WHY IS THE WEYL QUANTIZATION THE BEST?

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Basic classical mechanics – phase space $\mathbb{R}^{2d}$ with generic variables (x^i, p_j).

Basic quantum mechanics – Hilbert space $L^2(\mathbb{R}^d)$ with self-adjoint operators $\hat{x}^i$, $\hat{p}_j := \frac{\hbar}{\mathrm{i}}\frac{\partial}{\partial x^j}$, where $\hbar$ is a small parameter.

A linear transformation which to a complex function b on $\mathbb{R}^{2d}$ associates an operator $\mathrm{Op}^{\bullet}(b)$ on $L^2(\mathbb{R}^d)$ is often called a quantization of the symbol b.

Desirable properties:

(1) $\mathrm{Op}^\bullet(1) = \mathbb{1}$, $\mathrm{Op}^\bullet(x_i) = \hat{x}_i, \ \ \mathrm{Op}^\bullet(p_j) = \hat{p}_j;$

(2) $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(-y\hat{p}+w\hat{x})}\mathrm{Op}^\bullet(b)\mathrm{e}^{\frac{\mathrm{i}}{\hbar}(y\hat{p}-w\hat{x})} = \mathrm{Op}^\bullet\big(b(\cdot - y, \cdot - w)\big).$

(3) $\frac{1}{2}\big(\mathrm{Op}^\bullet(b)\mathrm{Op}^\bullet(c) + \mathrm{Op}^\bullet(c)\mathrm{Op}^\bullet(b)\big) \approx \mathrm{Op}^\bullet(bc);$

(4) $[\big(\mathrm{Op}^\bullet(b), \mathrm{Op}^\bullet(c)] \approx \mathrm{i}\hbar\mathrm{Op}^\bullet(\{b, c\});$

Let us strengthen the desirable property (1) to

$$\mathrm{Op}^\bullet(f(x)) = f(\hat{x}), \quad \mathrm{Op}^\bullet(g(p)) = g(\hat{p}).$$

The so-called x,p-quantization is determined by the additional condition

$$\mathrm{Op}^{x,p}(f(x)g(p)) = f(\hat{x})g(\hat{p}).$$

It is defined by

$$\big(\mathrm{Op}^{x,p}(b)\Psi\big)(x) = (2\pi\hbar)^{-d}\int \mathrm{d}p \int \mathrm{d}y b(x,p)\mathrm{e}^{\frac{\mathrm{i}(x-y)p}{\hbar}}\Psi(y).$$

In terms of its distributional kernel one can write

$$\mathrm{Op}^{x,p}(b)(x,y) = (2\pi\hbar)^{-d} \int \mathrm{d}p b(x,p) \mathrm{e}^{\frac{\mathrm{i}(x-y)p}{\hbar}}$$

We also have the closely related p, x-quantization,

$$\mathrm{Op}^{p,x}(b)(x,y) = (2\pi\hbar)^{-d} \int \mathrm{d}p b(y,p) \mathrm{e}^{\frac{\mathrm{i}(x-y)p}{\hbar}}$$

We have

$$\mathrm{Op}^{x,p}(b)^* = \mathrm{Op}^{p,x}(\bar{b}).$$

The Weyl quantization (or the Weyl-Wigner-Moyal quantization) is a compromise between the two above quantizations:

$$\mathrm{Op}(b)(x,y) = (2\pi\hbar)^{-d}\int \mathrm{d}p b\Big(\frac{x+y}{2},p\Big)\mathrm{e}^{\frac{\mathrm{i}(x-y)p}{\hbar}}.$$

If $\mathrm{Op}(b)=B$, the function b is often called the Wigner function or the Weyl symbol of the operator B:

$$b(x,p) = \int B\Big(x+\frac{z}{2},x-\frac{z}{2}\Big)\mathrm{e}^{-\frac{\mathrm{i}zp}{\hbar}}\mathrm{d}z.$$

We have

$$\mathrm{Op}(b)^* = \mathrm{Op}(\overline{b}).$$

Hermann Weyl

Eugene Wigner

Fix a normalized vector $\Psi \in L^2(\mathbb{R}^d)$. Define

$$\Psi_{(y,w)} := \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(-y\hat{p}+w\hat{x})}\Psi, \quad y, w \in \mathbb{R}^d \oplus \mathbb{R}^d,$$

sometimes called the family of coherent states associated with Ψ.

We have a continuous decomposition of identity

$$(2\pi\hbar)^{-d} \int |\Psi_{(y,w)})(\Psi_{(y,w)}| \mathrm{d}y \mathrm{d}w = \mathbb{1}.$$

Let b be a function on te phase space. We define its contravariant quantization by

$$\mathrm{Op}^{\mathrm{ct}}(b) := (2\pi\hbar)^{-d}\int |\Psi_{(x,p)})(\Psi_{(x,p)}| b(x,p)\mathrm{d}x\mathrm{d}p.$$

If $B = \mathrm{Op}^{\mathrm{ct}}(b)$, then b is called the contravariant symbol of B.

Let $b \geq 0$. Then $\mathrm{Op}^{\mathrm{ct}}(b) \geq 0$.

Let $B \in B(\mathcal{H})$. Then we define its covariant symbol by

$$b(x,p) := \left(\Psi_{(x,p)} | B \Psi_{(x,p)}\right).$$

B is then called the covariant quantization of b and is denoted by

$$\mathrm{Op}^{\mathrm{cv}}(b) = B.$$

Let $\mathrm{Op}^{\mathrm{cv}}(b) \geq 0$. Then $b \geq 0$.

Introduce complex coordinates

$$
\begin{aligned}
a_i &= (2\hbar)^{-1/2}(x_i + \mathrm{i}p_i), \\
a_i^* &= (2\hbar)^{-1/2}(x_i - \mathrm{i}p_i).
\end{aligned}
$$

and operators

$$
\begin{aligned}
\hat{a}_i &= (2\hbar)^{-1/2}(\hat{x}_i + \mathrm{i}\hat{p}_i), \\
\hat{a}_i^* &= (2\hbar)^{-1/2}(\hat{x}_i - \mathrm{i}\hat{p}_i).
\end{aligned}
$$

Consider a polynomial function on the phase space:

$$w(x,p) = \sum_{\alpha,\beta} w_{\alpha,\beta} x^\alpha p^\beta.$$

It is easy to describe the x, p and p, x quantizations of w in terms of ordering the positions and momenta:

$$\begin{aligned} \mathrm{Op}^{x,p}(w) &= \sum_{\alpha,\beta} w_{\alpha,\beta} \hat{x}^\alpha \hat{p}^\beta, \\ \mathrm{Op}^{p,x}(w) &= \sum_{\alpha,\beta} w_{\alpha,\beta} \hat{p}^\beta \hat{x}^\alpha. \end{aligned}$$

The Weyl quantization amounts to the full symmetrization of $\hat{x}_i$ and $\hat{p}_j$.

We can also rewrite the polynomial in terms of a_i, a_i^*. Thus we obtain

$$w(x,p) = \sum_{\gamma,\delta} \tilde{w}_{\gamma,\delta} a^{*\gamma} a^{\delta} =: \tilde{w}(a^*, a).$$

Then we can introduce the Wick quantization

$$\mathrm{Op}^{a^*,a}(w) = \sum_{\gamma,\delta} \tilde{w}_{\gamma,\delta} \hat{a}^{*\gamma} \hat{a}^{\delta}$$

and the anti-Wick quantization

$$\mathrm{Op}^{a,a^*}(w) = \sum_{\gamma,\delta} \tilde{w}_{\gamma,\delta} \hat{a}^{\delta} \hat{a}^{*\gamma}.$$

Consider the Gaussian vector $\Omega(x) = (\pi\hbar)^{-\frac{d}{4}} \mathrm{e}^{-\frac{1}{2\hbar}x^2}$. It is killed by the annihilation operators:

$$\hat{a}_i \Omega = 0.$$

Theorem

(1) The Wick quantization coincides with the covariant quantization for Gaussian coherent states.

(2) The anti-Wick quantization coincides with the contravariant quantization for Gaussian coherent states.

For Gaussian states one uses several alternative names of the covariant and contravariant symbol of an operator.

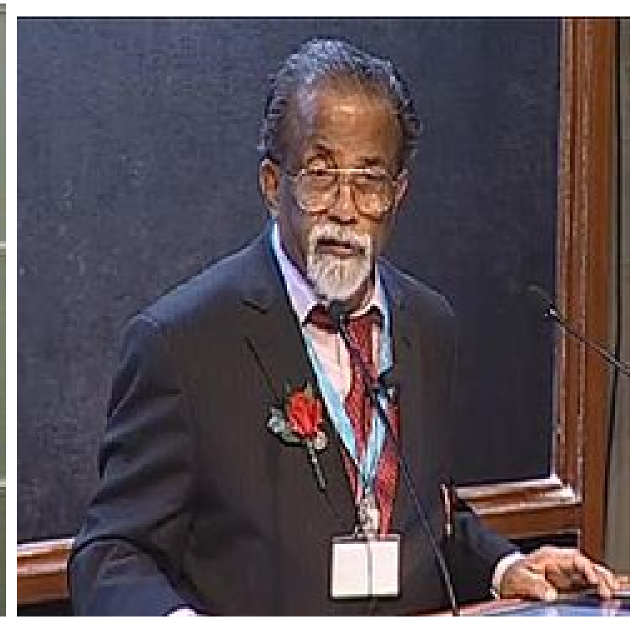
For contravariant symbol: anti-Wick symbol, Glauber-Sudarshan function, P-function.

For covariant symbol: Wick symbol, Husimi or Husimi-Kano function, Q-function.

We will use the terms Wick/anti-Wick quantization/symbol.

Gian-Carlo Wick

Roy J. Glauber



George Sudarshan

5 most natural quantizations in the Berezin diagram:

$$
\begin{array}{ccccc}
 & & \text{anti-Wick quantization} & & \\
 & & \Big\downarrow \mathrm{e}^{-\frac{\hbar}{4}(D_x^2+D_p^2)} & & \\
p,x \text{ quantization} & \overset{\mathrm{e}^{\frac{\mathrm{i}\hbar}{2}D_x\cdot D_p}}{\longrightarrow} & \text{Weyl-Wigner quantization} & \overset{\mathrm{e}^{\frac{\mathrm{i}\hbar}{2}D_x\cdot D_p}}{\longrightarrow} & x,p \text{ quantization} \\
 & & \Big\downarrow \mathrm{e}^{-\frac{\hbar}{4}(D_x^2+D_p^2)} & & \\
 & & \text{Wick quantization} & &
\end{array}
$$

The x, p- and p, x quantizations are invariant wrt the group $GL(\mathbb{R}^d)$ of linear transformations of the configuration space.

The Wick and anti-Wick quantizations are invariant wrt the unitary group $U(\mathbb{C}^d)$, generated by all harmonic oscillators whose ground state is the given Gaussian state.

The Weyl-Wigner quantization is invariant with respect to the symplectic group $Sp(\mathbb{R}^{2d})$.

For $\mathrm{Op}(b)\mathrm{Op}(c) = \mathrm{Op}(d)$ we have

$$d(x,p) = \mathrm{e}^{\frac{\mathrm{i}}{2}\hbar(D_{p_1}D_{x_2} - D_{x_1}D_{p_2})} b(x_1,p_1)c(x_2,p_2)\Big|_{\substack{x := x_1 = x_2, \\ p := p_1 = p_2.}},$$

where $D_y := \frac{1}{\mathrm{i}}\partial_y$. Often one denotes d by $b * c$. It is called the star or the Moyal product of b and c.

Consequences:

$$\frac{1}{2}\big(\mathrm{Op}(b)\mathrm{Op}(c)+\mathrm{Op}(c)\mathrm{Op}(b)\big) \;=\; \mathrm{Op}(bc)+O(\hbar^2),$$

$$[\mathrm{Op}(b),\mathrm{Op}(c)] \;=\; \mathrm{i}\hbar\mathrm{Op}(\{b,c\})+O(\hbar^3),$$

if $\mathrm{supp} b\cap\mathrm{supp} c=\emptyset,$ then $\mathrm{Op}(b)\mathrm{Op}(c) \;=\; O(\hbar^\infty).$

Let h be a nice function. Let $x(t), p(t)$ solve the Hamilton equations with the Hamiltonian h and the initial conditions $x(0), p(0)$. Then $r_t(x(0), p(0)) = \big(x(t), p(t)\big)$ defines a symplectic transformation. Formally,

$$\mathrm{e}^{\frac{-\mathrm{i}t}{\hbar}\mathrm{Op}(h)}\mathrm{Op}(b)\mathrm{e}^{\frac{\mathrm{i}t}{\hbar}\mathrm{Op}(h)} = \mathrm{Op}(b \circ r_t) + O(\hbar^2).$$

Under various assumptions this asymptotics can be made rigorous, and then it is called the Egorov Theorem.

If h is a quadratic polynomial, the transformation r_t is linear and there is no error term in the Egorov Theorem. The operators $\mathrm{e}^{\frac{\mathrm{i}t}{\hbar}\mathrm{Op}(h)}$ generate a group, which is the double covering of the symplectic group called the metaplectic group.

If $b, c \in L^2(\mathbb{R}^{2d})$, then (rigorously)

$$\mathrm{Tr}\mathrm{Op}(b)^*\mathrm{Op}(c) = (2\pi\hbar)^{-d} \int \overline{b(x,p)}c(x,p)\mathrm{d}x\mathrm{d}p.$$

Setting $b = 1$ we obtain (heuristically)

$$\mathrm{Tr}\mathrm{Op}(c) = (2\pi\hbar)^{-d} \int c(x,p)\mathrm{d}x\mathrm{d}p.$$

Formally, $\mathrm{Op}(b)^n = \mathrm{Op}(b^n) + O(\hbar^2)$. Hence for polynomial functions

$$f\big(\mathrm{Op}(b)\big) = \mathrm{Op}\big(f \circ b + O(\hbar^2)\big).$$

One can expect this to be true for a larger class of nice functions.

Consequently,

$$\begin{aligned} \mathrm{Tr} f\big(\mathrm{Op}(b)\big) &= \mathrm{Tr}\mathrm{Op}\big(f \circ b + O(\hbar^2)\big) \\ &= (2\pi\hbar)^{-d} \int f\big(b(x,p)\big) \mathrm{d}x \mathrm{d}p + O(\hbar^{-d+2}). \end{aligned}$$

For a bounded from below self-adjoint operator H set

$$N_\mu(H) := \mathrm{Tr}\mathbb{1}_{]-\infty,\mu]}(H),$$

which is the number of eigenvalues $\leq \mu$ of H counting multiplicity.

Then setting $f = \mathbb{1}_{]-\infty,\mu]}$, we obtain

$$N_\mu\big(\mathrm{Op}(h)\big) \;=\; (2\pi\hbar)^{-d}\int_{h(x,p)\leq\mu} \mathrm{d}x\mathrm{d}p + O(\hbar^{-d+2}).$$

In practice the error term $O(\hbar^{-d+2})$ may be too optimistic and one gets something worse (but hopefully at least $o(\hbar^{-d})$).

For example, if $V - \mu > 0$ outside a compact set, then

$$\begin{aligned} & N_\mu(-\hbar^2\Delta + V(x)) \\ \simeq \; & (2\pi\hbar)^{-d} c_d \int_{V(x)\leq\mu} |V(x) - \mu|_-^{\frac{d}{2}} \mathrm{d}x + o(\hbar^{-d}), \end{aligned}$$

which is often called the Weyl asymptotics.

Aspects of quantization.

Fundamental formalism

- used to define a quantum theory from a classical theory;
- underlying the emergence of classical physics from quantum physics.

Technical parametrization

- of operators used to prove theorems about PDE's;
- of observables in quantum optics and signal processing.

Elements of quantization should belong to standard curriculum!

Example: standard courses at

FACULTY OF PHYSICS, UNIVERSITY OF WARSAW.

Quantum Mechanics 1. (nonrelativistic theory);

Quantum Mechanics $1\frac{1}{2}$. (quantization, quantum information);

Quantum Mechanics 2A (relativistic theory);

Quantum Mechanics 2B (many body theory);

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