Algebraic methods in Quantum Physics

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1 Observables in quantum systems

1.1 States and observables

Let us describe basic framework of quantum mechanics. To avoid technical complications, in the first part of this section we will assume that the Hilbert space $\mathcal{H}$ describing a quantum system is finite dimensional, so that it can be identified with $\mathbb{C}^N$, for some N.

In basic courses on Quantum Mechanics we learn that a quantum state is described by a *density matrix* ρ and a *yes/no experiment* by an *orthogonal projection* P. The probability of the affirmative outcome of such an experiment equals

$$\mathrm{Tr}(\rho P).$$

Two orthogonal projections P_1 and P_2 are simultaneously measurable iff they commute.

We say that a family of orthogonal projections $P_1, \dots, P_n$ is an *orthogonal partition of unity on* $\mathcal{H}$ iff

$$\sum_{i=1}^{n} P_i = \mathbb{1}, \quad P_i P_j = \delta_{ij} P_j, \quad i, j = 1, \dots n.$$

Clearly, all elements of an orthogonal partition of unity commute with one another. Therefore, in principle, one can design an experiment that measures simultaneously all of them.

If $P_1, \dots, P_n$ is an orthogonal partition of unity, then setting $\mathcal{H}_i := \mathrm{Ran} P_i$, $i = 1, \dots, n$, we obtain an *orthogonal direct sum decomposition* $\mathcal{H} = \overset{n}{\underset{i=1}{\oplus}} \mathcal{H}_i$.

Thus specifying an ortogonal partition of unity is equivalnt to specifying an orthogonal direct sum decomposition.

Let $a_1, \ldots, a_n$ be a sequence of distinct real numbers, which we interpret as the outcomes of the experiment. We introduce a *self-adjoint operator* A by

$$A := \sum_{i=1}^{n} a_i P_i. \tag{1.1}$$

Clearly, the average outcome of the experiment is

$$\sum_{i=1}^{n} a_i \mathrm{Tr}\rho P_i = \mathrm{Tr}\rho A. \tag{1.2}$$

We call (1.2) the *expectation value of the observable A in the state ρ.* Clearly, $P_i = 1\!\!1_{\{a_i\}}(A)$ are the spectral *spectral projections of A onto its eigenvalues.*

Conversely, to any self-adjoint operator we can associate an orthogonal partition of unity given by its *spectral projections of A*:

$$1\!\!1_{\{a\}}(A), \quad a \in \sigma(A). \tag{1.3}$$

By *measuring the observable A* we mean measuring the partition of unity (1.3).

1.2 Superselection sectors

Let us start with a simple example of $\mathcal{H} = \mathbb{C}^2$ with basis $|\uparrow), |\downarrow)$. Introduce the Pauli matrices:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{1.4}$$

Pure states can be parametrized by θ, ϕ:

$$|\theta, \phi) := \cos\theta|\uparrow) + \mathrm{e}^{\mathrm{i}\phi}\sin\theta|\downarrow). \tag{1.5}$$

Measuring the Pauli matrices we obtain

$$(\theta, \phi|\sigma_1|\theta, \phi) = \sin 2\theta \cos\phi, \tag{1.6}$$
$$(\theta, \phi|\sigma_2|\theta, \phi) = \sin 2\theta \sin\phi, \tag{1.7}$$
$$(\theta, \phi|\sigma_3|\theta, \phi) = \cos 2\theta. \tag{1.8}$$

Suppose now that we cannot measure ϕ. This means we cannot measure σ_1 and σ_2, and only σ_3. Thus the observables consist of $\mathrm{Span}(1\!\!1, \sigma_3)$.

Note also that on these observables the pure state $|\theta, \phi)$ yields the same measurement as the density matrix

$$\rho_\theta := \cos^2\theta|\uparrow)(\uparrow| + \sin^2\theta|\downarrow)(\downarrow|. \tag{1.9}$$

Let us generalize this to any finite dimension. In the previous subsection we assumed that all orthogonal projections on $\mathcal{H}$, hence all self-adjoint operators on $\mathcal{H}$, correspond to possible experiments. We say that all self-adjoint elements of $B(\mathcal{H})$ are *observable*.

Sometimes this is not the case. We are going to describe several situations where only a part of self-adjoint operators are observable.

It may happen that the Hilbert space $\mathcal{H}$ has a distinguished direct sum decomposition

$$\mathcal{H} = \overset{n}{\underset{i=1}{\oplus}} \mathcal{H}_n \tag{1.10}$$

such that only self-adjoint operators that preserve each subspace $\mathcal{H}_i$ are measurable. We say then that $\mathcal{H}_i$, $i = 1, \dots, n$, are *superselection sectors*.

Let Q_i denote the orthogonal projection onto $\mathcal{H}_i$. Then linear combinations of Q_i can be measured simultaneously with all other observables. We say that they are *classical observables*.

If we choose an o.n. basis of $\mathcal{H}$ compatible with (1.10), then only block diagonal self-adjoint matrices are observable. States are also described by block diagonal matrices.

Superselection sectors arise typically when we have a strictly conserved quantity, this means a self-adjoint operator Q that commutes with all possible dynamics. For instance, the *total charge* of the system usually determines a superselection sector. Another example of a superselection sector is the *fermionic parity*: states of an *even* and *odd number of fermions* form two superselection sectors.

1.3 Composite quantum systems

Suppose that two quantum systems are described by Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$. Then the *composite system* is described by the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$. Observables of the first system are described by self-adjoint elemens of $B(\mathcal{H}_1) \otimes \mathbb{1}_{\mathcal{H}_2}$, whereas observables of the second system are described by self-adjoint elements of $\mathbb{1}_{\mathcal{H}_1} \otimes B(\mathcal{H}_2)$. Note that they commute, so that one can simultaneously measure them. From the point of view of the first system only self-adjoint elements of $B(\mathcal{H}_1) \otimes \mathbb{1}_{\mathcal{H}_2}$ are observable. Again, we have a situation where not all self-adjoint elements of $B(\mathcal{H})$ are observable.

Let $\mathcal{H}_1 = \mathbb{C}^p$ with an o.n. basis $e_1, \dots, e_p$ and $\mathcal{H}_2 = \mathbb{C}^q$ with an o.n. basis $f_1, \dots, f_q$. Then $e_i \otimes f_j$ $i = 1, \dots, p$, $j = 1, \dots, q$ is an o.n. basis of $\mathcal{H}_1 \otimes \mathcal{H}_2$. Matrices in $B(\mathbb{C}^p) \otimes \mathbb{1}_{\mathbb{C}^q}$ have the form

$$\begin{bmatrix} A & 0 & & \\ 0 & A & & \\ & & & \\ & & & A \end{bmatrix}, \quad A \in B(\mathbb{C}^p),$$

and matrices in $\mathbb{1}_{\mathcal{H}_1} \otimes B(\mathcal{H}_2)$ have the form

$$\begin{bmatrix} b_{11}\mathbb{1} & b_{12}\mathbb{1} & & \\ b_{21}\mathbb{1} & b_{22}\mathbb{1} & & \\ & & & \\ & & & b_{qq}\mathbb{1} \end{bmatrix}, \quad [b_{ij}] \in B(\mathbb{C}^q),$$

According to what we described above, the most general structure of the set of observables in finite dimension is as follows. Consider the Hilbert space $\mathcal{H} = \mathbb{C}^N$, $N = \sum_{i=1}^{n} p_i q_i$,

$$\mathcal{H} = \mathop{\oplus}_{i=1}^{n} \mathbb{C}^{p_i} \otimes \mathbb{C}^{q_i},$$

and the set

$$\mathfrak{A} := \mathop{\oplus}_{i=1}^{n} B(\mathbb{C}^{p_i}) \otimes \mathbb{1}_{q_i}.$$

Note that $\mathfrak{A}$ is a vector space closed wrt the multiplication and the Hermitian conjugation. It is an example of what mathematicians call a $*$-*algebra* represented on a Hilbert space.

As discussed before, in the finite dimensional case, observables of a quantum system are described by the self-adjoint part of a certain $*$-subalgebra of $B(\mathcal{H})$.

2 Algebras

2.1 Algebras

Let $\mathbb{K}$ be a field. We will consider only $\mathbb{K} = \mathbb{C}$, and sometimes $\mathbb{K} = \mathbb{R}$. Let $\mathfrak{A}$ be a vector space over $\mathbb{K}$. We say that $\mathfrak{A}$ is an *algebra* if it is equipped with an operation

$$\mathfrak{A} \times \mathfrak{A} \ni (A, B) \mapsto AB \in \mathfrak{A}$$

satisfying

$$\begin{array}{ll} A(B+C) = AB + AC, & (B+C)A = BA + CA, \\ (\alpha\beta)(AB) = (\alpha A)(\beta B), & A, B, C \in \mathfrak{A}, \quad \alpha, \beta \in \mathbb{K}. \end{array}$$

If in addition

$$A(BC) = (AB)C,$$

we say that it is an *associative algebra*. In practice by an algebra we will usually mean an associative algebra.

$\mathfrak{B} \subset \mathfrak{A}$ is called a *subalgebra* if it is a linear subspace of $\mathfrak{A}$ and $A, B \in \mathfrak{B} \Rightarrow AB \in \mathfrak{B}$. Clearly, a subalgebra is also an algebra.

Let $\mathcal{V}$ be a vector space. Clearly, the set of linear maps in $\mathcal{V}$, denoted by $L(\mathcal{V})$, is an algebra. A subalgebra of $L(\mathcal{V})$ is called a a *concrete algebra.*

$\mathfrak{A}$ is called a *commutative algebra* iff $A, B \in \mathfrak{A}$ implies $AB = BA$.

If $\mathfrak{A}_1$, $\mathfrak{A}_2$ are algebras, then their *direct sum* $\mathfrak{A}_1 \oplus \mathfrak{A}_2$ is also an algebra.

2.2 Identity and idempotents

An *identity* of an algebra $\mathfrak{A}$ is an element $\mathbb{1} \in \mathfrak{A}$ such that

$$A = \mathbb{1}A = A\mathbb{1}, \quad A \in \mathfrak{A}.$$

Any algebra has at most one identity. In fact, if $\mathbb{1}_1, \mathbb{1}_2$ are identities, then

$$\mathbb{1}_1 = \mathbb{1}_1 \mathbb{1}_2 = \mathbb{1}_2.$$

We say that $\mathfrak{A}$ is *unital* if it possesses an identity. In what follows, for $\lambda \in \mathbb{C}$ we will often simply write λ instead of $\lambda\mathbb{1}$.

We can always adjoin identity to an algebra $\mathfrak{A}$. We set $\mathfrak{A}_{\mathbb{1}} := \mathfrak{A} \oplus \mathbb{C}$ as a vector space with the multiplication

$$(A, \lambda)(B, \mu) := (AB + \lambda B + \mu A, \lambda\mu). \tag{2.1}$$

Then $\mathfrak{A}$ is embedded in $\mathfrak{A}_{\mathbb{1}}$ and $(0, 1)$ is the identity of $\mathfrak{A}_{\mathbb{1}}$.

Note that the above construction is usuful mostly if $\mathfrak{A}$ does not have its own identity. However, it can be always performed.

$P \in \mathfrak{A}$ is called an *idempotent* (or sometimes a *projection*) iff $P^2 = P$. $P\mathfrak{A}P$ is a subalgebra called a *reduced algebra*.

If $\mathfrak{A} \subset L(\mathcal{V})$ is a concrete algebra and $E \in \mathfrak{A}$ is its identity, then E is an idempotent in $L(\mathcal{V})$. We can then restrict $\mathfrak{A}$ to $\mathrm{Ran}E$.

An idempotent P is called *finite discrete* iff $P\mathfrak{A}P$ is finite dimensional. It is called *abelian* iff $P\mathfrak{A}P$ is commutative.

2.3 Commutant

Fix an algebra $\mathfrak{A}$. Let $\mathfrak{B}$ be a subset of $\mathfrak{A}$. Consider the family $\{\mathfrak{A}_j, \mid j \in J\}$ of all subalgebras of $\mathfrak{A}$ containing $\mathfrak{B}$. This set is non-empty, because $\mathfrak{A}$ is one of its elements. Then

$$\mathrm{Alg}(\mathfrak{B}) := \bigcap_{j \in J} \mathfrak{A}_j \tag{2.2}$$

is the smallest subalgebra of $\mathfrak{A}$ containing $\mathfrak{B}$. $\mathrm{Alg}(\mathfrak{B})$ will be called the *algebra generated by* $\mathfrak{B}$.

For instance, if $A \in \mathfrak{A}$, then $\mathrm{Alg}\{A\}$ is spanned by

$$A, A^2, A^3, \dots. \tag{2.3}$$

More generally, $\mathrm{Alg}\{A_1, \dots, A_n\}$ is spanned by monomials

$$A_{i_1} \cdots A_{i_k}, \quad i_1, \dots, i_k \in \{1, \dots, n\}. \tag{2.4}$$

Example 2.1. *Consider the algebra* $B(\mathbb{C}^2)$ *(of 2x2 matrices). Then* $\mathrm{Alg}(\sigma_3)$ *is the algebra of diagonal matrices and* $\mathrm{Alg}(\sigma_3, \sigma_1) = B(\mathbb{C}^2)$.

The *relative commutant of* $\mathfrak{B}$ *in* $\mathfrak{A}$ is defined as

$$\mathfrak{B}' \cap \mathfrak{A} := \{A \in \mathfrak{A} \ : \ AB = BA, \ B \in \mathfrak{B}\}$$

If there is no risk of confusion (it is clear which $\mathfrak{A}$ we have in mind), we will write $\mathfrak{B}'$ instead of $\mathfrak{B}' \cap \mathfrak{A}$ and call it the *commutant*. (Typically it is clear from the context that $\mathfrak{A} = B(\mathcal{H})$).

Theorem 2.2. (1) *A commutant is always a subalgebra containing the identity of* $\mathfrak{A}$.

(2) $\mathfrak{B}' = \mathrm{Alg}(\mathfrak{B})'$.

(3) $\mathfrak{B}'' \supset \mathrm{Alg}(\mathfrak{B})$.

(4) $\mathfrak{B}' = \mathfrak{B}''' = \dots$.

(5) $\mathfrak{B} \subset \mathfrak{B}'' = \mathfrak{B}'''' = \dots$.

Proof. The following inclusions are easy:

$$\mathfrak{B}_1 \subset \mathfrak{B}_2 \quad \Rightarrow \quad \mathfrak{B}_1' \supset \mathfrak{B}_2', \tag{2.5}$$

$$\mathfrak{B} \subset \mathfrak{B}''. \tag{2.6}$$

(2.5) together with (2.6) imply $\mathfrak{B}' \supset \mathfrak{B}^{'''}$. But (2.6) applied to $\mathfrak{B}'$ yields $\mathfrak{B}' \subset \mathfrak{B}^{'''}$. Thus $\mathfrak{B}' = \mathfrak{B}'''$. Now (4) and (5) follow. □

The *center* of an algebra $\mathfrak{B}$ equals

$$\mathfrak{Z}(\mathfrak{B}) = \{A \in \mathfrak{B} \ : \ AB = BA, \ B \in \mathfrak{B}\}.$$

Clearly, $\mathfrak{Z}(\mathfrak{B}) = \mathfrak{B} \cap \mathfrak{B}'$.

If $\mathfrak{A}$ is an algebra and $P \in \mathfrak{Z}(A)$ is an idempotent, then clearly $P\mathfrak{A} = P\mathfrak{A}P$ is a subalgebra. $\mathfrak{A}$ is naturally isomorphic to $P\mathfrak{A} \oplus (1-P)\mathfrak{A}$.

2.4 Homomorphisms

Let $\mathfrak{A}$, $\mathfrak{B}$ be algebras. A map $\phi : \mathfrak{A} \to \mathfrak{B}$ is called a *homomorphism* if it is linear and preserves the multiplication, that means it satisfies

(1) $\phi(\lambda A) = \lambda\phi(A)$;

(2) $\phi(A+B) = \phi(A) + \phi(B)$;

(3) $\phi(AB) = \phi(A)\phi(B)$.

If $\mathcal{V}$ is a vector space, then a homomorphism of $\mathfrak{A}$ into $L(\mathcal{V})$ is called a *representation* of $\mathfrak{A}$ in $\mathcal{V}$.

If $\mathfrak{A}$ is a unital algebra and $\phi : \mathfrak{A} \to \mathfrak{B}$ is a homomorphism, then $\phi(\mathbb{1})$ is an idempotent in $\mathfrak{B}$. ϕ is called *unital* iff

$$\phi(\mathbb{1}) = \mathbb{1}.$$

2.5 Ideals

$\mathfrak{B}$ is a *left ideal* of an algebra $\mathfrak{A}$ iff it is a linear subspace of $\mathfrak{A}$ and $A \in \mathfrak{A}$, $B \in \mathfrak{B} \Rightarrow AB \in \mathfrak{B}$. Similarly we define a *right ideal*.

If $A \in \mathfrak{A}$, then $\mathfrak{A}A$ is a left ideal.

$\mathfrak{B}$ is called a *two-sided ideal* if it is a left and right ideal. In what follows we will write an *ideal* instead of a two-sided ideal.

We say that an ideal $\mathfrak{I}$ is *proper* iff $\mathfrak{I} \neq \mathfrak{A}$. We say that an ideal $\mathfrak{I}$ is *nontrivial* iff $\mathfrak{I} \neq \mathfrak{A}$ and $\mathfrak{I} \neq \{0\}$.

Theorem 2.3. *The zero set (kernel) of a homomorphism is an ideal. If* $\mathfrak{I}$ *is an ideal of* $\mathfrak{A}$*, then* $\mathfrak{A}/\mathfrak{I}$ *has a natural structure of an algebra. The map*

$$\mathfrak{A} \ni A \mapsto A + \mathfrak{I} \in \mathfrak{A}/\mathfrak{I}$$

is a homomorphism whose kernel equals $\mathfrak{I}$*.*

$\mathfrak{I}$ is a maximal ideal if it is a proper ideal such that if $\mathfrak{K}$ is a proper ideal containing $\mathfrak{I}$, then $\mathfrak{I} = \mathfrak{K}$. Let $\mathrm{I}(\mathfrak{A})$, $\mathrm{MI}(\mathfrak{A})$ and $\mathrm{MI}_1(\mathfrak{A})$ denote the set of ideals, maximal ideals and ideals of codimesion 1 in $\mathfrak{A}$. Clearly,

$$\mathrm{MI}_1(\mathfrak{A}) \subset \mathrm{MI}(\mathfrak{A}) \subset \mathrm{I}(\mathfrak{A}).$$

Theorem 2.4. *If* $\mathfrak{A}$ *is unital and* $\mathfrak{I} \subset \mathfrak{A}$ *is a proper ideal, then there exists a maximal ideal containing* $\mathfrak{I}$*.*

Proof. Let $\{\mathfrak{I}_j \mid j \in J\}$ be the family of proper ideals containing $\mathfrak{I}$. $\mathbb{1} \notin \mathfrak{I}_j$ for all of them. By Kuratowski-Zorn lemma this family possesses maximal elements. □

Theorem 2.5. *Let* $\mathfrak{A}$ *be a commutative unital algebra. Let* $A \in \mathfrak{A}$ *be non-invertible. Then*

(1) $\mathfrak{I} := \{AB \ : \ B \in \mathfrak{A}\}$ *is a proper ideal;*

(2) *There exists a maximal ideal containing* A*;*

Proof. Clearly, $\mathfrak{I}$ is an ideal and $\mathbb{1} \notin \mathfrak{I}$. This shows (1). (2) follows from Theorem 2.4. □

We say that an algebra is simple if it has no nontrivial ideals.

Theorem 2.6. *Let* $\mathfrak{A}$ *be an algebra with a maximal ideal* $\mathfrak{I}$*. Then* $\mathfrak{A}/\mathfrak{I}$ *is simple.*

Theorem 2.7. *Let* $\pi : \mathfrak{A} \to \mathfrak{B}$ *be a homomorphism. Then*

(1) *If* $\mathfrak{I}$ *is an ideal in* $\mathfrak{B}$*, then* $\pi^{-1}(\mathfrak{I})$ *is an ideal in* $\mathfrak{A}$ *containing* $\mathrm{Ker}\pi$*. Thus we obtain a map*

$$\mathrm{I}(\mathfrak{B}) \ni \mathfrak{I} \mapsto \pi^{-1}(\mathfrak{I}) \in \{\mathfrak{I} \in \mathrm{I}(\mathfrak{A}) \ : \ \mathrm{Ker}\pi \subset \mathfrak{I}\}. \tag{2.7}$$

(2) *If* π *is surjective, then (2.7) is bijective.*

(3) *(2.7) maps* $\mathrm{MI}(\mathfrak{B})$ *into* $\mathrm{MI}(\mathfrak{A})$*.*

(4) *(2.7) maps* $\mathrm{MI}_1(\mathfrak{B})$ *into* $\mathrm{MI}_1(\mathfrak{A})$*.*

2.6 Left regular representation

The so-called *left regular representation*

$$\mathfrak{A} \ni A \mapsto \lambda(A) \in L(\mathfrak{A})$$

is defined by

$$\lambda(A)B := AB, \quad A, B \in \mathfrak{A}.$$

If $\mathfrak{A}$ is unital, then λ is injective. If $\mathfrak{A}$ is not unital, then λ can be extended to a representation

$$\mathfrak{A} \ni A \mapsto \lambda_{\mathbb{1}}(A) \in L(\mathfrak{A}_{\mathbb{1}})$$

in the obvious way, which is injective.

In any case, we see that every algebra is isomorphic to a concrete algebra.

2.7 Banach algebras

Recall that if $\mathcal{V}$ is a Banach space with norm $\|\cdot\|$, and A is a linear operator on $\mathcal{V}$, then one defines

$$\|A\| := \sup\{\|Av\| \mid v \in \mathcal{V}, \quad \|v\| = 1\} \tag{2.8}$$

If $\|A\| < \infty$, we say that A is bounded. The set of bounded operators on $\mathcal{V}$ is denoted $B(\mathcal{V})$. It is a Banach space with the norm $\|\cdot\|$ satisfying

$$\|AB\| \leq \|A\|\|B\|. \tag{2.9}$$

This motivates the following definition. We say that $\mathfrak{A}$ is a *Banach algebra* if it is an algebra over $\mathbb{C}$ or $\mathbb{R}$ equipped with a norm $\|\cdot\|$ such that $\mathfrak{A}$ is complete in this norm (in other words, $(\mathfrak{A}, \|\cdot\|)$ is a Banach space) and

$$\|AB\| \leq \|A\|\|B\|, \quad A, B \in \mathfrak{A}. \tag{2.10}$$

If $\mathcal{V}$ is a Banach space, then $B(\mathcal{V})$ equipped with the operator norm is a Banach algebra. More generally, every closed subalgebra of $B(\mathcal{V})$ is a Banach algebra.

If $\mathfrak{A}$ is a Banach algebra and $\mathfrak{C} \subset \mathfrak{A}$, then $\mathrm{Ban}(\mathfrak{C})$ denotes the smallest Banach algebra generated by $\mathcal{C}$.

2.8 Invertible elements

Let $\mathfrak{A}$ be an algebra. $A \in \mathfrak{A}$ is *left invertible* in $\mathfrak{A}$ iff there exists an element $B \in \mathfrak{A}$, called a *left inverse* of A, such that $BA = 1$. It is called *right invertible* iff there exists $C \in \mathfrak{A}$ such that $AC = 1$.

Theorem 2.8. *If $\mathfrak{I} \subset \mathfrak{A}$ is a proper left, resp. right ideal, then no elements of $\mathfrak{I}$ are left, resp. right invertible.*

Theorem 2.9. *Let $A \in \mathfrak{A}$. TFAE:*

(1) *A is left and right invertible.*

(2) *There exists a unique* $B \in \mathfrak{A}$ *such that* $AB = BA = \mathbb{1}$

Proof. Let B, C be a left and right inverse of A. Then

$$B = B\mathbb{1} = BAC = \mathbb{1}C = C.$$

□

If the above conditions are satisfied, then we say that A is *invertible*, (in $\mathfrak{A}$) and the element B, called the *inverse* of A, is denoted A^{-1}

Theorem 2.10. *1. If A is invertible and B is a left or right inverse of A, then $B = A^{-1}$.*

2. If A, B are invertible, then

$$(AB)^{-1} = B^{-1}A^{-1}, \quad A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}.$$

Theorem 2.11. *Let $\mathfrak{A}$ be a Banach algebra and $A, B \in \mathfrak{A}$ be invertible such that*

$$\|BA^{-1}\| < 1.$$

Then $A + B$ is invertible and

$$(A + B)^{-1} = \sum_{j=0}^{\infty}(-1)^j A^{-1}(BA^{-1})^j.$$

Moreover,

$$\|(A + B)^{-1}\| \leq \|A^{-1}\|(1 - \|BA^{-1}\|)^{-1},$$

$$\|A^{-1} - (A + B)^{-1}\| \leq \|A^{-1}BA^{-1}\|(1 - \|BA^{-1}\|)^{-1}.$$

In particular, invertible elements form an open subset of $\mathfrak{A}$ on which the inverse is a continuous function.

2.9 Spectrum

We assume that $\mathbb{K} = \mathbb{C}$. Let $\mathfrak{A}$ be a unital algebra. Let $A \in \mathfrak{A}$. We define the resolvent set of A as

$$\rho(A) := \{z \in \mathbb{C} \ : \ z\mathbb{1} - A \ \text{ is invertible } \}.$$

We define the spectrum of A as $\sigma(A) := \mathbb{C}\backslash\rho(A)$. (Or $\rho_{\mathfrak{A}}(A)$, resp. $\sigma_{\mathfrak{A}}(A)$ if we want to stress the dependence on the algebra).

Theorem 2.12. *Let $\mathfrak{A}$ be a unital Banach algebra and $A \in \mathfrak{A}$. Then*

(1) *If $\|(\lambda\mathbb{1} - A)^{-1}\| = c$, then $\{z : \ |z - \lambda| < c^{-1}\} \subset \rho(A)$.*

(2) $\|(z\mathbb{1} - A)^{-1}\| \geq (\mathrm{dist}(z, \sigma(A)))^{-1}$.

(3) $\{|z| > \|A\|\}$ *is contained in* ρA.

(4) *$\sigma(A)$ is a compact subset of $\mathbb{C}$.*

(5) *$(z\mathbb{1} - A)^{-1}$ is analytic on $\rho(A)$.*

(6) *$(z\mathbb{1} - A)^{-1}$ cannot be analytically extended beyond $\rho(A)$.*

(7) *$\sigma(A) \neq \emptyset$*

Proof. (1) For $|z - \lambda| < c^{-1}$, we have $\|(z - \lambda)(\lambda\mathbb{1} - A)^{-1}\| = |z - \lambda|c < 1$ Hence we can apply Theorem 2.11. This implies (2)

(3) We check that $\sum_{n=0}^{\infty} z^{-n-1}A^n$ is convergent for $|z| > \|A\|$ and equals $(z\mathbb{1} - A)^{-1}$.

(4) follows from (1) and (3).

(5) We check that the resolvent is differentiable in the complex sense:

$$h^{-1}\big((z + h - A)^{-1} - (z - A)^{-1}\big) = -(z + h - A)^{-1}(z - A)^{-1} \to -(z - A)^{-2}.$$

(6) follows from (2).

(7) $(z\mathbb{1} - A)^{-1}$ is an analytic function tending to zero at infinity. Hence it cannot be analytic everywhere, unless it is zero, which is impossible. □

Theorem 2.13. *Let $\mathfrak{B}$ be a closed subalgebra of a Banach algebra $\mathfrak{A}$ and $\mathbb{1}, A \in \mathfrak{B}$.*

(1) *$\rho_{\mathfrak{B}}(A)$ is an open and closed subset of $\rho_{\mathfrak{A}}(A)$ containing a neighborhood of ∞.*

(2) *The connected components of $\rho_{\mathfrak{A}}(A)$ and of $\rho_{\mathfrak{B}}(A)$ containing a neighborhood of infinity coincide.*

(3) *If $\rho_{\mathfrak{A}}(A)$ is connected, then $\rho_{\mathfrak{A}}(A) = \rho_{\mathfrak{B}}(A)$.*

Proof. $\rho_{\mathfrak{B}}(A)$ is open in $\mathbb{C}$. Hence also in $\rho_{\mathfrak{A}}(A)$.

Let $z_0 \in \rho_{\mathfrak{A}}(A)$ and $z_n \in \rho_{\mathfrak{B}}(A)$, $z_n \to z_0$. Then $(z_n - A)^{-1} \to (z_0 - A)^{-1}$ in $\mathfrak{A}$, hence also in $\mathfrak{B}$. Therefore, $z_0 \in \rho_{\mathfrak{B}}(A)$. Hence $\rho_{\mathfrak{B}}(A)$ is closed in $\rho_{\mathfrak{A}}(A)$. This proves 1.

(2) and (3) follow immediately from (1). □

Theorem 2.14 (Gelfand-Mazur)**.** *Let $\mathfrak{A}$ be a unital Banach algebra such that all non-zero elements are invertible. Then $\mathfrak{A} = \mathbb{C}$.*

Proof. Let $A \in \mathfrak{A}$. We know that $\sigma(A) \neq \emptyset$. Hence, there exists $\lambda \in \sigma(A)$. Thus $\lambda\mathbb{1} - A$ is not invertible. Hence $\lambda\mathbb{1} - A = 0$. Hence $A = \lambda\mathbb{1}$. □

2.10 Spectral radius

Spectral radius of $A \in \mathfrak{A}$ is defined as

$$\mathrm{sr}A := \sup_{\lambda \in \sigma A} |\lambda|.$$

Lemma 2.15. *Let a sequence of reals* (c_n) *satisfy*

$$c_n + c_m \geq c_{n+m}.$$

Then

$$\lim_{n\to\infty} \frac{c_n}{n} = \inf \frac{c_n}{n}.$$

Proof. Fix $m \in \mathbb{N}$. Let $n = mq + r$, $r < m$. We have

$$c_n \leq qc_m + c_r.$$

So

$$\frac{c_n}{n} \leq \frac{qc_m}{n} + \frac{c_r}{n}.$$

Hence

$$\limsup_{n\to\infty} \frac{c_n}{n} \leq \frac{c_m}{m}.$$

Thus,

$$\limsup_{n\to\infty} \frac{c_n}{n} \leq \inf \frac{c_m}{m}.$$

□

Theorem 2.16. *Let* $\mathfrak{A}$ *be a Banach algebra and* $A \in \mathfrak{A}$. *Then*

$$\lim_{n\to\infty} \|A^n\|^{\frac{1}{n}}$$

exists and equals $\mathrm{sr}A$. *Besides,* $\mathrm{sr}A \leq \|A\|$.

Proof. Let

$$c_n := \log \|A^n\|.$$

Then

$$c_n + c_m \geq c_{n+m}$$

Hence there exists

$$\lim_{n\to\infty} \frac{c_n}{n}.$$

Consequently, there exists

$$r := \lim_{n\to\infty} \|A^n\|^{1/n}.$$

By the Cauchy criterion, the series

$$\sum_{n=0}^{\infty} A^n z^{-1-n}. \tag{2.11}$$

is absolutely convergent for $|z| > r$, and divergent for $|z| < r$. We easily check that (2.11) equals $(z - A)^{-1}$. □

2.11 Characters on an algebra

Let $\mathfrak{A}$ be an algebra. A nonzero homomorphism of $\mathfrak{A}$ into $\mathbb{C}$ is called a character. We define $\mathrm{Char}(\mathfrak{A})$ to be the set of characters of $\mathfrak{A}$. If $\phi \in \mathrm{Char}(\mathfrak{A})$, then $\mathrm{Ker}\phi$ is clearly an ideal of codimension 1. Thus we obtain a map

$$\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \mathrm{Ker}\phi \in \mathrm{MI}_1(\mathfrak{A}). \tag{2.12}$$

Theorem 2.17. (1) *If $\mathfrak{A}$ is unital and $\mathfrak{I}$ is an ideal of codimension 1, then there exists a unique character ϕ such that $\mathfrak{I} = \mathrm{Ker}\phi$. Therefore, (2.12) is then a bijection.*

(2) *For a general $\mathfrak{A}$, for any $\phi \in \mathrm{Char}(\mathfrak{A})$ there exists a unique extension of ϕ to a character $\phi_{\mathbb{1}}$ on $\mathfrak{A}_{\mathbb{1}}$. It is given by $\phi_{\mathbb{1}}(\lambda\mathbb{1} + A) = \lambda + \phi(A)$.*

(3) *There exists a unique $\phi_\infty \in \mathrm{Char}(\mathfrak{A}_{\mathbb{1}})$ such that $\mathrm{Ker}\phi_\infty = \mathfrak{A}$.*

Proof. (1) If $\mathfrak{A}$ is unital then every character maps $\mathbb{1}$ to 1. Hence, for $A \in \mathfrak{I}$ and $\lambda \in \mathbb{C}$, seting $\phi(A + \lambda\mathbb{1}) := \lambda$ we obtain the unique character with $\mathrm{Ker}\phi = \mathfrak{I}$. □

For any $A \in \mathfrak{A}$ let $\hat{A}$ be the function

$$\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \hat{A}(\phi) := \phi(A) \in \mathbb{C}. \tag{2.13}$$

$\mathrm{Char}(\mathfrak{A})$ is endowed with the weakest topology such that (2.13) is continuous for any $A \in \mathfrak{A}$. Thus a net (ϕ_α) in $\mathrm{Char}(\mathfrak{A})$ converges to $\phi \in \mathrm{Char}(\mathfrak{A})$ iff for any $A \in \mathfrak{A}$, $\phi_\alpha(A) \to \phi(A)$.

Theorem 2.18.

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C(\mathrm{Char}(\mathfrak{A})) \tag{2.14}$$

is a homomorphism. Moreover, the range of (2.14) separates points and for every element of $\mathrm{Char}(\mathfrak{A})$ there exists A such that $\hat{A}(\phi) \neq 0$. Thus $\mathrm{Char}(\mathfrak{A})$ is a Tikhonov space. Moreover, the map

$$\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \phi_{\mathbb{1}} \in \mathrm{Char}(\mathfrak{A}_{\mathbb{1}})\backslash\{\phi_\infty\}$$

is a homeomorphism.

Proof. Let $A, B \in \mathfrak{A}$, $\phi \in \mathrm{Char}(\mathfrak{A})$. Then

$$\hat{A}(\phi)\hat{B}(\phi) = \phi(A)\phi(B) = \phi(AB) = \widehat{AB}(\phi).$$

If $\phi \neq \psi$ are characters, then there exists $A \in \mathfrak{A}$ such that $\phi(A) \neq \psi(A)$, or $\hat{A}(\phi) \neq \hat{A}(\psi)$.

If ϕ is a character, then there exists $A \in \mathfrak{A}$ such that $\phi(A) \neq 0$, or $\hat{A}(\phi) \neq 0$. □

Theorem 2.19. *Let $\pi : \mathfrak{A} \to \mathfrak{B}$ be a homomorphism. Then*

$$\mathrm{Char}(\mathfrak{B}) \ni \phi \mapsto \pi^{\#}(\phi) \in \mathrm{Char}(\mathfrak{A}), \tag{2.15}$$

defined for $\psi \in \mathrm{Char}(\mathfrak{B})$ by $(\pi^{\#}\psi)(A) := \psi(\pi(A))$, is continuous.

Proof. Let (ψ_i) be a net in $\mathrm{Char}(\mathfrak{B})$ converging to $\psi \in \mathrm{Char}(\mathfrak{B})$. Let $A \in \mathfrak{A}$. Then

$$\pi^{\#}(\psi_i)(A) = \psi_i(\pi(A)) \to \psi(\pi(A)) = \pi^{\#}(\psi)(A).$$

Hence $\pi^{\#}(\psi_i) \to \pi^{\#}(\psi)$. □

2.12 Characters on a Banach algebra

Theorem 2.20. *Let $\mathfrak{A}$ be a unital Banach algebra.*

(1) *Let $\mathfrak{I}$ be a maximal ideal in $\mathfrak{A}$. Then $\mathfrak{I}$ is closed.*

(2) *Let ϕ be a character on $\mathfrak{A}$. Then it is continuous and $\|\phi\| = 1$.*

(3) $\mathrm{Char}(\mathfrak{A})$ *is a compact Hausdorff space.*

(4) *The Gelfand transform*

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C(\mathrm{Char}(A))$$

is a norm decreasing unital homomorphism of Banach algebras.

Proof. (1) Invertible elements do not belong to any proper ideal. But a neighborhood of 1 consists of invertible elements. Hence the closure of any proper ideal does not contain 1.

By the continuity of operations, the closure of an ideal is an ideal. Hence if $\mathfrak{I}$ is any proper ideal, then $\mathfrak{I}^{\mathrm{cl}}$ is also a proper ideal.

(2) $\mathrm{Ker}\phi$ is a maximal ideal. Hence it is closed. Hence ϕ is continuous.

Suppose that $\|\phi\| > 1$. Then for some $A \in \mathfrak{A}$, $\|A\| < 1$ we have $|\phi(A)| > 1$. Now $A^n \to 0$ and $|\phi(A^n)| = |\phi(A)|^n \to \infty$, which means that ϕ is not continuous.

(3) and (4) follow easily from (2). □

Theorem 2.21. *Let $\mathfrak{A}$ be a Banach algebra.*

(1) *Let ϕ be a character on $\mathfrak{A}$. Then it is continuous and $\|\phi\| \leq 1$.*

(2) $\mathrm{Char}(\mathfrak{A})$ *is a locally compact Hausdorff space.*

(3) *The Gelfand transform*

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C_\infty(\mathrm{Char}(A))$$

is a norm decreasing homomorphism of Banach algebras.

Theorem 2.22. *Let $\mathfrak{A}$ be a commutative unital Banach algebra. Then every maximal ideal in $\mathfrak{A}$ has codimension 1. Hence* $\mathrm{MI}_1(\mathfrak{A}) = \mathrm{MI}(\mathfrak{A})$.

Proof. Let $\mathfrak{I}$ be an ideal of $\mathfrak{A}$ of codimension > 1. Then $\mathfrak{A}/\mathfrak{I}$ s a commutative Banach algebra of dimension > 1. In particular, $\mathfrak{A}/\mathfrak{I}$ is not $\mathbb{C}$. By the Gelfand-Mazur theorem, that is Thm 2.14, $\mathfrak{A}/\mathfrak{I}$ contains non-invertible elements. Every such an element is contained in a proper ideal $\mathfrak{K}$.

Let $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$ be the canonical homomorphism. By theorem 3.1, $\pi^{-1}(\mathfrak{K})$ is a proper ideal of $\mathfrak{A}$ containing $\mathfrak{I}$. Hence $\mathfrak{I}$ is not maximal. □

Theorem 2.23. *Let $\mathfrak{A}$ be an algebra and $A \in \mathfrak{A}$. Then*

(1) $\sigma_{\mathfrak{A}}(A) \supset \{\phi(A) \ : \ \phi \in \mathrm{Char}(\mathfrak{A})\}$.

(2) $\mathrm{Char}(\mathfrak{A}) \ni \phi \mapsto \phi(A) \in \sigma_{\mathfrak{A}}(A)$ *is a continuous map.*

(3) *If in addition $\mathfrak{A}$ is a commutative unital Banach algebra, then*

$$\sigma_{\mathfrak{A}}(A) = \{\phi(A) \ : \ \phi \in \mathrm{Char}(\mathfrak{A})\},$$

and hence

$$\mathrm{sr}(A) = \sup\{|\hat{A}(\phi)| \ : \ \phi \in \mathrm{Char}(\mathfrak{A})\} = \|\hat{A}\|.$$

Proof. If $\mathfrak{A}$ is non-unital, then we adjoin the identity and extend all the characters to $\mathfrak{A}_{\mathbb{1}}$.

Let $\phi \in \mathrm{Char}(\mathfrak{A})$ and $\phi(A) = \lambda$. Then $\phi(A - \lambda\mathbb{1}) = 0$. Hence $A - \lambda\mathbb{1}$ belongs to a proper ideal. Hence it is not invertible. Hence $\lambda \in \sigma(A)$, which proves (1).

Let $z \in \sigma(A)$ and $\mathfrak{A}$ be a Banach commutative algebra. Then $z\mathbb{1} - A$ is not invertible. Hence, by Thm 2.5, there exists a maximal ideal containing $z\mathbb{1} - A$. Therefore, by Thm 2.22, this ideal has codimension 1. By Thm 2.17, there exists $\phi \in \mathrm{Char}(\mathfrak{A})$ that vanishes on this ideal. Thus it satisfies $\phi(z\mathbb{1} - A) = 0$. Hence $z = \phi(A) = \hat{A}(\phi)$. This proves (3). □

Theorem 2.24. *Let $\mathfrak{A}$ be a commutative unital Banach algebra. Let $A \in \mathfrak{A}$. The following conditions are equivalent:*

(1) *A belongs to the intersection of all maximal ideals;*

(2) *For all $\phi \in \mathrm{Char}(\mathfrak{A})$ we have $\phi(A) = 0$*

(3) $\hat{A} = 0$*;*

(4) $\mathrm{sr}(A) = 0$*;*

(5) $\limsup \|A^n\|^{1/n} = 0$.

The set of $A \in \mathfrak{A}$ satisfying the conditions of Theorem 2.24 is called the radical of $\mathfrak{A}$. It is a closed ideal of $\mathfrak{A}$.

2.13 Problems

Problem 2.25. *We say that $\mathfrak{A}$ is a* division algebra *if all its nonzero elements are invertible. Prove that all division algebras over $\mathbb{R}$ are isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (the quaternions).*

Problem 2.26. *Prove that $L(\mathbb{R}^n)$, $L(\mathbb{C}^n)$ and $L(\mathbb{H}^n)$ are simple algebras over $\mathbb{R}$.*

Problem 2.27. *Prove that the algebra of upper-triangular matrices is isomorphic to that of lower-triagular ones.*

Problem 2.28. *Describe all ideals in the algebra of upper triangular 2×2 matrices.*

3 C^*-algebras

3.1 Operators on Hilbert spaces

From now on we assume that $\mathbb{K} = \mathbb{C}$.

Let $\mathcal{H}$ be a Hilbert space with scalar product $(\cdot|\cdot)$. Bounded operators on $\mathcal{H}$ are equipped with the Hermitian conjugation

$$B(\mathcal{H}) \ni A \mapsto A^* \in B(\mathcal{H}). \tag{3.1}$$

It is defined by

$$(v|Aw) = (A^*v|w), \quad v, w \in \mathcal{H}. \tag{3.2}$$

(3.1) is an antilinear map satisfying

$$A^{**} = A, \quad (AB)^* = B^*A^*, \quad \|A^*A\| = \|A\|^2. \tag{3.3}$$

3.2 $*$-algebras

In what follows we will try to incorporate the Hermitian conjugation into the theory of algebras. We will introduce the concept of a C^*-algebra, which are special kinds of Banach algebras possessing the $*$-operation. Among C^*-algebras especially important are von Neumann algebras and their abstract versions, which go under the name of W^*-algebras.

Let us start, however, with a concept of a $*$-algebra that does not use a norm. Such an approach should suffice in finite dimension.

We say that an algebra $\mathfrak{A}$ is a *$*$-algebra* if it is equipped with an antilinear map $\mathfrak{A} \ni A \mapsto A^* \in \mathfrak{A}$ such that $(AB)^* = B^*A^*$, $A^{**} = A$ and

$$A \neq 0 \text{ implies } A^*A \neq 0. \tag{3.4}$$

Note that the condition (3.4) removes an "unwanted" trivial examples.

Let $\mathfrak{A}$ be a $*$-algebra. We say that a subset $\mathfrak{B}$ of $\mathfrak{A}$ is *$*$-invariant* (or *self-adjoint*) if $A \in \mathfrak{A}$ implies $A^* \in \mathfrak{A}$. Every $*$-invariant subalgebra of $\mathfrak{A}$ is a $*$-algebra.

Theorem 3.1. *If* $\mathbb{1} \in \mathfrak{A}$*, then* $\mathbb{1}^* = \mathbb{1}$.

If $\mathcal{H}$ is a Hilbert space, then $B(\mathcal{H})$ equipped with the hermitian conjugation is a $*$-algebra.

If $\mathfrak{A}$, $\mathfrak{B}$ are $*$-algebras, then a homomorphism $\pi : \mathfrak{A} \to \mathfrak{B}$ satisfying $\pi(A^*) = \pi(A)^*$ is called a *$*$-homomorphism*.

The following theorem is a version of the Wederburn-Artin Theorem:

Theorem 3.2. (1) *Every finite dimensional $*$-algebra $\mathfrak{A}$ is $*$-isomorphic to*

$$\mathop{\oplus}_{i=1}^{n} B(\mathbb{C}^{p_i}),$$

for some $p_1, \ldots, p_n$

(2) *If in addition $\mathfrak{A}$ is a subalgebra of $B(\mathbb{C}^N)$ and contains the identity on $\mathbb{C}^N$, then there exist $q_1, \dots, q_n$ with $N = \sum_{i=1}^{n} p_i q_i$, and a basis of $\mathbb{C}^N$ such that*

$$\mathfrak{A} = \mathop{\oplus}_{i=1}^{n} B(\mathbb{C}^{p_i}) \otimes 1\!\!1_{q_i}. \tag{3.5}$$

3.3 Von Neumann algebras

Recall from Subsect. 2.3 that if $\mathfrak{B} \subset B(\mathcal{H})$, then the *commutant* of $\mathfrak{B}$ is defined as

$$\mathfrak{B}' := \{A \in B(\mathcal{H}) \ : \ AB = BA, \ B \in \mathfrak{B}\}.$$

Theorem 3.3. *If $\mathfrak{B}$ is $*$-invariant, then so is $\mathfrak{B}'$.*

Proof. Let $A \in \mathfrak{B}'$. Then for any $B \in \mathfrak{B}$, we have $AB = BA$. Hence $B^*A^* = A^*B^*$. But $\mathfrak{B}$ is $*$-invariant. Hence $CA^* = A^*C$ for any $C \in \mathfrak{B}$. □.

We say that $\mathfrak{A} \subset B(\mathcal{H})$ is a *von Neumann algebra* if it is $*$-invariant and $\mathfrak{A} = \mathfrak{A}''$. Clearly, von Neumann algebras are $*$-algebras.

It is easy to see that all $*$-subalgebras of $B(\mathbb{C}^N)$ containing $1\!\!1_N$ are von Neumann algebras. Indeed, if $\mathfrak{A}$ is given by (3.5), then $\mathfrak{A}$ is obviously $*$-invariant and

$$\mathfrak{A}' = \mathop{\oplus}_{i=1}^{n} 1\!\!1_{p_i} \otimes B(\mathbb{C}^{q_i}).$$

So, $\mathfrak{A}'' = \mathfrak{A}$.

Theorem 3.4. *Let $\mathfrak{B}$ be a $*$-invariant subset of $B(\mathcal{H})$. Then $\mathfrak{B}''$ is the smallest von Neuman algebra containing $\mathfrak{B}$.*

Proof. By Thm 2.5, $\mathfrak{B}'' = \mathfrak{B}''''$ and is $*$-invariant, hence $\mathfrak{B}''$ is a von Neumann algebra.

If $\mathfrak{B} \subset \mathfrak{A}$, then clearly again using Thm 2.5, $\mathfrak{B}'' \subset \mathfrak{A}''$. But if $\mathfrak{A}$ is a von Neumann algebra, then $\mathfrak{A}'' = \mathfrak{A}$. So $\mathfrak{B}'' \subset \mathfrak{A}$. □

We will say that $\mathfrak{B}''$ is the von Neumann algebra generated by $\mathfrak{B}$.

3.4 Normal states on a von Neumann algebra

Let $\mathfrak{A}$ be a von Neumann algebra in $B(\mathcal{H})$. Let ρ be a positive operator on $\mathcal{H}$ with $\mathrm{Tr}\rho = 1$. In physics such operators are called *density matrices.* In the theory operator algebras, one introduces a linear map

$$\mathfrak{A} \ni A \mapsto \omega(A) := \mathrm{Tr}\rho A \in \mathbb{C}. \tag{3.6}$$

Note that ω is a linear functional on $\mathfrak{A}$ satisfying

$$A \geq 0 \quad \Rightarrow \quad \omega(A) \geq 0, \tag{3.7}$$
$$\omega(1\!\!1) = 1. \tag{3.8}$$

Such functionals will be called *normal states*.

Note that the density matrix is not uniquely defined for *and*a given functional ω.

Example 3.5. *Let $\mathcal{V}$ and $\mathcal{W}$ be Hilbert spaces with an o.n. bases $\{e_i \mid i \in I\}$, resp. $\{f_i \mid i \in I\}$. Set*

$$\rho := \sum_i \lambda_i |e_i)(e_i|, \tag{3.9}$$

$$\rho_1 := \sum_i \lambda_i |e_i \otimes f_1)(e_i \otimes f_1|, \tag{3.10}$$

$$\Omega := \sum \sqrt{\lambda_i} e_i \otimes f_i. \tag{3.11}$$

Then for $A \in B(\mathcal{V})$,

$$\mathrm{Tr} A\rho = \mathrm{Tr} A \otimes \mathbb{1}\, \rho_1 = (\Omega | A \otimes \mathbb{1}\, \Omega). \tag{3.12}$$

In particular, for the algebra $B(\mathcal{V})$ represented on $\mathcal{V} \otimes \mathcal{W}$, every normal state can be purified.

3.5 C^*-algebras

(3.3) motivates the following definition:

We say that a Banach algebra is a C^**-algebra* if it is equipped with an antilinear map $\mathfrak{A} \ni A \mapsto A^* \in \mathfrak{A}$ such that $(AB)^* = B^* A^*$, $A^{**} = A$ and

$$\|A^* A\| = \|A\|^2, \quad A \in \mathfrak{A}. \tag{3.13}$$

Note that in a C^*-algebra the condition (3.4) is always automatically satisfied, so every C^*-algebra is a $*$-algebra in the sense of the definition from Subsection 3.2.

Theorem 3.6. *In a C^*-algebra we have $\|A\| = \|A^*\|$*

Proof. $\|A\|^2 = \|A^* A\| \leq \|A^*\| \|A\|$, hence $\|A\| \leq \|A^*\|$. Replacing A with A^* we obtrain $\|A^*\| \leq \|A\|$. □

Theorem 3.7. *If $\mathbb{1} \in \mathfrak{A}$, then $\|\mathbb{1}\| = 1$.*

Proof. We know that $\mathbb{1} = \mathbb{1}^*$ by Thm 3.1. Hence $\|\mathbb{1}\|^2 = \|\mathbb{1}^* \mathbb{1}\| = \|\mathbb{1}\|$. □

If $\mathcal{H}$ is a Hilbert space, then every closed $*$-subalgebra of $B(\mathcal{H})$ is a C^*-algebra. They are called *concrete C^*-algebras.*

A concrete C^*-algebra is called nondegenerate if for $\Phi \in \mathcal{H}$, $A\Phi = 0$ for all $A \in \mathfrak{A}$ implies $\Phi = 0$.

If $\mathfrak{A}$ is not necessarily non-degenerate, and $\mathcal{H}_1 := \{\Phi \in \mathcal{H} \ : \ Ax = 0, \ A \in \mathfrak{A}\}$, then $\mathfrak{A}$ restricted to $\mathcal{H}_1^\perp$ is nondegenerate.

In particular, all von Neumann algebras are concrete nondegenerate unital C^*-algebras.

3.6 Examples of infinite dimensional $*$-algebras

Here are a few examples of infinite dimensional $*$-algebras:

(1) (i) Finite rank operators on a Hilbert space $\mathcal{H}$.
(ii) Compact operators on $\mathcal{H}$.
(iii) Bounded operators on $\mathcal{H}$, that is, $B(\mathcal{H})$.

(2) (i) Compactly supported multiplication operators on $l^2(\mathbb{N})$; algebra isomorphic to $c_{\mathrm{c}}(\mathbb{N})$.
(ii) Vanishing at infinity multiplication operators on $l^2(\mathbb{N})$; algebra isomorphic to $c_\infty(\mathbb{N})$.
(iii) Bounded multiplication operators on $l^2(\mathbb{N})$. This algebra is isomorphic to $l^\infty(\mathbb{N})$.

(3) (i) Multiplication operators by continuous compactly supported functions on $L^2(\mathbb{R})$, algebra isomorphic to $C_{\mathrm{c}}(\mathbb{R})$.
(ii) Multiplication operators by continuous vanishing at infinity functions on $L^2(\mathbb{R})$, algebra isomorphic to $C_\infty(\mathbb{R})$.
(iii) Bounded multiplication operators on $L^2(\mathbb{R})$. This algebra is isomorphic to $L^\infty(\mathbb{R})$.

(i)'s are $*$-algebras, but not C^*-algebras. (ii)'s are C^*-algebras but not von Neumann algebra. They are the closures of (i)'s. (iii)'s are von Neumann algebras–they are the double commutants of (ii)'s.

For instance, the von Neumann algebra generated by finite rank or compact operators is the whole $B(\mathcal{H})$.

Physically, if we know that self-adjoint operators $A_1, \dots, A_n$ are observables, then as the observable algebra it is natural to take

$$\mathfrak{A} = \{A_1, \dots, A_n\}''.$$

Observables are often described by *unbounded self-adjoint operators*. This is not a serious problem. What is relevant for quantum measurements are spectral projections, which are bounded. Thus by saying that an algebra $\mathfrak{A} \subset B(\mathcal{H})$ is generated by (possibly unbounded) self-adjoitn operators $A_1, \dots, A_n$ we will mean that it is generated by spectral projections of these operators (or, equivalently, by their bounded Borel function).

1. Consider the operators $\hat{\phi}_i$, $i = 1, 2, 3$ on $L^2(\mathbb{R}^3)$. They are self-adjoint and commute. They have simple joint spectrum. The von Neumann algebra generated by $\hat{\phi}_i$, $i = 1, 2, 3$ is equal to the operators of multiplication by functions in $L^\infty(\mathbb{R}^3)$.

2. Consider in addition the operators $\hat{\pi}_i := \mathrm{i}^{-1}\partial_{x_i}$, $i = 1, 2, 3$ on $L^2(\mathbb{R}^3)$. The von Neumann algebra generated by $\hat{\phi}_i$, $\hat{\pi}_i$, $i = 1, 2, 3$, coincides with $B(L^2(\mathbb{R}^3))$.

3.7 Special elements of a $*$-algebra

$A \in \mathfrak{A}$ is called normal if $AA^* = A^*A$. It is called self-adjoint if $A^* = A$. $\mathfrak{A}_{\rm sa}$ denotes the set of self-adjoint elements of $\mathfrak{A}$

$P \in \mathfrak{A}$ is called an orthoprojector if it is a self-adjoint idempotent. $\mathrm{Proj}(\mathfrak{A})$ denotes the set of projectors of $\mathfrak{A}$.

Theorem 3.8. *Let $P^* = P$ and $P^2 = P^3$. Then P is an orthoprojector.*

$U \in \mathfrak{A}$ is called a partial isometry iff U^*U is an orthoprojector. If this is the case, then UU^* is also an orthoprojector. U^*U is called the right support of U and UU^* is called the left support of U.

U is called an isometry if $U^*U = 1$.

U is called a unitary if $U^*U = UU^* = 1$. $U(\mathfrak{A})$ denotes the set of unitary elements of $\mathfrak{A}$.

U is called a partial isometry iff U^*U and UU^* are orthoprojectors.

We can actually weaken the above condition:

Theorem 3.9. *Let either U^*U or UU^* be an orthoprojector. Then U is a partial isometry.*

3.8 Spectrum of elements of C^*-algebras

Theorem 3.10. *Let $A \in \mathfrak{A}$ be normal. Then*

$$\mathrm{sr}(A) = \|A\|.$$

Proof.

$$\|A^2\|^2 = \|A^{2*}A^2\| = \|(A^*A)^2\| = \|A^*A\|^2 = \|A\|^4.$$

Thus $\|A^{2^n}\| = \|A\|^{2^n}$. Hence, using the formula for the spectral radius of A we get $\|A^{2^n}\|^{2^{-n}} = \|A\|$. $\square$

Theorem 3.11. (1) *Let $V \in \mathfrak{A}$ be isometric. Then $\sigma(V) \subset \{|z| \leq 1\}$.*

(2) *$U \in \mathfrak{A}$ is unitary $\Rightarrow$ U is normal and $\sigma(U) \subset \{z \ : \ |z| = 1\}$.*

(3) *$A \in \mathfrak{A}$ is self-adjoint $\Rightarrow$ A is normal and $\sigma(A) \subset \mathbb{R}$.*

Proof. (1) We have $\|V\|^2 = \|V^*V\| = \|1\| = 1$. Hence, $\sigma(V) \subset \{|z| \leq 1\}$.

(2) Clearly, U is normal.

U is an isometry, hence $\sigma(U) \subset \{|z| \leq 1\}$.

U^{-1} is also an isometry, hence $\sigma(U^{-1}) \subset \{|z| \leq 1\}$. This implies $\sigma(U) \subset \{|z| \geq 1\}$.

(3) For $|\lambda^{-1}| > \|A\|$, $1 + \mathrm{i}\lambda A$ is invertible. We check that $U := (1 - \mathrm{i}\lambda A)(1 + \mathrm{i}\lambda A)^{-1}$ is unitary. Hence, by (2).$\Rightarrow$, $\sigma(U) \subset \{|z| = 1\}$. By the spectral mapping theorem, $\sigma(A) \subset \mathbb{R}$. $\square$

Note that in (2) and (3) we can actually replace $\Rightarrow$ $\Leftrightarrow$, which will be proven later.

3.9 Invariance of spectrum in C^*-algebras

Lemma 3.12. *Let A be invertible in $\mathfrak{A}$. Then A^{-1} belongs to $C^*(1, A)$.*

Proof. First assume that A is self-adjoint. Then $\sigma_{\mathfrak{A}}(A) \subset \mathbb{R}$. Hence $\rho_{\mathfrak{A}}(A)$ is connected. But $\mathfrak{C} := C^*(\mathbb{1}, A) = \mathrm{Ban}(\mathbb{1}, A)$. Hence, by Theorem 2.13,

$$\rho_{\mathfrak{C}}(A) = \rho_{\mathfrak{A}}(A) \tag{3.14}$$

A is invertible iff $0 \in \rho_{\mathfrak{A}}(A)$. By (3.14), this means that $0 \in \rho_{\mathfrak{C}}(A)$ and hence $A^{-1} \in \mathfrak{C}$.

Next assume that A be an arbitrary invertible element of $\mathfrak{A}$. Clearly, A^* is invertible in $\mathfrak{A}$ and $(A^*)^{-1} = (A^{-1})^*$. Likewise, A^*A is invertible in $\mathfrak{A}$ and $(A^*A)^{-1} = (A^*)^{-1}A^{-1}$. But A^*A is self-adjoint and hence $(A^*A)^{-1} \in C^*(\mathbb{1}, A^*A) \subset C^*(\mathbb{1}, A)$. Next we check that $A^{-1} = (A^*A)^{-1}A^*$. □

Theorem 3.13. *Let $\mathfrak{B} \subset \mathfrak{A}$ be C^*-algebras and $A, \mathbb{1} \in \mathfrak{B}$. Then $\sigma_{\mathfrak{B}}(A) = \sigma_{\mathfrak{A}}(A)$.*

Proof. By Lemma 3.12, $\sigma_{\mathfrak{A}}(A) = \sigma_{\mathfrak{C}}(A)$, where $\mathfrak{C} := C^*(\mathbb{1}, A)$. But $\mathfrak{C} \subset \mathfrak{B} \subset \mathfrak{A}$. □

Motivated by the above theorem, when speaking about C^*-algebras, we will write $\sigma(A)$ instead of $\sigma_{\mathfrak{A}}(A)$.

3.10 Spectral theorem for self-adjoint operators

Theorem 3.14. *Let $\mathfrak{A}$ be a unital C^*-algebra and $A \in \mathfrak{A}$ be self-adjoint. Then there exists a unique continuous isomorphism*

$$C(\sigma(A)) \ni f \mapsto f(A) \in C^*(\mathbb{1}, A) \subset \mathfrak{A},$$

such that

(1) $\mathrm{id}(A) = A$ *if* $\mathrm{id}(z) = z$.
Moreover, we have

(2) *If f is a polynomial, then $f(A)$ coincides with $f(A)$ defined by the holomorphic calculus.*

(3) $\sigma(f(A)) = f(\sigma(A))$.

(4) $g \in C(f(\sigma(A))) \Rightarrow g \circ f(A) = g(f(A))$.

(5) $\|f(A)\| = \sup |f|$.

3.11 Gelfand theory for commutative C^*-algebras

Theorem 3.15. *Let $\mathfrak{A}$ be a C^*-algebra and ϕ a character on $\mathfrak{A}$. Then ϕ is a $*$-homomorphism and $\|\phi\| = 1$.*

Proof. Adjoin the unit if needed. Let $A = A^*$. Let $\tilde{\phi} := \phi\Big|_{C^*(\mathbb{1},A)}$. Then $\tilde{\phi}$ is a character on the commutative C^*-algebra $C^*(\mathbb{1}, A)$. Hence $\tilde{\phi}(A) \in \sigma(A) \subset \mathbb{R}$. Thus $\phi(A) \in \mathbb{R}$.

Let $A \in \mathfrak{A}$ be arbitrary. Then $\mathrm{Re}A := \frac{1}{2}(A + A^*)$ and $\mathrm{Im}A := \frac{1}{2\mathrm{i}}(A^*_A)$ are self-adjoint. Hence, $\phi(\mathrm{Re}A)$, $\phi(\mathrm{Im}A) \in \mathbb{R}$. By linearity, this implies

$$\phi(A^*) = \overline{\phi(A)}. \tag{3.15}$$

□

Theorem 3.16. *Let $\mathfrak{A}$ be a unital commutative C^*-algebra. Then the Gelfand transform*

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C(\mathrm{Char}(\mathfrak{A}))$$

is a $*$-isomorphism.

Proof. **Step 1** We already know that it is a norm-decreasing homomorphism by Theorem 2.20.

Step 2 Using (3.15) we see that the Gelfand transform is a $*$-homomorphism.

Step 3 Every $A \in \mathfrak{A}$ is normal. Hence $\|A\| = \mathrm{sr}(A)$ by Theorem 3.10. But we know that $\|\hat{A}\| = \mathrm{sr}(A)$. This show that the Gelfand transform is isometric.

Step 4 We know that the image of the Gelfand transform is dense in $C(\mathrm{Char}(\mathfrak{A}))$ and $\mathfrak{A}$ is complete. We proved also that it is isometric. Hence it is bijective. □

Theorem 3.17. (1) *$U \in \mathfrak{A}$ is unitary $\Leftrightarrow$ U is normal and $\sigma(U) \subset \{z \;:\; |z| = 1\}$.*

(2) *$A \in \mathfrak{A}$ is self-adjoint $\Leftrightarrow$ A is normal and $\sigma(A) \subset \mathbb{R}$.*

Proof. $\Rightarrow$ was proven before.

(1)$\Leftarrow$. Consider the algebra $\mathfrak{C} := C^*(\mathbb{1}, U)$. By the normality of U, it is commutative. Let $\phi \in \mathrm{Char}(\mathfrak{C})$. Then $\phi(U^*)\phi(U) = \overline{\phi(U)}\phi(U) = 1$. Hence $\sigma(U) \subset \{|z| = 1\}$. Hence $U^*U = 1$.

(2)$\Leftarrow$. Consider the algebra $\mathfrak{C} := C^*(\mathbb{1}, A)$. By the normality of A, it is commutative. Let $\phi \in \mathrm{Char}(\mathfrak{C})$, Then $\phi(A) \in \sigma(A) \subset \mathbb{R}$. Hence $\phi(A^*) = \phi(A)$. Hence $A^* = A$. □

Theorem 3.18. *Let $\mathfrak{A}$ be a commutative C^*-algebra. Then the Gelfand transform*

$$\mathfrak{A} \ni A \mapsto \hat{A} \in C_\infty(\mathrm{Char}(\mathfrak{A}))$$

is a $*$-isomorphism.

3.12 Fuglede's theorem

Theorem 3.19. *Let $A, B \in \mathfrak{A}$ and let B be normal. Then $AB = BA$ implies $AB^* = B^*A$.*

Proof. For $\lambda \in \mathbb{C}$, the operator $U(\lambda) := \mathrm{e}^{\lambda B^* - \overline{\lambda} B} = \mathrm{e}^{-\overline{\lambda} B}\mathrm{e}^{\lambda B^*}$ is unitary. Moreover, $A = \mathrm{e}^{\overline{\lambda} B} A \mathrm{e}^{-\overline{\lambda} B}$. Hence

$$\mathrm{e}^{-\lambda B^*} A \mathrm{e}^{\lambda B^*} = U(-\lambda) A U(\lambda) \tag{3.16}$$

is a uniformly bounded analytic function. Hence is constant. Differentiating it wrt λ we get $[A, B^*] = 0$. □

3.13 Functional calculus for normal operators

Theorem 3.20. *Let $\mathfrak{A}$ be a unital C^*-algebra. Let $A \in \mathfrak{A}$ be normal. Then there exists a unique continuous unital $*$-isomorphism*

$$C(\sigma(A)) \ni f \mapsto f(A) \in C^*(\mathbb{1}, A) \subset \mathfrak{A},$$

such that

(1) $\mathrm{id}(A) = A$ *if* $\mathrm{id}(z) = z$.

Moreover, we have

(2) *If* $f \in \mathrm{Hol}(\sigma(A))$*, then* $f(A)$ *coincides with* $f(A)$ *defined by the holomorphic functional calculus.*

(3) $\sigma(f(A)) = f(\sigma(A))$.

(4) $g \in C(f(\sigma(A))) \Rightarrow g \circ f(A) = g(f(A))$.

(5) $\|f(A)\| = \sup |f|$.

Proof. If f is a polynomial, that is $f(z) = \sum a_{nm} z^n \overline{z}^m$, we set

$$f(A) := \sum a_{nm} A^n A^{*m}.$$

$C^*(\mathbb{1}, A)$ is a commutative algebra. Let ϕ be a character on $C^*(\mathbb{1}, A)$. Then we easily check that $\phi(f(A)) = f(\phi(A))$. Hence $\sigma(f(A)) = f(\sigma(A))$.

Clearly, $f(A)$ is normal. Hence

$$\|f(A)\| = \mathrm{sr}(f(A)) = \sup |f|.$$

Therefore, on polynomials the map $f \to f(A)$ is isometric. Since polynomials are dense in a complete metric space $C(\sigma(A))$ and polynomials in A, A^* are dense in a complete metric space $C^*(\mathbb{1}, A)$, there is exactly one continuous extension of this map to the whole $C(\sigma(A))$, which is an isometric bijection of $C(\sigma(A))$ to $C^*(\mathbb{1}, A)$.

Clearly, on polynomials, the map $f \mapsto f(A)$ is a $*$-homomorphism. Since the multiplication, and involution are continuous both in $C(\sigma(A))$ and $C^*(\mathbb{1}, A)$, this map is a homomorphism on $C(\sigma(A))$. □

If $\mathfrak{A}$ is not unital, either we can adjoin the identity and consider the algebra $\mathfrak{A}_{\mathbb{1}}$, or we can use the following version of the above theorem:

Theorem 3.21. *Let $\mathfrak{A}$ be a C^*-algebra. Let $A \in \mathfrak{A}$ be normal. Then there exists a unique continuous $*$-isomorphism*

$$C_\infty(\sigma(A)\backslash\{0\}) \ni f \mapsto f(A) \in C^*(A) \subset \mathfrak{A},$$

such that $\mathrm{id}(A) = A$ *if* $\mathrm{id}(z) = z$.

3.14 Positive elements

If $A \in B(\mathcal{H})$ then we say that $A \geq 0$ iff

$$(v|Av) \geq 0, \quad v \in \mathcal{H}. \tag{3.17}$$

In a C^*-algebra we do not have a Hilbert space at our disposal, therefore we need to define the postivity differently.

Let $A \in \mathfrak{A}$. We say that A is positive iff A is self-adjoint and $\sigma(A) \subset [0, \infty[$. $\mathfrak{A}_+$ will denote the set of positive elements in $\mathfrak{A}$. We will write $A \geq B$ iff $A - B \in \mathfrak{A}_+$. We will write $A > B$ iff $A \geq B$ and $A \neq B$.

Lemma 3.22. *Let A be self-adjoint and $\|A\| \leq 2\lambda$. Then $\|\lambda\mathbb{1} - A\| \leq \lambda$ iff $A \geq 0$.*

Theorem 3.23. (1) *$A \in \mathfrak{A}_+$ and $\lambda \geq 0$ implies $\lambda A \in \mathfrak{A}_+$.*

(2) *$A, B \in \mathfrak{A}_+$ implies $A + B \in \mathfrak{A}_+$.*

(3) *$A, -A \in \mathfrak{A}_+$ implies $A = 0$.*

(4) *$\mathfrak{A}_+$ is closed.*

In other words, $\mathfrak{A}_+$ is a closed pointed cone.

Proof. (2) We use by Lemma 3.22 with $\lambda := \|A\| + \|B\|$:

$$\big\|\|A\| + \|B\| - A - B\big\| \leq \big\|\|A\| - A\big\| + \big\|\|B\| - B\|\big\| \leq \|A\| + \|B\|.$$

Hence, $A + B \geq 0$.

(3) $\sigma(A), \sigma(-A) \subset [0, \infty[$ implies $\sigma(A) = \{0\}$. But A is self-adjoint. Hence $A = 0$.

(4) Let $A_n \to A$. Then $\|A_n\| \to \|A\|$. $A_n \in \mathfrak{A}_+$ iff $\|A_n - \|A_n\|\| \leq \|A_n\|$. By taking the limit, $\|A - \|A\|\| \leq \|A\|$. Hence $A \in \mathfrak{A}_+$. □

Theorem 3.24. *Let $A \in \mathfrak{A}_+$ and $n \in \mathbb{N}\backslash\{0\}$. Then there exists a unique $B \in \mathfrak{A}_+$ such that $B^n = A$.*

Proof. $[0,\infty[\ni\lambda\mapsto\lambda^{1/n}$ is a continuous function. Hence $B:=A^{1/n}$ is well defined by spectral theorem and satisfies the requirements of the theorem.

Let $B\in\mathfrak{A}_+$, $B^n=A$. Clearly,

$$BA=B^{n+1}=AB. \tag{3.18}$$

Let $\mathfrak{C}:=C^*(\mathbb{1},B,A)$. By (3.18), $\mathfrak{C}$ is commutative. If $\phi\in\mathrm{Char}(\mathfrak{C})$, then $\phi(A)=\phi(B^n)=\phi(B)^n$. Moreover, $\phi(B)>0$. Hence $\phi(B)=\phi(A)^{1/n}$. Hence B is uniquely determined, and equals $A^{1/n}$. □

Theorem 3.25 (Jordan decomposition of a self-adjoint operator.)**.** *Let $A\in\mathfrak{A}$ be self-adjoint. Then there exist unique $A_+,A_-\in\mathfrak{A}_+$ such that $A_+A_-=A_-A_+=0$ and $A=A_+-A_-$.*

Proof. The functions $|x|_+:=\max(x,0)$ and $|x|_-:=\max(-x,0)$ are continuous. Hence A_+ and A_- can be defined as $|A|_+$ and $|A|_-$ by the functional calculus.

Assume that A_- and A_+ satisfy the conditions of the theorem. Then

$$A^2=A_-^2+A_+^2=(A_++A_-)^2.$$

By the uniqueness of the positive square root, $|A|=A_++A_-$. Hence $A_+=\frac{1}{2}(|A|+A)$ and $A_-=\frac{1}{2}(|A|-A)$. □

Theorem 3.26. *Let $A\in\mathfrak{A}$. The following conditions are equivalent*

(1) $A\geq 0$.

(2) *There exists $B\in\mathfrak{A}$ such that $A=B^*B$.*

Proof. (1) $\Rightarrow$ (2) is contained in Theorem 3.24. In fact, $A=(\sqrt{A})^2$.

Let us prove (1) $\Leftarrow$ (2). Clearly, B^*B is self-adjoint. Let $B^*B=A_+-A_-$ be its Jordan decomposition.

Clearly

$$(BA_-)^*(BA_-)=A_-(A_+-A_-)A_-=-A_-^3\in-\mathfrak{A}_+.$$

Let $BA_-=S+\mathrm{i}T$. Then

$$\begin{aligned}(BA_-)(BA_-)^* &= S^2+T^2+\mathrm{i}(TS-ST)\\ &= -(BA_-)^*(BA_-)+2(S^2+T^2)\in\mathfrak{A}_+,\end{aligned}$$

using the fact that $\mathfrak{A}_+$ is a convex cone.

But

$$\sigma\big((BA_-)^*(BA_-)\big)\cup\{0\}=\sigma\big((BA_-)(BA_-)^*\big)\cup\{0\}.$$

Hence $\sigma\big((BA_-)^*(BA_-)\big)=\{0\}$. Consequently, $(BA_-)^*(BA_-)=0$. Consequently, $A_-^3=0$. By the uniqueness of the positive third root, $A_-=0$. □

Theorem 3.27. (1) *Let A be self-adjoint, then $-\|A\| \leq A \leq \|A\|$.*

In what follows, let $0 \leq B \leq A$. Then

(2) $\|B\| \leq \|A\|$,

(3) *If $D^*D \leq 1$, then $DD^* \leq 1$.*

(4) $0 \leq C^*BC \leq C^*AC$.

(5) $0 \leq (\lambda + A)^{-1} \leq (\lambda + B)^{-1}, \quad 0 < \lambda$.

(6) $B(\lambda + B)^{-1} \leq A(\lambda + A)^{-1}$.

(7) $0 \leq B^\theta \leq A^\theta, \quad 0 \leq \theta \leq 1$,

Proof. (1) $\sigma(A) \subset [-\|A\|, \|A\|]$. Hence $\|A\| - A \geq 0$ and $\|A\| + A \geq 0$.

(2) By (1), $A \leq \|A\|$. Hence, $B \leq \|A\|$. Hence $\sigma(B) \subset [0, \|A\|]$. Therefore, $\|B\| \leq \|A\|$.

(3) Clearly, $\|D^*D\| \leq 1$. Hence $\|DD^*\| \leq 1$. Hence, by (1), $DD^* \leq 1$.

(4) $C^*(A - B)C = \left((A - B)^{\frac{1}{2}}C\right)^*(A - B)^{\frac{1}{2}}C \geq 0$.

(5) Clearly, $\lambda + A \geq \lambda + B \geq \lambda$. Hence $\lambda + A$ and $\lambda + B$ are positive invertible. By (4), applied with $C = (\lambda + A)^{-\frac{1}{2}}$, for $D := (\lambda + B)^{\frac{1}{2}}(\lambda + A)^{-\frac{1}{2}}$ we have $1 \geq D^*D$. Hence $1 \geq DD^*$.

(6) follows immediately from (5).

(7). We use (6) and

$$A^\theta = c_\theta \int_0^\infty \lambda^{\theta - 1} A(\lambda + A)^{-1} \mathrm{d}\lambda.$$

□

3.15 Linear functionals

Let ω be a linear functional on $\mathfrak{A}$. The adjoint functional ω^* is defined by

$$\omega^*(A) := \overline{\omega(A^*)}.$$

We say that ω is self-adjoint iff $\omega^* = \omega$, or equivalently, if $\omega(A) \in \mathbb{R}$ for A self-adjoint.

We say that ω is positive iff

$$\omega(A) \geq 0, \qquad A \in \mathfrak{A}_+.$$

The set of continuous functionals over $\mathfrak{A}$ will be denoted $\mathfrak{A}^\#$. The set of continuous positive functionals over $\mathfrak{A}$ will be denoted $\mathfrak{A}_+^\#$.

Theorem 3.28. *If ω is a positive functional, then it is self-adjoint and*

$$|\omega(A^*B)|^2 \leq \omega(A^*A)\omega(B^*B). \tag{3.19}$$

Proof. If A is self-adjoint, then we can decompose A as $A = -A_- + A_+$ with A_-, A_+ positive. Now $\omega(A_\pm) \geq 0$. Hence $\omega(A) = \omega(A_+) - \omega(A_-) \in \mathbb{R}$.

To prove (3.19), we note that for any $\lambda \in \mathbb{C}$,

$$\omega\big((A + \lambda B)^*(A + \lambda B)\big) \geq 0.$$

□

Theorem 3.29. *Let ω be a linear functional on a unital C^*-algebra $\mathfrak{A}$. The following conditions are equivalent:*

(1) *ω is positive*

(2) *ω is continuous and $\|\omega\| = \omega(\mathbb{1})$*

Proof. (1)⇒(2). **Step 1** Let $A \in \mathfrak{A}_+$. Then $A \leq \|A\|$. Hence $|\omega(A)| = \omega(A) \leq \|A\|\omega(\mathbb{1})$.

Step 2 Let $B \in \mathfrak{A}$. Then, by (3.19), using the positivity of B^*B and Step 1, we get

$$|\omega(B)|^2 \leq \omega(\mathbb{1})\omega(B^*B) \leq \omega(\mathbb{1})^2\|B^*B\| = \omega(\mathbb{1})^2\|B\|^2.$$

Hence $\|\omega\|^2 \leq \omega(\mathbb{1})^2$.

(1)⇐ (2). It is enough to assume that $\|\omega\| = 1$.

Step 1 Let A be self-adjoint. Let $\alpha, \beta, \gamma \in \mathbb{R}$ and $\omega(A) = \alpha + \mathrm{i}\beta$. It is enough to assume that $\omega(\mathbb{1}) = \|\omega\| = 1$. Clearly,

$$\|\gamma\mathbb{1} - \mathrm{i}A\| = \sqrt{\gamma^2 + \|A\|^2}, \quad \omega(\gamma\mathbb{1} - \mathrm{i}A) = \gamma + \beta - \mathrm{i}\alpha.$$

But

$$|\omega(\gamma\mathbb{1} - \mathrm{i}A)|^2 \leq \|\gamma\mathbb{1} - \mathrm{i}A\|^2.$$

Hence

$$(\gamma + \beta)^2 + \alpha^2 \leq \gamma^2 + \|A\|^2.$$

For large $|\gamma|$, this is possible only if $\beta = 0$. Hence ω is self-adjoint.

Step 2 Let $A \in \mathfrak{A}_+$. Then $\big\|\|A\| - A\big\| \leq \|A\|$. Therefore,

$$\big|\|A\|\omega(\mathbb{1}) - \omega(A)\big| \leq \|A\|.$$

But $\omega(\mathbb{1}) = 1$, and $\omega(A)$ is real. Hence $\omega(A) \geq 0$. □

Theorem 3.30. *Let ω be a linear functional on a non-unital C^*-algebra. The following conditions are equivalent:*

(1) *ω is positive*

(2) *ω is continuous and for some positive approximate identity $\{E_\alpha\}$ of $\mathfrak{A}$*

$$\|\omega\| = \lim_\alpha \omega(E_\alpha^2).$$

(3) *ω is continuous and if the functional $\omega_{\mathbb{1}} : \mathfrak{A}_{\mathbb{1}} \to \mathbb{C}$ is given by $\omega_{\mathbb{1}}(\lambda + A) := \lambda\|\omega\| + \omega(A)$, then $\omega_{\mathbb{1}}$ is a positive functional on $\mathfrak{A}_{\mathbb{1}}$*

Moreover, $\omega_{\mathbb{1}}$ is the unique functional on $\mathfrak{A}_{\mathbb{1}}$ that extends ω and satisfies $\|\omega\| = \|\omega_{\mathbb{1}}\|$.

Proof. (1)⇒(2). **Step 1.** We want to show that

$$c := \sup\{\omega(A) \ : \ 0 \le A \le 1\}$$

is finite. Suppose that it is not true, $0 \le A_n \le 1$ and $\omega(A_n) \to \infty$. Then we will find $\lambda_n \ge 0$ such that $\sum \lambda_n < \infty$ and $\sum \lambda_n \omega(A_n) = \infty$. But $A := \sum \lambda_n A_n$ is convergent and, for any n,

$$\sum_{j=1}^{n} \lambda_j \omega(A_j) \le \omega(A) < \infty,$$

which is a contradiction.

Step 2. If $A \in \mathfrak{A}$, then $A = \sum_{j=0}^{3} \mathrm{i}^j A_j$ with $A_j \in \mathfrak{A}_+$ and $\|A_j\| \le \|A\|$. Hence

$$|\omega(A)| \le \sum_{j=0}^{3} \omega(A_j) \le 4c\|A\|.$$

Hence ω is continuous.

Step 3. Let E_α be a positive approximate unit. $\omega(E_\alpha)$ is an increasing bounded net, so $c := \lim_\alpha \omega(E_\alpha)$ exists. Since $\|E_\alpha\| \le 1$, we have $c \le \|\omega\|$.

Step 4 Let $A \in \mathfrak{A}$. Then

$$|\omega(E_\alpha A)|^2 \le \omega(E_\alpha^2)\omega(A^*A) \le \omega(E_\alpha^2)\|\omega\|\|A^*A\| \le c\|\omega\|\|A\|^2.$$

Moreover, $E_\alpha A \to A$ and ω is continuous, hence the left hand side goes to $|\omega(A)|^2$. Hence $|\omega(A)|^2 \le c\|\omega\|\|A\|^2$. Therefore, $\|\omega\| \le c$.

(2)⇒(3). It is obvious that $\|\omega_{\mathbb{1}}\| \ge \|\omega\|$. Let us prove the converse inequality.

Let E_α be a positive approximative unit. We have

$$\omega_{\mathbb{1}}(\lambda + A) = \lim_\alpha \omega(\lambda E_\alpha + E_\alpha A).$$

Hence

$$\begin{aligned} |\omega_{\mathbb{1}}(\lambda + A)| &= \lim\nolimits_\alpha |\omega(\lambda E_\alpha + E_\alpha A)| \le \lim\nolimits_\alpha \|\omega\|\|\lambda E_\alpha + E_\alpha A\| \\ &\le \|\omega\| \limsup\nolimits_\alpha \|E_\alpha\|\|\lambda + A\| = \|\omega\|\|\lambda + A\|. \end{aligned}$$

Hence, $\|\omega_{\mathbb{1}}\| \le \|\omega\|$.

Thus we proved that $\|\omega\| = \|\omega_{\mathbb{1}}\|$. Therefore, $\omega_{\mathbb{1}}(1) = \|\omega_{\mathbb{1}}\|$. Therefore, ω is positive by the previous theorem.

(3)⇒(1) is obvious. □

A positive functional over $\mathfrak{A}$ satisfying $\|\omega\| = 1$ will be called a state. For a unital algebra it is equivalent to $\omega(\mathbb{1}) = 1$. For a non-unital algebra it is equivalent to $1 = \sup\{\omega(A) \ : \ A \leq 1\}$. The set of states on a C^*-algebra $\mathfrak{A}$ will be denoted $\mathbb{E}(\mathfrak{A})$.

If ω is a positive functional on $\mathfrak{A}$, then

$$\omega_{\mathbb{1}}(A+\lambda) := \omega(A) + \lambda\|\omega\| \quad A \in \mathfrak{A}, \ \lambda \in \mathbb{C},$$

defines a state on $\mathfrak{A}_{\mathbb{1}}$ extending ω with $\|\omega\| = \|\omega_{\mathbb{1}}\|$.

If ϕ is a positive functional on $\mathfrak{A}_{\mathbb{1}}$, then

$$\phi(A+\lambda) = \theta\omega(A) + \lambda\|\phi\|, \quad A \in \mathfrak{A}, \ \lambda \in \mathbb{C},$$

where $0 \leq \theta \leq \|\phi\|$, and ω is a state on $\mathfrak{A}$.

3.16 The GNS representation

Recall that $(\mathcal{H}, \pi)$ is a $*$-representation of a C^*-algebra $\mathfrak{A}$ iff $\pi : \mathfrak{A} \to B(\mathcal{H})$ is a homomorphism and $\pi(A^*) = \pi(A)^*$. Let $(\mathcal{H}, \pi)$ be a $*$-representation of $\mathfrak{A}$, $\Omega \in \mathcal{H}$ and $\omega \in \mathfrak{A}_+^\#$. We say that Ω is a vector representative of ω iff

$$\omega(A) = (\Omega|\pi(A)\Omega).$$

We say that Ω is cyclic iff $\pi(\mathfrak{A})\Omega$ is dense in $\mathcal{H}$. $(\mathcal{H}, \pi, \Omega)$ is called a cyclic $*$-representation iff $(\pi, \mathcal{H})$ is a $*$-representation and Ω is a cyclic vector.

Theorem 3.31. *Let ω be a state on $\mathfrak{A}$. Then there exists a cyclic $*$-representation $(\mathcal{H}_\omega, \pi_\omega, \Omega_\omega)$ such that Ω_ω is a vector representative of ω. Such a representation is unique up to a unitary equivalence.*

Proof. We adjoin the unit if needed.

For $A, B \in \mathfrak{A}$, $\omega(A^*B)$ is a pre-Hilbert scalar product on $\mathfrak{A}$, that means, it is a sesquillinear form satisfying $\omega(A^*A) \geq 0$ for all $A \in \mathfrak{A}$. Define

$$\mathfrak{N}_\omega := \{A \in \mathfrak{A} \ : \ \omega(A^*A) = 0\}. \tag{3.20}$$

The scalar product on $\mathfrak{A}/\mathfrak{N}_\omega$ is well defined:

$$\big(A + \mathfrak{N}_\omega | B + \mathfrak{N}_\omega\big) := \omega(A^*|B). \tag{3.21}$$

Let $\mathcal{H}_\omega$ be the completion of $\mathfrak{A}/\mathfrak{N}_\omega$.

Let $B \in \mathfrak{N}_\omega$, $A \in \mathfrak{A}$. Clearly,

$$B^*A^*AB \leq \|A^*A\|B^*B. \tag{3.22}$$

Therefore,

$$\omega\big((AB)^*AB\big) = \omega(B^*A^*AB \leq \|A^*A\|\omega(B^*B). \tag{3.23}$$

Hence, $\mathfrak{N}_\omega$ is a left ideal. It is clearly closed.

The left regular representation

$$\mathfrak{A} \ni A \mapsto \lambda(A) \in L(\mathfrak{A}), \quad \lambda(A)B := AB,$$

preserves $\mathfrak{N}_\omega$. Hence we can define the representation π_ω on $\mathfrak{A}/\mathfrak{N}_\omega$ by

$$\pi_\omega(A)(B + \mathfrak{N}_\omega) := AB + \mathfrak{N}_\omega.$$

We have

$$\|\pi_\omega(A)(B + \mathfrak{N}_\omega)\|^2 = \|AB + \mathfrak{N}_\omega\|^2 = \omega(B^*A^*AB) \tag{3.24}$$

$$\leq \|A^*A\|\omega(B^*B) = \|A\|^2\|B + \mathfrak{N}_\omega\|^2. \tag{3.25}$$

Hence $\|\pi_\omega(A)\| \leq \|A\|$ and π_ω extends to a bounded linear map on $\mathcal{H}_\omega$.

We set $\Omega_\omega := \mathbb{1} + \mathfrak{N}_\omega$. Clearly, $\pi_\omega(A)\Omega_\omega = A + \mathfrak{N}_\omega$, hence Ω_ω is cyclic. □

Note that $\pi_\omega(\mathfrak{A})$ is a C^*-algebra inside $B(\mathcal{H}_\omega)$. It generates the von Neumann algebra $\pi_\omega(\mathfrak{A})'' \subset B(\mathcal{H}_\omega)$.

Below $\mathbb{C}^n$ for $n \in \mathbb{N}$ will have the usual meaning, and for $n = \infty$, $\mathbb{C}^n = l^2(\mathbb{N})$. We write e_j, $\quad j = 1, 2, \dots$ for its canonical o.n. basis.

Example 3.32. *Consider the C^*-algebra of compact operators $K(\mathcal{H})$ with an o.n. basis $\{f_j\}_{j\in\mathbb{N}}$. Consider the state ω, which on $A \in K(\mathcal{H})$ acts as*

$$\omega(A) := \sum_{j=1}^{n} \lambda_j (f_j|Af_j), \quad \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j > 0, \quad n \in \mathbb{N} \cup \{\infty\}. \tag{3.26}$$

Then the GNS Hilbert space can be identified with $\mathcal{H} \otimes \mathbb{C}^n$, and the GNS vector is

$$\Omega := \sum_{j=1}^{n} \sqrt{\lambda_j} f_j \otimes e_j. \tag{3.27}$$

The GNS representation is

$$\pi(A) := A \otimes \mathbb{1}. \tag{3.28}$$

The corresponding von Neumann algebra is $B(\mathcal{H}) \otimes \mathbb{1}$.

Example 3.33. *Consider the C^*-algebra $C[0,1]$. Let $x_1, x_2, \dots$ be a sequence of distinct numbers from $[0,1]$. Consider the state ω, which on $F \in C[0,1]$ acts as*

$$\omega(F) := \sum_{j=1}^{n} \lambda_j F(x_j), \quad \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j > 0, \quad n \in \mathbb{N} \cup \{\infty\}. \tag{3.29}$$

Then the GNS Hilbert space can be identified with $\mathbb{C}^n$, and the GNS vector is

$$\Omega := \sum_{j=1}^{n} \sqrt{\lambda_j} e_j. \tag{3.30}$$

The GNS representation is

$$\pi(F) := \sum_{j=1}^{n} F(x_j)|e_j)(e_j|. \tag{3.31}$$

The corresponding von Neumann algebra is isomorphic to $l^\infty(\mathbb{N})$.

Example 3.34. *Consider the C^*-algebra $C[0,1]$. Let λ be a continuous function on $[0,1]$ with support $[a,b] \subset [0,1]$. Consider the state ω, which on $F \in C[0,1]$ acts as*

$$\omega(F) := \int_0^1 \lambda(x)F(x)\mathrm{d}x, \quad \int_0^1 \lambda(x)\mathrm{d}x = 1, \quad \lambda \geq 0. \tag{3.32}$$

Then the GNS Hilbert space can be identified with the space $L^2[a,b]$, and the GNS vector is

$$\Omega(x) := \sqrt{\lambda(x)}. \tag{3.33}$$

The GNS representation is

$$\pi(F) := F(x)\Big|_{x\in[a,b]}. \tag{3.34}$$

The corresponding von Neumann algebra is $L^\infty[a,b]$, acting as multiplication operators on $L^2[a,b]$.

4 W^*-algebras

4.1 Introduction

Let $\mathcal{V}$ be a Hilbert space and v_n be a sequence of vectors in $\mathcal{V}$.

(1) We say that v_n is norm convergent to v if $\lim_{j\to\infty} \|v_j - v\| = 0$.

(2) We say that v_n is weakly convergent to v if $\lim_{j\to\infty}(w|v_j - v) = 0$ for every $w \in \mathcal{V}$.

Let (A_j) be a sequence of operators in $B(\mathcal{V},\mathcal{W})$.

(1) We say that (A_j) is norm convergent to A iff $\lim\limits_{j\to\infty} \|A_j - A\| = 0$. In this case we write

$$\lim_{j\to\infty} A_j = A.$$

(2) We say that (A_j) is strongly convergent to A iff $\lim\limits_{j\to\infty} \|A_j v - Av\| = 0$, $v \in \mathcal{V}$. In this case we write

$$\mathrm{s-}\lim_{j\to\infty} A_j = A.$$

(3) We say that (A_j) is weakly convergent to A iff $\lim\limits_{j\to\infty} |(w|A_j v) - (w|Av)| = 0$, $v \in \mathcal{V}$, $w \in \mathcal{W}$. In this case we write

$$\mathrm{w-}\lim_{j\to\infty} A_j = A.$$

The above definitions are related to three distinct topologies on $B(\mathcal{V},\mathcal{W})$: the norm topology, the strong operator topology and the weak operator topology. The first is generated by the operator norm. The other two are examples of locally convex topologies, and they are not given by any norm.

The famous Von Neumann's Density Theorem says that von Neumann algebras are precisely weakly (or strongly) closed $*$-algebras in $B(\mathcal{H})$ containing the identity.

In order to state and prove this theorem we need to recall the concept of a topology.

4.2 Topological spaces

If X is a set, then 2^X will denote the family of all subsets of X.

$(X,\mathcal{T})$ is a topological space iff X is a set and $\mathcal{T}\subset 2^X$ satisfies

(1) $\emptyset, X\in\mathcal{T}$;

(2) $A_i\in\mathcal{T},\ i\in I,\ \Rightarrow \underset{i\in I}{\cup} A_i\in\mathcal{T}$;

(3) $A_1,\dots,A_n\in\mathcal{T}\ \Rightarrow\ \overset{n}{\underset{i=1}{\cap}} A_i\in\mathcal{T}$.

Elements of $\mathcal{T}$ are called sets open in X. We will call $\mathcal{T}$ "a topology".

A set $A\subset X$ is called closed in X iff $X\backslash A$ is open.

If $\mathcal{T}$, $\mathcal{S}$ are topologies on X, then we say that $\mathcal{T}$ is weaker than $\mathcal{S}$ iff $\mathcal{T}\subset\mathcal{S}$.

If $\mathcal{T}=2^X$, then we say that $\mathcal{T}$ is discrete.

If $\mathcal{T}=\{\emptyset,X\}$, then we say that $\mathcal{T}$ is antidiscrete.

If $\mathcal{B}\subset 2^X$, then there exists the weakest topology containing $\mathcal{B}$.

If $A\subset X$, then the closure of A, denoted $A^{\rm cl}$, is the smallest closed set containing A.

Let $d: X\times X\to[0,\infty[$ be a metric. For any $x\in X$ and $r>0$ define the open ball of radius r and center x:

$$B(x,r):=\{y\in X\mid d(x,y)<r\}. \tag{4.35}$$

Then the weakest topology containing the set of all open balls

$$\{B(x,r)\mid x\in X,\quad r>0\} \tag{4.36}$$

is called the topology generated by the metric d.

4.3 Locally convex vector spaces

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. A function $p:\mathcal{V}\to[0,\infty[$ is called a seminorm if

$$p(\lambda v)=|\lambda|p(v),\quad p(v+w)\le p(v)+p(w),\quad v,w\in\mathcal{V},\quad \lambda\in\mathbb{C}. \tag{4.37}$$

If in addition

$$p(v)=0\quad\Rightarrow\quad v=0, \tag{4.38}$$

then it is called a norm. Every norm p defines a metric on X by

$$d(v,w):=p(v-w). \tag{4.39}$$

Suppose that we have a family of seminorms $\{p_j\}_{j\in J}$ on a vector space $\mathcal{V}$. For each $v \in \mathcal{V}$, $r > 0$ and each seminorm p_j of them we define a "generalized ball"

$$B_j(v,r) := \{w \in \mathcal{V} \mid p_j(v,w) < r\}. \tag{4.40}$$

Then the wekest topology containing the set of all open balls

$$\{B_j(v,r) \mid v \in X, \quad r > 0, \quad j \in J\} \tag{4.41}$$

is called the topology generated by the family of seminorms $\{p_j\}_{j\in J}$.

We say that $\mathcal{V}$ is a locally convex vector space if there exists a family of seminorms $\{p_j\}_{j\in J}$ that generates its topology and such that for each $v \in \mathcal{V}$ there exists $j \in J$ and $p_j(v) > 0$

Example 4.1. *Let $\mathcal{V}$ be a Hilbert space. For any $v \in \mathcal{V}$ define the seminorm*

$$p_v(w) := |(v|w)|. \tag{4.42}$$

Then the topology generated by these seminorms is called the weak topology on $\mathcal{V}$.

For instance, if $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence in $\mathcal{V}$ (that means $(e_i|e_j) = \delta_{ij}$), then $\mathrm{w-}\lim_{n\to\infty} e_n = 0$, but in the norm topology the sequence (e_n) has no limit.

Example 4.2. *Here is a more general example. Suppose that $\mathcal{V}$ is a vector space. Let $\mathcal{V}^{\#\mathrm{alg}}$ denote its algebraic dual, that is, the psace of linear functional on $\mathcal{V}$. Let $\mathcal{W}$ be a subspace of $\mathcal{V}^{\#\mathrm{alg}}$. Then $\sigma(\mathcal{V},\mathcal{W})$ topology is defined by the seminorms*

$$p_w(v) := |\langle w|v\rangle|, \qquad w \in \mathcal{W}. \tag{4.43}$$

Then a linear functional w is $\sigma(\mathcal{V},\mathcal{W})$ continuous iff $w \in \mathcal{W}$.

If $\mathcal{V}$ is a Banach space, let $\mathcal{V}^{\#}$ denote the space of continuous linear functionals on $\mathcal{V}$. Clearly, $\mathcal{V}$ can be identified as a subspace of $\mathcal{V}^{\#\#}$.

Then the topology $\sigma(\mathcal{V},\mathcal{V}^{\#})$ is called weak. The topology $\sigma(\mathcal{V}^{\#},\mathcal{V})$ is called #-weak.

4.4 Topologies on $B(\mathcal{H})$

Let $\mathcal{H}$ be a Hilbert space. We define a number of locally convex topologies on $B(\mathcal{H})$ by specifying families of seminorms for a given operator $A \in B(\mathcal{H})$:

weak op. topology:	$\vert(\Psi\vert A\Phi)\vert$,	$\Phi, \Psi \in \mathcal{H}$;
σ-weak topology:	$\vert\sum(\Phi_n\vert A\Psi_n)\vert$,	$\sum\Vert\Phi_n\Vert^2, \sum\Vert\Psi_n\Vert^2 < \infty$;
strong topology:	$\Vert A\Phi\Vert$,	$\Phi \in \mathcal{H}$;
σ-strong topology:	$(\sum\Vert A\Psi_n)\Vert^2)^{1/2}$,	$\sum\Vert\Psi_n\Vert^2 < \infty$;
$*$-strong op. topology:	$\Vert A\Phi\Vert + \Vert A^*\Phi\Vert$,	$\Phi \in \mathcal{H}$;
$\sigma*$-strong topology:	$(\sum(\Vert A\Psi_n)\Vert^2 + \Vert A^*\Psi_n\Vert^2)^{1/2}$,	$\sum\Vert\Psi_n\Vert^2 < \infty$;.

Below we give different, equivalent families of seminorms. All the families below are partially ordered:

weak op. topology:	$\vert\mathrm{Tr}\rho A\vert$,	$\rho \in B_+^{\mathrm{fin}}(\mathcal{H})$;
σ-weak topology:	$\vert\mathrm{Tr}\rho A\vert$,	$\rho \in B_+^1(\mathcal{H})$;
strong op. topology:	$(\mathrm{Tr}A^*A\rho)^{1/2}$,	$\rho \in B_+^{\mathrm{fin}}(\mathcal{H})$;
σ-strong topology:	$(\mathrm{Tr}A^*A\rho)^{1/2}$,	$\rho \in B_+^1(\mathcal{H})$;
$*$-strong op. topology:	$(\mathrm{Tr}(A^*A + A^*A)\rho)^{1/2}$,	$\rho \in B_+^{\mathrm{fin}}(\mathcal{H})$;
$\sigma*$-strong topology:	$(\mathrm{Tr}(A^*A + AA^*)\rho)^{1/2}$,	$\rho \in B_+^1(\mathcal{H})$.

All these topologies coincide on projections and on unitary operators. The following relations are immediate:

Theorem 4.3. (1) *We have the following relations:*

$$\begin{array}{ccccccc} \text{weak} & < & \text{strong} & < & *\text{−strong} & & \\ \wedge & & \wedge & & \wedge & & \\ \sigma\text{−weak} & < & \sigma\text{−strong} & < & \sigma*\text{−strong} & < & \text{norm} \end{array}$$

(2) *On $B(\mathcal{H})_1$ (the unit ball in $B(\mathcal{H})$)*

(i) *the weak and the σ-weak topologies coincide;*
(ii) *the strong and the σ-strong topologies coincide;*
(iii) *the $*$-strong and the $\sigma*$-strong topologies coincide,*

Clearly, the σ-weak topology coincides with the weak$^\#$ topology in the terminology of Banach spaces.

The weak topology defined above does not coincide with the weak topology in the terminology of Banach spaces.

Theorem 4.4. *Let $\psi \in B(\mathcal{H})^\#$.*

(1) *TFAE*

  (i) ψ *is weakly continuous;*
  (ii) ψ *is strongly continuous*
  (iii) ψ *is $*$-strongly continuous;*
  (iv) ψ *is given by* $\psi(A) = \mathrm{Tr}\gamma A$ *with* $\gamma \in B^{\mathrm{fin}}(\mathcal{H})$.

(2) *TFAE*

  (i) ψ *is σ-weakly continuous;*
  (ii) ψ *is σ-strongly continuous*
  (iii) ψ *is $\sigma*$-strongly continuous;*
  (iv) ψ *is given by* $\psi(A) = \mathrm{Tr}\gamma A$ *with* $\gamma \in B^1(\mathcal{H})$.

Proof. The following implications are obvious: (4)⇔(1)⇒(2)⇒(3). Let us prove (3)⇒(4).

Thus let ψ be a σ^*-s continuous functional. Then there exists $\rho \in B^1_+(\mathcal{H})$ such that

$$|\psi(A)| \leq (\sum \mathrm{Tr}A^*A\rho + A^*A\rho)^{1/2}.$$

Diagonalizing ρ we obtain an orthogonal sequence Φ_n with $\sum \|\Phi_n\|^2 < \infty$ such that

$$|\psi(A)| \leq (\sum \|A^*\Phi_n\|^2 + \|A\Phi_n\|^2)^{1/2}.$$

Set $\tilde{\mathcal{H}} := \sum_{n\neq 0} \mathcal{H}_n$, where $\mathcal{H}_n = \mathcal{H}$, $\mathcal{H}_{-n} = \overline{\mathcal{H}}$, for $n = 1, 2, \dots$.

Moreover, set $\tilde{\Phi} = (\tilde{\Phi}_n)$, where $\tilde{\Phi}_n = \Phi_n$, $\tilde{\Phi}_{-n} = \overline{\Phi}_n$, $n = 1, 2, \dots$. For $A \in B(\mathcal{H})$, let $\tilde{A} = \oplus \tilde{A}_n$, with $\tilde{A}_n = A$, $\tilde{A}_{-n} = \overline{A}^*$, for $n = 1, 2, \dots$. Clearly,

$$|\psi(A)| \leq \|\tilde{A}\tilde{\Phi}\|. \tag{4.44}$$

Let $\mathcal{K}$ be the subspace of $\tilde{\mathcal{H}}$ defined as the closure of $\{\tilde{A}\tilde{\Phi} \ : \ A \in B(\mathcal{H})\}$. Now for $\Xi := \tilde{A}\tilde{\Phi}$ we set

$$\tilde{\psi}(\Xi) := \psi(A). \tag{4.45}$$

Using (4.44), we easily see that $\tilde{\psi}$ is well-defined on $\mathcal{K}$ and bounded by 1.

Hence, there exists $\Psi = (\Psi_n) \in \mathcal{K}$ such that $\tilde{\psi}(\Xi) = (\Psi|\Xi)$, so that

$$\begin{aligned}
\psi(A) &= (\Psi|\tilde{A}\tilde{\Phi}) && (4.46)\\
&= \sum_{n=1}^{\infty}(\Psi_{-n}|\overline{A}^*\overline{\Phi}_n) + \sum_{n=1}^{\infty}(\Psi_n|A\Phi_n) && (4.47)\\
&= \sum_{n=1}^{\infty}(\Phi_n|A\overline{\Psi}_{-n}) + \sum_{n=1}^{\infty}(\Psi_n|A\Phi_n). && (4.48)
\end{aligned}$$

□

Linear functionals satisfying the conditions of Theorem 4.4 are often called normal functionals. The space of normal functionals on $\mathfrak{M}$ will be denoted $\mathfrak{M}_\#$. The set of normal states on $\mathfrak{M}$ will be denoted $\mathbb{E}(\mathfrak{M})$.

Theorem 4.5. *Let $\mathcal{K} \subset B(\mathcal{H})$ be a convex set. Consider the following statements:*

1. *$\mathcal{K}$ is closed (a) σ-weakly, (b) σ-strongly (c) $\sigma*$-strongly;*
2. *for any $r > 0$, $\mathcal{K} \cap B(\mathcal{H})_r$ is (a) weakly, (b) strongly, (c) $*$-strongly.*
3. *$\mathcal{K}$ is closed (a) σ-weakly, (b) σ-strongly (c) $\sigma*$-strongly;*
4. *for any $r > 0$, $\mathcal{K} \cap B(\mathcal{H})_r$ is closed; (a) σ-weakly, (b) σ-strongly (c) $\sigma*$-strongly;*

Then within each group, (a)$\Leftrightarrow$(b)$\Leftrightarrow$(c). Moreover, (1)$\Leftarrow$(2)$\Leftrightarrow$(3)$\Leftrightarrow$(4).

Proof. Within each group the equivalence is obvious, because the set of continuous linear functionals is the same.

(2)$\Leftrightarrow$(4) is obvious, because the respective topologies coincide for (a), (b) and (c).

(3a)$\Leftrightarrow$(4a) follows by the Krein-Shmulian Theorem. □

Theorem 4.6. *We have:*

$B(\mathcal{H}) \ni A \mapsto AB, BA$	*are weakly and σ-weakly continuous,*
$B(\mathcal{H})_r \times B(\mathcal{H}) \ni (A, B) \mapsto AB$	*is strongly and σ-strongly continuous,*
$B(\mathcal{H})_r \times B(\mathcal{H})_r \ni (A, B) \mapsto AB$	*is $*$-strongly and $\sigma*$-strongly continuous.*

Theorem 4.7. *$*$ is weakly, σ-weakly, $*$-strongly and $\sigma*$-strongly continuous.*

4.5 Monotone convergence

Theorem 4.8. *Let $\{A_\lambda \ : \ \lambda \in \Lambda\}$ be a uniformly bounded family of self-adjoint operators in $B(\mathcal{H})$. . Then there exists the smallest self-adjoint operator A such that $A_\lambda \leq A$. We will denote it $\mathrm{lub} A_\lambda$ (the least upper bound).*

Proof. Let $\|A_\lambda\| \leq c$. For each $v \in \mathcal{V}$, $(v|A_\lambda v)$ is an increasing net bounded by $c\|v\|^2$. Hence it is convergent. Using the polarization identity we obtain the convergence of $(v|A_\lambda w)$. Thus we obtain a sesquilinear form

$$\lim_\lambda (v|A_\lambda w) \tag{4.49}$$

It is bounded by c, hence it is given by a bounded operator, which we denote by A, so that (4.49) equals $(v|Aw)$. It is evident that A is the smallest self-adjoint operator greater than A_λ. □

Theorem 4.9. *Let $(A_\lambda \ : \ \lambda \in \Lambda)$ be an increasing net of self-adjoint operators, which is uniformly bounded. Then*

$$\mathrm{lub}_\lambda A = \mathrm{s}-\lim_\lambda A_\lambda.$$

Proof. Since $A - A_\lambda \geq 0$, we have

$$(A - A_\lambda)^2 = (A - A_\lambda)^{\frac{1}{2}}(A - A_\lambda)(A - A_\lambda)^{\frac{1}{2}} \leq \|A - A_\lambda\|(A - A_\lambda).$$

Besides, $\|A - A_\lambda\| \leq 2c$. Now

$$\|(A - A_\lambda)v\|^2 = (v|(A - A_\lambda)^2 v) \leq \|A - A_\lambda\|(v|(A - A_\lambda)v) \to 0.$$

□

Theorem 4.10. (1) *Let (A_λ) be a net weakly convergent to zero such that $0 \leq A_\lambda \leq C$. Then (A_λ) is strongly convergent to 0.*

(2) *Let (A_λ) be a net weakly convergent to A such that $A_\lambda \geq A \geq C > 0$. Then (A_λ^{-1}) is strongly convergent to A^{-1}.*

(3) *Let (A_λ) and (B_λ) be nets weakly convergent to A and B. Let $\|A_\lambda\|$ be bounded. Then $(A_\lambda B_\lambda)$ is strongly convergent to AB.*

4.6 Commutant

Let $\mathfrak{N} \subset B(\mathcal{H})$. We define the commutant of $\mathfrak{N}$:

$$\mathfrak{N}' := \{A \in B(\mathcal{H}) \ : \ AB = BA, \ B \in \mathfrak{N}\}.$$

Theorem 4.11. (1) *$\mathfrak{N}_1 \subset \mathfrak{N}_2$ implies $\mathfrak{N}_1' \supset \mathfrak{N}_2'$;*

(2) *$(\mathfrak{N}_1 \cap \mathfrak{N}_2)' = \mathfrak{N}_1' \cup \mathfrak{N}_2'$;*

(3) *$(\mathfrak{N}_1 \cup \mathfrak{N}_2)' = \mathfrak{N}_1' \cap \mathfrak{N}_2'$;*

(4) *$\mathfrak{N} \subset \mathfrak{N}'' = \mathfrak{N}^{(iv)} = \cdots$;*

(5) *$\mathfrak{N}' = \mathfrak{N}''' = \cdots$;*

(6) *$\mathfrak{N}$ is a weakly closed algebra;*

(7) *if $\mathfrak{N}$ is $*$-invariant, then so is $\mathfrak{N}'$.*

Proof. (3) $\mathfrak{N} \subset \mathfrak{N}''$ is obvious. By the same argument, $\mathfrak{N}' \subset \mathfrak{N}'''$. $\mathfrak{N} \subset \mathfrak{N}''$ together with (1) implies $\mathfrak{N}' \supset \mathfrak{N}'''$. Hence $\mathfrak{N}' = \mathfrak{N}'''$.

(4) Let A_α be a net in $\mathfrak{N}'$ weakly convergent to $A \in B(\mathcal{H})$. Let $B \in \mathfrak{N}$. Then $A_\alpha B = BA_\alpha$ and

$$(\Phi|AB\Psi) = \lim_\alpha (\Phi|A_\alpha B\Psi) = \lim_\alpha (\Phi|BA_\alpha\Psi) = (\Phi|BA\Psi).$$

□

Let $\mathcal{K}$ be another Hilbert space.

$$\begin{aligned} \mathfrak{N} \otimes 1\!\!1 \ &:= \{A \otimes 1\!\!1 \ : \ A \in \mathfrak{N}\}, \\ \mathfrak{N} \otimes B(\mathcal{K}) \ &:= \{B \in B(\mathcal{H} \otimes \mathcal{K}) \ : \ 1\!\!1 \otimes (v_1| \ B \ 1\!\!1 \otimes |v_2) \in \mathfrak{N}, \ \ v_1 v_2 \in \mathcal{K}\}. \end{aligned}$$

Theorem 4.12. (1) *$(\mathfrak{N} \otimes 1\!\!1)' = \mathfrak{N}' \otimes B(\mathcal{K})$;*

(2) *$(\mathfrak{N} \otimes B(\mathcal{K}))' = \mathfrak{N}' \otimes 1\!\!1$;*

(3) *$(\mathfrak{N} \otimes 1\!\!1)'' = \mathfrak{N}'' \otimes B(\mathcal{K})$.*

Proof. The inclusions $\supset$ are clear. Let us show the converse inclusions.
(1) Let $B \in (\mathfrak{N} \otimes 1\!\!1)'$ and $A \in \mathfrak{N}$. Then

$$\begin{aligned} A\ 1\!\!1\otimes(v_1|\ B\ 1\!\!1\otimes|v_2) &= 1\!\!1\otimes(v_1|\ A\otimes 1\!\!1\ B\ 1\!\!1\otimes|v_2) \\ &= 1\!\!1\otimes(v_1|\ B\ A\otimes 1\!\!1\ 1\!\!1\otimes|v_2) = 1\!\!1\otimes(v_1|\ B\ 1\!\!1\otimes|v_2)A. \end{aligned}$$

Hence $1\!\!1\otimes(v_1|\ B\ 1\!\!1\otimes|v_2) \in \mathfrak{N}'$. Therefore, $B \in \mathfrak{N}' \otimes B(\mathcal{K})$.

(2) Let $B \in B(\mathcal{H} \otimes \mathcal{K})$. Let e_i, $i \in I$, be an orthonormal basis of $\mathcal{K}$. Set $B_{ij} := 1\!\!1\otimes(e_i|\ B\ 1\!\!1\otimes|e_j)$ and $E_{ij} := |e_i)(e_j|$. We have

$$1\!\!1\otimes(e_i| \quad [B, E_{ij}]\ 1\!\!1\otimes|e_j) = B_{ii} - B_{jj}. \tag{4.50}$$

Thus $B \in (1\!\!1\otimes B(\mathcal{K}))'$ iff $B_{ij} = 0$, $B_{ii} = B_{jj}$, $i \neq j$. Hence there exists $A \in B(\mathcal{H})$ such that $B = A \otimes 1\!\!1$.

Now $B = A \otimes 1\!\!1 \in \mathfrak{N}'\otimes 1\!\!1$ iff $A \in \mathfrak{N}'$.

(3) follows immediately from (1) and (2). □

4.7 Von Neumann's Density Theorem

Recall that we say that a concrete algebra $\mathfrak{A} \subset B(\mathcal{H})$ is nondegenerate if $\Phi \in \mathcal{H}$ and $A\Phi = 0$ for all $A \in \mathfrak{A}$ implies $\Phi = 0$.

The aim of this subsection is to prove the following theorem:

Theorem 4.13. *Let $\mathfrak{A}$ be a nondegenerate $*$-algebra in $B(\mathcal{H})$. Then $\mathfrak{A}$ is σ-s dense in $\mathfrak{A}''$.*

If $\mathcal{K}$ is a subspace of $\mathcal{H}$, then $[\mathcal{K}]$ will denote the orthogonal projection onto $\mathcal{K}^{\rm cl}$.

Lemma 4.14. *Let $\mathfrak{A}$ be a $*$-algebra in $B(\mathcal{X})$ and $\Psi \in \mathcal{X}$. Then*

(1) $[\mathfrak{A}\Psi] \in \mathfrak{A}'$;

(2) *If $\mathfrak{A}$ is nondegenerate, then* $\Psi \in (\mathfrak{A}\Psi)^{\rm cl}$;

(3) *If $\mathfrak{A}$ is nondegenerate, then* $(\mathfrak{A}\Psi)^{\rm cl} = (\mathfrak{A}''\Psi)^{\rm cl}$.

Proof. (1) Let $A \in \mathfrak{A}$. Then $A(\mathfrak{A}\Psi)^{\rm cl} \in (\mathfrak{A}\Psi)^{\rm cl}$. Therefore

$$A[\mathfrak{A}\Psi] = [\mathfrak{A}\Psi]A[\mathfrak{A}\Psi]$$

By conjugation,

$$[\mathfrak{A}\Psi]A^* = [\mathfrak{A}\Psi]A^*[\mathfrak{A}\Psi].$$

Since $\mathfrak{A}$ is $*$-invariant, we can replace A with A^* in the last equality. Thus

$$A[\mathfrak{A}\Psi] = [\mathfrak{A}\Psi]A.$$

(2) Let $A \in \mathfrak{A}$. Using (1) in the second step, we obtain

$$A\Psi = [\mathfrak{A}\Psi]A\Psi = A[\mathfrak{A}\Psi]\Psi.$$

Thus $A(\mathbb{1} - [\mathfrak{A}\Psi])\Psi = 0$. By the nondegeneracy of $\mathfrak{A}$,

$$\Psi = [\mathfrak{A}\Psi]\Psi.$$

(3) Let $A \in \mathfrak{A}''$. Using first (2), and then (1), we obtain

$$A\Psi = A[\mathfrak{A}\Psi]\Psi = [\mathfrak{A}\Psi]A\Psi \in (\mathfrak{A}\Psi)^{\mathrm{cl}}. \tag{4.51}$$

Hence, $\mathfrak{A}''\Psi \subset (\mathfrak{A}\Psi)^{\mathrm{cl}}$. Taking the closure, we obtain $(\mathfrak{A}''\Psi)^{\mathrm{cl}} \subset (\mathfrak{A}\Psi)^{\mathrm{cl}}$. The converse inclusion is obvious. □

Proof of Theorem 4.13. Let $\mathcal{K}$ be a separable Hilbert space with an orthonormal basis $(e_j)_{j\in\mathbb{N}}$.

Let $\rho \in B_+^1(\mathcal{H})$. Then $\rho = \sum_{j=1}^\infty |\Phi_j)(\Phi_j|$ for some orthogonal family (Φ_j) with $\sum_{j=1}^\infty \|\Phi_j\|^2 < \infty$. Let $\Psi := \sum_{j=1}^\infty \Phi_j \otimes e_j \in \mathcal{H} \otimes \mathcal{K}$.

$\mathfrak{A}$ is nondegenerate, hence so is $\mathfrak{A} \otimes \mathbb{1}$. Therefore, we can apply Lemma 4.14 with $\mathfrak{A}$ replaced with $\mathfrak{A} \otimes \mathbb{1}$ and $\mathcal{X}$ replaced with $\mathcal{H} \otimes \mathcal{K}$. It implies that

$$(\mathfrak{A}\otimes\mathbb{1}\ \Psi)^{\mathrm{cl}} = (\mathfrak{M} \otimes \mathbb{1}\ \Psi)^{\mathrm{cl}}.$$

Hence, for any $\epsilon > 0$ and $A \in \mathfrak{M}$ we can find $B \in \mathfrak{A}$ such that

$$\epsilon > \|B\otimes\mathbb{1}\Psi - A\otimes\mathbb{1}\Psi\|^2 = \sum_{j=0}^\infty \|(B-A)\Phi_j\|^2 = \mathrm{Tr}(B-A)^*(B-A)\rho.$$

Hence $A \in \mathfrak{A}^{\sigma\mathrm{scl}}$. □

4.8 Concrete W^*-algebras

Theorem 4.15. *Let $\mathfrak{M}$ be a $*$-subalgebra of $B(\mathcal{H})$. Then TFAE:*

(1) *$\mathfrak{M}$ is (a) weakly closed; (b) strongly closed; (c) $*$-strongly closed;*

(2) *$(\mathfrak{M})_1$ is (a) weakly closed; (b) strongly closed; (c) $*$-strongly closed;*

(3) *$\mathfrak{M}$ is (a) σ-weakly closed; (b) σ-strongly closed; (c) $\sigma*$-strongly closed;*

(4) *$(\mathfrak{M})_1$ is (a) σ-weakly closed; (b) σ-strongly closed; (c) $\sigma*$-strongly closed;*

If $\mathfrak{M}$ satisfies the above conditions, then we say that $\mathfrak{M}$ is a concrete W^*-algebra.

Theorem 4.16. *Let $\mathfrak{M} \subset B(\mathcal{H})$. TFAE:*

(1) *$\mathfrak{M}$ is a von Neumann algebra, that is, it is $*$-invariant and $\mathfrak{M}'' = \mathfrak{M}$;*

(2) *$\mathfrak{M}$ is a concrete W^*-algebra and $\mathbb{1}_\mathcal{H} \in \mathfrak{M}$;*

(3) *$\mathfrak{M}$ is a nondegenerate concrete W^*-algebra in $B(\mathcal{H})$.*

Proof of Theorem 4.15 and 4.16. In Theorem 4.15, the equivalence within each group is clear, also the implications (1)⇐(2)⇔(3)⇔(4).

If $\mathfrak{N} \subset B(\mathcal{H})$, then $W^*(\mathfrak{N})$ will denote the smallest concrete W^*-algebra containing $\mathfrak{N}$. Note that for any $*$-invariant $\mathfrak{N} \subset B(\mathcal{H})$ containing $\mathbb{1}$, $W^*(\mathfrak{N}) = \mathfrak{N}''$.

Theorem 4.17. *Let $\mathfrak{Z}$ be the center of the von Neumann algebra $\mathfrak{M}$. Then $W^*(\mathfrak{M} \cup \mathfrak{M}') = \mathfrak{Z}'$.*

4.9 Kaplansky's density theorem

We say that $f \in C(\mathbb{R})$ is strongly continuous iff

$$B_{\mathrm{h}}(\mathcal{H}) \ni A \mapsto f(A) \in B(\mathcal{H})$$

is strongly continuous.

Theorem 4.18. *If $f \in C(\mathbb{R})$, $|f(t)| \leq a|t| + b$, then f is strongly continuous.*

Proof. Step 1. $t \mapsto t$ is strongly continuous.
Step 2. $t \mapsto (t-z)^{-1}$ with $\mathrm{Im}z \neq 0$ is strongly continuous. In fact, If $A_i \to A$ strongly, then

$$(A_i - z)^{-1}\Phi - (A-z)^{-1}\Phi = (A_i - z)^{-1}(A - A_i)(A-z)^{1}\Phi \to 0.$$

Step 3. Functions in $C_\infty(\mathbb{R})$ are strongly continuous. In fact, the uniformly closed algebra generated by $(t-z)^{-1}$ is $C_\infty(\mathbb{R})$.
Step 4. If h, g are strongly continuous and h is bounded, then hg is strongly continuous. In fact,

$$h(A_i)g(A_i)\Phi - h(A_)g(A)\Phi = h(A_i)(g(A_i) - g(A))\Phi + (h(A_i) - h(A))g(A)\Phi \to 0$$

Step 5. Let $f \in C(\mathbb{R})$ and $|f(t)| \leq a|t| + b$. Then $f\frac{1}{(t^2+1)} =: g \in C_\infty(\mathbb{R})$, and hence is strongly continuous. Using Step 1 and Step 4, gt is strongly continuous. It is clearly bounded. Thus $(gt)t = gt^2$ is strongly continuous. Hence $f = gt^2 + g$ is strongly continuous. □

Theorem 4.19 (Kaplansky's density theorem)**.** *Let $\mathfrak{A}$ be a $*$-algebra in a W^*-algebra and $\mathfrak{A}$ is σ-weakly dense in $\mathfrak{M}$. Then*

(1) *$(\mathfrak{A})_1$ is $\sigma*$-strongly dense in $(\mathfrak{M})_1$.*

(2) *$(\mathfrak{A}_{\mathrm{h}})_1$ is σ-strongly dense in $(\mathfrak{M}_{\mathrm{h}})_1$.*

(3) *$(\mathfrak{A}_+)_1$ is σ-strongly dense in $(\mathfrak{M}_+)_1$.*

Proof. Let $A \in (\mathfrak{M}_+)_1$. Then there exists a net (A_i) in $\mathfrak{A}$ convergent to A. Replacing A_i with $\frac{1}{2}(A_i + A_i^*)$ we can assume that A_i are self-adjoint. Let

$$f(t) = \begin{cases} 0 & t < 0 \\ t & 0 \leq t \leq 1 \\ 1 & 1 < t. \end{cases}$$

Then f is strongly continuous. Hence

$$f(A_i) \to f(A) = A$$

strongly. This proves (3). The proof of (2) is similar.

To prove (1), note that the unit ball of $B(\mathbb{C}^2)\otimes\mathfrak{A}$ is strongly dense in $B(\mathbb{C}^2)\otimes\mathfrak{M}$. Let $A\in(\mathfrak{M})_1$. Then

$$B := \begin{bmatrix} 0 & A \\ A^* & 0 \end{bmatrix}$$

is contained in the unit ball of $B(\mathbb{C}^2)\otimes\mathfrak{M}_{\mathrm{h}}$. Hence there exists a sequence

$$B_i := \begin{bmatrix} B_i^{11} & B_i^{12} \\ B_i^{21} & B_i^{22} \end{bmatrix}$$

in the unit ball $B(\mathbb{C}^2)\otimes\mathfrak{A}_{\mathrm{h}}$ strongly convergent to B. Then B_i^{12} belongs to the unit ball of $\mathfrak{A}$ and is $*$-strongly convergent to A. □

4.10 Functional calculus

Theorem 4.20. *Let $\mathfrak{M}$ be a von Neumann algebra. Let $A\in\mathfrak{M}$ be self-adjoint. Then there exists a unique σ^*-strongly continuous unital $*$-homomorphism*

$$L^\infty_{\mathrm{Borel}}(\sigma(A)) \ni f \mapsto f(A)\in\mathfrak{M}, \tag{4.52}$$

which on $C(\sigma(A))$ coincides with the previously defined $f(A)$. It satisfies

(1) *If $f_\alpha\in L^\infty_{\mathrm{Borel}}(\sigma(A))$ is uniformly bounded sequence converging pointwise to f then $f_\alpha(A)\to f(A)$ σ^*-strongly.*

(2) $\|f(A)\|\leq\sup|f|$;

(3) $\sigma(f(A))\subset f(\sigma(A))$;

(4) $g\in L^\infty_{\mathrm{Borel}}(f(\sigma(A))\Rightarrow g\circ f(A)=g(f(A))$.

(5) *The image of (4.52) equals $W^*(1,A)$.*

Note that we constructed a homomorphism of $L^\infty_{\mathrm{Borel}}(\sigma(A))$ onto $W^*(1,A)$.

Theorem 4.21. *$\mathfrak{M}$ is generated by its projectors.*

5 Tensor product

In this chapter we describe the terminology and notation of multilinear algebra. We will concentrate on the infinite dimensional case, where it is often natural to use the structure of Hilbert spaces.

We will consider two setups: that of vector spaces and that of Hilbert spaces. If $\mathcal{X},\mathcal{Y}$ are vector spaces, then $L(\mathcal{X},\mathcal{Y})$ will denote the set of linear operators from $\mathcal{X}$ to $\mathcal{Y}$. If $\mathcal{X},\mathcal{Y}$ are Hilbert spaces, then $B(\mathcal{X},\mathcal{Y})$ will denote the set of bounded operators from $\mathcal{X}$ to $\mathcal{Y}$.

5.1 Vector and Hilbert spaces

Let $\mathcal{V}$ be a vector space. A set $\{e_i \ : \ i\in I\}\subset\mathcal{V}$ is called linearly independent if for any finite subset $\{e_{i_1},\dots,e_{i_n}\}\subset\{e_i \ : \ i\in I\}$

$$c_1e_{i_1}+\cdots+c_ne_{i_n}=0 \quad\Rightarrow\quad c_1=\cdots=c_n=0. \tag{5.1}$$

$\{e_i \ : \ i\in I\}$ is a Hamel basis (or simply a basis) of $\mathcal{V}$ if it is a maximal linearly independent set. It means that it is linearly independent and if we add any $v\in\mathcal{V}$ to $\{e_i \ : \ i\in I\}\subset\mathcal{V}$ then it is not linearly independent any more. Note that every $v\in\mathcal{V}$ can be written as a finite linear combination $v=\sum_{i\in I}\lambda_ie_i$ in a unique way.

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$ or $\mathbb{R}$ equipped with a scalar product $(v|w)$ (positive, nondegenerate, sesquilinear form). It defines a metric on $\mathcal{V}$ by

$$\|v-w\|:=\sqrt{(v-w|v-w)}. \tag{5.2}$$

We say that $\mathcal{V},(\cdot|\cdot)$ is a Hilbert space if $\mathcal{V}$ is complete.

If $\mathcal{V},(\cdot|\cdot)$ is not necessarily complete, then we can always complete it, that is find a larger complete space $\mathcal{V}^{\rm cpl},(\cdot|\cdot)$ in which $\mathcal{V}$ is embedded as a dense subspace. $\mathcal{V}^{\rm cpl}$ is uniquely defined and is called the completion of $\mathcal{V}$.

For instance, if we take $C_{\rm c}(\mathbb{R})$, $C_{\rm c}^\infty(\mathbb{R})$ or $\mathcal{S}(\mathbb{R})$ with the usual scalar product $(f|g)=\int\overline{f(x)}g(x){\rm d}x$, then its completion is $L^2(\mathbb{R})$.

If $\mathcal{V}$ is a Hilbert space, then $\{e_i \ : \ i\in I\}$ is called an orthonormal basis (o.n.b.) if it is a maximal orthonormal set. Note that every $v\in\mathcal{V}$ can be written as a linear combination $v=\sum_{i\in I}\lambda_ie_i$, where $\sum_{i\in I}|\lambda_i|^2<\infty$, in a unique way

Note that in a finite dimensional Hilbert space every orthonormal basis is a basis. This is not true in infinite dimensional Hilbert spaces.

5.2 Direct sum

Let $(\mathcal{V}_i)_{i\in I}$ be a family of vector spaces. The *algebraic direct sum of* $\mathcal{V}_i$ will be denoted

$$\mathop{\oplus}\limits_{i\in I}^{\rm al}\mathcal{V}_i, \tag{5.3}$$

It consists of sequences $(v_i)_{i\in I}$, which are zero for all but a finite number of elements.

If $(\mathcal{V}_i)_{i\in I}$ is a family of Hilbert spaces, then $\overset{\mathrm{al}}{\underset{i\in I}{\oplus}} \mathcal{V}_i$ has a natural scalar product.

$$\Big((y_i)_{i\in I}\Big|(v_i)_{i\in I}\Big) = \sum_{i\in I}(y_i|v_i). \tag{5.4}$$

The *direct sum of $\mathcal{V}_i$ in the sense of Hilbert spaces* is defined as

$$\underset{i\in I}{\oplus} \mathcal{V}_i := \left(\overset{\mathrm{al}}{\underset{i\in I}{\oplus}} \mathcal{V}_i\right)^{\mathrm{cpl}}.$$

If I is finite, then $\overset{\mathrm{al}}{\underset{i\in I}{\oplus}} \mathcal{V}_i = \underset{i\in I}{\oplus} \mathcal{V}_i$

Let $(\mathcal{V}_i)$, $(\mathcal{W}_i)$, $i \in I$, be families of vector spaces. If $a_i \in L(\mathcal{V}_i, \mathcal{W}_i)$, $i \in I$, then their *direct sum* is denoted $\underset{i\in I}{\oplus} a_i$ and belongs to $L\left(\overset{\mathrm{al}}{\underset{i\in I}{\oplus}} \mathcal{V}_i, \overset{\mathrm{al}}{\underset{i\in I}{\oplus}} \mathcal{W}_i\right)$. It is defined as

$$\Big(\underset{i\in I}{\oplus} a_i\Big)(v_i)_{i\in I} = (a_i v_i)_{i\in I} \tag{5.5}$$

Let $\mathcal{V}_i$, $\mathcal{W}_i$, $i \in I$ be families of Hilbert spaces, and $a_i \in B(\mathcal{V}_i, \mathcal{W}_i)$ with $\sup_{i\in I} \|a_i\| < \infty$. Then the operator $\underset{i\in I}{\oplus} a_i$ is bounded. Its extension in $B\left(\underset{i\in I}{\oplus} \mathcal{V}_i, \underset{i\in I}{\oplus} \mathcal{W}_i\right)$ will be denoted by the same symbol.

5.3 Tensor product

Let $\mathcal{V}, \mathcal{W}$ be vector spaces. The *algebraic tensor product of $\mathcal{V}$ and $\mathcal{W}$* will be denoted $\mathcal{V} \overset{\mathrm{al}}{\otimes} \mathcal{W}$. Here is one of its definitions

Let $\mathcal{Z}$ be the space of finite linear combinations of vectors (v, w), $v \in \mathcal{V}$, $w \in \mathcal{W}$. In $\mathcal{Z}$ we define the subspace $\mathcal{Z}_0$ spanned by

$$\begin{array}{cc} (\lambda v, w) - \lambda(v, w), & (v, \lambda w) - \lambda(v, w), \\ (v_1 + v_2, w) - (v_1, w) - (v_2, w), & (v, w_1 + w_2) - (v, w_1) - (v, w_2). \end{array}$$

We set $\mathcal{V} \overset{\mathrm{al}}{\otimes} \mathcal{W} := \mathcal{Z}/\mathcal{Z}_0$. If $v \in \mathcal{V}$, $w \in \mathcal{W}$, we define $v \otimes w := (v, w) + \mathcal{Z}_0$.

Remark 5.1. *Note that (v, w) above is just a symbol and not an element of $\mathcal{V} \oplus \mathcal{W}$. Elements of the space $\mathcal{Z}$ have the form*

$$\sum_{j=1}^{n} \lambda_n (v_n, w_n). \tag{5.6}$$

In particular, in general

$$(v_1, w_1) + (v_2, w_2) \not\sim (v_1 + v_2, w_1 + w_2), \tag{5.7}$$

$$\lambda(v, w) \not\sim (\lambda v, \lambda w). \tag{5.8}$$

$\mathcal{V}\overset{\mathrm{al}}{\otimes}\mathcal{W}$ is a vector space and $\otimes$ is an operation satisfying

$$\begin{aligned} (\lambda v)\otimes w &= \lambda v\otimes w, & v\otimes(\lambda w) &= \lambda v\otimes w,\\ (v_1+v_2)\otimes w &= v_1\otimes w + v_2\otimes w, & v\otimes(w_1+w_2) &= v\otimes w_1 + v\otimes w_2. \end{aligned}$$

Vectors of the form $v\otimes w$ are called *simple tensors*. Not all elements of $\mathcal{V}\otimes\mathcal{W}$ are simple tensors, but they span $\mathcal{V}\overset{\mathrm{al}}{\otimes}\mathcal{W}$.

If $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are bases of $\mathcal{V}$, resp. $\mathcal{W}$, then $\{e_i\otimes f_j\}_{(i,j)\in I\times J}$ is a basis of $\mathcal{V}\overset{\mathrm{al}}{\otimes}\mathcal{W}$,

Suppose that $\mathcal{V},\mathcal{W},\mathcal{X}$ are vector spaces. Then it is easy to see that

$$(\mathcal{V}\otimes\mathcal{W})\otimes\mathcal{X} \quad \text{is naturally isomorphic to} \quad \mathcal{V}\otimes(\mathcal{W}\otimes\mathcal{X}). \tag{5.9}$$

This can be seen by comparing the bases. We will use this identification without a comment, and thus we will drop the parentheses in (5.9).

If $\mathcal{V}$, $\mathcal{W}$ are Hilbert spaces, then $\mathcal{V}\overset{\mathrm{al}}{\otimes}\mathcal{W}$ has a unique scalar product such that

$$(v_1\otimes w_1|v_2\otimes w_2) := (v_1|v_2)(w_1|w_2), \quad v_1,v_2\in\mathcal{V}, \quad w_1,w_2\in\mathcal{W}.$$

To see this it is enough to choose o.n.b's $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ in $\mathcal{V}$, resp. $\mathcal{W}$. Then every element of $\mathcal{V}\overset{\mathrm{al}}{\otimes}\mathcal{W}$ can be written as an (infinite) linear combination of $e_i\otimes f_j$ and we can use them as an orthonormal set defining this scalar product.

We set

$$\mathcal{V}\otimes\mathcal{W} := (\mathcal{V}\overset{\mathrm{al}}{\otimes}\mathcal{W})^{\mathrm{cpl}},$$

and call it the *tensor product of* $\mathcal{V}$ *and* $\mathcal{W}$ *in the sense of Hilbert spaces*. If $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$ are o.n.b's of $\mathcal{V}$, resp. $\mathcal{W}$, then $\{e_i\otimes f_j\}_{(i,j)\in I\times J}$ is an o.n.b. of $\mathcal{V}\otimes\mathcal{W}$,

If one of the Hilbert spaces $\mathcal{V}$ or $\mathcal{W}$ is finite dimensional, then $\mathcal{V}\overset{\mathrm{al}}{\otimes}\mathcal{W} = \mathcal{V}\otimes\mathcal{W}$.

5.4 Tensor product of operators

Let $\mathcal{V}_1,\mathcal{V}_2,\mathcal{W}_1,\mathcal{W}_2$ be vector spaces. If $a\in L(\mathcal{V}_1,\mathcal{V}_2)$ and $b\in L(\mathcal{W}_1,\mathcal{W}_2)$, then there exists a unique operator $a\otimes b\in L(\mathcal{V}_1\overset{\mathrm{al}}{\otimes}\mathcal{W}_1,\mathcal{V}_2\overset{\mathrm{al}}{\otimes}\mathcal{W}_2)$ such that on simple tensors we have

$$(a\otimes b)(y\otimes w) = (ay)\otimes(bw). \tag{5.10}$$

To see this it is enough to choose bases $(e_i)_{i\in I}$ in $\mathcal{V}_1$ and $(f_j)_{j\in J}$ in $\mathcal{W}_1$ and to define $a\otimes b$ on the basis $(e_i\otimes f_j)_{(i,j)\in I\times J}$ by

$$(a\otimes b)e_i\otimes f_j := (ae_i)\otimes(bf_j). \tag{5.11}$$

Then we check that thus defined operator satisfies (5.10) and is unique. It is called the *tensor product of* a *and* b.

If $\mathcal{V}_1,\mathcal{V}_2,\mathcal{W}_1,\mathcal{W}_2$ are Hilbert spaces and $a\in B(\mathcal{V}_1,\mathcal{V}_2)$, $b\in B(\mathcal{W}_1,\mathcal{W}_2)$, then $a\otimes b$ is bounded. It extends uniquely to an operator in $B(\mathcal{V}_1\otimes\mathcal{W}_1,\mathcal{V}_2\otimes\mathcal{W}_2)$, denoted by the same symbol.

To prove the boundedness of $a \otimes b = a \otimes \mathbb{1}\ \mathbb{1} \otimes b$, it is sufficient to consider the operator $a \otimes \mathbb{1}$ from $\mathcal{V}_1 \overset{\mathrm{al}}{\otimes} \mathcal{W}$ to $\mathcal{V}_2 \overset{\mathrm{al}}{\otimes} \mathcal{W}$. Let $e_1, e_2, \dots$ and $f_1, f_2 \dots$ be orthonormal bases in $\mathcal{V}_1$, $\mathcal{W}$ resp. Consider a vector $\sum c_{ij} \mathrm{e}_i \otimes f_j$.

$$\begin{aligned}
\Big\| a \otimes \mathbb{1} \sum_i c_{ij} e_i \otimes f_j \Big\|^2 &= \sum_j \Big\| \sum_i c_{ij} a e_i \Big\|^2 \\
\leq \sum_j \|a\|^2 \Big\| \sum_i c_{ij} e_i \Big\|^2 &\leq \sum_j \|a\|^2 \sum_i |c_{ij}|^2 \\
&= \|a\|^2 \Big\| \sum_{ij} c_{ij} e_i \otimes f_j \Big\|^2 .
\end{aligned}$$

5.5 Infinite tensor product of grounded Hilbert spaces

A pair $(\mathcal{H}, \Omega)$ consisting of a Hilbert space and a vector $\Omega \in \mathcal{H}$ of norm 1 is called a grounded Hilbert space. Let $(\mathcal{H}_1, \Omega_1), (\mathcal{H}_2, \Omega_2), \dots$ be a sequence of grounded Hilbert spaces. We introduce an isometric identification

$$\overset{n}{\underset{i=1}{\otimes}} \mathcal{H}_i \ni \Psi \mapsto \Psi \otimes \Omega_{n+1} \in \overset{n+1}{\underset{i=1}{\otimes}} \mathcal{H}_i .$$

We define

$$\overset{\infty}{\underset{i=1}{\otimes}} (\mathcal{H}_i, \Omega_i) := \left(\bigcup_{n=1}^{\infty} \overset{n}{\underset{i=1}{\otimes}} \mathcal{H}_i \right)^{\mathrm{cpl}} . \tag{5.12}$$

The image of $\Psi \in \mathcal{H}_n$ will be denoted by

$$\Psi \otimes \overset{\infty}{\underset{j=n+1}{\otimes}} \Omega_j .$$

Choose an o.n. basis $\{e_{ij}\}_{j \in \mathcal{J}_i}$ in each $\mathcal{H}_i$ such that $e_{i1} = \Omega_i$. Then the vectors

$$e_{1j_1} \otimes e_{2j_2} \otimes \cdots , \quad j_i \in \mathcal{J}_i, \tag{5.13}$$

where only for a finite number of $i \in \mathbb{N}$ we have $j_i \neq 1$, form an o.n. basis of (5.12).

Theorem 5.2. *Let $\Phi_i \in \mathcal{H}_i$, $i = 1, 2, \dots$. Suppose that for some N*

$$\lim_{n \to \infty} \prod_{i=N}^{n} (\Omega_i | \Phi_i) \tag{5.14}$$

exists and is nonzero. Then in (5.12) *there exists the limit of*

$$\Psi_n := \overset{n}{\underset{i=1}{\otimes}} \Phi_i \otimes \overset{\infty}{\underset{j=n+1}{\otimes}} \Omega_j \tag{5.15}$$

Proof. First note that

$$\lim_{m\to\infty}\left|\sup_{n>m}\prod_{i=m+1}^{n}(\Omega_i|\Phi_i)-1\right|=0. \tag{5.16}$$

Then we compute that for $m<n$

$$\|\Psi_n-\Psi_m\|^2=2-2\mathrm{Re}\prod_{i=m+1}^{n}(\Phi_j|\Omega).$$

By (5.16), Ψ_n is Cauchy. Besides, it belongs to $\otimes_{i=1}^{n}\mathcal{H}_i$. Hence it possesses a limit in (5.12). □

$\lim_{n\to\infty}\Psi_n$ will be denoted by

$$\mathop{\otimes}_{i=1}^{\infty}\Phi_i.$$

5.6 UHF algebras

Let $n_1,n_2,\dots$ be positive integers. Set $\mathcal{H}_j:=\mathbb{C}^{n_j}$. We introduce the identifications

$$B(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_k)\ni A\mapsto A\otimes 1\!\!1_{\mathcal{H}_{k+1}}\in B(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_{k+1}).$$

Define

$$\mathrm{UHF}_0=\mathrm{UHF}_0(n_1,n_2,\dots):=\bigcup_{k=1}^{\infty}B(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_k), \tag{5.17}$$

$$\mathrm{UHF}=\mathrm{UHF}(n_1,n_2,\dots):=\mathrm{UHF}_0(n_1,n_2,\dots)^{\mathrm{cpl}}. \tag{5.18}$$

The image of $A\in B(\mathcal{H}_1\otimes\cdots\otimes\mathcal{H}_k)$ in UHF will be denoted $A\otimes\mathop{\otimes}\limits_{j=k+1}^{\infty}1\!\!1_j$. Note that $B(\mathcal{H}_j)$ can be considered to be commuting subalgebras of $\mathrm{UHF}(n_1,n_2,\cdots)$:

$$B(\mathcal{H}_j)\ni B\mapsto\mathop{\otimes}_{j=1}^{k-1}1\!\!1_j\otimes B\otimes\mathop{\otimes}_{j=k+1}^{\infty}1\!\!1_j\in\mathrm{UHF}. \tag{5.19}$$

Let $p_1,p_2,\dots$ be prime numbers in the ascending order. Let $\alpha_1,\alpha_2,\dots\in\{0,1,2,\dots,\infty\}$. The expression of the form

$$p_1^{\alpha_1}p_2^{\alpha_2}\cdots$$

will be called a supernatural number. Note that the usual natural numbers are contained in the set of supernatural numbers. We can multiply supernatural numbers in the obvious way.

Let $p_1^{\alpha_1}p_2^{\alpha_2}\cdots$ be a supernatural number. We define $\mathbb{Q}_n$ to be the set of rational numbers of the form $\frac{q}{p_1^{k_1}p_2^{k_2}\dots}$ where $k_j\leq\alpha_j$. Note that $\mathbb{Q}_n$ is an abelian subgroup of $\mathbb{Q}$.

We say that a positive linear functional on a C^*-algebra $\mathfrak{A}$ is a trace iff $\tau(AB)=\tau(BA)$.

Theorem 5.3. (1) *Let* $(n_1, n_2, \dots)$ *and* $(n_1', n_2', \dots)$ *be two sequences of integers. Then* $\mathrm{UHF}(n_1, n_2, \dots)$ *is isomorphic to* $\mathrm{UHF}(n_1', n_2', \dots)$ *iff we have the equality of supernatural numbers*

$$n_1 n_2 \cdots = n_1' n_2' \cdots$$

Therefore, we will write $\mathrm{UHF}(n)$ *with* $n := n_1 n_2 \cdots$ *instead of* $\mathrm{UHF}(n_1, n_2, \cdots)$.

(2) *On* $\mathrm{UHF}(n)$ *there exists a unique tracial state* τ.

(3) $\{\tau(P) \ : \ P \in \mathrm{Proj}(\mathrm{UHF}(n))\} = \mathbb{Q}_n \cap [0, 1]$.

(4) $P_1, P_2 \in \mathrm{UHF}(n)$ *are unitarily equivalent iff* $\tau(P_1) = \tau(P_2)$.

(5) $\mathrm{UHF}(n)$ *is a simple* C^* *algebra.*

Lemma 5.4. *Let* $\mathfrak{A}$ *be a* C^**-algebra and* $A = A^* \in \mathfrak{A}$ *and* $\|A - A^2\| \leq \epsilon < \frac{1}{4}$. *Then there exists a function* f *continuous on* $\sigma(A)$ *such that* $f(A)$ *is a projection and*

$$\|A - f(A)\| \leq \frac{1 - \sqrt{1 - 4\epsilon}}{2}.$$

Proof. $|x(1-x)| = \left|\frac{1}{4} - (\frac{1}{2} - x)^2\right| \leq \epsilon$ implies, for $\epsilon < \frac{1}{4}$ $\frac{1-\sqrt{1+4\epsilon}}{2} \leq x \leq \frac{1-\sqrt{1-4\epsilon}}{2} < \frac{1}{2}$ or $\frac{1}{2} < \frac{1+\sqrt{1-4\epsilon}}{2} \leq x \leq \frac{1+\sqrt{1+4\epsilon}}{2}$. Hence $f(x) := \begin{cases} 0, & x < \frac{1}{2}, \\ 1, & x > \frac{1}{2}. \end{cases}$ is continuous on $\sigma(A)$. Clearly, $|x - f(z)| < \frac{1-\sqrt{1-4\epsilon}}{2}$ on $\sigma(A)$. □

Proof of Thm 5.3. (1) We inductively define a $*$-homomorphism of

$$\rho_n : B(\mathcal{H}_1) \otimes \cdots \otimes B(\mathcal{H}_n) \to B(\mathcal{H}_1') \otimes \cdots \otimes B(\mathcal{H}_{N_n}') \tag{5.20}$$

for N_n big enough, such that ρ_{n+1} extends ρ_n. Thus we construct a *isomorphism $\rho : UHF_0(n_1, n_2, \dots) \to UHF_0(n_1', n_2', \dots)$. Clearly, it extends to a $*$-isomorphism $\rho : UHF(n_1, n_2, \dots) \to UHF(n_1', n_2', \dots)$.

(2) On $B(\mathcal{H}_i \otimes \cdots \otimes \mathcal{H}_k)$ there exists a unique tracial state $\tau(\cdot) = \frac{1}{n_1 \cdots n_k}\mathrm{Tr}$. It can be extended to a tracial state on $\mathrm{UHF}(n_1, n_2, \dots)$.

(3) It is easy to see that $\{\tau(P) \ : \ P \in \mathrm{Proj}(\mathrm{UHF}_0(n))\} = \mathbb{Q}_n \cap [0, 1]$.

Let $P \in \mathrm{Proj}(\mathrm{UHF}(n_1, n_2, \dots))$ and $\frac{1}{4} > \epsilon > 0$. There exists $A \in \mathrm{UHF}_0(n_1, n_2, \dots)$ such that $\|P - A\| < \epsilon$. We have

$$\|P - 2^{-1}(A + A^*)\| \leq \frac{1}{2}\|P - A\| + \frac{1}{2}\|P - A^*\| < \epsilon.$$

Hence, we can assume that A is self-adjoint. By Lemma 5.4, there exists $Q = f(A)$ – a projection in $\mathrm{UHF}_0(n_1, n_2, \dots)$ – such that $\|A - Q\| < \frac{(1-\sqrt{1-4\epsilon})}{2}$. Hence, $\|P - Q\| < \epsilon + \frac{(1-\sqrt{1-4\epsilon})}{2} < 1$. Therefore, there exists a unitary U such that $Q = UPU^*$. Therefore, $\tau(Q) = \tau(P)$.

(4)$\Rightarrow$ is obvious. $\Leftarrow$ is obvious on $UHF_0(n)$. Using (3) this extends to $UHF(n)$. □

5.7 States and representation on $\mathrm{UHF}(n)$

If ω_i is a state on $\mathbb{C}^{n_i}$, then on $\mathrm{UHF}(n_1 n_2 \cdots)$ we can define the state

$$\omega := \mathop{\otimes}_{j=1}^{\infty} \omega_j. \tag{5.21}$$

Theorem 5.5. (1) *(5.21) is a pure state iff ω_j is pure.*

(2) *Suppose ω_j are pure and given by $|\Omega_j)(\Omega_j|$ for normalized vectors $\Omega_j \in \mathcal{H}_j$. Then the GNS representation for ω is is unitarily equivalent to the representation in $\mathop{\otimes}_{j=1}^{\infty}(\mathcal{H}_j, \Omega_j)$ such that $A_n \in B(\mathcal{H}_n) \subset \mathrm{UHF}(n_1 n_2 \cdots)$ is represented as*

$$\pi_\omega(A_n) = \mathop{\otimes}_{i=1}^{n-1} \mathbb{1}_{\mathcal{H}_i} \otimes A_n \otimes \mathop{\otimes}_{j=n+1}^{\infty} \mathbb{1}_{\mathcal{H}_j}.$$

(3) *Let $\Phi_1, \Phi_2, \dots,$ define a state ϕ. Then ω is unitarily equivalent to ϕ iff the product $\prod_{j=1}^{\infty}(\Omega_j|\Phi_j)$ is convergent.*

Theorem 5.6. (1) *(5.21) is always factorial.*

(2) *Let $\gamma_1, \gamma_2, \dots$ be nondegenerate density matrices on $\mathcal{H}_j$ and let ω_i be the corresponding states on $B(\mathcal{H}_j)$. Let ω be the corresponding state on $\mathrm{UHF}(n_1 n_2 \cdots)$. Then the GNS representation for ω is unitarily equivalent to the representation in the space*

$$\mathop{\otimes}_{j=1}^{\infty}(B^2(\mathcal{H}_j), \sqrt{\gamma_j})$$

such that $A_n \in B(\mathcal{H}_n) \subset \mathrm{UHF}(n_1 n_2 \cdots)$ is represented as the multiplication on the left by

$$\pi(A_n) = \mathop{\otimes}_{i=1}^{n-1} \mathbb{1}_{\mathcal{H}_i} \otimes A_n \otimes \mathop{\otimes}_{j=n+1}^{\infty} \mathbb{1}_{\mathcal{H}_j}.$$

(3) *Let and $\gamma_1', \gamma_2', \dots$ be another sequence of density matrices with $\omega_1', \omega_2', \dots$ the corresponding states on $B(\mathcal{H}_1), B(\mathcal{H}_2), \dots$. Let ω' be the corresponding state on $\mathrm{UHF}(n_1 n_2 \cdots)$. Then the corresponding GNS representations are quasiequivalent iff $\prod_{j=1}^{\infty} \mathrm{Tr}\sqrt{\gamma_j}\sqrt{\gamma_j'}$ is convergent.*

5.8 Hyperfinite factors

Let ω be a state as in (5.21). Introduce the hyperfinite W^*-algebra

$$\mathrm{HF}(\omega) := \pi_\omega(\mathrm{UHF}(n))''.$$

Clearly, the state $(\Omega| \cdot \Omega)$ on $\mathrm{HF}(\omega)$ is a normal state on $\mathrm{HF}(\omega)$ such that $(\Omega|\pi_\omega(A)\Omega) = \omega(A)$. We will denote it also by ω.

Clearly, $\mathop{\otimes}_{j=1}^{n} B^2(\mathcal{H}_j) \simeq B^2\left(\mathop{\otimes}_{j=1}^{n} \mathcal{H}_j\right)$ sits inside $\mathrm{HF}(\omega)$. On $B^2\left(\mathop{\otimes}_{j=1}^{n} \mathcal{H}_j\right)$ the modular conjugation is just the hermitian conjugation and the natural cone is

Theorem 5.7. *Let τ_ω be the modular dynamics corresponding to ω. Then τ_ω^t is inner iff*

$$\prod_{j=1}^{\infty} \mathrm{Tr}(\omega_j|\omega^{\mathrm{i}t}) \tag{5.22}$$

is convergent. If this is the case, then τ_ω^t is implemented by $\bigotimes_{j=1}^{\infty} \omega_j^{\mathrm{i}t}$.

For example, consider $n = 2^\infty$ and

$$\omega_j = (e^{h_j} + \mathrm{e}^{-h_j}) \begin{bmatrix} \mathrm{e}^{h_j} & 0 \\ 0 & e^{-h_j} \end{bmatrix}.$$

Then (5.22) equals

$$\prod_{j=1}^{\infty} (\mathrm{e}^{-h_j} + \mathrm{e}^{h_j})^{-1} (\mathrm{e}^{-h_j - \mathrm{i}h_j t} + \mathrm{e}^{h_j + ih_j t}).$$

6 Second quantization

6.1 Fock spaces

Let $\mathcal{Y}$ be a vector space. Let S_n denote the *permutation group of n elements* and $\sigma \in S_n$. $\Theta(\sigma)$ is defined as the unique operator in $L(\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y})$ such that

$$\Theta(\sigma) y_1 \otimes \cdots \otimes y_n = y_{\sigma^{-1}(1)} \otimes \cdots \otimes y_{\sigma^{-1}(n)}. \tag{6.1}$$

To see that $\Theta(\sigma)$ is well defined we first choose a basis $\{e_i\}_{i\in I}$ of $\mathcal{Y}$. Then we define $\Theta(\sigma)$ on the corresponding basis of $\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$:

$$\Theta(\sigma) e_{i_1} \otimes \cdots \otimes e_{i_n} = e_{i_{\sigma^{-1}(1)}} \otimes \cdots \otimes e_{i_{\sigma^{-1}(n)}}.$$

Then we extend by linearity $\Theta(\sigma)$ to the whole $\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$. It is easy to see that the operator defined in this way satisfies (6.1).

We can check that

$$S_n \ni \sigma \mapsto \Theta(\sigma) \in L(\overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}) \tag{6.2}$$

is a group representation.

We say that a tensor $\Psi \in \overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}$ is symmetric, resp. antisymmetric if

$$\Theta(\sigma)\Psi = \Psi, \tag{6.3}$$

$$\text{resp.} \qquad \Theta(\sigma)\Psi = \mathrm{sgn}(\sigma)\Psi. \tag{6.4}$$

We define the *symmetrization/antisymmetrization projections*

$$\Theta_{\mathrm{s}}^n := \frac{1}{n!}\sum_{\sigma\in S_n} \Theta(\sigma), \qquad \Theta_{\mathrm{a}}^n := \frac{1}{n!}\sum_{\sigma\in S_n} \mathrm{sgn}\sigma\Theta(\sigma).$$

They project onto symmetric/antisymmetric tensors.

We will often write s/a to denote either s or a.

If $\mathcal{Y}$ is a Hilbert space, then $\Theta(\sigma)$ is unitary and $\Theta_{\mathrm{s/a}}^n$ are orthogonal projections.

Let $\mathcal{Y}$ be a vector space. The *algebraic n-particle bosonic/fermionic space* is defined as

$$\overset{\mathrm{al}}{\otimes}{}^n_{\mathrm{s/a}}\mathcal{Y} := \Theta_{\mathrm{s/a}}^n \overset{\mathrm{al}}{\otimes}{}^n \mathcal{Y}.$$

The *algebraic bosonic/fermionic Fock space* or the *symmetric/antisymmetric tensor algebra* is

$$\overset{\mathrm{al}}{\Gamma}_{\mathrm{s/a}}(\mathcal{Y}) := \overset{\mathrm{al}}{\underset{n=0}{\overset{\infty}{\oplus}}} \overset{\mathrm{al}}{\otimes}{}^n_{\mathrm{s/a}}\mathcal{Y}.$$

The *vacuum vector* is $\Omega := 1 \in \otimes^0_{\mathrm{s/a}}\mathcal{Y} = \mathbb{C}$.

If $\mathcal{Y}$ is a Hilbert space, then the *n-particle bosonic/fermionic space* is defined as

$$\otimes^n_{\mathrm{s/a}}\mathcal{Y} := \Theta^n_{\mathrm{s/a}} \otimes^n \mathcal{Y}.$$

The *bosonic/fermionic Fock space* is

$$\Gamma_{\mathrm{s/a}}(\mathcal{Y}) := \underset{n=0}{\overset{\infty}{\oplus}} \otimes^n_{\mathrm{s/a}}\mathcal{Y}.$$

6.2 Creation/annihilation operators

For $z \in \mathcal{Y}$ we define the *creation operator*

$$\hat{a}^*(z)\Psi := \Theta_{\mathrm{s/a}}^{n+1}\sqrt{n+1}z \otimes \Psi, \quad \Psi \in \otimes_{\mathrm{s/a}}^n \mathcal{Y},$$

and the *annihilation operator* $\hat{a}(z) := (\hat{a}^*(z))^*$. (We often omit the hat).

We will sometimes write $(z|$ and $|z)$ for the following operators

$$\mathcal{V} \ni v \mapsto (z|v := (z|v) \in \mathbb{C}, \tag{6.5}$$

$$\mathbb{C} \ni \lambda \mapsto \lambda|z) := \lambda z \in \mathcal{V}. \tag{6.6}$$

Then on $\otimes_{\mathrm{s/a}}^n \mathcal{Y}$ we have

$$a^*(z) = \Theta_{\mathrm{s/a}}^{n+1}\sqrt{n+1}|z) \otimes \mathbb{1}^{n\otimes}, \tag{6.7}$$

$$a(z) = \sqrt{n}(z| \otimes \mathbb{1}^{(n-1)\otimes}. \tag{6.8}$$

Above we used the *compact notation* for creation/annihilation operators popular among mathematicians. Physicists commonly prefer the *traditional notation*, which is longer and less canonical.

One version of the traditional notation uses a fixed basis $\{e_i\}_{i\in I}$ of $\mathcal{Z}$ and set $a_i^* := a^*(e_i)$, $a_i := a(e_i)$. Then if $z = \sum_i z_i e_i$, we have

$$a^*(z) = \sum_i z_i a_i^*, \quad a(z) = \sum_i \overline{z}_i a_i, \tag{6.9}$$

$$[a_i, a_j^*]_\mp = \delta_{ij}, \quad [a_i, a_j]_\mp = 0. \tag{6.10}$$

Alternatively, one often identifies $\mathcal{Z}$ with, say, $L^2(\mathbb{R}^d, \mathrm{d}\xi)$. If z equals a function $\Xi \ni \xi \mapsto z(\xi)$, then

$$a^*(z) = \int z(\xi) a_\xi^* \mathrm{d}\xi, \quad a(z) = \int \overline{z}(\xi) a_\xi \mathrm{d}\xi.$$

Note that formally

$$[a(\xi), a^*(\xi')]_\mp = \delta(\xi - \xi'), \quad [a(\xi), a(\xi')]_\mp = 0. \tag{6.11}$$

The space $\otimes_{\mathrm{s/a}}^n \mathcal{Z}$ can then be identified with the space of symmetric/antisymmetric square integrable functions $L^2(\mathbb{R}^{nd})$, and then

$$\big(a(\xi)\Phi\big)(\xi_1', \dots, \xi_{n-1}') = \sqrt{n}\Phi(\xi, \xi_1', \dots, \xi_{n-1}'). \tag{6.12}$$

6.3 Integral kernel of an operator

Every linear operator A on $\mathbb{C}^n$ can be represented by a matrix $[A_i^j]$.

One would like to generalize this concept to infinite dimensional spaces (say, Hilbert spaces) and continuous variables instead of a discrete variables i, j. Suppose that a given vector space is represented, say, as $L^2(\mathbb{R}^d)$, or more generally,

$L^2(X)$ where X is a certain space with a measure. One often uses the representation of an operator A in terms of its *integral kernel* $\mathbb{R}^d\times\mathbb{R}^d \ni (x,y)\mapsto A(x,y)$, so that

$$A\Psi(x)=\int A(x,y)\Psi(y)\mathrm{d}y.$$

Note that strictly speaking $A(\cdot,\cdot)$ does not have to be a function. E.g. in the case $X=\mathbb{R}^d$ it could be a distribution, hence one often says the *distributional kernel* instead of the *integral kernel*. Sometimes $A(\cdot,\cdot)$ is ill-defined anyway. At least formally, we have

$$AB(x,y)=\int A(x,z)B(z,y)\mathrm{d}z,$$

$$A^*(x,y)=\overline{A(y,x)}.$$

Here is a situation where there is a good mathematical theory of integral/distributional kernels:

Theorem 6.1 (The Schwartz kernel theorem)**.** *B is a continuous linear transformation from $\mathcal{S}(\mathbb{R}^d)$ to $\mathcal{S}'(\mathbb{R}^d)$ iff there exists a distribution $B(\cdot,\cdot)\in\mathcal{S}'(\mathbb{R}^d\oplus\mathbb{R}^d)$ such that*

$$(\Psi|B\Phi)=\int\overline{\Psi(x)}B(x,y)\Phi(y)\mathrm{d}x\mathrm{d}y,\quad \Psi,\Phi\in\mathcal{S}(\mathbb{R}^d).$$

Note that $\Leftarrow$ is obvious. The distribution $B(\cdot,\cdot)\in\mathcal{S}'(\mathbb{R}^d\oplus\mathbb{R}^d)$ is called the *distributional kernel of the transformation B*. All bounded operators on $L^2(\mathbb{R}^d)$ satisfy the Schwartz kernel theorem.

Examples:

(1) $\mathrm{e}^{-\mathrm{i}xy}$ is the kernel of the Fourier transformation

(2) $\delta(x-y)$ is the kernel of identity.

(3) $\partial_x\delta(x-y)$ is the kernel of ∂_x.

6.4 Second quantization of operators

For a contraction q on $\mathcal{Z}$ the operator $q^{\otimes n}$ commutes with $\Theta(\sigma)$, $\sigma\in S_n$. Therefore, it preserves $\otimes^n_{\mathrm{s/a}}\mathcal{Z}$. We define the operator $\Gamma(q)$ on $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$ by

$$\Gamma(q)\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}}=q\otimes\cdots\otimes q\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}}.$$

$\Gamma(q)$ is called the *second quantization of* q.

Similarly, for an operator h on $\mathcal{Z}$ the operator $h\otimes 1^{(n-1)\otimes}+\cdots+1^{(n-1)\otimes}\otimes h$ preserves $\otimes^n_{\mathrm{s/a}}\mathcal{Z}$. We define the operator $\mathrm{d}\Gamma(h)$ by

$$\mathrm{d}\Gamma(h)\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}}=h\otimes 1^{(n-1)\otimes}+\cdots+1^{(n-1)\otimes}\otimes h\Big|_{\otimes^n_{\mathrm{s/a}}\mathcal{Z}}.$$

$\mathrm{d}\Gamma(h)$ is called the *(infinitesimal) second quantization of* h.

Note the identities

$$\Gamma(\mathrm{e}^{\mathrm{i}th}) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}, \quad \Gamma(q)\Gamma(r) = \Gamma(qr), \quad [\mathrm{d}\Gamma(h), \mathrm{d}\Gamma(k)] = \mathrm{d}\Gamma([h,k]),$$
$$\Gamma(q)\mathrm{d}\Gamma(h)\Gamma(q^{-1} = \mathrm{d}\Gamma(qhq^{-1}). \tag{6.13}$$

Let $\{e_i \mid i \in I\}$ be an orthonormal basis of $\mathcal{Z}$. Write $\hat{a}_i := \hat{a}(e_i)$. Let h be an operator on $\mathcal{Z}$ given by the matrix $[h_{ij}]$. Then

$$\mathrm{d}\Gamma(h) = \sum_{ij} h_{ij}\hat{a}_i^*\hat{a}_j. \tag{6.14}$$

Let us prove it in the bosonic case. Let $\Phi \in \Gamma_\mathrm{s}^n(\mathcal{Z})$.

$$\hat{a}_i^*\hat{a}_j\Phi = n\Theta_\mathrm{s}^n|e_i) \otimes \mathbb{1}^{(n-1)\otimes}(e_j| \otimes \mathbb{1}^{(n-1)\otimes}\Phi \tag{6.15}$$
$$= n\Theta_\mathrm{s}^n|e_i)(e_j| \otimes \mathbb{1}^{(n-1)\otimes}\Phi \tag{6.16}$$
$$= \frac{1}{(n-1)!}\sum_{\sigma\in S_n}\Theta(\sigma)|e_i)(e_j| \otimes \mathbb{1}^{(n-1)\otimes}\Theta(\sigma)^{-1}\Phi \tag{6.17}$$
$$= \sum_{k=1}^{n}\mathbb{1}^{(k-1)\otimes}|e_i)(e_j| \otimes \mathbb{1}^{(n-k)\otimes}\Phi. \tag{6.18}$$

More generally, if the integral kernel of an operator h is $h(x,y)$, then

$$\mathrm{d}\Gamma(h) = \int h(x,y)\hat{a}_x^*\hat{a}_y\mathrm{d}x\mathrm{d}y. \tag{6.19}$$

For instance, if h is the multiplication operator by $h(\xi)$, then $\mathrm{d}\Gamma(h) = \int h(\xi)\hat{a}_\xi^*\hat{a}_\xi\mathrm{d}\xi$.

6.5 Symmetric/antisymmetric tensor product

Let $\Psi \in \otimes_{\mathrm{s/a}}^p\mathcal{Z}$, $\Phi \in \otimes_{\mathrm{s/a}}^q\mathcal{Z}$. We set

$$\Psi \otimes_{\mathrm{s/a}} \Phi := \Theta_{\mathrm{s/a}}^{p+q}\Psi \otimes \Phi. \tag{6.20}$$

Note that

$$z \otimes \cdots \otimes z = z \otimes_\mathrm{s} \cdots \otimes_\mathrm{s} z. \tag{6.21}$$

If there are n terms, it is often written as $z^{n\otimes}$. In the antisymmetric case one usually prefers

$$\Psi \wedge \Phi := \frac{(p+q)!}{p!q!}\Psi \otimes_\mathrm{a} \Phi. \tag{6.22}$$

The operations $\otimes_\mathrm{s}$, $\otimes_\mathrm{a}$, $\wedge$ are associative. We have

$$y_1 \wedge \cdots \wedge y_n = \sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(n)}, \tag{6.23}$$
$$y_1 \otimes_\mathrm{a} \cdots \otimes_\mathrm{a} y_n = \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}(\sigma)y_{\sigma(1)} \otimes \cdots \otimes y_{\sigma(n)}. \tag{6.24}$$

Let $\{e_i\}_{i\in I}$ be a linearly ordered orthonormal basis in $\mathcal{Z}$. Then

$$\sqrt{n!}e_{i_1}\otimes_{\mathrm{a}}\cdots\otimes_{\mathrm{a}} e_{i_n},\quad i_1<\cdots<i_n, \tag{6.25}$$

forms an o.n.b of $\otimes_{\mathrm{a}}^n(\mathcal{Z})$.

$$\frac{\sqrt{n!}}{\sqrt{k_1!\cdots k_n!}}e_{i_1}^{\otimes k_1}\otimes_{\mathrm{s}}\cdots\otimes_{\mathrm{s}} e_{i_m}^{\otimes k_m},\quad k_1+\cdots+k_m=n, \tag{6.26}$$

forms an o.n.b of $\otimes_{\mathrm{s}}^m(\mathcal{Z})$.

If $\dim\mathcal{Z}=d$, then

$$\dim\otimes_{\mathrm{s}}^n\mathcal{Z}=\frac{(d+n-1)!}{(d-1)!n!},\quad \dim\otimes_{\mathrm{a}}^n\mathcal{Z}=\frac{d!}{n!(d-n)!}. \tag{6.27}$$

6.6 Exponential law

Let $\mathcal{Z},\mathcal{W}$ be Hilbert spaces. We can treat them as subspaces of $\mathcal{Z}\oplus\mathcal{W}$. Let $\Phi\in\otimes_{\mathrm{s/a}}^n\mathcal{Z}$, $\Psi\in\otimes_{\mathrm{s/a}}^m\mathcal{W}$. We can identify $\Phi\otimes\Psi$ with

$$U\Phi\otimes\Psi:=\sqrt{\frac{(n+m)!}{n!m!}}\Phi\otimes_{\mathrm{s/a}}\Psi\in\otimes_{\mathrm{s/a}}^{n+m}(\mathcal{Z}\oplus\mathcal{W}). \tag{6.28}$$

Theorem 6.2. *The map* (6.28) *extends to a unitary map*

$$U:\Gamma_{\mathrm{s/a}}(\mathcal{Z})\otimes\Gamma_{\mathrm{s/a}}(\mathcal{W})\to\Gamma_{\mathrm{s/a}}(\mathcal{Z}\oplus\mathcal{W}). \tag{6.29}$$

It satisfies

$$U\Omega\otimes\Omega=\Omega, \tag{6.30}$$
$$\mathrm{d}\Gamma(h\oplus g)U=U\big(\mathrm{d}\Gamma(h)\otimes\mathbb{1}+\mathbb{1}\otimes\mathrm{d}\Gamma(g)\big), \tag{6.31}$$
$$\Gamma(p\oplus q)U=U\Gamma(p)\otimes U\Gamma(q), \tag{6.32}$$
$$a^*(z\oplus w)U=U\big(a^*(z)\otimes\mathbb{1}+\mathbb{1}\otimes a^*(w)\big), \tag{6.33}$$
$$a(z\oplus w)U=U\big(a(z)\otimes\mathbb{1}+\mathbb{1}\otimes a(w)\big),\quad \textit{in the bosonic case}, \tag{6.34}$$
$$a^*(z\oplus w)U=U\big(a^*(z)\otimes\mathbb{1}+(-1)^N\otimes a^*(z)\big), \tag{6.35}$$
$$a(z\oplus w)U=U\big(a(z)\otimes\mathbb{1}+(-1)^N\otimes a(z)\big),\quad \textit{in the fermionic case.} \tag{6.36}$$

Proof. Let us prove the unitarity of this map in the symmetric case:

$$\Phi\otimes_{\mathrm{s}}\Psi=\frac{1}{(n+m)!}\sum_{\sigma\in S_{n+m}}\Theta(\sigma)\Phi\otimes\Psi \tag{6.37}$$
$$=\frac{n!m!}{(n+m)!}\sum_{[\sigma]\in S_{n+m}/S_n\times S_m}\Theta(\sigma)\Phi\otimes\Psi. \tag{6.38}$$

The terms in the sum on the right are mutually orthogonal. The maps $\Theta(\sigma)$ are unitary. The number of cosets in $S_{n+m}/S_n \times S_m$ is $\frac{(n+m)!}{n!m!}$. Therefore

$$\begin{aligned}(\Phi \otimes_{\rm s} \Psi|\Phi' \otimes_{\rm s} \Psi') &= \Big(\frac{n!m!}{(n+m)!}\Big)^2 \sum_{[\sigma]\in S_{n+m}/S_n\times S_m} \big(\Theta(\sigma)\Phi\otimes\Psi|\Theta(\sigma)\Phi'\otimes\Psi'\big) \\ &= \frac{n!m!}{(n+m)!}(\Phi\otimes\Psi|\Phi'\otimes\Psi'). \end{aligned} \tag{6.39}$$

□

6.7 Wick quantization

In this subsection we introduce the Wick quantization and Wick symbol. We will do this using a fixed orthonormal basis of the one-particle Hilbert space $\mathcal{Z}$. (We could also use a continuus variable representation, e.g. $L^2(\mathbb{R}^d)$, the reader can easily figure it out). Later, at (6.47), we will give an equivalent, more elegant but maybe less intuitive, basis-independent definition.

Let $e_i, \quad i = 1, 2, \dots$ be an o.n. basis of $\mathcal{Z}$. Let $b \in B(\otimes^k\mathcal{Z}, \otimes^m\mathcal{Z})$. It can be written as

$$b = \sum b_{i_m,\dots,i_1;j_k,\dots,j_1} |e_{i_m}\otimes\cdots\otimes e_{i_1})(e_{j_k}\otimes\cdots\otimes e_{j_1}|, \tag{6.40}$$

where $b_{i_m,\dots,i_1;j_k,\dots,j_1}$ are complex numbers–matrix elements of the operator b. We define the Wick quantization of b as an operator on $\Gamma^{\rm fin}_{\rm s/a}(\mathcal{Z})$ defined by

$$b(\hat{a}^*,\hat{a}) := \sum b_{i_m,\dots,i_1;j_k,\dots,j_1} \hat{a}^*_{i_1}\cdots\hat{a}^*_{i_m}\hat{a}_{j_k}\cdots\hat{a}_{j_1}. \tag{6.41}$$

Note that the notation $b(\hat{a}^*,\hat{a})$ suggests the the Wick quantization is "function of $\hat{a}^*,\hat{a}$", which in some sense is true. However, $b(\hat{a}^*,\hat{a})$ should be understood as an "indivisible symbol".

Note also that we inverted the order for creation operators in (6.41). This is irrelevant for bosons, where the order does not matter—it is convenient for fermions.

Note that a part of the information contained in b is irrelevant for $b(\hat{a}^*,\hat{a})$. In fact, by the (anti-)commutation relations, $b(\hat{a}^*,\hat{a})$ depends only on $\Theta^m_{\rm s/a} b \Theta^k_{\rm s/a}$. Thus we have an alternative definition: if $\tilde{b} \in B(\otimes^k_{\rm s/a}\mathcal{Z}, \otimes^m_{\rm s/a}\mathcal{Z})$ such that

$$\tilde{b} = \sum \tilde{b}_{i_m,\dots,i_1;j_k,\dots,j_1} |e_{i_m}\otimes\cdots\otimes e_{i_1})(e_{j_k}\otimes\cdots\otimes e_{j_1}|, \tag{6.42}$$

then

$$\tilde{b}(\hat{a}^*,\hat{a}) := \sum \tilde{b}_{i_m,\dots,i_1;j_k,\dots,j_1} \hat{a}^*_{i_1}\cdots\hat{a}^*_{i_m}\hat{a}_{j_k}\cdots\hat{a}_{j_1}. \tag{6.43}$$

Note that now the matrix elements of $\tilde{b}$ are automatically symmetric/antisymmetric in the first m/last k indices

With the operator $b(\hat{a}^*,\hat{a})$ we can associate the multilinear map

$$b(a^*,a) := \sum b_{i_m,\dots,i_1;j_k,\dots,j_1} a_{i_1}^* \cdots a_{i_m}^* a_{j_k} \cdots a_{j_1}, \tag{6.44}$$

where a_i^*, a_j are treated as commuting/anticommuting (classical) variables. This multilinear map $b(a^*,a)$ is called the *Wick symbol* of the operator $b(\hat{a}^*,\hat{a})$. It depends only on $\Theta_{\mathrm{s/a}}^m b \Theta_{\mathrm{s/a}}^k$. In the bosonic case it is a usual polynomial in the variables a^*, a. In the anticomuting case one also uses the term "polynomial" (in anticommuting variables) for such multilinear maps and their linear combinations.

It is not difficult to see that given an operator B, or actually a quadratic form on $\Gamma_{\mathrm{s/a}}^{\mathrm{fin}}(\mathcal{Z})$, there exist unique $\tilde{b}_{m,k} \in B(\otimes_{\mathrm{s/a}}^k \mathcal{Z}, \otimes_{\mathrm{s/a}}^m \mathcal{Z})$, $m,k = 0,\dots\infty$ such that

$$B = \sum_{m,k=0}^{\infty} \tilde{b}_{m,k}(\hat{a}^*,\hat{a}). \tag{6.45}$$

The polynomial

$$\sum_{m,k=0}^{\infty} \tilde{b}_{m,k}(a^*,a). \tag{6.46}$$

is called the *Wick symbol of B*.

Here is an equivalent definition of $b(\hat{a}^*,\hat{a})$ for $b \in B(\otimes^k \mathcal{Z}, \otimes^m \mathcal{Z})$. Its only nonzero matrix elements are between $\Phi \in \otimes_{\mathrm{s/a}}^{p+m} \mathcal{Z}$, $\Psi \in \otimes_{\mathrm{s/a}}^{p+k} \mathcal{Z}$, and equal

$$(\Phi|b(\hat{a}^*,\hat{a})\Psi) \quad = \quad \frac{\sqrt{(m+p)!(k+p)!}}{p!}(\Phi|b \otimes 1_{\mathcal{Z}}^{\otimes p}\Psi). \tag{6.47}$$

To see this we compute:

$$\begin{aligned}
&\big(\Phi|\hat{a}_{i_1}^* \cdots \hat{a}_{i_m}^* \hat{a}_{j_k} \cdots \hat{a}_{j_1}\Psi\big) && (6.48)\\
=&\big(\hat{a}_{i_m} \cdots \hat{a}_{i_1}\Phi|\hat{a}_{j_k} \cdots \hat{a}_{j_1}\Psi\big) && (6.49)\\
=&\sqrt{(m+p)\cdots(p+1)(k+p)\cdots(p+1)} && (6.50)\\
&\times \Big((e_{i_1}| \otimes \cdots \otimes (e_{i_m}| \otimes \mathbb{1}^{\otimes p}\Phi \Big| (e_{j_1}| \otimes \cdots \otimes (e_{j_k}| \otimes \mathbb{1}^{\otimes p}\Psi\Big). && (6.51)
\end{aligned}$$

6.8 Wick symbol and coherent states

In the bosonic case, we have the identities

$$\mathrm{e}^{-\hat{a}^*(w)+\hat{a}(w)}\hat{a}(v)\mathrm{e}^{\hat{a}^*(w)-\hat{a}(w)} = \hat{a}(v) + (v|w), \tag{6.52}$$

$$\mathrm{e}^{-\hat{a}^*(w)+\hat{a}(w)}\hat{a}^*(v)\mathrm{e}^{\hat{a}^*(w)-\hat{a}(w)} = \hat{a}^*(v) + (w|v). \tag{6.53}$$

We also introduce the coherent state corresponding to $w \in \mathcal{Z}$:

$$\Omega_w := \mathrm{e}^{\hat{a}^*(w)-\hat{a}(w)}\Omega. \tag{6.54}$$

Note that $\hat{a}(v)\Omega_w = (v|w)\Omega_w$. If $b \in B(\mathcal{Z}^{\otimes k}, \mathcal{Z}^{\otimes m})$, then we have the identity

$$\frac{(\Omega_w|b(\hat{a}^*, \hat{a})\Omega_z)}{(\Omega_w|\Omega_z)} = \big(w^{\otimes m}|bz^{\otimes k}\big). \tag{6.55}$$

Using the polynomial interpretation of b and treating $w = \sum_i w_i e_i$, $z = \sum_i z_i e_i$, as classical variables and writing w_i^* for the complex conjugate of w_i, we can rewrite (6.55) as

$$b(w^*, z). \tag{6.56}$$

6.9 Particle number preserving operators

If $m = k$, then the operator $b(\hat{a}^*, \hat{a})$ preserves the number of particles and (6.47). For $\Phi \in \otimes_{\mathrm{s/a}}^n \mathcal{Z}$, $\Psi \in \otimes_{\mathrm{s/a}}^n \mathcal{Z}$ it can be rewritten as

$$(\Phi|b(\hat{a}^*, \hat{a})\Psi) \quad = \quad \frac{n!}{(n-m)!}(\Phi|b \otimes 1_{\mathcal{Z}}^{\otimes(n-m)}\Psi). \tag{6.57}$$

But $\frac{n!}{(n-m)!m!}$ is the number of m-element subsets of $\{1, 2, \dots, n\}$. Therefore in the obvious notation, we can rewrite (6.57) as

$$\frac{1}{m!} b(\hat{a}^*, \hat{a}) = \sum_{1 \le i_1 < \cdots < i_m \le n} b_{i_1, \dots, i_m}. \tag{6.58}$$

In particular, for $m = 2$ we can write

$$\frac{1}{2} b(\hat{a}^*, \hat{a}) = \sum_{1 \le i < j \le n} b_{ij}. \tag{6.59}$$

Finally, for $m = 1$, on $\otimes_{\mathrm{s/a}}^n \mathcal{Z}$ we have

$$b(\hat{a}^*, \hat{a}) = \sum_{1 \le i \le n} b_i = \mathrm{d}\Gamma(b). \tag{6.60}$$

Set

$$\tilde{b} := \frac{1}{m!} \sum_{\sigma \in S_m} \Theta(\sigma) b \Theta(\sigma^{-1}). \tag{6.61}$$

We have

$$\Theta(\sigma)\tilde{b} = \tilde{b}\Theta(\sigma). \tag{6.62}$$

In fact, (6.62) equals $\mathrm{sgn}(\sigma)\tilde{b}$. Moreover,

$$\Theta_{\mathrm{s/a}}^m \tilde{b} \Theta_{\mathrm{s/a}}^m = \Theta_{\mathrm{s/a}}^m b \Theta_{\mathrm{s/a}}^m. \tag{6.63}$$

Therefore, when considering Wick polynomials in the number preserving case we can always assume (6.62).

6.10 Examples

Consider the Schrödinger Hamiltonian of n identical particles on $L^2(\mathbb{R}^{dN})$

$$H_n = -\sum_{i=1}^{n} \Delta_i + \sum_{1\leq i<j\leq n} V(x_i - x_j), \tag{6.64}$$

$$P_n = \sum_{i=1}^{n} \frac{1}{\mathrm{i}} \partial_{x_i}, \tag{6.65}$$

In the momentum representation

$$\begin{aligned} H_n &= \sum_{i=1}^{n} p_i^2 \\ &\quad + (2\pi)^{-d} \sum_{1\leq i<j\leq N} \delta(p_i' + p_j' - p_j - p_i) \hat{V}(p_i' - p_i). \\ P_n &= \sum_{i=1}^{n} p_i. \end{aligned}$$

Consider the 2nd quantization of $L^2(\mathbb{R}^d)$. We have the position representation, with the generic variables x, y and the momentum representation with the generic variables k, k'. We can pass from one representation to the other by

$$a^*(k) = (2\pi)^{-\frac{d}{2}} \int a^*(x) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x, \quad a^*(x) = (2\pi)^{-\frac{d}{2}} \int a^*(k) \mathrm{e}^{\mathrm{i}kx} \mathrm{d}k, \tag{6.66}$$

$$a(k) = (2\pi)^{-\frac{d}{2}} \int a(x) \mathrm{e}^{\mathrm{i}kx} \mathrm{d}x, \quad a(x) = (2\pi)^{-\frac{d}{2}} \int a(k) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}k. \tag{6.67}$$

In the 2nd quantized notation we can rewrite all this as

$$\begin{aligned} H := \mathop{\oplus}_{n=0}^{\infty} H_n &= -\int a_x^* \Delta_x a_x \mathrm{d}x \qquad (6.68) \\ &\quad + \frac{1}{2} \int\int \mathrm{d}x \mathrm{d}y V(x-y) a_x^* a_y^* a_y a_x \\ &= \int p^2 a_p^* a_p \mathrm{d}p \qquad (6.69) \\ &\quad + \frac{1}{2} (2\pi)^{-d} \int\int\int \mathrm{d}p \mathrm{d}q \mathrm{d}k \hat{V}(k) a_{p+k}^* a_{q-k}^* a_q a_p \\ P := \mathop{\oplus}_{n=0}^{\infty} P_n &= \int a_x^* \frac{1}{\mathrm{i}} \partial_x a_x \mathrm{d}x \qquad (6.70) \\ &= \int p a_p^* a_p \mathrm{d}p. \qquad (6.71) \end{aligned}$$

Consider $L^2([0,L]^d) \simeq L^2\left(\frac{2\pi}{L}\mathbb{Z}^d\right)$ and its 2nd quantization. Again we use x, y in the position representation with periodic boundary conditions and k, k'

in the momentum representation. We can pass from one representation to the other by

$$a^*(k) = L^{-\frac{d}{2}} \int a(x) \mathrm{e}^{-\mathrm{i}kx} \mathrm{d}x, \qquad a^*(x) = L^{-\frac{d}{2}} \sum_k a(k) \mathrm{e}^{\mathrm{i}kx}, \tag{6.72}$$

$$a(k) = L^{-\frac{d}{2}} \int a(x) \mathrm{e}^{\mathrm{i}kx} \mathrm{d}x, \qquad a(x) = L^{-\frac{d}{2}} \sum_k a(k) \mathrm{e}^{-\mathrm{i}kx}. \tag{6.73}$$

Here are the analogs of (6.69) and (6.71):

$$\begin{aligned} H = & \sum_p p^2 a_p^* a_p \\ & + \frac{1}{2L^d} \sum_p \sum_q \sum_k \hat{V}(k) a_{p+k}^* a_{q-k}^* a_q a_p, \\ P = & \sum_p p a_p^* a_p. \end{aligned}$$

6.11 Problems

Problem 6.3. *On $\mathbb{C}^2$ consider $h = \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix}$ Find the spectrum of $\mathrm{d}\Gamma(h)$*

1. *in the bosonic case, that is, on $\Gamma_{\mathrm{s}}(\mathbb{C}^2)$;*
2. *in the fermionic case, that is, on $\Gamma_{\mathrm{a}}(\mathbb{C}^2)$.*

Problem 6.4. *Find the spectrum of $H = a^* a + \frac{\overline{\lambda}}{2} a^2 + \frac{\lambda}{2} a^{*2}$.*

Hint: For $|\lambda| < 1$ set

$$\mu = \frac{1}{\lambda}(1 - \sqrt{1 - \lambda^2}), \quad b = \frac{1}{\sqrt{1 - \mu^2}}(a + \mu a^*), \tag{6.74}$$

Then $[b, b^*] = 1$ and

$$\begin{aligned} H = & \frac{1}{2}(a^* a + a a^* + \overline{\lambda} a^2 + \lambda a^{*2}) - \frac{1}{2} \\ = & \frac{(1 + \mu^2)}{2(1 - \mu^2)}(b^* b + b b^*) - \frac{1}{2} \\ = & \frac{(1 + \mu^2)}{(1 - \mu^2)} b^* b + \frac{\mu^2}{1 - \mu^2}. \end{aligned}$$

Problem 6.5. *Compute*

$$\Gamma(q) a^*(z) \Gamma(q^{-1}), \quad \Gamma(q) a(z) \Gamma(q^{-1}); \tag{6.75}$$

$$[\mathrm{d}\Gamma(h), a^*(z)], \quad [\mathrm{d}\Gamma(h), a(z)]. \tag{6.76}$$

Answers

$$a^*(qz), \qquad a(q^{*-1}z), \tag{6.77}$$
$$a^*(hz), \qquad -a(h^*z). \tag{6.78}$$

In the next two problem we consider the Fock space $\Gamma_{\mathrm{s}}(\mathbb{C})$.

Problem 6.6. *Compute*

$$\mathrm{e}^{t\hat{a}^*\hat{a}}\hat{a}^*\mathrm{e}^{-t\hat{a}^*\hat{a}}, \qquad \mathrm{e}^{t\hat{a}^*\hat{a}}a\mathrm{e}^{-t\hat{a}^*\hat{a}}, \tag{6.79}$$
$$\mathrm{e}^{\frac{t}{2}(\hat{a}^*-\hat{a}^2)}a^*\mathrm{e}^{\frac{t}{2}(-\hat{a}^*+\hat{a}^2)}, \qquad \mathrm{e}^{\frac{t}{2}(\hat{a}^*-\hat{a}^2)}a\mathrm{e}^{\frac{t}{2}(-\hat{a}^*+\hat{a}^2)}. \tag{6.80}$$

Answers

$$\mathrm{e}^{t}\hat{a}^*, \qquad \mathrm{e}^{-t}\hat{a}, \tag{6.81}$$
$$-\hat{a}\sinh t + \hat{a}^*\cosh t \qquad \hat{a}\cosh t - a^*\sinh t. \tag{6.82}$$

Problem 6.7. *Find the Wick symbols of*

1. $(\hat{a}^* + \hat{a})^3$*;*
2. $\hat{a}^2\hat{a}^{*2}$*;*
3. $\mathrm{e}^{(t\hat{a}^*-\bar{t}\hat{a})}$*;*
4. $\mathrm{e}^{t\hat{a}^*\hat{a}} = \Gamma(\mathrm{e}^t)$*;*
5. $\mathrm{e}^{\frac{t}{2}(\hat{a}^{*2}-\hat{a}^2)}$.

Answers:

1. $\hat{a}^{*3} + 3\hat{a}^{*2}\hat{a} + 3\hat{a}^*\hat{a}^2 + \hat{a}^3 + 3\hat{a}^* + 3\hat{a}$;
2. $\hat{a}^{*2}a^2 + 4\hat{a}^*\hat{a} + 2$;
3. $\mathrm{e}^{t\hat{a}^*}\mathrm{e}^{-\frac{|t|^2}{2}}\mathrm{e}^{-\bar{t}\hat{a}}$,
4. $\mathrm{e}^{\hat{a}^*(\mathrm{e}^t-1)\hat{a}}$,
5. $\mathrm{e}^{\frac{\tanh t}{2}\hat{a}^{*2}}\mathrm{e}^{\frac{-\ln\cosh t}{2}}\mathrm{e}^{\hat{a}^*(\frac{1}{\cosh t}-1)\hat{a}}\mathrm{e}^{\frac{-\tanh t}{2}\hat{a}^2}$

7 Slater determinants and the Hartree-Fock method

7.1 Fermionic Fock space

Let $\mathcal{W}$ be a Hilbert space. We consider the fermionic Fock space $\Gamma_{\mathrm{a}}(\mathcal{W})$. Suppose that $e_1, e_2, \dots$ is an o.n. basis of $\mathcal{W}$. We use two conventions:

$$a_i := a(e_i), \quad a_i^* := a^*(e_i).$$

Then

$$[a_i, a_j]_+ = 0, \quad [a_i, a_j^*]_+ = \delta_{i,j}, \quad a_i\Omega = 0. \tag{7.83}$$

Alternatively, we use creation/annihilation operators parametrized with $w = \sum w_i e_i \in \mathcal{W}$ writing

$$a(w) = \sum \overline{w}_i a_i, \quad a^*(w) = \sum w_i a_i^*,$$

We can write

$$[a(w), a(w')]_+ = 0, \quad [a(w), a_j^*(w')]_+ = (w|w'), \quad a(w)\Omega = 0. \tag{7.84}$$

7.2 Slater determinants

Let $\mathcal{Z}$ be a finite dimensional subspace of $\mathcal{W}$. Without loss of generality we can assume that it is spanned by $e_1, \dots, e_m$. Then

$$a^*(e_m)\cdots a^*(e_1)\Omega = \frac{1}{\sqrt{m!}} \sum_{\sigma\in S_m} \mathrm{sgn}\sigma e_{\sigma(m)} \otimes \cdots \otimes e_{\sigma(1)} \tag{7.85}$$

$$= \sqrt{m!} e_m \otimes_{\mathrm{a}} \cdots \otimes_{\mathrm{a}} e_1 = \frac{1}{\sqrt{m!}} e_m \wedge \cdots \wedge e_1 \tag{7.86}$$

is a normalized vector. Such vectors are called *Slater determinants*. If $f_1, \dots, f_m$ is another basis of $\mathcal{Z}$, so that $e_i = \sum_j c_{ij} f_j$, then

$$a^*(e_m)\cdots a^*(e_1)\Omega = \det[c_{ij}] a^*(f_m)\cdots a^*(f_1)\Omega.$$

Let π denote the orthogonal projection on the space $\mathcal{Z}$. Note that the state

$$\omega_\pi(A) := \big(a^*(e_m)\cdots a^*(e_1)\Omega | A a^*(e_m)\cdots a^*(e_1)\Omega\big) \tag{7.87}$$

depends only on the space $\mathcal{Z}$ (or equivalently on π).

Suppose now that $e_j, \quad j = 1, 2, \dots$ is an o.n. basis of $\mathcal{W}$. Then the vectors $a_{i_m}^* \cdots a_{i_1}^* \Omega$, $i_1 < \cdots < i_m$ form an orthonormal basis of $\otimes_{\mathrm{a}}^m \mathcal{W}$.

7.3 Changing the vacuum

Let us introduce a new notation for the old creation/annihilation operators. Set

$$\tilde{a}_i := \begin{cases} a_i^* & i \le m, \\ a_j & j > m; \end{cases} \quad \tilde{a}_i^* := \begin{cases} a_i & i \le m, \\ a_j^* & j > m. \end{cases}.$$

Then $\tilde{a}_i, \tilde{a}_i^*, i = 1, \dots$ satisfy the usual anticommutation relations

$$[\tilde{a}_i, \tilde{a}_j]_+ = 0, \quad [\tilde{a}_i, \tilde{a}_j^*]_+ = \delta_{i,j}, \tag{7.88}$$

with the vacuum $\tilde{\Omega} := a_m^* \cdots a_1^* \Omega$:

$$\tilde{a}_i \tilde{\Omega} = 0. \tag{7.89}$$

Let us introduce the complex conjugation on the space $\mathcal{Z}$:

$$\mathcal{Z} \ni w = \sum w_n e_n \mapsto \overline{w} := \sum \overline{w}_i e_i \in \mathcal{Z}.$$

Then we can set

$$\tilde{a}(w) := \sum_{i=1}^{n} \tilde{a}_i \overline{w}_i + \sum_{j=n+1}^{\infty} w_j \tilde{a}_j, \tag{7.90}$$

$$\tilde{a}^*(w) := \sum_{i=1}^{n} \tilde{a}_i^* \overline{w}_i + \sum_{j=n+1}^{\infty} w_j \tilde{a}_j^*. \tag{7.91}$$

Written more compactly, and denoting the complex conjugation by $\mathcal{C}w = \overline{w}$, we can write this as

$$\tilde{a}(w) = a^*(\mathcal{C}\pi w) + a\big((\mathbb{1} - \pi)w\big), \quad \tilde{a}^*(w) = a(\mathcal{C}\pi w) + a^*\big((\mathbb{1} - \pi)w\big). \tag{7.92}$$

Then $\tilde{a}(w)$, $\tilde{a}^*(w)$ satisfy the usual commutation relations with vacuum $\tilde{\Omega}$

$$[\tilde{a}(w), \tilde{a}(w')]_+ = 0, \quad [\tilde{a}(w), \tilde{a}_j^*(w')]_+ = (w|w'), \quad \tilde{a}(w)\tilde{\Omega} = 0. \tag{7.93}$$

Thus in the new representation the 1-particle space is $\mathcal{C}\pi\mathcal{W} \oplus (\mathbb{1} - \pi)\mathcal{W}$ and not $\mathcal{W}$.

We can implement this change (up to the sign for odd dim $\mathcal{Z}$) by the unitary transformation $U : \Gamma_{\mathrm{a}}(\mathcal{W}) \to \Gamma_{\mathrm{a}}\big(\mathcal{C}\pi\mathcal{W} \oplus (\mathbb{1} - \pi)\mathcal{W}\big)$ defined by

$$U := \prod_{i=1}^{m} (-a_i + a_i^*). \tag{7.94}$$

In fact,

$$Ua_i^* U^* = (-1)^m \tilde{a}_i, \tag{7.95}$$

$$Ua_i U^* = (-1)^m \tilde{a}_i, \tag{7.96}$$

$$U\Omega = \tilde{\Omega}. \tag{7.97}$$

In fact

$$(-a + a^*)a(-a + a^*)^* = -a^* a a^* = -a^*(aa^* + a^* a) = -a^*, \tag{7.98}$$

$$(-a + a^*)a^*(-a + a^*)^* = -aa^* a = -a(aa^* + a^* a) = -a. \tag{7.99}$$

In this construction $\mathcal{Z}$ is often called the space of antiparticles.

The operators $\tilde{a}_i$, $\tilde{a}_i^*$ for $i = 1, \ldots, m$ are often denoted by a different letter, say, b_i, b_i^*.

7.4 Free fermionic Hamiltonians

Consider $H = \mathrm{d}\Gamma(h)$, where h is a self-adjoint operator on $\mathcal{W}$. For simplicity, assume that H has discrete spectrum and is bounded from below. We can diagonalize h in an o. n. basis $e_1, e_2, \dots$, so that

$$h = \sum_i \lambda_i |e_i)(e_i|,$$

where λ_i $i = 1, 2, \dots$ is an increasing sequence.

It is easy to see that $\mathrm{d}\Gamma(h)$ possesses a unique ground state iff $0 \notin \sigma(h)$. Indeed, let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m < 0 < \lambda_{m+1} \leq \dots$. Then the ground state of $\mathrm{d}\Gamma(h)$ is given by

$$\Phi := a_m^* \cdots a_1^* \Omega,$$

so that

$$H\Phi = E\Phi, \quad E = \lambda_1 + \cdots + \lambda_m.$$

Setting $b_i := a_i^*$, $b_i^* := a_i$ for $i \leq m$, the Hamiltonian H can be rewritten as

$$H = \sum_i \lambda_i a_i^* a_i = \sum_{i \leq m} |\lambda_i| b_i^* b_i + \sum_{i > m} \lambda_i a_i^* a_i + E.$$

The constant E is usually dropped and we use the renormalized Hamiltonian

$$H_{\mathrm{ren}} = \sum_{i \leq m} |\lambda_i| b_i^* b_i + \sum_{i > m} \lambda_i a_i^* a_i.$$

Example 7.1. *Consider the free Fermi gas with the chemical potential μ in volume L.*

$$H = \sum_{k \in \frac{2\pi}{L}\mathbb{Z}^d} (k^2 - \mu) a_k^* a_k.$$

The ground state is called the "Fermi sea": $\prod_{k^2 < \mu} a_k^* \Omega$. It has the energy

$$E = \sum_{k^2 < \mu} (k^2 - \mu).$$

The renormalized Hamiltonian is

$$H_{\mathrm{ren}} = \sum_{k^2 < \mu} |k^2 - \mu| b_k^* b_k + \sum_{k^2 \geq \mu} |k^2 - \mu| a_k^* a_k.$$

7.5 CAR algebra

Consider symbols a_i, a_i^*, $i = 1, \dots, n$. We can form the $*$-algebra spanned by monomials

$$a_{j_1}^{?} \cdots a_{j_k}^{?} \tag{7.100}$$

where ? is either empty or $*$. The product is the concatenation, the involution is putting $*$, where $a_i^{**} = a_i$, and reversing the order.

Next we impose the relations

$$[a_i, a_j]_+ = 0, \quad [a_i, a_j^*]_+ = \delta_{i,j}, \quad i, j = 1, \dots, n. \tag{7.101}$$

We obtain a $*$-algebra, which we denote $\mathrm{CAR}(\mathbb{C}^n)$.

Theorem 7.2. *The $*$-algebra $\mathrm{CAR}(\mathbb{C}^n)$ is $*$-isomorphic to $B(\otimes^n \mathbb{C}^2)$. More precisely, if we identify*

$$\Gamma_{\mathrm{a}}(\mathbb{C}^n) \simeq \mathop{\otimes}_{j=1}^{n} \Gamma_{\mathrm{a}}(\mathbb{C}e_j) \simeq \otimes^n \mathbb{C}^2, \tag{7.102}$$

then a_j, a_j^* are given by the usual annihilation/creation operators.

Note that the above $*$-isomorphism endows $\mathrm{CAR}(\mathbb{C}^n)$ with a norm which satisfies all axioms of a C^*-algebra. Therefore, $\mathrm{CAR}(\mathbb{C}^n)$ becomes a C^*-algebra.

We have the embedding

$$\mathrm{CAR}(\mathbb{C}^n) \simeq B(\otimes^n \mathbb{C}^2) \ni A \mapsto A \otimes 1\!\!1_{\mathbb{C}^2} \in B(\otimes^{n+1} \mathbb{C}^2) \simeq \mathrm{CAR}(\mathbb{C}^{n+1}). \tag{7.103}$$

Consider a Hilbert space $\mathcal{W}$, possibly infinite dimensional. Let $a(w)$, $a^*(w)$ be symbols satisfying

$$[a(w), a(w')]_+ = 0, \quad [a(w), a_j^*(w')]_+ = (w|w'). \tag{7.104}$$

Let $\mathrm{CAR}_0(\mathcal{W})$ denote the $*$-algebra generated by these symbols, as above. Clearly, for any finite dimensional $\mathcal{Z} \subset \mathcal{W}$ of dimension n, $\mathrm{CAR}(\mathcal{Z})$ is isomorphic to $B(\otimes^n \mathbb{C}^2)$. Thus $\mathrm{CAR}_0(\mathcal{W})$ is endowed by a unique norm. Let $\mathrm{CAR}(\mathcal{W}) := \mathrm{CAR}_0(\mathcal{W})^{\mathrm{cpl}}$.

If we fix an o.n. basis $e_1, e_2, \dots$ of $\mathcal{W}$, and we identify $\mathbb{C}^n$ with $\mathrm{Span}(e_1, \dots e_n)$, then

$$\bigcup_{n=1}^{\infty} \mathrm{CAR}(\mathbb{C}^n) \simeq \bigcup_{n=1}^{\infty} B(\otimes^n \mathbb{C}^2) \tag{7.105}$$

is dense in $\mathrm{CAR}_0(\mathcal{W})$. Now (7.105) is $\mathrm{UHF}_0(2^\infty)$, whose completion is $\mathrm{UHF}(2^\infty)$. Hence

$$\mathrm{CAR}(\mathcal{W}) \simeq \mathrm{UHF}(2^\infty). \tag{7.106}$$

The $*$-algebra $\mathrm{CAR}(\mathcal{W})$ has an obvious representation on $\Gamma_{\mathrm{a}}(\mathcal{W})$. We will denote this representation by ρ, so that

$$\rho(a(w)) = a(w), \quad \rho(a^*(w)) = a^*(w), \tag{7.107}$$

where on the left we have "abstract symbols", and on the right "concrete operators in $B(\Gamma_{\mathrm{a}}(\mathcal{W}))$". We will see that often other representations are preferable.

7.6 Antiparticles of any dimension

Now let $\mathcal{Z}$ is a closed subspace of $\mathcal{W}$ of any dimension. Let π be the projection onto $\mathcal{Z}$. We choose an antiunitary involution $\mathcal{Z} \ni z \mapsto \mathcal{C}z \in \mathcal{Z}$ called "charge conjugation". Then we set

$$\tilde{\rho}\big(a(w)\big) := a^*\big(\mathcal{C}\pi w\big) + a\big(\mathbb{1} - \pi)w\big) \in B(\Gamma_{\mathrm{a}}(\mathcal{C}\mathcal{Z} \oplus \mathcal{Z}^\perp), \tag{7.108}$$

$$\tilde{\rho}\big(a^*(w)\big) := a\big(\mathcal{C}\pi w\big) + a^*\big(\mathbb{1} - \pi)w\big) \in B(\Gamma_{\mathrm{a}}(\mathcal{C}\mathcal{Z} \oplus \mathcal{Z}^\perp) \tag{7.109}$$

satisfy the usual anticommutation relations. Therefore, they extend to a $*$-representation

$$\tilde{\rho} : \mathrm{CAR}(\mathcal{W}) \to B(\Gamma_{\mathrm{a}}(\mathcal{C}\mathcal{Z} \oplus \mathcal{Z}^\perp) \tag{7.110}$$

If $\mathcal{Z}$ is infinite dimensional, there is no unitary operator U that intertwines the two kinds of representations of CAR. More precisely, if $U : \Gamma_{\mathrm{a}}(\mathcal{W}) \to \Gamma_{\mathrm{a}}(\mathcal{C}\mathcal{Z} \oplus \mathcal{Z}^\perp)$ and

$$U\rho(A) = \tilde{\rho}(A)U, \quad A \in \mathrm{CAR}(\mathcal{W}), \tag{7.111}$$

then $U = 0$.

In particular, there exists no vector killed by $\tilde{\rho}(a(w))$.

7.7 Fermionic positive energy quantization

Seppose now that h is a self-adjoint operator on $\mathcal{W}$. Then on $\mathrm{CAR}(\mathcal{W})$ we have a 1-parameter $*$-automorphism group given by

$$\alpha_t(a(w)) := a(\mathrm{e}^{\mathrm{i}th}w), \quad \alpha_t(a^*(w)) := a^*(\mathrm{e}^{\mathrm{i}th}w), \quad w \in \mathcal{W}. \tag{7.112}$$

In the basic representation

$$\rho : \mathrm{CAR}(\mathcal{W}) \to B\big(\Gamma_{\mathrm{a}}(\mathcal{W})\big) \tag{7.113}$$

we have for $H := \mathrm{d}\Gamma(h)$

$$\rho\big(\alpha_t(A)\big) = \mathrm{e}^{\mathrm{i}tH}\rho(A)\mathrm{e}^{-\mathrm{i}tH}. \tag{7.114}$$

Unfortunately, if h is not positive, then neither is$H$ and one can argue that H is not physical.

Assume for simplicity that $\mathbb{1}_0(h) = 0$. Set $\Lambda_\pm := \mathbb{1}_{[0,\infty[}(\pm h)$. Choose a conjugation $\mathcal{C}$ on $\Lambda_-\mathcal{W}$. Then we can change the representation to $\Gamma_{\mathrm{a}}(\mathcal{C}\Lambda_-\mathcal{W} \oplus \Lambda_+\mathcal{W})$. The new renormalized Hamiltonian is

$$\tilde{H} := \mathrm{d}\Gamma(-\mathcal{C}\Lambda_- h\mathcal{C} \oplus \Lambda_+ h), \tag{7.115}$$

which is positive. We have

$$\tilde{\rho}\big(\alpha_t(A)\big) = \mathrm{e}^{\mathrm{i}t\tilde{H}}\tilde{\rho}(A)\mathrm{e}^{-\mathrm{i}t\tilde{H}}. \tag{7.116}$$

Example 7.3. *In infinite volume the Hamiltonian of free Fermi gas is*

$$H = \int (k^2 - \mu) a_k^* a_k \mathrm{d}k.$$

E is infinite and the Slater determinant is ill defined. However, we can change the representation of CAR replacing H with

$$H_{\mathrm{ren}} = \int_{k^2<\mu} |k^2 - \mu| b_k^* b_k \mathrm{d}k + \int_{k^2\geq\mu} |k^2 - \mu| a_k^* a_k \mathrm{d}k.$$

Example 7.4. *Consider the Dirac Hamiltonian*

$$h := \vec{\alpha}\vec{p} + \beta m + V(x).$$

It is a self-adjoint operator on $L^2(\mathbb{R}^3 \otimes \mathbb{C}^4)$ The naive quantization of h, that is $\mathrm{d}\Gamma(h)$, acts on the space $\Gamma_{\mathrm{a}}\big(L^2(\mathbb{R}^3 \otimes \mathbb{C}^4)\big)$. It is however physically meaningless—it yields an operator unbounded from below. Formally, the ground state of $\mathrm{d}\Gamma(h)$ is the Slater determinant with all negative energy states present. This state is called the Dirac sea*.*

In practice, we change the representation of CAR. Set

$$\Lambda^\pm := 1\!\!1_{[0,\infty[}(\pm h).$$

The physical one particle space is

$$\mathcal{C}\Lambda^- L^2(\mathbb{R}^3 \otimes \mathbb{C}^4) \oplus \Lambda^+ L^2(\mathbb{R}^3 \otimes \mathbb{C}^4),$$

where $\mathcal{C}$ is an antilinear map, usually the charge conjugation.

7.8 Expectation values of Slater determinants

Theorem 7.5. *Let b be an operator on $\otimes^m \mathcal{W}$. Let π be an orthogonal projection onto a subspace of $\mathcal{W}$ and ω_π the corresponding Slater determinant state. Then*

$$\omega_\pi\big(b(a^*, a)\big) = \sum_{\sigma \in S_m} \mathrm{Tr}\, b\, \pi^{\otimes m}\, \Theta(\sigma) \mathrm{sgn}(\sigma).$$

Proof. It is enough to check this assuming that

$$b = |e_{i_1} \otimes \cdots \otimes e_{i_m})(e_{j_1} \otimes \cdots \otimes e_{j_m}|,$$

corresponding to

$$b(a^*, a) = a_{i_1}^* \cdots a_{i_m}^* a_{j_m} \cdots a_{j_1}.$$

Now

$$\omega_\pi\big(b(a^*, a)\big) \tag{7.117}$$

$$=(a_1^* \cdots a_n^* \Omega | a_{i_1}^* \cdots a_{i_m}^* a_{j_m} \cdots a_{j_1}\ a_1^* \cdots a_n^* \Omega) \tag{7.118}$$

is nonzero only if $i_1, \ldots, i_m$ are distinct,

$$\{i_1, \ldots, i_m\} = \{j_1, \ldots, j_m\} \subset \{1, \ldots, n\}.$$

Then it is $\mathrm{sgn}\sigma$, where σ is the unique permutation that maps $\{j_1, \ldots, j_m\}$ onto $\{i_1, \ldots, i_m\}$. Clearly,

$$\Theta(\sigma) e_{j_1} \otimes \cdots \otimes e_{j_m} = e_{i_1} \otimes \cdots \otimes e_{i_m}. \tag{7.119}$$

Thus (7.118) is

$$\mathrm{sgn}(\sigma) \mathrm{Tr}\, \pi^{\otimes m} |\mathrm{e}_{i_1} \otimes \cdots \otimes e_{i_m})(e_{j_m} \otimes \cdots \otimes e_{j_1}| \Theta(\sigma). \tag{7.120}$$

□

In particular, we have the cases $n = 1, 2$:

$$\omega_\pi\big(\mathrm{d}\Gamma(h)\big) = \mathrm{Tr}\pi h, \tag{7.121}$$

$$\omega_\pi\big(b(a^*, a)\big) = \mathrm{Tr}\, b\, \pi \otimes \pi (\mathbb{1} - \tau), \tag{7.122}$$

where $\tau : \mathcal{W} \otimes \mathcal{W} \to \mathcal{W} \otimes \mathcal{W}$ is the transposition of the factors in the tensor product.

7.9 The Hartree-Fock method

Let h be a self-adjoint operator on $\mathcal{W}$ and b on $\mathcal{W} \otimes \mathcal{W}$. We assume that $\tau b \tau = b$. Consider the particle number preserving operator

$$H = \mathrm{d}\Gamma(h) + \frac{1}{2} b(a^*, a) \tag{7.123}$$

$$= \sum h_{ij} a_i^* a_j + \frac{1}{2} \sum b_{i_2 i_1 j_2 j_1} a_{i_1}^* a_{i_2}^* a_{j_2} a_{j_1}, \tag{7.124}$$

where in the second line we recall the standard definitions of Wick quantizations using an arbitrary o.n. basis. We would like to find the ground state energy of H in the n-body sector.

The Hartree-Fock functional is the expectation value of H in a Slater determinant:

$$\mathcal{E}_{\mathrm{HF}}(\pi) := \omega_\pi(H) = \mathrm{Tr} h \pi + \frac{1}{2} \mathrm{Tr}\, b\, \pi \otimes \pi\, (\mathbb{1} - \tau) \tag{7.125}$$

$$\sum h_{ij} \pi_{ji} + \frac{1}{2} \sum \big(b_{i_2 i_1 j_2 j_1} \pi_{i_2 j_2} \pi_{i_1 j_1} - b_{i_2 i_1 j_2 j_1} \pi_{i_2 j_1} \pi_{i_1 j_2}\big). \tag{7.126}$$

The ground state energy of H is clearly estimated from above by its Hartree-Fock energy

$$E_{\mathrm{HF}} := \inf\{\mathcal{E}_{\mathrm{HF}}(\pi) \ :\ \pi \text{ is an } n\text{-dimensional orthogonal projection}\}.$$

If a minimizer of $\mathcal{E}_{\rm HF}$ exists, we denote it by $\pi_{\rm HF}$. We define the Hartree-Fock Hamiltonian (called also the Fock Hamiltonian) by its expectation value in a trace class matrix γ:

$$\mathrm{Tr} h_{\rm HF}\gamma := \mathrm{Tr} h\gamma + \mathrm{Tr}\, b\, \pi_{\rm HF}\otimes\gamma\, (\mathbb{1}-\tau).$$

Notice the absence of $\frac{1}{2}$.

Theorem 7.6. *$\pi_{\rm HF}$ is a projection onto n lowest lying levels of $h_{\rm HF}$*

Proof. Every orthogonal projection has the kernel

$$\pi(x,y) = \sum_{i=1}^{n} \overline{\phi_i(x)}\phi_i(y),$$

where $\phi_1,\dots,\phi_n$ is an orthonormal basis of $\mathrm{Ran}\pi$. The Hartree-Fock functional can be written as

$$\begin{aligned}\mathcal{E}_{\rm HF}(\pi) =: \mathcal{E}(\phi_1,\dots\phi_n) &= \sum_i (\phi_i|h\phi_i)\\ &+ \frac{1}{2}\sum_{ij}(\phi_i\otimes\phi_j|b\,\phi_i\otimes\phi_j) - \frac{1}{2}\sum_{ij}(\phi_i\otimes\phi_j|b\,\phi_j\otimes\phi_i).\end{aligned}$$

Using the method of Lagrange multipliers, $E_{\rm HF}$ is given as the infimum of

$$\mathcal{E}_{\rm HF}(\phi_1,\dots,\phi_n) - \sum_{ij}\epsilon_{ij}\big((\phi_i|\phi_j)-\delta_{ij}\big),$$

where we may assume that the matrix ϵ_{ij} is Hermitian. Writing $\phi_i+\delta\phi_i$, $\epsilon_{ij}+\delta\epsilon_{ij}$ for the variations, we find

$$\delta\mathcal{E}_{\rm HF} = \sum_i \big(\phi_i|h_{\rm HF}\delta\phi_i\big) + \big(\delta\phi_i|h_{\rm HF}\phi_i\big) \tag{7.127}$$

$$- \sum_{ij}\epsilon_{ij}(\phi_i|\delta\phi_j) - \sum_{ij}\epsilon_{ij}(\delta\phi_i|\phi_j) \tag{7.128}$$

$$+ \sum_{ij}\delta\epsilon_{ij}\big((\phi_i|\phi_j)-\delta_{ij}\big). \tag{7.129}$$

Comparing the coefficients at $\delta\phi_i$ on the right of the scalar product and on the left of the scalar product independently, we obtain

$$h_{\rm HF}\phi_i = \sum_j \epsilon_{ij}\phi_j.$$

We can diagonalize the matrix $[\epsilon_{ij}]$ with a unitary transformation, so that $\epsilon_{ij} = \delta_{ij}\epsilon_i$, and we obtain

$$h_{\rm HF}\phi_i = \epsilon_i\phi_i.$$

Thus the minimizing sequence $\phi_1, \dots, \phi_n$ can consist of normalized eigenvectors of h_{HF}.

Now assume that there is an eigenvector of h_{HF}, say ψ orthogonal to $\phi_1, \dots \phi_n$ and with an eigenvalue β lower than one of the eigenvalues $\epsilon_1, \dots, \epsilon_n$. For instance,

$$h_{\mathrm{HF}}\psi = \beta\psi, \quad \beta < \epsilon_1.$$

Then we can consider a variation $\phi_1 + \delta\phi_1 := \sqrt{1-t^2}\phi_1 + t\psi$. This variation is tangent to the constraints. Besides,

$$\begin{aligned}&\delta\mathcal{E}_{\mathrm{HF}}(\phi_1 + \delta\phi_1, \phi_2, \dots, \phi_n)\\ =&\frac{\delta^2}{\delta\phi_1^2}\mathcal{E}_{\mathrm{HF}}\delta\phi_1\delta\phi_1 + \frac{\delta^2}{\delta\overline{\phi}_1^2}\mathcal{E}_{\mathrm{HF}}\delta\overline{\phi}_1\delta\overline{\phi}_1 + \frac{\delta^2}{\delta\overline{\phi}_1\delta\phi_1}\mathcal{E}_{\mathrm{HF}}\delta\overline{\phi}_1\delta\phi_1.\end{aligned}$$

The first two terms are zero because of the operator $\mathbb{1} - \tau$. The second equals

$$-t^2(\phi_1|h_{\mathrm{HF}}\phi_1) + t^2(\psi|h_{\mathrm{HF}}\psi) = t^2(-\epsilon_1 + \beta),$$

hence is negative. □

Note that the Hartree-Fock energy is in general not equal to the sum of the lowest n eigenvalues of H_{HF}.

7.10 Hartree-Fock method for atomic systems

Suppose now that $V(x) = V(-x)$ and

$$\begin{aligned}H = &-\int a_x^*\Delta_x a_x \mathrm{d}x + \int a_x^* W(x) a_x \mathrm{d}x &(7.130)\\ &+\frac{1}{2}\int\int a_x^* a_y^* V(x-y) a_x a_y \mathrm{d}x\mathrm{d}y. &(7.131)\end{aligned}$$

Let π be an n-dimensional projection. We set

$$\rho(x) := \pi(x,x), \quad \rho_{\mathrm{HF}}(x) := \pi_{\mathrm{HF}}(x,x).$$

Then

$$\begin{aligned}\mathcal{E}_{\mathrm{HF}}(\pi) = &-\int \delta(x-y)\Delta_x\pi(x,y)\mathrm{d}x\mathrm{d}y + \int \delta(x-y)W(x)\pi(x,y)\mathrm{d}x\mathrm{d}y\\ +\frac{1}{2}\int &V(x-y)\delta(x-x')\delta(y-y')\big(\pi(x,x')\pi(y,y') - \pi(x,y')\pi(y,x')\big)\mathrm{d}x\mathrm{d}y\mathrm{d}x'\mathrm{d}y'\\ = &\int \partial_x\partial_y\pi(x,y)\Big|_{x=y}\mathrm{d}x + \int W(x)\rho(x)\mathrm{d}x &(7.132)\\ +&\frac{1}{2}\int\int V(x-y)\rho(x)\rho(y)\mathrm{d}x\mathrm{d}y - \frac{1}{2}\int\int V(x-y)|\pi(x,y)|^2\mathrm{d}x\mathrm{d}y,\\ H_{\mathrm{HF}} = &-\Delta + W(x) + \int \rho_{\mathrm{HF}}(y)V(x-y)\mathrm{d}y - T_{\mathrm{ex}}, &(7.133)\end{aligned}$$

where T_{ex} is a nonlocal operator with the kernel

$$T_{\mathrm{ex}}(x,y) = V(x-y)\pi_{\mathrm{HF}}(x,y).$$

Above we used the following identities involving integral kernels of operators A,B on $L^2(\mathbb{R}^d)$ and C,D on $L^2(\mathbb{R}^d)\otimes L^2(\mathbb{R}^d)\simeq L^2(\mathbb{R}^d\times\mathbb{R}^d)$:

$$\mathrm{Tr}AB = \int A(x,y)B(y,x)\mathrm{d}x\mathrm{d}y, \tag{7.134}$$

$$\mathrm{Tr}CD = \int C(x,y;x'y')D(x',y';x,y)\mathrm{d}x\mathrm{d}y\mathrm{d}x'\mathrm{d}y'. \tag{7.135}$$

A semiclassical argument implies that the first term in (7.132), that is the kinetic energy, can be approximated by

$$(2\pi)^{-d}\frac{d}{d+2}c_d^{-2/d}\int \rho^{\frac{d+2}{d}}(x)\mathrm{d}x, \tag{7.136}$$

where c_d is the volume of a unit ball in d dimensions. We also expect that the last term, that is the exchange energy is relatively small. This leads to the so-called Thomas-Fermi functional, which depends only on the density:

$$\begin{aligned}\mathcal{E}_{\mathrm{TF}}(\rho) :=&(2\pi)^{-d}\frac{d}{d+2}c_d^{-2/d}\int \rho^{\frac{d+2}{d}}(x)\mathrm{d}x\\ &+\int W(x)\rho(x)\mathrm{d}x + \frac{1}{2}\int\int V(x-y)\rho(x)\rho(y)\mathrm{d}x\mathrm{d}y.\end{aligned}$$

8 Squeezed states

8.1 1-mode squeezed vector

Consider $\Gamma_{\mathrm{s}}(\mathbb{C})$.

Theorem 8.1. *Let $|c|<1$. Then*

$$\Omega_c := (1-|c|^2)^{\frac{1}{4}}\mathrm{e}^{\frac{c}{2}a^{*2}}\Omega$$

is a normalized vector satisfying

$$(a-ca^*)\Omega_c = 0. \tag{8.137}$$

Proof. Expanding in power series and using the Lie identity we obtain

$$\begin{aligned}\left(\mathrm{e}^{\frac{c}{2}a^{*2}}\Omega|\mathrm{e}^{\frac{c}{2}a^{*2}}\Omega\right) &= \sum_{n=0}^{\infty}\frac{|c|^{2n}(2n)!}{(n!)^2 2^{2n}}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n|c|^{2n}(-\frac{1}{2})(-\frac{1}{2}-1)\cdots(-\frac{1}{2}-n)}{n!} = \left(1-|c|^2\right)^{-\frac{1}{2}},\\ \mathrm{e}^{\frac{c}{2}a^{*2}}a\mathrm{e}^{-\frac{c}{2}a^{*2}} &= a+\frac{c}{2}[a^{*2},a] = a-ca^*,\end{aligned}$$

The first identity shows that Ω_c is normalized. The second implies $\mathrm{e}^{-\frac{c}{2}a^{*2}}(a - ca^*)\mathrm{e}^{\frac{c}{2}a^{*2}} = a$, which yields (8.137). $\square$

Theorem 8.2. *Set*
$$U_t := \mathrm{e}^{\frac{t}{2}(-a^{*2}+a^2)}.$$
Then
$$U_t a U_t^{-1} = a\cosh t + a^*\sinh t, \tag{8.138}$$
$$U_t a^* U_t^{-1} = a^*\cosh t + a\sinh t, \tag{8.139}$$
$$U_t = \frac{1}{\sqrt{\cosh t}}\mathrm{e}^{-\frac{\tanh t}{2}a^{*2}}\Gamma\Big(\frac{1}{\cosh t}\Big)\mathrm{e}^{\frac{\tanh t}{2}a^2}, \tag{8.140}$$
$$\Omega_{\tanh t} = U_t\Omega. \tag{8.141}$$

Proof. We have
$$[-\frac{1}{2}a^{*2} + a^2, a] = a^*, \quad [-\frac{1}{2}a^{*2} + a^2, a^*] = a. \tag{8.142}$$
Hence
$$\frac{\mathrm{d}}{\mathrm{d}t}U_t a U_t^{-1} = U_t a^* U_{-t}, \quad \frac{\mathrm{d}}{\mathrm{d}t}U_t a U_t^{-1} = U_t a^* U_{-t}. \tag{8.143}$$
This shows (8.138) and (8.139).

Using the identity concerning the derivative of $\Gamma(\mathrm{e}^h) = \mathrm{e}^{ha^*a}$ contained in (8.146), we next differentiate in t the right hand side of (8.140) obtaining
$$-\frac{\sinh t}{2\cosh t}U_t - \frac{1}{2\cosh^2 t}a^{*2}U_t - \frac{\sinh t}{\cosh^2 t}a^*U_t a + \frac{1}{2\cosh^2 t}U_t a^2 \tag{8.144}$$
$$=\Big(- \frac{1}{2\cosh^2 t}a^{*2} + \frac{1}{2\cosh^2 t}(\cosh t a + \sinh t a^*)^2$$
$$- \frac{\sinh t}{\cosh^2 t}a^*(\cosh t a + \sinh t a^*) - \frac{\sinh t}{2\cosh t}\Big)U_t$$
$$=\frac{1}{2}(-a^{*2} + a^2)U_t = \frac{\mathrm{d}}{\mathrm{d}t}U_t. \tag{8.145}$$
This shows (8.140). $\square$

Lemma 8.3. *Let $t \mapsto h(t)$ be a complex function. Then*
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{h(t)a^*a} = \dot{h}(t)\mathrm{e}^{h(t)}a^*\mathrm{e}^{h(t)a^*a}a. \tag{8.146}$$

Proof.
$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{ha^*a} = \dot{h}\mathrm{e}^{ha^*a}a^*a$$
$$= \dot{h}\mathrm{e}^{ha^*a}a^*\mathrm{e}^{-ha^*a}\mathrm{e}^{ha^*a}a = \dot{h}\mathrm{e}^h a^*\mathrm{e}^{ha^*a}a, \tag{8.147}$$
where we used $\mathrm{e}^{ha^*a}a^*\mathrm{e}^{-ha^*a} = \mathrm{e}^h a^*$. $\square$

8.2 Many-mode squeezed vector

Suppose c is a symmetric complex matrix on $\mathbb{C}^n$. One can show that then there exists an orthonormal basis such that c is diagonal where all terms on the diagonal are nonnegative. Therefore, we have the many-mode generalizations of the results of the previous subsection to $\Gamma_{\rm s}(\mathbb{C}^n)$:

Theorem 8.4. *Let c be a symmetric $n \times n$ matrix such that $\|c\| < 1$. Then*

$$\Omega_c := \det(1 - |c|^2)^{\frac{1}{4}} \mathrm{e}^{\frac{1}{2} c_{ij} a_i^* a_j^*} \Omega$$

is a normalized vector satisfying

$$(a_i - c_{ij} a_j^*)\Omega_c = 0. \tag{8.148}$$

where we write $|c| := \sqrt{c^* c}$.

Theorem 8.5. *Let θ be a symmetric $n \times n$ matrix. Set*

$$U_\theta := \mathrm{e}^{\frac{1}{2}(-\theta_{ij} a_i^* a_j^* + \overline{\theta}_{ij} a_j a_i)}.$$

Then

$$U_\theta a_i U_\theta^{-1} = (\overline{\cosh|\theta|})_{ij} a_j + \Big(\theta \frac{\sinh|\theta|}{|\theta|}\Big)_{ij} a_j^*, \tag{8.149}$$

$$U_\theta a_i^* U_\theta^{-1} = (\cosh|\theta|)_{ij} a_j^* + \Big(\overline{\theta} \frac{\sinh\overline{|\theta|}}{\overline{|\theta|}}\Big)_{ij} a_j, \tag{8.150}$$

$$U_\theta = \frac{1}{\sqrt{\det\cosh|\theta|}} \mathrm{e}^{-\left(\theta \frac{\tanh|\theta|}{2|\theta|}\right)_{ij} a_i^* a_j^*} \Gamma\Big(\frac{1}{\cosh|\theta|}\Big) \mathrm{e}^{\left(\overline{\theta} \frac{\tanh\overline{|\theta|}}{2\overline{|\theta|}}\right)_{ij} a_j a_i}, \tag{8.151}$$

$$U_\theta \Omega = \Omega_{\frac{\tanh|\theta|}{|\theta|}\theta}. \tag{8.152}$$

8.3 Single-mode gauge-invariant squeezed vector

Consider $\Gamma_{\rm s}(\mathbb{C}^2)$. The creation/annihilation of first mode are denoted a^*, a, of the second b^*, b.

We assume that in our space there is a "charge operator"

$$Q := a^* a - b^* b,$$

and we are interested mostly in gauge invariant states, that is satisfying $Q = 0$.

Theorem 8.6. *Let $|c| < 1$. Then*

$$\Omega^c := (1 - |c|^2)^{\frac{1}{2}} \mathrm{e}^{c a^* b^*} \Omega$$

is a normalized vector satisfying

$$(a - c b^*)\Omega^c = 0, \tag{8.153}$$

$$(b - c a^*)\Omega^c = 0. \tag{8.154}$$

Proof.

$$\left(\mathrm{e}^{ca^*b^*}\Omega|\mathrm{e}^{ca^*b^*}\Omega\right) = \sum_{n=0}^{\infty}\frac{|c|^{2n}(n!)^2}{(n!)^2} = \left(1-|c|^2\right)^{-1}.$$

Using

$$\mathrm{e}^{-ca^*b^*}a\mathrm{e}^{ca^*b^*} = a - c[a^*b^*, a] = a + cb^*,$$

we obtain (8.154). □

Remark 8.7. *Clearly,*

$$\mathrm{e}^{ca^*b^*} = \exp\left(\frac{c}{4}(a^*+b^*)^2 - \frac{c}{4}(a^*-b^*)^2\right).$$

Hence a single mode gauge-invariant squeezed vector can be also understood as a 2-mode squeezed state. However, it is often simple to deal with it directly.

Theorem 8.8. *Set*

$$U^t := \mathrm{e}^{t(-a^*b^*+ab)}.$$

Then

$$U^t a U^{-t} = a\cosh t + b^*\sinh t, \tag{8.155}$$
$$U^t a^* U^{-t} = a^*\cosh t + b\sinh t, \tag{8.156}$$
$$U^t b U^{-t} = b\cosh t + a^*\sinh t, \tag{8.157}$$
$$U^t b^* U^{-t} = b^*\cosh t + a\sinh t, \tag{8.158}$$
$$U^t = \frac{1}{\cosh t}\mathrm{e}^{-\tanh t a^*b^*}\Gamma\left(\frac{1}{\cosh t}\right)\mathrm{e}^{\tanh t ba}, \tag{8.159}$$
$$\Omega^{-\tanh t} = U^t\Omega. \tag{8.160}$$

Proof. We compute

$$\frac{\mathrm{d}}{\mathrm{d}t}U^t = (-a^*b^* + ba)U^t = -\frac{1}{\cosh^2 t}a^*b^*U^t + \frac{1}{\cosh^2 t}U^t ba - \frac{\sinh t}{\cosh^2 t}\left(a^*U^t a + b^*U^t b\right) - \frac{\sinh t}{\cosh t}U^t.$$

9 Bose gas and superfluidity

n identical *bosonic* particles are described by the Hilbert space

$$\mathcal{H}_n := L^2_{\mathrm{s}}\left((\mathbb{R}^d)^n\right) = \otimes^n_{\mathrm{s}} L^2(\mathbb{R}^d),$$

the *Schrödinger Hamiltonian*

$$H_n = -\sum_{i=1}^{n} \Delta_i + \lambda \sum_{1\le i<j\le n} V(x_i - x_j)$$

and the *momentum* $P_n := -\sum\limits_{i=1}^{n} \mathrm{i}\partial_{x_i}$. We have $P_n H_n = H_n P_n$, which expresses the *translational invariance* of our system.
The *potential* V is a real function on $\mathbb{R}^d$ that decays at infinity and satisfies $V(x) = V(-x)$.

We enclose these particles in a box of size L with fixed density $\rho := \frac{n}{L^d}$ and n large. Instead of the more physical Dirichlet boundary conditions, to keep translational invariance we impose the *periodic boundary conditions*, replacing the original V by the *periodized potential*

$$V^L(x) \quad := \quad \sum_{n\in\mathbb{Z}^d} V(x + Ln) = \frac{1}{L^d} \sum_{\mathrm{p}\in(2\pi/L)\mathbb{Z}^d} \mathrm{e}^{\mathrm{i}px} \hat{V}(\mathrm{p}),$$

well defined on the torus $[-L/2, L/2[^d$. (Note that above we used the *Poisson summation formula*).

The original Hilbert space is replaced by

$$\mathcal{H}_n^L := L_\mathrm{s}^2\Big(\big([-L/2, L/2[^d\big)^n\Big) = \otimes_\mathrm{s}^n \big(L^2([-L/2, L/2[^d)\big).$$

We have a new Hamiltonian

$$H_n^L = -\sum_{i=1}^{n} \Delta_i^L + \lambda \sum_{1\le i<j\le n} V^L(x_i - x_j)$$

and a new momentum $P_n^L := -\sum\limits_{i=1}^{n} \mathrm{i}\partial_{x_i}^L$.

Because of the periodic boundary conditions we still have

$$P_n^L H_n^L = H_n^L P_n^L.$$

In the sequel we drop the superscript L.

We use the second quantized formalism

$$\mathcal{H} = \mathop{\oplus}_{n=0}^{\infty} \mathcal{H}_n = \Gamma_\mathrm{s}\Big(L^2[0, L]^d\Big)$$
$$\simeq \Gamma_\mathrm{s}\Big(l^2\big(\frac{2\pi}{L}\mathbb{Z}^d\big)\Big).$$

The Hamiltonian and the momentum in second quantized notation are

$$H := \overset{\infty}{\underset{n=0}{\oplus}} H_n = -\int a_x^* \Delta_x a_x \mathrm{d}x + \frac{\lambda}{2}\int\int \mathrm{d}x\mathrm{d}y a_x^* a_y^* V(x-y) a_y a_x$$

$$= \sum_p p^2 a_p^* a_p + \frac{\lambda}{2L^d}\sum_{p,q,k} \hat{V}(k) a_{p+k}^* a_{q-k}^* a_q a_p,$$

$$P := \overset{\infty}{\underset{n=0}{\oplus}} P_n = \int a_x^* \frac{1}{\mathrm{i}}\partial_x a_x \mathrm{d}x$$

$$= \sum_p p a_p^* a_p.$$

9.1 Bogoliubov's approximation in the canonical formalism

We assume that the potential is *repulsive*, more precisely,

$$\hat{V} \geq 0, \quad V \geq 0.$$

The Hamiltonian H commutes with N. We are interested in its low energy part for large eigenvalues n of the number of particle operator N.

We expect that for low energies most particles will be spread evenly over the whole box staying in the *zeroth mode*, so that $N \simeq N_0 := a_0^* a_0$. (The Bose statistics does not prohibit to occupy the same state). Following the arguments of N. N. Bogoliubov from 1947, we drop all terms in the Hamiltonian involving more than two creation/annihilation operators of a nonzero mode. We obtain

$$\begin{aligned} H \quad &\approx \quad \frac{\lambda \hat{V}(0)}{2L^d} a_0^* a_0^* a_0 a_0 + \sum_{k\neq 0}\Big(k^2 + a_0^* a_0 \frac{\lambda}{L^d}\big(\hat{V}(k) + \hat{V}(0)\big)\Big) a_k^* a_k \\ &\quad + \sum_{k\neq 0} \frac{\lambda}{2L^d}\hat{V}(k)\Big(a_0^* a_0^* a_k a_{-k} + a_k^* a_{-k}^* a_0 a_0\Big) \\ &= \quad \frac{\lambda \hat{V}(0)\rho}{2}(N-1) + H_{\mathrm{bg}} + R, \end{aligned}$$

where we set

$$\begin{aligned} \rho \quad &= \quad \frac{N}{L^d}, \\ H_{\mathrm{bg}} \quad &:= \quad \sum_{k\neq 0}\big(k^2 + \lambda\rho\hat{V}(k)\big) a_k^* a_k \\ &\quad + \frac{1}{2}\sum_{k\neq 0}\lambda\rho\hat{V}(k)\big(a_k^* a_{-k}^* + a_k a_{-k}\big), \\ R \quad &= \quad -\frac{\lambda\hat{V}(0)}{2L^d}(N-N_0)(N-N_0-1) \\ &\quad + \sum_{k\neq 0}\frac{\lambda}{2L^d}\hat{V}(k)\Big((a_0^* a_0^* - N) a_k a_{-k} + a_k^* a_{-k}^*(a_0 a_0 - N)\Big). \end{aligned}$$

We used

$$
\begin{aligned}
a_0^* a_0^* a_0 a_0 &= N_0(N_0 - 1) \\
&= N(N-1) - 2N_0(N - N_0) - (N - N_0)(N - N_0 - 1).
\end{aligned}
$$

We argue that R is small because

$$
a_0^* a_0^* \approx a_0 a_0 \approx N_0 \approx N.
$$

A *Bogoliubov transformation*, is a linear transformation of creation/annihilation operators preserving the commutation relations. If we demand in addition that it should commute with translations, it should have the form

$$
\tilde{a}_p := c_p a_p + s_p a_{-p}^*, \tag{9.161}
$$

$$
\tilde{a}_p^* := c_p a_p^* + s_p a_{-p}, \quad p \neq 0, \tag{9.162}
$$

where

$$
c_p^2 - s_p^2 = 1, \quad p \neq 0.
$$

We are looking for a Bogoliubov transformation that diagonalizes the quadratic Hamiltonian $H_{\rm bg}$:

$$
\begin{aligned}
H_{\rm bg} &= E_{\rm bg} + \sum_{p \neq 0} \omega(p) \tilde{a}_p^* \tilde{a}_p, \\
P_{\rm bg} &= \sum_{p \neq 0} p \tilde{a}_p^* \tilde{a}_p,
\end{aligned}
$$

This is realized by

$$
\begin{aligned}
c_p &:= \frac{\sqrt{|p|^2 + 2\lambda\rho\hat{V}(p)} + |p|}{2\sqrt{\omega(p)}}, \\
s_p &:= \frac{\sqrt{|p|^2 + 2\lambda\rho\hat{V}(p)} - |p|}{2\sqrt{\omega(p)}}, \\
\omega(p) &:= |p|\sqrt{|p|^2 + 2\lambda\rho\hat{V}(p)}, \\
E_{\rm bg} &:= -\frac{1}{2} \sum_{p \neq 0} \left(|p|^2 + \lambda\rho\hat{V}(p) - |p|\sqrt{|p|^2 + 2\lambda\rho\hat{V}(p)} \right).
\end{aligned}
$$

$\omega(p)$ is called the *Bogoliubov dispersion relation* and $E_{\rm bg}$ the *Bogoliubov energy*.

Let us show some computations:

$$
\begin{aligned}
& A(a_k^* a_k + a_{-k}^* a_{-k}) + B(a_k^* a_{-k}^* + a_{-k} a_k) \\
=& (C a_k^* + S a_{-k})(C a_k + S a_{-k}^*) + (C a_k^* + S a_{-k})(C a_k + S a_{-k}^*) - 2S^2,
\end{aligned}
$$

where

$$
\begin{aligned}
C &:= \frac{1}{2}(\sqrt{A+B} + \sqrt{A-B}), \\
S &:= \frac{1}{2}(\sqrt{A+B} - \sqrt{A-B}).
\end{aligned}
$$

To obtain c_k, s_k we divide C, S by the square root of

$$C^2 - S^2 = \sqrt{A^2 - B^2}.$$

Note that $c_p = \cosh \beta_p$, $s_p = \sinh \beta_p$, where

$$\tanh(\beta_p) := \frac{|p|^2 + \lambda\rho\hat{V}(p) - |p|\sqrt{|p|^2 + 2\lambda\rho\hat{V}(p)}}{\lambda\rho\hat{V}(p)},$$

Set

$$U = \exp\Big(\sum_{p\neq 0} \frac{\beta_p}{2} \left(-a_p^* a_{-p}^* + a_p a_{-p} \right) \Big).$$

Then U is unitary and

$$\begin{aligned} \tilde{a}_p &= U a_p U^*, \\ \tilde{a}_p^* &= U a_p^* U^*, \\ H_{\mathrm{bg}} &= E_{\mathrm{bg}} + U \sum_{p\neq 0} \omega(p) a_p^* a_p U^*, \\ P &= U \sum_{p\neq 0} p a_p^* a_p U^*. \end{aligned}$$

The ground state of the Bogoliubov Hamiltonian is a squeezed state in the non-zero mode sector:

$$\frac{a_0^{*n}}{\sqrt{n!}} U\Omega.$$

The Bogoliubov dispersion relation depends on λ and ρ only through $\lambda\rho = \frac{\lambda n}{L^d}$.

The Bogoliubov Hamiltonian depends on L only through the choice of the lattice spacing $\frac{2\pi}{L}$.

We expect that the low energy part of the excitation spectra of H_n and H_{bg} are close to one another for large n, hoping that then $n - n_0$ is small. We expect some kind of uniformity wrt L.

Note that formally we can even take the limit $L \to \infty$ obtaining

$$\begin{aligned} H_{\mathrm{bg}} - E_{\mathrm{bg}} &= (2\pi)^{-d} \int \omega(p) \tilde{a}_p^* \tilde{a}_p \mathrm{d}p, \\ P &= (2\pi)^{-d} \int p \tilde{a}_p^* \tilde{a}_p \mathrm{d}p. \end{aligned}$$

9.2 Bogoliubov's approximation in the grand-canonical approach

For a *chemical potential* $\mu > 0$, we define the *grand-canonical Hamiltonian*

$$\begin{aligned} H_\mu := H - \mu N &= \sum_p (p^2 - \mu) a_p^* a_p \\ &\quad + \frac{\lambda}{2L^d} \sum_{p,q,k} \hat{V}(k) a_{p+k}^* a_{q-k}^* a_q a_p. \end{aligned}$$

We will mostly set $\lambda = 1$.

If E_μ is the ground state energy of H_μ, then it is realized in the sector n satisfying

$$\partial_\mu E_\mu \quad = \quad -n.$$

In what follows we drop the subscript μ.

For $\alpha \in \mathbb{C}$, we define the *displacement* or *Weyl operator* of the zeroth mode: $W_\alpha := \mathrm{e}^{-\alpha a_0^* + \overline{\alpha} a_0}$. Let $\Omega_\alpha := W_\alpha \Omega$ be the corresponding *coherent vector*. Note that $P\Omega_\alpha = 0$. The *expectation of the Hamiltonian* in Ω_α is

$$(\Omega_\alpha | H \Omega_\alpha) = -\mu |\alpha|^2 + \frac{\hat{V}(0)}{2L^d} |\alpha|^4.$$

It is minimized for $\alpha = \mathrm{e}^{\mathrm{i}\tau} \frac{\sqrt{L^d \mu}}{\sqrt{\hat{V}(0)}}$, where τ is an *arbitrary phase*.

We apply the *Bogoliubov translation to the zero mode* of H by $W(\alpha)$. This means making the substitution

$$\begin{aligned} a_0 = \tilde{a}_0 + \alpha, \quad & a_0^* = \tilde{a}_0^* + \overline{\alpha}, \\ a_k = \tilde{a}_k, \qquad & a_k^* = \tilde{a}_k^*, \qquad k \neq 0. \end{aligned}$$

Note that

$$\tilde{a}_k = W_\alpha^* a_k W_\alpha, \quad \tilde{a}_k^* = W_\alpha^* a_k^* W_\alpha,$$

and thus the operators with and without tildes satisfy the same commutation relations. We drop the tildes.

Here is the translated Hamiltonian:

$$\begin{aligned} H \quad := \quad & -L^d \frac{\mu^2}{2\hat{V}(0)} \\ & + \sum_k \Big(\frac{1}{2} k^2 + \hat{V}(k) \frac{\mu}{\hat{V}(0)} \Big) a_k^* a_k \\ & + \sum_k \hat{V}(k) \frac{\mu}{2\hat{V}(0)} \left(\mathrm{e}^{-\mathrm{i}2\tau} a_k a_{-k} + \mathrm{e}^{\mathrm{i}2\tau} a_k^* a_{-k}^* \right) \\ & + \sum_{k,k'} \frac{\hat{V}(k)\sqrt{\mu}}{\sqrt{\hat{V}(0) L^d}} (\mathrm{e}^{-\mathrm{i}\tau} a_{k+k'}^* a_k a_{k'} + \mathrm{e}^{\mathrm{i}\tau} a_k^* a_{k'}^* a_{k+k'}) \\ & + \sum_{k_1+k_2=k_3+k_4} \frac{\hat{V}(k_2 - k_3)}{2L^d} a_{k_1}^* a_{k_2}^* a_{k_3} a_{k_4}. \end{aligned}$$

If we (temporarily) replace the potential $V(x)$ with $\lambda V(x)$, where λ is a (small) positive constant, the translated Hamiltonian can be rewritten as

$$H^\lambda = \lambda^{-1} H_{-1} + H_0 + \sqrt{\lambda} H_{\frac{1}{2}} + \lambda H_1.$$

Thus the 3rd and 4th terms are in some sense small, which suggests dropping them. Thus

$$H \approx -L^d \frac{\mu^2}{2\hat{V}(0)} + \mu(\mathrm{e}^{\mathrm{i}\tau} a_0^* + \mathrm{e}^{-\mathrm{i}\tau} a_0)^2 + H_{\mathrm{bg}},$$

where

$$\begin{aligned} H_{\mathrm{bg}} &= \sum_{k\neq 0} \Big(\frac{1}{2}k^2 + \hat{V}(k)\frac{\mu}{\hat{V}(0)}\Big) a_k^* a_k \\ &\quad + \sum_{k\neq 0} \hat{V}(k)\frac{\mu}{2\hat{V}(0)} \left(\mathrm{e}^{-\mathrm{i}2\tau} a_k a_{-k} + \mathrm{e}^{\mathrm{i}2\tau} a_k^* a_{-k}^*\right) \end{aligned}$$

Then we proceed as before obtaining the *Bogoliubov dispersion relation*

$$\omega(p) = |p| \sqrt{|p|^2 + 2\mu \frac{\hat{V}(p)}{\hat{V}(0)}}.$$

and the *Bogoliubov energy*

$$E_{\mathrm{bg}} := -\frac{1}{2}\sum_{p\neq 0}\left(|p|^2 + \mu\frac{\hat{V}(p)}{\hat{V}(0)} - |p|\sqrt{|p|^2 + 2\mu\frac{\hat{V}(p)}{\hat{V}(0)}}\right)$$

Thus, as compared with the canonical approach, we have μ in place of $\lambda\rho$.

Note that the grand-canonical Hamiltonian H_μ is invariant wrt the $U(1)$ *symmetry* $\mathrm{e}^{\mathrm{i}\tau N}$. The parameter α has an arbitrary phase. Thus we *broke the symmetry* when translating the Hamiltonian.

The *zero mode* is not a harmonic oscillator – it has continuous spectrum and it can be interpreted as a kind of a *Goldstone mode*.

9.3 Landau's argument for superfluidity

A translation invariant system such as homogeneous Bose gas is described by a family of commuting self-adjoint operators (H, P), where $P = (P_1, \ldots, P_d)$ is the momentum. If the translation invariance is on $\mathbb{R}^d$, then the momentum spectrum is $\mathbb{R}^d$. If it is in a box with periodic boundary conditions then $\mathrm{e}^{\mathrm{i}P_i L} = \mathbb{1}$, therefore the momentum spectrum is $\frac{2\pi}{L}\mathbb{Z}^d$.

We can define its *energy-momentum spectrum* $\sigma(H, P)$.

$$\sigma(H,P) \subset \begin{cases} \mathbb{R}\times\mathbb{R}^d, & L = \infty, \\ \mathbb{R}\times\frac{2\pi}{L}\mathbb{Z}^d, & L < \infty. \end{cases}$$

By general arguments the momentum of the ground state of a Bose gas is zero. Let E denote the *ground state energy* of H. We define the critical velocity by

$$c_{\mathrm{crit}} := \sup\{c \ : \ H \geq E + c|P|\}.$$

Suppose that our n-body system is described with (H, P) with critical velocity c_{crit}. We add to H a perturbation u travelling at a speed w:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t = \Big(H + \lambda\sum_{i=1}^{n} u(x_i - \mathrm{w}t)\Big)\Psi_t.$$

We go to the moving frame:

$$\Psi_t^{\mathrm{w}}(x_1, \dots, x_n) := \Psi_t(x_1 - \mathrm{w}t, \dots, x_n - \mathrm{w}t).$$

We obtain a Schrödinger equation with a time-independent Hamiltonian

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t^{\mathrm{w}} = \Big(H - \mathrm{w}P + \lambda\sum_{i=1}^{n} u(x_i)\Big)\Psi_t^{\mathrm{w}}.$$

Let Ψ_{gr} be the *ground state* of H. Is it *stable against a travelling perturbation*? We need to consider the *tilted Hamiltonian* $H - \mathrm{w}P$.

If $|\mathrm{w}| < c_{\mathrm{crit}}$, then $H - \mathrm{w}P \geq E$ and Ψ_{gr} is still a ground state of $H - \mathrm{w}P$. So Ψ_{gr} is stable.

If $|\mathrm{w}| > c_{\mathrm{crit}}$, then $H - \mathrm{w}P$ is unbounded from below. So Ψ_{gr} is not stable any more.

10 Fermionic Gaussian states

10.1 1-mode particle-antiparticle vector

Consider $\Gamma_{\mathrm{a}}(\mathbb{C}^2)$. The creation/annihilation of first mode are denoted a^*, a, of the second b^*, b.

We assume that in our space there is a "charge operator"

$$Q := a^*a - b^*b,$$

and we are interested mostly in states with $Q = 0$.

Theorem 10.1. *Let $c \in \mathbb{C}$. Then*

$$\Omega^c := (1 + |c|^2)^{-\frac{1}{2}} \mathrm{e}^{ca^*b^*}\Omega = (1 + |c|^2)^{-\frac{1}{2}}\big(\Omega + ca^*b^*\Omega\big)$$

is a normalized vector satisfying

$$\begin{aligned}(a - cb^*)\Omega^c &= 0,\\ (b + ca^*)\Omega^c &= 0.\end{aligned}$$

Theorem 10.2. *Set*

$$U^t := \mathrm{e}^{t(-a^*b^* + ba)}.$$

Then

$$U^t a U^{-t} = a\cos t + b^* \sin t, \tag{10.163}$$

$$U^t a^* U^{-t} = a^* \cos t + b\sin t, \tag{10.164}$$

$$U^t b U^{-t} = b\cos t - a^* \sin t, \tag{10.165}$$

$$U^t b^* U^{-t} = b^* \cos t - a\sin t, \tag{10.166}$$

$$U^t = \cos t \mathrm{e}^{-\tan t a^* b^*}\Gamma\Big(\frac{1}{\cos t}\Big)\mathrm{e}^{\tan t ba}, \tag{10.167}$$

$$\Omega^{-\tan t} = U^t\Omega. \tag{10.168}$$

Proof. First we derive (10.163)-(10.166). Then we compute

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}U^t &= (-a^*b^* + ba)U^t \\ &= -\frac{1}{\cos^2 t}a^*b^*U^t + \frac{1}{\cos^2 t}U^t ba + \frac{\sin t}{\cos^2 t}\Big(a^*U^t a + b^* U^t b\Big) - \frac{\sin t}{\cos t}U^t.\end{aligned}$$

□

10.2 Fermionic oscillator

Let

$$H = (a^* + a)(b^* + b).$$

Theorem 10.3. *We have* $H^2 = -\mathbb{1}$, $H^* = -H$

$$\begin{aligned}\mathrm{e}^{tH} &= \cos t\mathbb{1} + \sin tH, \\ \mathrm{e}^{tH}(a^* + a)\mathrm{e}^{-tH} &= \cos 2t(a^* + a) - \sin 2t(b^* + b), \\ \mathrm{e}^{tH}(b^* + b)\mathrm{e}^{-tH} &= \cos 2t(b^* + b) + \sin 2t(a^* + a), \\ \mathrm{e}^{tH}(a^* - a)\mathrm{e}^{-tH} &= a^* - a, \\ \mathrm{e}^{tH}(b^* - b)\mathrm{e}^{-tH} &= b^* - b, \\ \Omega^{\tan t} &= \mathrm{e}^{tH}\Omega.\end{aligned}$$

In particular,

$$\begin{aligned}\mathrm{e}^{\pm\frac{\pi}{2}H} &= \pm H, & & \\ Ha^*H^{-1} &= -a, & HaH^{-1} &= -a^*, \\ Hb^*H^{-1} &= -b, & HbH^{-1} &= -b^*.\end{aligned}$$

11 Fermi gas and superconductivity

11.1 Fermi gas

We consider fermions with spin $\frac{1}{2}$ described by the Hilbert space

$$\mathcal{H}_n := \otimes_{\mathrm{a}}^n \left(L^2(\mathbb{R}^d, \mathbb{C}^2)\right).$$

We use the chemical potential from the beginning and we do not to assume the locality of interaction, so that the Hamiltonian is

$$H_n = -\sum_{i=1}^{n} \big(\Delta_i - \mu\big) + \lambda \sum_{1\leq i<j\leq n} v_{ij}.$$

The interaction will be given by a 2-body operator on $\otimes^2 \left(L^2(\mathbb{R}^d, \mathbb{C}^2)\right)$ given by

$$(v\Phi)_{i_1,i_2}(x_1,x_2) \quad = \quad \int\int v(x_1,x_2,x_3,x_4)\Phi_{i_1,i_2}(x_3,x_4)\mathrm{d}x_3\mathrm{d}x_4.$$

We will assume that v is invariant wrt the exchange of particles, Hermitian, real and translation invariant:

$$\begin{aligned} v(x_1,x_2,x_3,x_4) &= v(x_2,x_1,x_4,x_3) \\ &= \overline{v(x_1,x_2,x_3,x_4)} \\ &= \overline{v(x_4,x_3,x_2,x_1)} \\ &= v(x_1+y,x_2+y,x_3+y,x_4+y). \end{aligned}$$

By the invariance wrt the exchange of particles v preserves $\otimes^2_{\mathrm{a}} \left(L^2(\mathbb{R}^d, \mathbb{C}^2)\right)$. By translation invariance, v can be written as

$$\begin{aligned} v(x_1,x_2,x_3,x_4) = (2\pi)^{-4d} \int & \mathrm{e}^{\mathrm{i}k_1x_1+\mathrm{i}k_2x_2-\mathrm{i}k_3x_3-\mathrm{i}k_4x_4} q(k_1,k_2,k_3,k_4) \\ & \times \delta(k_1+k_2-k_3-k_4)\mathrm{d}k_1\mathrm{d}k_2\mathrm{d}k_3\mathrm{d}k_4, \end{aligned}$$

where q is a function defined on the subspace $k_1+k_2 = k_3+k_4$. An example of such interaction is a local 2-body potential $V(x)$ such that $V(x) = V(-x)$, which corresponds to

$$\begin{aligned} v(x_1,x_2,x_3,x_4) &\quad = \quad V(x_1-x_2)\delta(x_1-x_4)\delta(x_2-x_3), \\ q(k_1,k_2,k_3,k_4) &\quad = \quad \int \mathrm{d}\mathrm{p}\hat{V}(\mathrm{p})\delta(k_1-k_4-\mathrm{p})\delta(k_2-k_3+\mathrm{p}). \end{aligned}$$

Similarly, as before, we periodize the interaction

$$\begin{aligned} & v^L(x_1,x_2,x_3,x_4) \\ = & \sum_{\mathrm{n}_1,\mathrm{n}_2,\mathrm{n}_3\in\mathbb{Z}^d} v(x_1+\mathrm{n}_1L, x_2+\mathrm{n}_2L, x_3+\mathrm{n}_3L, x_4) \\ = & \frac{1}{L^{3d}} \sum_{k_1+k_2=k_3+k_4} \mathrm{e}^{\mathrm{i}k_1\cdot x_1+\mathrm{i}k_2x_2-\mathrm{i}k_3x_3-\mathrm{i}k_4x_4} q(k_1,k_2,k_3,k_4), \end{aligned}$$

where $k_i \in \frac{2\pi}{L}\mathbb{Z}^d$. The Hamiltonian

$$H^{L,n} = \sum_{1\leq i\leq n} \big(-\Delta_i^L - \mu\big) + \sum_{1\leq i<j\leq n} v_{ij}^L$$

acts on $\mathcal{H}^{n,L} := \otimes_{\mathrm{a}}^n \left(L^2([-L/2,L/2]^d,\mathbb{C}^2)\right)$. We drop the superscript L.

We will denote the spins by $i =\uparrow,\downarrow$. It is convenient to put all the n-particle spaces into a single Fock space

$$\overset{\infty}{\underset{n=0}{\oplus}} \mathcal{H}^n = \Gamma_{\mathrm{a}}\big(L^2([L/2,L/2]^d,\mathbb{C}^2)\big)$$

and rewrite the Hamiltonian and momentum in the language of 2nd quantization:

$$\begin{aligned} H := \overset{\infty}{\underset{n=0}{\oplus}} H^n = & \sum_i \int a_{x,i}^*(\Delta_x-\mu)a_{x,i_2}\mathrm{d}x \\ & +\frac{1}{2}\sum_{i_1,i_2}\int\int a_{x_1,i_1}^* a_{x_2,i_2}^* v(x_1,x_2,x_3,x_4)a_{x_3,i_2}a_{x_4,i_1}\mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3\mathrm{d}x_4, \\ P := \overset{\infty}{\underset{n=0}{\oplus}} P^n = & -\sum_i \mathrm{i}\int a_{x,i}^*\nabla_x a_{x,i}\mathrm{d}x. \end{aligned}$$

In the momentum representation,

$$\begin{aligned} H &= \sum_i\sum_k (k^2-\mu)a_{k,i}^*a_{k,i} \\ &+\frac{1}{2L^d}\sum_{i_1,i_2}\sum_{k_1+k_2=k_3+k_4} q(k_1,k_2,k_3,k_4)a_{k_1,i_1}^*a_{k_2,i_2}^*a_{k_3,i_2}a_{k_4,i_1}, \\ P &= \sum_i\sum_k k a_{k,i}^*a_{k,i}. \end{aligned}$$

We also have the generators of the spin $su(2)$.

$$S_x = \frac{1}{2}\sum_k (a_{k\uparrow}^*a_{k\downarrow}+a_{k\downarrow}^*a_{k\uparrow}), \tag{11.169}$$

$$S_y = \frac{\mathrm{i}}{2}\sum_k (a_{k\uparrow}^*a_{k\downarrow}-a_{k\downarrow}^*a_{k\uparrow}), \tag{11.170}$$

$$S_z = \frac{1}{2}\sum_k (a_{k\uparrow}^*a_{k\uparrow}-a_{k\downarrow}^*a_{k\downarrow}). \tag{11.171}$$

The Hamiltonian is invariant with respect to the spin $su(2)$.

11.2 Hartree-Fock-Bogoliubov approximation with BCS ansatz

We try to compute the excitation spectrum of the Fermi gas by approximate methods. We look for a minimum of the energy among Gaussian states. We assume that a minimizer is invariant wrt translations and the spin $su(2)$. We use the Hartree-Fock-Bogoliubov approximation with the Bardeen-Cooper-Schrieffer ansatz.

For a sequence $\frac{2\pi}{L}\mathbb{Z}^d \ni k \mapsto \theta_k$ such that $\theta_k = \theta_{-k}$, set

$$U_\theta := \prod_k \mathrm{e}^{\frac{1}{2}\theta_k(-a^*_{k\uparrow}a^*_{-k\downarrow}+a_{-k\downarrow}a_{k\uparrow}-a^*_{-k\uparrow}a^*_{k\downarrow}+a_{k\downarrow}a_{-k\uparrow})}.$$

(Note the double counting for $k \neq 0$). We are looking for a minimizer of the form $U_\theta\Omega$.

Note that U_θ commutes with P and the spin $su(2)$. Therefore, $U_\theta\Omega$ is translation and $su(2)$ invariant.

We want to compute

$$(U_\theta\Omega|HU_\theta\Omega) = (\Omega|U^*_\theta HU_\theta\Omega).$$

To do this we can use the fact that U_θ implements Bogoliubov rotations:

$$\begin{aligned}
U^*_\theta a^*_{k\uparrow}U_\theta &= \cos\theta_k a^*_{k\uparrow} + \sin\theta_k a_{-k\downarrow},\\
U^*_\theta a_{k\uparrow}U_\theta &= \cos\theta_k a_{k\uparrow} + \sin\theta_k a^*_{-k\downarrow},\\
U^*_\theta a^*_{k\downarrow}U_\theta &= \cos\theta_k a^*_{k\downarrow} - \sin\theta_k a_{-k\uparrow},\\
U^*_\theta a_{k\downarrow}U_\theta &= \cos\theta_k a_{k\downarrow} - \sin\theta_k a^*_{-k\uparrow},
\end{aligned}$$

Afteer inserting this into $U^*_\theta HU_\theta$ be can Wick order the obtained expression.

In practice, this is usually presented differently. One makes the substitution

$$\begin{aligned}
a_{k\uparrow} &= \cos\theta_k b^*_{k\uparrow} + \sin\theta_k b_{-k\downarrow},\\
a_{k\uparrow} &= \cos\theta_k b_{k\uparrow} + \sin\theta_k b^*_{-k\downarrow},\\
a^*_{k\downarrow} &= \cos\theta_k b^*_{k\downarrow} - \sin\theta_k b_{-k\uparrow},\\
a_{k\downarrow} &= \cos\theta_k b_{k\downarrow} - \sin\theta_k b^*_{-k\uparrow},
\end{aligned}$$

in the Hamiltonian. Note that

$$\begin{aligned}
U_\theta a^*_{k\uparrow}U^*_\theta &= b^*_{k\uparrow},\\
U_\theta a_{k\uparrow}U^*_\theta &= b_{k\uparrow},\\
U_\theta a^*_{k\downarrow}U^*_\theta &= b^*_{k\downarrow},\\
U_\theta a_{k\downarrow}U^*_\theta &= b_{k\downarrow}.
\end{aligned}$$

Then one Wick orders wrt the operators B^*, b. Our Hamiltonian becomes

$$\begin{aligned}
H = B &+ \sum_k D(k)\big(b^*_{k\uparrow}b_{k\uparrow} + b^*_{k\downarrow}b_{k\downarrow}\big)\\
&+ \frac{1}{2}\sum_k O(k)\big(b^*_{k\uparrow}b^*_{-k\downarrow} + b^*_{-k\uparrow}b^*_{k\downarrow}\big) + \frac{1}{2}\sum_k \overline{O}(k)\big(b_{-k\downarrow}b_{k\uparrow} + b_{k\downarrow}b_{-k\uparrow}\big)\\
&+ \text{terms higher order in } b\text{'s}.
\end{aligned}$$

Note that

$$(\Omega_\theta|H\Omega_\theta) = B.$$

By the Beliaev Theorem, minimizing B is equivalent to $O(k) = 0$.

If we choose the Bogoliubov transformation according to the minimization procedure, the Hamiltonian equals

$$H = B + \sum_k D(k)\big(b_{k\uparrow}^* b_{k\uparrow} + b_{k\downarrow}^* b_{k\downarrow}\big) + \text{terms higher order in } b\text{'s}$$

with

$$\begin{aligned} B &= \sum_k (k^2 - \mu)(1 - \cos 2\theta_k) \\ &\quad + \frac{1}{4L^d} \sum_{k,k'} \alpha(k,k') \sin 2\theta_k \sin 2\theta_{k'} \\ &\quad + \frac{1}{4L^d} \sum_{k,k'} \beta(k,k')(1 - \cos 2\theta_k)(1 - \cos 2\theta_{k'}). \end{aligned}$$

Here,

$$\begin{aligned} \alpha(k,k') &:= \frac{1}{2}\big(q(k,-k,-k',k') + q(-k,k,-k',k')\big), \\ \beta(k,k') &= 2q(k,k',k',k) - q(k',k,k',k). \end{aligned}$$

In particular, in the case of local potentials we have

$$\begin{aligned} \alpha(k,k') &:= \frac{1}{2}\big(\hat{V}(k-k') + \hat{V}(k+k')\big), \\ \beta(k,k') &= 2\hat{V}(0) - \hat{V}(k-k'). \end{aligned}$$

The condition $\partial_{\theta_k} B = 0$, or equivalently $O(k) = 0$, has many solutions. We can have

$$\sin 2\theta_k = 0, \qquad \cos 2\theta_k = \pm 1,$$

They correspond to *Slater determinants* and have a fixed number of particles. The solution of this kind minimizing B, is called the *normal* or *Hartree-Fock solution*.

Under some conditions the global minimum of B is reached by a non-normal configuration satisfying

$$\sin 2\theta_k = -\frac{\delta(k)}{\sqrt{\delta^2(k) + \xi^2(k)}}, \qquad \cos 2\theta_k = \frac{xi(k)}{\sqrt{\delta^2(k) + \xi^2(k)}},$$

where

$$\begin{aligned} \delta(k) &= \frac{1}{2L^d} \sum_{k'} \alpha(k,k') \sin 2\theta_{k'}, \\ \xi(k) &= k^2 - \mu + \frac{1}{2L^d} \sum_{k'} \beta(k,k')(1 - \cos 2\theta_{k'}), \end{aligned}$$

and at least some of $\sin 2\theta_k$ are different from 0. It is sometimes called a *superconducting solution*.

For a superconducting solution we get

$$D(k) = \sqrt{\xi^2(k) + \delta^2(k)}.$$

Thus we obtain a positive dispersion relation. One can expect that it is strictly positive, since otherwise the two functions δ and ξ would have a coinciding zero, which seems unlikely. Thus we expect that the dispersion relation $D(k)$ has a *positive energy gap*.

Conditions guaranteeing that a superconducting solution minimizes the energy should involve some kind of negative definiteness of the quadratic form α – this is what we vaguely indicated by saying that the interaction is *attractive*. Indeed, multiply the definition of $\delta(k)$ with $\sin 2\theta_k$ and sum it up over k. We then obtain

$$\begin{aligned} &\sum_k \sin^2 2\theta_k \sqrt{\delta^2(k) + \xi^2(k)} \\ = \quad & -\frac{1}{2L^d} \sum_{k,k'} \sin 2\theta_k \alpha(k,k') \sin 2\theta_{k'}. \end{aligned}$$

The left hand side is positive. This means that the quadratic form given by the kernel $\alpha(k,k')$ has to be negative at least at the vector given by $\sin 2\theta_k$.

12 Quantum lattice systems

12.1 Equivalence of representations

Let $\mathfrak{A}$ be a C^*-algebra. Recall that a $*$-homomorphism $\pi : \mathfrak{A} \to B(\mathcal{H})$ is also called a *$*$-representation of $\mathfrak{A}$ on $\mathcal{H}$.*

Let $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ be two $*$-representation of $\mathfrak{A}$. We say that they are *unitarily equivalent* iff there exists a unitary $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\pi_2(A) = U\pi_1(A)U^*, \quad A \in \mathfrak{A}. \tag{12.172}$$

We say that they are *quasiequivalent* if there exists a $*$-isomorphism

$$\rho : \pi_1(\mathfrak{A})'' \to \pi_2(\mathfrak{A})'' \tag{12.173}$$

such that

$$\pi_2(A) = \rho\big(\pi_1(A)\big), \quad A \in \mathfrak{A}. \tag{12.174}$$

Clearly, unitary equivalence implies quasiequivalence. The converse in general is not true. E.g. the $*$-representations

$$B(\mathcal{H}) \ni A \mapsto A \otimes \mathbb{1}_n \in B(\mathcal{H} \otimes \mathbb{C}^n) \tag{12.175}$$

are quasiequivalent but not unitarily equivalent for distinct $n \in \mathbb{N}$.

In the algebraic approach to Quantum Physics we start from a C^*-algebra of observables $\mathfrak{A}$, which may have many inequivalent representations $\pi_i : \mathfrak{A} \to B(\mathcal{H}_i)$. The choice of the representation is dictated by the physical situation: the temperature of the system, its chemical potential, its phase. In many cases, e.g. in Quantum Field Theory, one usually chooses a ground state representation–a representation with a positive Hamiltonian.

12.2 Basic algebraic framework of quantum physics

Let us describe the basic steps of the algebraic description of a quantum system.

(1) We start with a C^*-algebra $\mathfrak{A}$, whose self-adjoint elements describe observables of our quantum system. We will denote by $\mathrm{Aut}(\mathfrak{A})$ the group of $*$-automorphisms of $\mathfrak{A}$. They describe symmetries of the system.

(2) The Heisenberg dynamics is described by a 1-parameter group of $*$-automorphisms, that is

$$\mathbb{R} \ni t \mapsto \rho_t \in \mathrm{Aut}(\mathfrak{A}), \quad \rho_0 = \mathrm{id}, \quad \rho_t\rho_s = \rho_{t+s}. \tag{12.176}$$

We usually assume that ρ is strongly continuous, that is,

$$\mathbb{R} \ni t \mapsto \rho_t(A) \text{ is norm continuous}, \quad A \in \mathfrak{A}. \tag{12.177}$$

we then say that $(\mathfrak{A}, \rho)$ is a C^*-dynamical system.

(3) We choose a state ω on $\mathfrak{A}$. Then we pass to the GNS representation generated by ω on the Hilbert space $\mathcal{H}_\omega$ with the cyclic vector Ω_ω:

$$\pi_\omega : \mathfrak{A} \to B(\mathcal{H}_\omega), \quad \omega(A) = \big(\Omega_\omega | \pi_\omega(A)\Omega_\omega\big). \tag{12.178}$$

(4) Assume that ω is time-invariant, that is

$$\omega(A) = \omega\big(\rho_t(A)\big), \quad t \in \mathbb{R}. \tag{12.179}$$

Then on the GNS Hilbert space we can define a unitary implementation of ρ_t. In fact, we set

$$U_{\omega,t}\pi_\omega(A)\Omega_\omega := \pi_\omega(\rho_t(A))\Omega_\omega. \tag{12.180}$$

We have (dropping the subscript ω for legibility)

$$\begin{aligned} \big(U_t\pi(A)\Omega|U_t\pi(B)\Omega\big) &= \big(\pi(\rho_t(A))\Omega|\pi(\rho_t(B))\Omega\big) \\ = \ \big(\Omega|\pi(\rho_t(A^*B)\Omega\big) &= \omega\big(\rho_t(A^*B)\big) \\ = \ \omega\big(A^*B\big) &= \big(\pi(A)\Omega|\pi(B)\Omega\big). \end{aligned} \tag{12.181}$$

Thus $U_{\omega,t}$ preserves the scalar product on $\pi_\omega(\mathfrak{A})\Omega$. But $\pi_\omega(\mathfrak{A})\Omega$ is dense in $\mathcal{H}$. Hence, $U_{\omega,t}$ is unitary. Clearly,

$$\mathbb{R} \ni t \mapsto U_{\omega,t} \tag{12.182}$$

is a one-parameter unitary group.

(5) Assume that the dynamics (12.177) is strongly continuous. Then (12.182) is strongly continuous as well, that is,

$$\mathbb{R} \ni t \mapsto U_{\omega,t}\Phi \in \mathcal{H}_\omega, \quad \Phi \in \mathcal{H}_\omega \tag{12.183}$$

is norm continuous. By the Stone Theorem, there exists a self-adjoint operator H_ω such that $U_{\omega,t} = \mathrm{e}^{-\mathrm{i}tH_\omega}$. Note that $H_\omega\Omega_\omega = 0$.

(6) We say that ω is a ground state of the system $(\mathfrak{A}, \rho)$ if $H \geq 0$.

Example 12.1. *Let $\mathfrak{A} = K(\mathcal{H})$ (the C^*-algebra of compact operators on $\mathcal{H}$). Consider the dynamics $\rho_t(A) := \mathrm{e}^{\mathrm{i}tH}A\mathrm{e}^{-\mathrm{i}tH}$. Let $\omega(A) = (e|Ae)$, where $\|e\| = 1$, $e \in \mathcal{H}$. Then $\mathcal{H}_\omega \simeq \mathcal{H}$. ω is time-invariant if $He = Ee$ (e is an eigenvector of H). Then $H_\omega = H - E$. ω is a ground state of $(\mathfrak{A}, \rho)$ iff $E = \inf\sigma(H)$.*

Example 12.2. *Let $e_1, e_2 \in \mathcal{H}$ be an orthonormal sequence, $\lambda_1, \lambda_2, \ldots$ satisfy $\lambda_i > 0$, $\lambda_1 + \lambda_2 + \cdots = 1$. Consider*

$$\omega(A) = \lambda_1(e_1|Ae_1) + \lambda_2(e_2|Ae_2) + \cdots . \tag{12.184}$$

Then

$$\mathcal{H}_\omega = \bigoplus_i \mathcal{H} \otimes \mathbb{C}e_i. \tag{12.185}$$

If e_1, e_2 are eigenvectors of H, so that $He_i = E_ie_i$, then ω is time invariant. We have

$$H_\omega = \bigoplus_i (H - E_i). \tag{12.186}$$

ω is a ground state if $E_1 = E_2 = \cdots = \inf\sigma(H)$.

12.3 Lattice systems

Choose $n, d \in \mathbb{N}$. To every $j = (j_1, \dots, j_d) \in \mathbb{Z}^d$ we associate a space $\mathcal{H}_j \simeq \mathbb{C}^n$ describing the "spin" at site j. Let $2^{\mathbb{Z}^d}_{\rm fin}$ denote the family of finite subsets of $\mathbb{Z}^d$. To every $\Lambda \in 2^{\mathbb{Z}^d}_{\rm fin}$ we associate the Hilbert space and the algebra of observables

$$\mathcal{H}_\Lambda := \underset{j\in\Lambda}{\otimes} \mathcal{H}_j, \tag{12.187}$$

$$\mathfrak{A}_\Lambda := B(\mathcal{H}_\Lambda). \tag{12.188}$$

If $\Lambda \subset \Lambda' \in 2^{\mathbb{Z}^d}_{\rm fin}$, then we have the identification

$$\mathfrak{A}_\Lambda \ni A \mapsto A \otimes \mathbb{1}_{\Lambda'\setminus\Lambda} \in \mathfrak{A}_{\Lambda'}. \tag{12.189}$$

The $*$-algebra of local observables is defined as

$$\mathfrak{A}_{\rm loc} := \bigcup_{\Lambda\in 2^{\mathbb{Z}^d}_{\rm fin}} \mathfrak{A}_\Lambda. \tag{12.190}$$

It is equipped with a norm satisfying the C^*-property. Its completion is called the C^*-algebra of quasilocal observables

$$\mathfrak{A} := \mathfrak{A}_{\rm loc}^{\rm cpl}. \tag{12.191}$$

Note that $\mathfrak{A} \simeq {\rm UHF}(n^\infty)$.

All spaces $\mathcal{H}_j$, $j \in \mathbb{Z}^d$ can be identified with $\mathbb{C}^n$. Therefore, for any $k \in \mathbb{Z}^d$, there exists an obvious unitary map $U_k : \mathcal{H}_j \to \mathcal{H}_{j+k}$. By tensoring, we obtain the unitary map $U_k : \mathcal{H}_\Lambda \to \mathcal{H}_{\Lambda+k}$ (which we denote by the same symbol). We also have the corresponding automorphism

$$\tau_k \mathfrak{A}_\Lambda \to \mathfrak{A}_{\Lambda+k}, \quad \tau_k(A) := U_k A U_k^*, \tag{12.192}$$

which extends to the automorphism of $\mathfrak{A}$. Thus we obtain an action of the group

$$\mathbb{Z}^d \ni k \mapsto \tau_k \in {\rm Aut}(\mathfrak{A}). \tag{12.193}$$

Suppose h is a self-adjoint operator on $\mathbb{C}^n$. If $\Lambda \in 2^{\mathbb{Z}^d}_{\rm fin}$, let us write h_j for h acting on $\mathcal{H}_j$, multiplied by the identity. Set

$$h_\Lambda := \sum_{j\in\Lambda} h_j. \tag{12.194}$$

It defines the automorphism

$$\rho_{\Lambda,t}(A) := {\rm e}^{{\rm i}th_\Lambda} A {\rm e}^{-{\rm i}th_\Lambda} \quad A \in \mathfrak{A}_\Lambda. \tag{12.195}$$

It is easy to see that

$$\rho_t(A) := \lim_{\Lambda\nearrow\mathbb{Z}^d} \rho_{\Lambda,t}(A), \quad A \in \mathfrak{A} \tag{12.196}$$

exists. We obtain the dynamics

$$\mathbb{R} \ni t \mapsto \rho_t \in \mathrm{Aut}(\mathfrak{A}), \tag{12.197}$$

which commutes with (12.193).

Suppose that γ is a density matrix on $\mathbb{C}^n$. Let us write γ_j for γ acting on $\mathcal{H}_j$ and

$$\gamma_\Lambda := \underset{j\in\Lambda}{\otimes} \gamma_j. \tag{12.198}$$

It defines a state

$$\omega_\Lambda(A) := \mathrm{Tr}\gamma_\Lambda A, \quad A \in \mathfrak{A}_\Lambda. \tag{12.199}$$

The following limit exists:

$$\omega(A) := \lim_{\Lambda \nearrow \mathbb{Z}^d} \omega_\Lambda(A), \quad A \in \mathfrak{A}. \tag{12.200}$$

The state ω is invariant wrt (12.193). If γ commutes with h, it is also invariant wrt (12.197).

Note that different choices of γ lead to inequivalent GNS representations.

12.4 KMS condition on lattice systems

Consider now a lattice quantum system on $\mathbb{Z}^d$ with the algebra $\mathfrak{A}$ and the Hamiltonian given by a self-adjoint operator h on $\mathbb{C}^n$, as in (12.194). For each $\Lambda \in 2^{\mathbb{Z}^d}_{\mathrm{fin}}$ we can define the Gibbs state

$$\omega_{\Lambda,\beta}(A) := \frac{\mathrm{Tr}\,\mathrm{e}^{-\beta H_\Lambda} A}{\mathrm{Tr}\mathrm{e}^{-\beta H_\Lambda}}. \tag{12.201}$$

Hence, this state satisfies the β-KMS condition wrt the dynamics ρ_Λ. There exists the limit

$$\omega_\beta(A) := \lim_{\Lambda \nearrow \mathbb{Z}^d} \omega_{\Lambda,\beta}(A). \tag{12.202}$$

ω_β satisfies the β-KMS condition wrt the dynamics ρ. Note that this state is not given by a density matrix. In fact,

$$\frac{\mathrm{e}^{-\beta H_\Lambda}}{\mathrm{Tr}\mathrm{e}^{-\beta H_\Lambda}} = \underset{j\in\Lambda}{\otimes} \frac{\mathrm{e}^{-\beta h_j}}{\mathrm{Tr}\mathrm{e}^{-\beta h_j}} \tag{12.203}$$

does not have a limit in any sense and $\lim_{\Lambda \nearrow \mathbb{Z}^d} \mathrm{Tr}\mathrm{e}^{-\beta H_\Lambda}$ typically converges to infinity.

This suggests that for lattice systems the β-KMS condition is a reasonable generalization of the β-Gibbs state.

13 KMS condition

13.1 Gibbs states

Let us go back to the basic setup of Quantum Physics: $\mathcal{H}$ is a Hilbert space, H is a self-adjoint operator called Hamiltonian, and the Heisenberg dynamics is

$$\rho_t(A) := \mathrm{e}^{\mathrm{i}tH} A \mathrm{e}^{-\mathrm{i}tH}, \quad A \in B(\mathcal{H}). \tag{13.204}$$

Let $\beta \in \mathbb{R}$. The β-Gibbs state is defined as

$$\omega_\beta(A) := \mathrm{Tr}\gamma_\beta A, \quad \gamma_\beta := \frac{\mathrm{e}^{-\beta H}}{\mathrm{Tr}\mathrm{e}^{-\beta H}}, \tag{13.205}$$

This is possible iff

$$\mathrm{Tr}\mathrm{e}^{-\beta H} < \infty \tag{13.206}$$

Note that e.g. if $\mathcal{H}$ is finite dimensional, then this is satisfied for all $\beta \in \mathbb{R}$. If the operator H is bounded from below but unbounded, then (13.206) can be true only for $\beta \geq 0$

The β-Gibbs state describes the system at temperature $T = \frac{1}{k_{\mathrm{B}}\beta}$. Note that this state is unique (if exists)—hence we cannot describe multiple phases at a given temperature using this setup.

Let $E_0 = \inf \sigma(H)$. Note that

$$\omega_\infty(A) := \lim_{\beta\to\infty} \omega_\beta(A) = \frac{\mathrm{Tr}\mathbb{1}_{\{E_0\}}(H)A}{\mathrm{Tr}\mathbb{1}_{\{E_0\}}(H)}. \tag{13.207}$$

If $\dim \mathbb{1}_{E_0}(H) = 1$, we obtain the pure state. Otherwise, we obtain the uniform combination of all ground states.

Gibbs states can be

Let $B^2(\mathcal{H})$ denote the set of Hilbert-Schmidt operators on $\mathcal{H}$, that is

$$B^2(\mathcal{H}) := \{A \in B(\mathcal{H}) \mid \mathrm{Tr}A^*A < \infty\}. \tag{13.208}$$

It can be treated as a Hilbert space equipped with the scalar product

$$(A|B) := \mathrm{Tr}A^*B, \quad A, B \in B^2(\mathcal{H}). \tag{13.209}$$

Consider the GNS representation π_β wrt ω_β, with the corresponding Hilbert space $\mathcal{H}_\beta$ and cyclic vector Ω_β. Note that we can identify $\mathcal{H}_\beta$ with $B^2(\mathcal{H})$ and $\Omega_\beta = \sqrt{\gamma_\beta}$. The representation acts as multiplication from the left:

$$\pi_\beta(A)B := AB, \quad A \in B(\mathcal{H}), \quad B \in B^2(\mathcal{H}). \tag{13.210}$$

On the GNS space we have a distinguished generator of the dynamics called the Liouvillean

$$LB := HB - BH, \quad B \in B^2(\mathcal{H}). \tag{13.211}$$

We have

$$\rho_t(\pi_\beta(A)) = \mathrm{e}^{\mathrm{i}tL}\pi_\beta(A)\mathrm{e}^{-\mathrm{i}tL}, \quad A \in B(\mathcal{H}); \tag{13.212}$$

$$L\Omega_\beta = 0. \tag{13.213}$$

Thus the GNS representation for various β can be viewed in the same Hilbert space, with the same Liouvillean, and only the cyclic vector is varied.

13.2 From a state to the dynamics

In physics a dynamics is typically more fundamental than a state. Therefore, we started from the Hamiltonian. One can proceed in the reverse direction. Let γ be a density matrix which is nondegenerate ($\mathrm{Ker}\gamma = \{0\}$), and ω is the corresponding state. Such states satisfy

$$A > 0 \ \Rightarrow \ \omega(A) > 0 \tag{13.214}$$

and are called faithful.

Define

$$H := -\beta^{-1}\ln\gamma \tag{13.215}$$

Then H is a self-adjoint operator. If we define the dynamics ρ_t by (13.204), then ω is the β-Gibbs state condition for ρ. Thus every faithful state is the β-Gibbs state for a certain unique dynamics.

13.3 KMS condition for Gibbs states

Suppose that $A, B \in B(\mathcal{H})$. In addition, assume that

$$\mathbb{R} \ni t \mapsto \rho_t(B) \tag{13.216}$$

extends to an analytic function

$$\mathbb{C} \ni z \mapsto \rho_z(B) = \mathrm{e}^{\mathrm{i}zH}B\mathrm{e}^{-\mathrm{i}zH}. \tag{13.217}$$

Such B are weakly dense in $B(\mathcal{H})$: e.g. we can assume that B is "localized in the energy", that is $B = \mathbb{1}_{[E_0,E]}(H)B\mathbb{1}_{[E_0,E]}(H)$ for some E. If dimension $\mathcal{H}$ is finite, this applies to all $B \in B(\mathcal{H})$.

Then

$$\omega_\beta(A\rho_{\mathrm{i}\beta}(B)) = \omega_\beta(BA). \tag{13.218}$$

In fact,

$$\begin{aligned}\omega_\beta(A\rho_{\mathrm{i}\beta}(B)) = &\quad \mathrm{Tr}\mathrm{e}^{-\beta H}A\mathrm{e}^{\mathrm{i}^2\beta H}B\mathrm{e}^{-\mathrm{i}^2\beta H} = \mathrm{Tr}A\mathrm{e}^{-\beta H}B \\ = &\ \ \mathrm{Tr}\mathrm{e}^{-\beta H}BA \quad = \omega_\beta(BA).\end{aligned} \tag{13.219}$$

13.4 KMS condition on C^*-algebras

Suppose now that $\mathfrak{A}$ is a C^*-algebra equipped with a strongly continuous dynamics

$$\mathbb{R} \ni t \mapsto \rho_t \in \mathrm{Aut}(\mathfrak{A}). \tag{13.220}$$

Lemma 13.1. *There exists a norm-dense $*$-subalgebra $\mathfrak{A}_{\mathrm{an}} \subset \mathfrak{A}$ such that for every $B \in \mathfrak{A}_{\mathrm{an}}$*

$$\mathbb{R} \ni t \mapsto \rho_t(B) \tag{13.221}$$

extends to an analytic function

$$\mathbb{C} \ni z \mapsto \rho_z(B). \tag{13.222}$$

Proof. For $z \in \mathbb{C}$ and $B \in \mathfrak{A}$ set

$$B_{z,\epsilon} := \int_{-\infty}^{+\infty} \rho_s(B) \mathrm{e}^{-\frac{(z-s)^2}{2\epsilon}} \frac{\mathrm{d}s}{\sqrt{2\pi\epsilon}}. \tag{13.223}$$

Then we check that

$$\rho_t(B_z) = B_{z+t}, \quad \lim_{\epsilon \searrow 0} B_{0,\epsilon} = B. \tag{13.224}$$

In particular, $\rho_t(B_{0,\epsilon}) = B_{t,\epsilon}$ extends to an analytic function $\mathbb{C} \ni z \mapsto B_{z,\epsilon} \in \mathfrak{A}$. $\square$

We say that a state ω on $\mathfrak{A}$ satisfies the β-KMS condition if

$$\omega\big(A\rho_{\mathrm{i}\beta}(B)\big) = \omega(BA), \quad A \in \mathfrak{A}, \quad B \in \mathfrak{A}_{\mathrm{an}}. \tag{13.225}$$

One can show that the KMS condition implies that the state is stationary.

13.5 KMS condition on UHF algebras

Recall that if $n_1, n_2, \dots$ are positive integers, $\mathcal{H}_j := \mathbb{C}^{n_j}$, with the identifications

$$B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k) \ni A \mapsto A \otimes \mathbb{1}_{\mathcal{H}_{k+1}} \in B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_{k+1}),$$

we define

$$\mathrm{UHF}_0 = \mathrm{UHF}_0(n_1, n_2, \dots) := \bigcup_{k=1}^{\infty} B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k), \tag{13.226}$$

$$\mathrm{UHF} = \mathrm{UHF}(n_1, n_2, \dots) := \mathrm{UHF}_0(n_1, n_2, \dots)^{\mathrm{cpl}}. \tag{13.227}$$

Let h_j be a self-adjoint operator on $\mathcal{H}_j$. Then we have a dynamics on UHF, which for $A \in B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k)$ is given by

$$\rho_t(A) := \mathop{\otimes}_{j=1}^{k} \mathrm{e}^{\mathrm{i}th_j} \; A \; \mathop{\otimes}_{j=1}^{k} \mathrm{e}^{-\mathrm{i}th_j} \tag{13.228}$$

$$= \exp\Big(\mathrm{i}t \mathop{\oplus}_{j=1}^{k} h_j\Big) A \exp\Big(-\mathrm{i}t \mathop{\oplus}_{j=1}^{k} h_j\Big). \tag{13.229}$$

Let $\beta \in \mathbb{R}$ and set

$$\gamma_{\beta,j} := \frac{\mathrm{e}^{-\beta h_j}}{\mathrm{Tr}\mathrm{e}^{-\beta h_j}}. \tag{13.230}$$

On UHF we define the state ω_β, which for $A \in B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k)$ is given by

$$\omega_\beta(A) = \mathrm{Tr} \bigotimes_{j=1}^{k} \gamma_{\beta,j}\ A. \tag{13.231}$$

It is easy to see that ω_β satisfies the β-KMS condition for ρ. It is stationary and faithful.

Conversely, let γ_j be nondegenerate density matrices on $\mathcal{H}_j$, $j = 1, 2, \ldots$. On UHF we define the state ω, which for $A \in B(\mathcal{H}_1 \otimes \cdots \otimes \mathcal{H}_k)$ is given by

$$\omega_\beta(A) = \mathrm{Tr} \bigotimes_{j=1}^{k} \gamma_j\ A. \tag{13.232}$$

Set $h_j := -\frac{\ln \gamma_j}{\beta}$ and defines the dynamics ρ as above. The ω is a β-KMS state for ρ.

13.6 Gibbs states for multiple observables

Suppose a quantum system is described by a Hilbert space $\mathcal{H}$ and a Hamiltonian H. Suppose that $\vec{O} = (O_0, \ldots, O_n)$ is a sequence of commuting self-adjoint oparators left invariant by the dynamics (equivalently, commuting with H) describing "easy to control observables". One can argue that the following density matrix can be used to describe such a quantum system:

$$\theta_{\vec{\beta}} = \frac{\mathrm{e}^{-\vec{\beta}\vec{O}}}{\mathrm{Tr}\mathrm{e}^{-\vec{\beta}\vec{O}}}, \tag{13.233}$$

where $\vec{\beta} = (\beta_0, \ldots, \beta_n) \in \mathbb{R}^{n+1}$ are the "inverse temperatures of various observables". The above density matrix is clearly time-invariant. But there are many more time-invariant matrices: in fact, for any function of n variables

$$\frac{f(\vec{O})}{\mathrm{Tr} f(\vec{O})} \tag{13.234}$$

is also time-invariant. Nevertheless, the exponential function is in some sense distinguished, and one can argue that one can limit oneself to density matrices of the form 13.233. Here is an argument which indicates good properties of (13.233).

Suppose we have two systems described by $\mathcal{H}^a, H^a$ and $\mathcal{H}^b, H^b$. Suppose that we "'put them in contact" obtaining a composite system $\mathcal{H}^a \otimes \mathcal{H}^b$ with the Hamiltonian $H^a \otimes \mathbb{1} + \mathbb{1} \otimes H^b$. Introduce the observables for the composite system $O_i := O_i^a \otimes \mathbb{1} + \mathbb{1} \otimes O_i^b$. Then,

$$\mathrm{e}^{-\vec{\beta}\vec{O}} = \mathrm{e}^{-\vec{\beta}\vec{O}^a} \otimes \mathrm{e}^{-\vec{\beta}\vec{O}^b}, \tag{13.235}$$

$$\mathrm{Tr}\mathrm{e}^{-\vec{\beta}\vec{O}} = \mathrm{Tr}\mathrm{e}^{-\vec{\beta}\vec{O}^a}\mathrm{Tr}\mathrm{e}^{-\vec{\beta}\vec{O}^b}, \tag{13.236}$$

$$\text{and hence } \theta_{\vec{\beta}} = \theta^a_{\vec{\beta}} \otimes \theta^b_{\vec{\beta}}. \tag{13.237}$$

Thus the ansatz (13.233) is stable with respect putting together non-interacting systems.

One introduuces the partition function

$$Z(\vec{\beta}) := \mathrm{Tr} \mathrm{e}^{-\vec{\beta}\vec{O}}. \tag{13.238}$$

One can use the partition function to compute the expectation values of the observables:

$$\langle O_i \rangle_{\vec{\beta}} = \mathrm{Tr}\theta_{\vec{\beta}} O_i = \frac{\mathrm{Tr} \mathrm{e}^{-\vec{\beta}\vec{O}} O_i}{\mathrm{Tr} \mathrm{e}^{-\vec{\beta}\vec{O}}} = -\partial_{\beta_i} \ln Z(\vec{\beta}). \tag{13.239}$$

Note that $\langle O_i \rangle_{\vec{\beta}}$ can often be used to parametrize $\theta_{\vec{\beta}}$ instead of β_i. In particular, if O_i is positive then $\langle O_i \rangle_{\vec{\beta}}$ is decreasing.

13.7 Chemical potential

The most common choice of the observables is $O_0 = H$, the Hamiltonian, and $O_i = N_i$ the number operators of various species of particles. Then $\beta_0 = \beta$ is called the inverse temperature, and $\beta_i = -\beta\mu_i$, $i = 1, \dots n$, where μ_i is called the chemical potential of the ith species. One introduces the "grand-canonical Hamiltonian"

$$H_{\mu_1,\dots,\mu_n} := H - \mu_1 N_1 - \dots - \mu_n N_n. \tag{13.240}$$

The grand-canonical Gibbs state is the usual Gibbs state of the grand-canonical Hamiltonian:

$$\theta_{\beta,\mu_1,\dots,\mu_n} = \frac{\mathrm{e}^{-\beta H_{\mu_1,\dots,\mu_n}}}{\mathrm{Tr} \mathrm{e}^{-\beta H_{\mu_1,\dots,\mu_n}}}. \tag{13.241}$$

14 Lattice models

14.1 Nearest neighbor's interactions

Let us consider $d = 2$ ($\mathcal{H}_i \simeq \mathbb{C}^2$). Denote the canonical basis of $\mathbb{C}^2$ by $e_\uparrow$, $e_\downarrow$. Introduce the Pauli matrices:

$$\sigma^x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^y = \begin{bmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \sigma^z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{14.242}$$

When they act on $\mathcal{H}_j$, they will be denoted $\sigma_j^x, \sigma_j^y, \sigma_j^z$. Here are some of the most popular models in statistical physics:

(1) **Ising Model.** For $\Lambda \in 2^{\mathbb{Z}^d}_{\mathrm{fin}}$ we set

$$H_\Lambda := J \sum_{\substack{|j-k|=1,\\ j,k\in\Lambda}} \sigma_j^z \sigma_k^z + h \sum_{j\in\Lambda} \sigma_j^z. \tag{14.243}$$

(2) **XX Model.** For $\Lambda \in 2^{\mathbb{Z}^d}_{\mathrm{fin}}$ we set

$$H_\Lambda := J \sum_{\substack{|j-k|=1,\\ j,k\in\Lambda}} (\sigma_j^x\sigma_k^x + \sigma_j^y\sigma_k^y) + h\sum_{j\in\Lambda}\sigma_j^z. \tag{14.244}$$

(3) **(Isotropic) Quantum Heisenberg Model or XXX Model.** For $\Lambda \in 2^{\mathbb{Z}^d}_{\mathrm{fin}}$ we set

$$H_\Lambda := J \sum_{\substack{|j-k|=1,\\ j,k\in\Lambda}} (\sigma_j^x\sigma_k^x + \sigma_j^y\sigma_k^y + \sigma_j^z\sigma_k^z) + h\sum_{j\in\Lambda}\sigma_j^z. \tag{14.245}$$

In all these models the local Hamiltonian defines the local dynamics

$$\rho_{\Lambda,t}(A) := \mathrm{e}^{\mathrm{i}tH_\Lambda} A \mathrm{e}^{-\mathrm{i}tH_\Lambda}, \quad A \in \mathfrak{A}_\Lambda. \tag{14.246}$$

Then one shows the existence of the dynamics in thermodynamic limit

$$\rho_t(A) := \lim_{\Lambda\nearrow\mathbb{Z}^d} \rho_{t,\Lambda}(A), \quad A \in \mathfrak{A}. \tag{14.247}$$

The parameter J is sometimes called *hopping*. If $J > 0$, we say that the model is antiferromagnetic, if $J < 0$ it is ferromagnetic. h is called the *external magnetc field*.

Let $\mathfrak{A}_\Lambda^{\mathrm{dg}}$ denote the algebra of diagonal elements of $\mathcal{A}_\Lambda$ (those commuting with σ_j^z). It is a commutative subalgebra of $\mathfrak{A}_\Lambda$. The operators $\sigma_j^z,\ j \in \Lambda$ belong to $\mathfrak{A}_\Lambda^{\mathrm{dg}}$. By taking

$$\mathfrak{A}_{\mathrm{loc}}^{\mathrm{dg}} := \bigcup_{\Lambda\in 2^{\mathbb{Z}^d}_{\mathrm{fin}}} \mathfrak{A}_\Lambda^{\mathrm{dg}}, \quad \mathfrak{A}^{\mathrm{dg}} := \left(\mathfrak{A}_{\mathrm{loc}}^{\mathrm{dg}}\right)^{\mathrm{cpl}}. \tag{14.248}$$

we obtain a commutative subalgebra of $\mathfrak{A}$. The dynamics of the Ising model preserves the commutative C^*-algebra $\mathfrak{A}^{\mathrm{dg}}$. (It is actually trivial on $\mathfrak{A}^{\mathrm{dg}}$) . Therefore, the Ising model is in reality a classical model, and putting it among genuinely quantum models may be considered artificial. Nevertheless, it can be studied by similar methods as quantum models.

14.2 Ground states of Ising model

Here are pure ground states of the Ising model for $h = 0$ and $d = 1$:

$$\omega^\uparrow := \Big(\bigotimes_j e_j^\uparrow \Big| \cdot \Big| \bigotimes_j e_j^\uparrow\Big), \tag{14.249}$$

$$\omega^\downarrow := \Big(\bigotimes_j e_j^\downarrow \Big| \cdot \Big| \bigotimes_j e_j^\downarrow\Big), \tag{14.250}$$

$$\omega_n^< := \Big(\bigotimes_{j<n} e_j^\downarrow \otimes \bigotimes_{j\geq n} e_j^\uparrow \Big| \cdot \Big| \bigotimes_{j<n} e_j^\downarrow \otimes \bigotimes_{j\geq n} e_j^\uparrow\Big), \tag{14.251}$$

$$\omega_n^> := \Big(\bigotimes_{j<n} e_j^\uparrow \otimes \bigotimes_{j\geq n} e_j^\downarrow \Big| \cdot \Big| \bigotimes_{j<n} e_j^\uparrow \otimes \bigotimes_{j\geq n} e_j^\downarrow\Big). \tag{14.252}$$

Of course, we can also take convex combinations of these states, obtaining mixed ground states. The GNS representations generated by $\omega_n^<$ for distinct n are all equivalent. The same is true for $\omega_n^>$.

However, the representations generated by $\omega^\uparrow$, $\omega^\downarrow$ $\omega_n^<$ and $\omega_n^>$ are inequivalent. Thus we have 4 inequivalent irreducible ground state represetnations.

14.3 Phases of models with nearest neighbor's interactions

One can show that for $\beta \in [0, +\infty]$ all the models described in Subsect. 14.1 have β-KMS states. For high temperatures they are unique: that means, there exists $0 < \beta_\mathrm{c} \leq +\infty$ such that for $0 \leq \beta < \beta_\mathrm{c}$ there exists a unique β-KMS state.

For instance, consider the ferromagnetic Ising model (with $J < 0$). For $h \neq 0$ and for all $\beta \in [0, \infty]$ we have a unique β-KMS state with most spins aligned with the magnetic field. Denote it by $\omega_{h,\beta}$.

For $h = 0$, $\beta < \beta_\mathrm{c}$ the state $\omega_{h,\beta}$ can be extended by continuity to $h = 0$. For $d = 1$ we have $\beta_\mathrm{c} = +\infty$. For $d \geq 2$ we have $\beta_\mathrm{c} < +\infty$, and for $\beta_\mathrm{c} < \beta \leq +\infty$ the limits

$$\omega_{\pm 0,\beta} := \lim_{\pm h \searrow 0} \omega_{h,\beta} \tag{14.253}$$

are different.

15 Thermal states of Fermi gases

15.1 Fermions at positive temperature

Consider first 1 degree of freedom. For $\lambda \in \mathbb{R}$ let the Hamiltonian be $H = \mathrm{d}\Gamma(\lambda) = \lambda a^* a$ on $\Gamma_\mathrm{a}(\mathbb{C})$. It is easy to compute the Gibbs density matrix θ_β, the partition function $Z(\beta)$, the average number $\langle N \rangle_\beta$ and average energy $\langle H \rangle_\beta$:

$$\theta_\beta = \frac{\mathrm{e}^{-\beta \mathrm{d}\Gamma(\lambda)}}{\mathrm{Tr} \mathrm{e}^{-\beta \mathrm{d}\Gamma(\lambda)}} = \frac{\Gamma(\mathrm{e}^{-\beta\lambda})}{1 + \mathrm{e}^{-\beta\lambda}}, \tag{15.254}$$

$$Z(\beta) = 1 + \mathrm{e}^{-\beta\lambda}, \tag{15.255}$$

$$\langle N \rangle_\beta = \frac{1}{1 + \mathrm{e}^{\beta\lambda}}, \tag{15.256}$$

$$\langle H \rangle_\beta = \frac{\lambda}{1 + \mathrm{e}^{\beta\lambda}}. \tag{15.257}$$

If we have many degrees of freedom, so that h is a self-adjoint operator on the space $\mathcal{Z}$ with λ_i on the diagonal and

$$H = \mathrm{d}\Gamma(h) = \sum \lambda_i a_i^* a_i, \tag{15.258}$$

then using $\Gamma_{\rm a}(\mathcal{Z}) = \bigotimes\limits_i \Gamma_{\rm a}(\mathbb{C})$ we obtain

$$\theta_\beta = \bigotimes_i \frac{\Gamma(\mathrm{e}^{-\beta\lambda_i})}{1+\mathrm{e}^{-\beta\lambda_i}} = \frac{\Gamma(\mathrm{e}^{-\beta h})}{\mathrm{Tr}\Gamma(\mathrm{e}^{-\beta h})}, \tag{15.259}$$

$$\ln Z(\beta) = \sum_i \ln(1+\mathrm{e}^{-\beta\lambda_i}) = \mathrm{Tr}\ln(1\!\!1+\mathrm{e}^{-\beta h}), \tag{15.260}$$

$$\langle N\rangle_\beta = \sum_i \frac{1}{1+\mathrm{e}^{\beta\lambda_i}} = \mathrm{Tr}\frac{1}{1\!\!1+\mathrm{e}^{\beta h}}, \tag{15.261}$$

$$\langle H\rangle_\beta = \sum_i \frac{\lambda_i}{1+\mathrm{e}^{\beta\lambda_i}} = \mathrm{Tr}\frac{h}{1\!\!1+\mathrm{e}^{\beta h}}. \tag{15.262}$$

15.2 Fermi gas

Consider now the Fermi gas in a box of sidelength L, that is on $L^2([0,L[^d)$ with periodic boundary conditions:

$$\begin{aligned} H^L &:= \mathrm{d}\Gamma\Big(-\frac{1}{2}\Delta^L - \mu 1\!\!1\Big) && (15.263)\\ &= \sum_{k\in\frac{2\pi}{L}\mathbb{Z}^d} a_k^* a_k \Big(\frac{1}{2}k^2 - \mu\Big). && (15.264)\end{aligned}$$

Then

$$\begin{aligned} \langle N\rangle_{\beta,\mu} &= \sum_{k\in\frac{2\pi}{L}\mathbb{Z}^d} \theta\Big(-\frac{k^2}{2}+\mu\Big), \quad \beta=+\infty && (15.265)\\ &= \sum_{k\in\frac{2\pi}{L}\mathbb{Z}^d} \frac{1}{1+\mathrm{e}^{\beta(\frac{k^2}{2}-\mu)}}, \quad \beta>0. && (15.266)\end{aligned}$$

Taking the limit $L\to\infty$ we obtain

$$\begin{aligned} H &:= \mathrm{d}\Gamma\Big(\frac{1}{2}\Delta - \mu 1\!\!1\Big) && (15.267)\\ &= \int a_k^* a_k \Big(\frac{1}{2}k^2-\mu\Big)\mathrm{d}k. && (15.268)\end{aligned}$$

Then

$$\begin{aligned} \langle n\rangle_{\beta,\mu} := \lim_{L\to\infty}\frac{\langle N\rangle_{\beta,\mu}}{L^d} &= \int \mathrm{d}k\theta\Big(-\frac{k^2}{2}+\mu)\Big), \quad \beta=+\infty; && (15.269)\\ &= \int \mathrm{d}k\frac{1}{1+\mathrm{e}^{\beta(\frac{k^2}{2}-\mu)}}, \quad 0\le\beta<\infty. && (15.270)\end{aligned}$$

Note that

$$\begin{aligned} [0,\infty[\ni\mu &\mapsto \langle n\rangle_{\beta,\mu}, \quad \beta=+\infty; && (15.271)\\ \mathbb{R}\ni\mu &\mapsto \langle n\rangle_{\beta,\mu}, \quad 0\le\beta<+\infty. && (15.272)\end{aligned}$$

are bijections.

15.3 CAR algebra

Consider space $\mathcal{Z}$ and symbols $a(z)$, $a^*(z)$, $z \in \mathcal{Z}$. We can form the $*$-algebra spanned by monomials

$$a^?(z_1)\cdots a^?(z_n) \tag{15.273}$$

where ? is either empty or $*$. The product is the concatenation, the involution is putting $*$, where $a^{**}(z) = a(z)$, and reversing the order.

Next we impose the relations

$$a(\lambda_1 z_1 + \lambda_2 z_2) = \overline{\lambda}_1 a(z_1) + \overline{\lambda}_2 a(z_2), \tag{15.274}$$

$$a^*(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 a^*(z_1) + \lambda_2 a^*(z_2), \tag{15.275}$$

$$[a(z_1), a(z_2)]_+ = [a^*(z_1), a^*(z_2)]_+ = 0 \tag{15.276}$$

$$[a(z_1), a^*(z_2)]_+ = (z_1|z_2). \tag{15.277}$$

We obtain a $*$-algebra, which we denote $\mathrm{CAR}_0(\mathcal{Z})$. Note that this algebra is spanned by "Wick-ordered" (or "normally ordered") monomials:

$$a^*(z_1)\cdots a^*(z_n)a(w_m)\cdots a(w_1). \tag{15.278}$$

It has an obvious representation on the Fock space $\Gamma_{\mathrm{a}}(\mathcal{Z})$

$$\pi : \mathrm{CAR}_0(\mathcal{Z}) \to B(\Gamma_{\mathrm{a}}(\mathcal{Z})) \tag{15.279}$$

which endows $\mathrm{CAR}_0(\mathcal{Z})$ with a norm satisfying the C^*-property. We define the C^*-algebra

$$\mathrm{CAR}(\mathcal{Z}) := \mathrm{CAR}_0(\mathcal{Z})^{\mathrm{cpl}}, \tag{15.280}$$

Actually, π extends to an isometric representation $\pi : \mathrm{CAR}(\mathcal{Z}) \to \pi\big(\mathrm{CAR}(\mathcal{Z})\big)^{\mathrm{cl}}$. Thus $\mathrm{CAR}(\mathcal{Z})$ is isomorphic to a subalgebra of $B(\Gamma_{\mathrm{a}}(\mathcal{Z}))$. But we prefer to view it as an abstract C^*-algebra, which may have many representations.

Let h be a self-adjoint operator on $\mathcal{Z}$. Then

$$\rho_t(a^*(z) = a^*(\mathrm{e}^{\mathrm{i}th}z), \quad \rho_t\big(a^*(z)\big) = a^*(\mathrm{e}^{\mathrm{i}th}z) \tag{15.281}$$

extends uniquely to a continuous $*$ automorphism. We obtain a 1-parameter group

$$\mathbb{R} \ni t \mapsto \rho_t \in \mathrm{Aut}(\mathrm{CAR}(\mathcal{Z})). \tag{15.282}$$

In the representation (15.279) it is generated by the Hamiltonian $\mathrm{d}\Gamma(h)$:

$$\pi(\rho_t(A)) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}\pi(A)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(h)}. \tag{15.283}$$

If $\mathcal{Z}$ is finite dimensional, then the Gibbs state for $\mathrm{d}\Gamma(h)$ is given by the density matrix

$$\theta := \frac{\Gamma(\gamma)}{\mathrm{Tr}\Gamma(\gamma)}, \quad \gamma = \mathrm{e}^{-\beta h}. \tag{15.284}$$

15.4 Fermionic quasifree states

Let ω be a state on $\mathrm{CAR}(\mathcal{Z})$. We say that it is gauge-invariant quasifree if

$$\omega\big(a^*(z_1)\cdots a^*(z_n)a(w_m)\cdots a(w_1)\big) \tag{15.285}$$

is nonzero only if $n = m$ and then it is

$$\sum_{\sigma\in S_n}\prod_{i=1}^{n}\omega\big(a^*(z_i)a(w_{\sigma(i)})\big)\mathrm{sgn}\sigma \tag{15.286}$$

Clearly the state is uniquely defined by the quadratic form

$$\omega\big(a^*(z)a(w)\big) = (w|\rho z), \tag{15.287}$$

$$\omega\big(a(w)a^*(z)\big) = \big(w|(\mathbb{1}-\rho)z\big). \tag{15.288}$$

Hence $0 \leq \rho \leq \mathbb{1}$ and we can set $\gamma = \frac{\rho}{\mathbb{1}-\rho}$, $\rho = \frac{\gamma}{\mathbb{1}+\gamma}$.

We will denote this state by ω_γ and rewrite this as

$$\omega_\gamma\big(a^*(z)a(w)\big) = \Big(w|\frac{1}{\mathbb{1}+\gamma^{-1}}z\Big), \tag{15.289}$$

$$\omega_\gamma\big(a(w)a^*(z)\big) = \Big(w|\frac{1}{\mathbb{1}+\gamma}z\Big). \tag{15.290}$$

If $\gamma = \mathrm{e}^{-\beta h}$, then ω_γ is β-KMS for the dynamics ρ_t. Let us check it on an example:

$$\omega_\gamma\Big(a(z)\rho_{\mathrm{i}\beta}\big(a^*(w)\big)\Big) = \omega_\gamma\Big(a(z)a^*\big(\mathrm{e}^{-\beta h}w\big)\Big) \tag{15.291}$$

$$= \Big(z|\frac{1}{\mathbb{1}+\mathrm{e}^{-\beta h}}\mathrm{e}^{-\beta h}w\Big) = \Big(z|\frac{1}{\mathbb{1}+\mathrm{e}^{\beta h}}w\Big) = \omega_\gamma\big(a^*(w)a(z)\big) \tag{15.292}$$

If $\mathcal{Z}$ is finite dimensional, then the state with the density matrix $\theta = \frac{\Gamma(\gamma)}{\mathrm{Tr}\Gamma(\gamma)}$ is quasifree with the covariance as above. In particular, the β-Gibbs state for $\mathrm{d}\Gamma(h)$ is quasifree with $\gamma = \mathrm{e}^{-\beta h}$. To see this, we diagonalize h. Then we have an orthonormal basis of $\mathcal{Z}$, and expand everything in this basis.

15.5 Araki-Wyss representation

Let us fix a conjugation $\mathcal{C}$ on $\mathcal{Z}$ and consider the Fock space $\Gamma_{\mathrm{a}}(\mathcal{Z}\oplus\mathcal{C}\mathcal{Z})$. Let γ be a positive operator on $\mathcal{Z}$. Define

$$a_\gamma^*(z) := a^*\Big((1+\gamma^{-1})^{-\frac{1}{2}}z, 0\Big) + a\Big(0, \mathcal{C}(1+\gamma)^{-\frac{1}{2}}z\Big), \tag{15.293}$$

$$a_\gamma(z) := a\Big((1+\gamma^{-1})^{-\frac{1}{2}}z, 0\Big) + a^*\Big(0, \mathcal{C}(1+\gamma)^{-\frac{1}{2}}z\Big). \tag{15.294}$$

We check that $(\Omega|\cdot\Omega)$ is quasifree and

$$[a_\gamma(z_1), a_\gamma(z_2)]_+ = [a_\gamma^*(z_1), a_\gamma^*(z_2)]_+ = 0 \tag{15.295}$$

$$[a_\gamma(z_1), a_\gamma^*(z_2)]_+ = (z_1|z_2), \tag{15.296}$$

$$(\Omega|a_\gamma(w)a_\gamma^*(z)\Omega) = \Big(w|\frac{1}{1\!\!1+\gamma^{-1}}z\Big), \tag{15.297}$$

$$(\Omega|a_\gamma^*(z)a_\gamma(w)\Omega) = \Big(w|\frac{1}{1\!\!1+\gamma}z\Big). \tag{15.298}$$

We can define a representation

$$\pi_\gamma : \mathrm{CAR}(\mathcal{Z}) \to B(\Gamma_{\mathrm{a}}(\mathcal{Z}\oplus\mathcal{C}\mathcal{Z}) \tag{15.299}$$

by setting

$$\pi_\gamma(a(z)) = a_\gamma(z), \quad \pi_\gamma(a^*(z)) = a_\gamma^*(z). \tag{15.300}$$

Note that

$$\omega_\gamma\big(\pi_\gamma(A)\big) = (\Omega|\pi_\gamma(A)\Omega). \tag{15.301}$$

The representation π_γ is the GNS representation for the state ω_γ and Ω is the corresponding cyclic vector.

16 Thermal states of Bose gases

16.1 Bosons at positive temperature

Consider first 1 degree of freedom. For $\lambda \in \mathbb{R}$ let the Hamiltonian be $H = \mathrm{d}\Gamma(\lambda) = \lambda a^* a$ on $\Gamma_{\mathrm{s}}(\mathbb{C})$. It is easy to compute the Gibbs density matrix θ_β, the partition function $Z(\beta)$, the average number $\langle N\rangle_\beta$ and average energy $\langle H\rangle_\beta$:

$$\theta_\beta = \frac{\mathrm{e}^{-\beta\mathrm{d}\Gamma(\lambda)}}{\mathrm{Tr}\mathrm{e}^{-\beta\mathrm{d}\Gamma(\lambda)}} = \Gamma(\mathrm{e}^{-\beta\lambda})(1-\mathrm{e}^{-\beta\lambda}), \tag{16.302}$$

$$Z(\beta) = \frac{1}{1-\mathrm{e}^{-\beta\lambda}}, \tag{16.303}$$

$$\langle N\rangle_\beta = \frac{1}{\mathrm{e}^{\beta\lambda}-1}, \tag{16.304}$$

$$\langle H\rangle_\beta = \frac{\lambda}{\mathrm{e}^{\beta\lambda}-1}. \tag{16.305}$$

If we have many degrees of freedom, so that h is a self-adjoint operator on the space $\mathcal{Z}$ with λ_i on the diagonal and

$$H = \mathrm{d}\Gamma(h) = \sum \lambda_i a_i^* a_i, \tag{16.306}$$

then using $\Gamma_{\rm s}(\mathcal{Z}) = \underset{i}{\otimes} \Gamma_{\rm s}(\mathbb{C})$ we obtain

$$\theta_\beta = \underset{i}{\otimes} \Gamma(\mathrm{e}^{-\beta\lambda_i})(\mathrm{e}^{-\beta\lambda_i} - 1) = \frac{\Gamma(\mathrm{e}^{-\beta h})}{\mathrm{Tr}\Gamma(\mathrm{e}^{-\beta h})}, \tag{16.307}$$

$$\ln Z(\beta) = -\sum_i \ln(\mathrm{e}^{-\beta\lambda_i} - 1) = \mathrm{Tr}\ln(\mathrm{e}^{-\beta h} - 1\!\!1), \tag{16.308}$$

$$\langle N \rangle_\beta = \sum_i \frac{1}{\mathrm{e}^{\beta\lambda_i} - 1} = \mathrm{Tr}\frac{1}{\mathrm{e}^{\beta h} - 1\!\!1}, \tag{16.309}$$

$$\langle H \rangle_\beta = \sum_i \frac{\lambda_i}{\mathrm{e}^{\beta\lambda_i} - 1} = \mathrm{Tr}\frac{h}{\mathrm{e}^{\beta h} - 1\!\!1}. \tag{16.310}$$

16.2 Bose gas

Consider now the Bose gas in a box of sidelength L, that is on $L^2([0,L[^d)$ with periodic boundary conditions and $\mu \leq 0$:

$$H^L := \mathrm{d}\Gamma\Big(-\frac{1}{2}\Delta^L - \mu 1\!\!1\Big) \tag{16.311}$$

$$= \sum_{k \in \frac{2\pi}{L}\mathbb{Z}^d} a_k^* a_k \Big(\frac{1}{2}k^2 - \mu\Big). \tag{16.312}$$

Then for $\beta > 0$,

$$\langle N \rangle_{\beta,\mu} = \sum_{k \in \frac{2\pi}{L}\mathbb{Z}^d} \frac{1}{\mathrm{e}^{\beta(\frac{k^2}{2} - \mu)} - 1}. \tag{16.313}$$

Taking the limit $L \to \infty$ we obtain

$$H := \mathrm{d}\Gamma\Big(\frac{1}{2}\Delta - \mu 1\!\!1\Big) \tag{16.314}$$

$$= \int a_k^* a_k \Big(\frac{1}{2}k^2 - \mu\Big)\mathrm{d}k. \tag{16.315}$$

Then

$$\langle n \rangle_{\beta,\mu} := \lim_{L\to\infty} \frac{\langle N \rangle_{\beta,\mu}}{L^d} = \int \mathrm{d}k \frac{1}{\mathrm{e}^{\beta(\frac{k^2}{2} - \mu)} - 1}.. \tag{16.316}$$

Set

$$\mu_d := \frac{|\mathbb{S}^{d-1}| 2^{\frac{d}{2}-1}}{(2\pi)^d \beta^{\frac{d}{2}}} \int_0^\infty \frac{s^{\frac{d}{2}-1}\mathrm{d}s}{\mathrm{e}^s - 1}. \tag{16.317}$$

Note that

$$]-\infty, 0[\ni \mu \mapsto \langle n \rangle_{\beta,\mu} \in]0, +\infty[, \quad d = 1, 2; \tag{16.318}$$

$$]-\infty, 0[\ni \mu \mapsto \langle n \rangle_{\beta,\mu} \in]0, \mu_d[, \quad d \geq 3 \tag{16.319}$$

are bijections. We see that at $d \geq 3$ we have a phase transition at a positive temperature.

16.3 CCR algebra

Consider space $\mathcal{Z}$ and symbols $a(z)$, $a^*(z)$, $z \in \mathcal{Z}$. We can form the $*$-algebra spanned by monomials

$$a^?(z_1)\cdots a^?(z_n) \tag{16.320}$$

where ? is either empty or $*$. The product is the concatenation, the involution is putting $*$, where $a^{**}(z) = a(z)$, and reversing the order.

Next we impose the relations

$$a(\lambda_1 z_1 + \lambda_2 z_2) = \overline{\lambda}_1 a(z_1) + \overline{\lambda}_2 a(z_2), \tag{16.321}$$

$$a^*(\lambda_1 z_1 + \lambda_2 z_2) = \lambda_1 a(z_1) + \lambda_2 a^*(z_2), \tag{16.322}$$

$$[a(z_1), a(z_2)] = [a^*(z_1), a^*(z_2)] = 0 \tag{16.323}$$

$$[a(z_1), a^*(z_2)] = (z_1|z_2). \tag{16.324}$$

We obtain a $*$-algebra, which we denote $\mathrm{CCR}_0(\mathcal{Z})$. Note that this algebra is spanned by "Wick-ordered" (or "normally ordered") monomials:

$$a^*(z_1)\cdots a^*(z_n)a(w_m)\cdots a(w_1). \tag{16.325}$$

It has an obvious representation on the finite-particle bosonic Fock space $\Gamma_{\mathrm{s}}^{\mathrm{fin}}(\mathcal{Z})$

$$\pi : \mathrm{CCR}_0(\mathcal{Z}) \to L(\Gamma_{\mathrm{a}}^{\mathrm{fin}}(\mathcal{Z})) \tag{16.326}$$

Unfortunately, the image of this representation consists typically of unbounded operators and we do not obtain a C^*-algebra.

Let h be a self-adjoint operator on $\mathcal{Z}$. Then

$$\rho_t(a^*(z) = a^*(\mathrm{e}^{\mathrm{i}th}z), \quad \rho_t\big(a^*(z)\big) = a^*(\mathrm{e}^{\mathrm{i}th}z) \tag{16.327}$$

extends uniquely to a continuous $*$ automorphism. We obtain a 1-parameter group

$$\mathbb{R} \ni t \mapsto \rho_t \in \mathrm{Aut}(\mathrm{CCR}_0(\mathcal{Z})). \tag{16.328}$$

In the representation (16.326) it is generated by the Hamiltonian $\mathrm{d}\Gamma(h)$:

$$\pi(\rho_t(A)) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}\pi(A)\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(h)}. \tag{16.329}$$

If $\mathcal{Z}$ is finite dimensional, then the Gibbs state for $\mathrm{d}\Gamma(h)$ is given by the density matrix

$$\theta := \frac{\Gamma(\gamma)}{\mathrm{Tr}\Gamma(\gamma)}, \quad \gamma = \mathrm{e}^{-\beta h}. \tag{16.330}$$

16.4 Bosonic quasifree states

Let ω be a state on $\mathrm{CAR}(\mathcal{Z})$. We say that it is gauge-invariant quasifree if

$$\omega\big(a^*(z_1)\cdots a^*(z_n)a(w_m)\cdots a(w_1)\big) \tag{16.331}$$

is nonzero only if $n = m$ and then it is

$$\sum_{\sigma\in S_n}\prod_{i=1}^{n}\omega\big(a^*(z_i)a(w_{\sigma(i)})\big) \tag{16.332}$$

Clearly the state is uniquely defined by the quadratic form

$$\omega\big(a^*(z)a(w)\big) = (w|\rho z), \tag{16.333}$$

$$\omega\big(a(w)a^*(z)\big) = \big(w|(1\!\!1+\rho)z\big). \tag{16.334}$$

Hence $0 \leq \rho$ and we can set $\gamma = \frac{\rho}{1\!\!1+\rho}$, $\rho = \frac{\gamma}{1\!\!1-\gamma}$.

We will denote this state by ω_γ and rewrite this as

$$\omega_\gamma\big(a^*(z)a(w)\big) = \Big(w|\frac{1}{\gamma^{-1}-1\!\!1}z\Big), \tag{16.335}$$

$$\omega_\gamma\big(a(w)a^*(z)\big) = \Big(w|\frac{1}{1\!\!1-\gamma}z\Big). \tag{16.336}$$

If $\gamma = \mathrm{e}^{-\beta h}$, then ω_γ is β-KMS for the dynamics ρ_t. Let us check it on an example:

$$\omega_\gamma\Big(a(z)\rho_{\mathrm{i}\beta}\big(a^*(w)\big)\Big) = \omega_\gamma\Big(a(z)a^*\big(\mathrm{e}^{-\beta h}w\big)\Big) \tag{16.337}$$

$$= \Big(z|\frac{1}{1\!\!1-\mathrm{e}^{-\beta h}}\mathrm{e}^{-\beta h}w\Big) = \Big(z|\frac{1}{\mathrm{e}^{\beta h}-1\!\!1}w\Big) = \omega_\gamma\big(a^*(w)a(z)\big) \tag{16.338}$$

If $\mathcal{Z}$ is finite dimensional, then the state with the density matrix $\theta = \frac{\Gamma(\gamma)}{\mathrm{Tr}\Gamma(\gamma)}$ is quasifree with the covariance as above. In particular, the β-Gibbs state for $\mathrm{d}\Gamma(h)$ is quasifree with $\gamma = \mathrm{e}^{-\beta h}$. To see this, we diagonalize h. Then we have an orthonormal basis of $\mathcal{Z}$, and expand everything in this basis.

16.5 Araki-Woods representation

Let us fix a conjugation $\mathcal{C}$ on $\mathcal{Z}$ and consider the Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z}\oplus\mathcal{C}\mathcal{Z})$. Let γ be a positive operator on $\mathcal{Z}$. Define

$$a_\gamma^*(z) := a^*\Big((\gamma^{-1}-1\!\!1)^{-\frac{1}{2}}z, 0\Big) + a\Big(0, \mathcal{C}(1-\gamma)^{-\frac{1}{2}}z\Big), \tag{16.339}$$

$$a_\gamma(z) := a\Big((\gamma^{-1}-1\!\!1)^{-\frac{1}{2}}z, 0\Big) + a^*\Big(0, \mathcal{C}(1\!\!1-\gamma)^{-\frac{1}{2}}z\Big). \tag{16.340}$$

We check that $(\Omega|\cdot\Omega)$ is quasifree and

$$[a_\gamma(z_1), a_\gamma(z_2)]_+ = [a_\gamma^*(z_1), a_\gamma^*(z_2)]_+ = 0 \tag{16.341}$$

$$[a_\gamma(z_1), a_\gamma^*(z_2)]_+ = (z_1|z_2), \tag{16.342}$$

$$(\Omega|a_\gamma(w)a_\gamma^*(z)\Omega) = \Big(w|\frac{1}{\gamma^{-1}-1\!\!1}z\Big), \tag{16.343}$$

$$(\Omega|a_\gamma^*(z)a_\gamma(w)\Omega) = \Big(w|\frac{1}{1\!\!1-\gamma}z\Big). \tag{16.344}$$

We can define a representation

$$\pi_\gamma : \mathrm{CCR}_0(\mathcal{Z}) \to L(\Gamma_\mathrm{s}(\mathcal{Z} \oplus \mathcal{C}\mathcal{Z}) \tag{16.345}$$

by setting

$$\pi_\gamma(a(z)) = a_\gamma(z), \quad \pi_\gamma(a^*(z)) = a_\gamma^*(z). \tag{16.346}$$

Note that

$$\omega_\gamma\big(\pi_\gamma(A)\big) = (\Omega|\pi_\gamma(A)\Omega). \tag{16.347}$$

The representation π_γ is the GNS representation for the state ω_γ and Ω is the corresponding cyclic vector.