \frac{2\pi}{q}[} \mathrm{sp} H_\theta.$$

This implies in particular, that the spectrum will typically consist of k disjoint bands.

6.2 Harper's equation

Let $\alpha \in [0, 2\pi[$. Consider the operator H_α on $l^2(\mathbb{Z}^2)$

$$\begin{aligned}(H_\alpha f)_{n,m} &= f_{n-1,m} + f_{n+1,m} \\ &\quad + \mathrm{e}^{-in\alpha} f_{n,m-1} + \mathrm{e}^{in\alpha} f_{n,m+1}.\end{aligned}$$

Note that this operator describes a particle on a 2-dimensional lattice in a magnetic field with flux α through a unit cell.

Introduce the unitary operator $\mathcal{F} : l^2(\mathbb{Z}^2) \to L^2(\mathbb{Z} \times [0, 2\pi[)$ given by

$$(\mathcal{F}f)_n(\phi) := \frac{1}{\sqrt{2\pi}} \sum_m f_{n,m} \mathrm{e}^{-im\phi}$$

with the inverse given by

$$(\mathcal{F}^* f)_{n,m} = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} f_n(\phi) \mathrm{e}^{im\phi} \mathrm{d}\phi.$$

For $\alpha, \phi \in [0, 2\pi[$ introduce the operator $H_{\alpha,\phi}$ (called sometimes Harper's operator)

$$(H_{\alpha,\phi} f)_n = f_{n-1} + f_{n+1} + 2\cos(\alpha n + \phi) f_n.$$

Theorem 6.2 *We have, for almost all ϕ, $(\mathcal{F} H_\alpha \mathcal{F}^* f)(\phi) = H_{\alpha,\phi} f(\phi)$. Hence*

$$\mathrm{sp} H_\alpha = \bigcup_{\phi \in [0,2\pi[} \mathrm{sp} H_{\alpha,\phi}.$$

The spectrum of H_α plotted as the function of α yields the famous Hofstadter butterfly. One can show that for irrational ϕ the spectrum of $H_{\alpha,\phi}$ does not depend on ϕ and is singular continuous.

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