Commun. Math. Phys. 279, 1–30 (2008)
Digital Object Identifier (DOI) 10.1007/s00220-008-0419-3

Communications in
Mathematical Physics

Extended Weak Coupling Limit for Pauli-Fierz Operators

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Received: 23 October 2006 / Accepted: 29 June 2007
Published online: 8 February 2008 –

Abstract: We consider the weak coupling limit for a quantum system consisting of a small subsystem and reservoirs. It is known rigorously since [10] that the Heisenberg evolution restricted to the small system converges in an appropriate sense to a Markovian semigroup. In the nineties, Accardi, Frigerio and Lu [1] initiated an investigation of the convergence of the unreduced unitary evolution to a singular unitary evolution generated by a Langevin-Schrödinger equation. We present a version of this convergence which is both simpler and stronger than the formulations which we know. Our main result says that in an appropriately understood weak coupling limit the interaction of the small system with environment can be expressed in terms of the so-called quantum white noise.

1. Introduction

One of the main goals of mathematical physics is to justify various approximate effective models used by physicists by deriving them as limiting cases of more fundamental theories. This paper is devoted to a class of such models that one sometimes calls quantum Langevin dynamics. We show that quantum Langevin dynamics arise naturally as the limit of a dynamics of a small system weakly interacting with a reservoir where not only the small system, but also the reservoir is taken into account. We will call this version of a weak coupling limit the *extended weak coupling limit*, to differentiate it from the better known *reduced weak coupling limit*, which involves only the dynamics reduced to the small system.

To our knowledge, the main idea of extended weak coupling limit first appeared in the literature in the work of Accardi, Frigerio and Lu in [1] under the name of *stochastic limit*.

Our approach is inspired by their work, nevertheless we think that it is both simpler and more powerful.

The reader may also find it useful to compare the present work with our previous paper [12], which describes the extended weak coupling limit on a relatively simple (and less physical) example of the *Friedrichs model*. [12], apart from presenting results, which we believe are mathematically interesting in their own right, can be viewed as a preparatory exercise for the present work.

1.1. Quantum Markov semigroups. Before we discuss quantum Langevin dynamics, we should recall a better known class of effective dynamics – that of *quantum Markov semigroups* (or, in other words, completely positive unity preserving time-continuous semigroups). They are often used as a phenomenological description of quantum systems. It is well known that every quantum Markov semigroup on $B(\mathcal{K})$, where $\mathcal{K}$ is a finite dimensional Hilbert space, can be written as e^{tL}, where L can be written in the so-called *Lindblad form* [24]

$$L(S) = -\mathrm{i}(\Upsilon S - S\Upsilon^*) + \nu^* S \nu, \qquad S \in \mathcal{B}(\mathcal{K}), \tag{1.1}$$

ν is an operator from $\mathcal{K}$ to $\mathcal{K} \otimes \mathfrak{h}$ for some auxiliary Hilbert space $\mathfrak{h}$ and Υ is an operator on $\mathcal{K}$ satisfying

$$-\mathrm{i}\Upsilon + \mathrm{i}\Upsilon^* = -\nu^*\nu. \tag{1.2}$$

Note that given L, the operators Υ and ν are not defined uniquely.

1.2. Reduced weak coupling limit. It is generally assumed that only reversible (unitary) dynamics appear in fundamental quantum physics. Nevertheless, in phenomenological approaches researchers often apply non-unitary quantum Markov semigroups to describe irreversible phenomena. A possible justification for their use is provided by the so-called weak coupling limit, an idea that goes back to Pauli and van Hove [22], and was made rigorous in an elegant work of E. B. Davies [10]. Davies proved that if a small quantum system is weakly coupled to the environment, then the reduced dynamics in the interaction picture, after rescaling the time as $\lambda^{-2}t$, converges to a quantum Markov semigroup defined on the observables of the small system.

To be more specific, consider a system given by a Hilbert space $\mathcal{H} := \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$, where $\mathcal{K}$ is a finite dimensional Hilbert space, $\mathcal{H}_{\mathrm{R}}$ is the 1-particle space of the reservoir and $\Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$ is the corresponding bosonic Fock space. The composite system is described by the dynamics generated by the self-adjoint operator

$$H_\lambda = K \otimes 1 + 1 \otimes \mathrm{d}\Gamma(H_{\mathrm{R}}) + \lambda(a^*(V) + a(V)). \tag{1.3}$$

Here K describes the Hamiltonian of the small system, $\mathrm{d}\Gamma(H_{\mathrm{R}})$ describes the dynamics of the reservoir expressed by the second quantization of a self-adjoint operator H_{R} on $\mathcal{H}_{\mathrm{R}}$, and $a^*(V)/a(V)$ describe the interaction between the small system and the reservoir, which we assume to be given by the creation/annihilation operators of an operator $V \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{H}_{\mathrm{R}})$.

The notation that we use to define H_λ is explained only in Sect. 2 and may be unfamiliar to some of the readers. Therefore let us describe the operators appearing in (1.3) with perhaps a better known (although less compact) notation. To this end it is convenient to identify $\mathcal{H}_{\mathrm{R}}$ with $L^2(\Xi, \mathrm{d}\xi)$, for some measure space $(\Xi, \mathrm{d}\xi)$, so that one can introduce a_ξ^*/a_ξ – the usual creation/annihilation operators describing bosonic excitations of the reservoir. Let H_{R} be the multiplication operator by a real function

$\Xi \ni \xi \mapsto x(\xi)$ and let $\Xi \ni \xi \mapsto v(\xi) \in \mathcal{B}(\mathcal{K})$ be the function describing the operator V. Then we have an alternative notation

$$\begin{aligned} \mathrm{d}\Gamma(H_\mathrm{R}) &= \int x(\xi) a_\xi^* a_\xi \mathrm{d}\xi, \\ a^*(V) &= \int v(\xi) a_\xi^* \mathrm{d}\xi, \\ a(V) &= \int v^*(\xi) a_\xi \mathrm{d}\xi. \end{aligned}$$

Operators of the form (1.3) are often used in quantum physics in phenomenological descriptions of a small quantum system interacting with an environment. Some varieties of (1.3) are known under such names as the spin-boson, Fröhlich, Nelson and polaron Hamiltonian. Following [11], we will call operators of the form (1.3) *Pauli-Fierz operators*. (Note, however, that some authors use this name in a slightly different meaning.)

The vacuum vector in $\Gamma_\mathrm{s}(\mathcal{H}_\mathrm{R})$ will be denoted by Ω. Let $I_\mathcal{K} : \mathcal{K} \to \mathcal{H}$ denote the isometric embedding, which maps a vector $\phi \in \mathcal{K}$ on $\phi \otimes \Omega \in \mathcal{H}$. Note that $I_\mathcal{K}^* : \mathcal{H} \to \mathcal{K}$ equals $1_\mathcal{K} \otimes \langle \Omega|$, and $I_\mathcal{K}^* \cdot I_\mathcal{K}$ is the conditional expectation from $\mathcal{B}(\mathcal{H})$ onto $\mathcal{B}(\mathcal{K})$.

One version of the result of Davies says that under some mild assumptions the following limit exists:

$$\begin{aligned} \Lambda_t(S) &:= \lim_{\lambda \searrow 0} \mathrm{e}^{\mathrm{i}\lambda^{-2} t K} I_\mathcal{K}^* \mathrm{e}^{-\mathrm{i}t\lambda^{-2} H_\lambda} S \otimes 1 \, \mathrm{e}^{\mathrm{i}t\lambda^{-2} H_\lambda} I_\mathcal{K} \mathrm{e}^{-\mathrm{i}\lambda^{-2} t K} \\ &= \lim_{\lambda \searrow 0} I_\mathcal{K}^* \mathrm{e}^{\mathrm{i}\lambda^{-2} t H_0} \mathrm{e}^{-\mathrm{i}t\lambda^{-2} H_\lambda} S \otimes 1 \, \mathrm{e}^{\mathrm{i}t\lambda^{-2} H_\lambda} \mathrm{e}^{-\mathrm{i}\lambda^{-2} t H_0} I_\mathcal{K}, \end{aligned} \tag{1.4}$$

and Λ_t is a quantum Markov semigroup. Thus we obtain a, possibly irreversible, quantum Markov semigroup as a limit of a family of reversible, physically realistic dynamics. We also obtain a concrete expression for the generator of Λ_t. More precisely, Υ and ν appearing in (1.1) are uniquely defined in terms of K, H_R and V.

In the literature on both the reduced and the extended weak coupling limit, one usually considers a nontrivial reference state for the reservoir, whereas we reduce our treatment to a vector state. This is justified since one can always represent the reservoir state as a vector state via the GNS construction. In particular, in the case of a thermal bosonic state, we can use the Araki-Woods representations of the CCR, so that the reservoir state is given by the Fock vacuum. The free reservoir Hamiltonian and the interaction are modified appropriately. For this reason, it is not always appropriate to call (1.3) a "Hamiltonian". In typical applications that we have in mind, the environment is a collection of heat baths at various positive temperatures, and then it is natural to take $\mathrm{d}\Gamma(H_\mathrm{R})$ to be the sum of their *Liouvilleans*. In this case, H_λ is not bounded from below, and it probably should not be called a "Hamiltonian". On the other hand, the name "Liouvillean" is not appropriate either, since on the small system K is actually the Hamiltonian, not the Liouvillean. Following the terminology introduced in [11], in such a case H_λ should be called a *semi-Liouvillean*.

1.3. Quantum Langevin dynamics. It is well known that a 1-parameter semigroup of contractions on a Hilbert space can be written as a compression of a unitary group. This unitary group is called a *dilation* of the semigroup.

A similar fact is true in the case of a quantum Markov semigroup. It has been noticed that every such semigroup can be written as

$$\Lambda_t(S) = I_{\mathcal{K}}^* \mathrm{e}^{-\mathrm{i}tZ}\, S \otimes 1\, \mathrm{e}^{\mathrm{i}tZ} I_{\mathcal{K}}, \quad S \in \mathcal{B}(\mathcal{K}). \tag{1.5}$$

Here, Z is a self-adjoint operator on a Hilbert space $\mathcal{Z} = \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{R}})$ for some 1-particle space $\mathcal{Z}_{\mathrm{R}}$ and $I_{\mathcal{K}} : \mathcal{K} \to \mathcal{Z}$ is defined analogously as before.

Unfortunately, in the literature there seems to be no consistent and uniform terminology for this dilation. A possible name for the unitary dynamics $\mathrm{e}^{\mathrm{i}tZ}$ seems to be a *Langevin-Schrödinger dynamics* or a *stochastic Schrödinger dynamics* for the semigroup Λ_t. The corresponding dynamics in the Heisenberg picture, that is $\mathrm{e}^{-\mathrm{i}tZ} \cdot \mathrm{e}^{\mathrm{i}tZ}$, will be called a *quantum Langevin dynamics* or a *quantum stochastic dynamics* for the semigroup Λ_t.

The first construction of a quantum Langevin dynamics was probably given by Hudson and Parthasaraty. In [23] they introduced the so-called *quantum stochastic differential equation* - a generalization of the usual stochastic differential equation known from the Ito calculus. The group $\mathrm{e}^{-\mathrm{i}tZ}$ is then given by the solution to this equation.

If the operators Υ and ν that appear in the generator of Λ_t written in the Lindblad form (1.1) are given, then there exists a canonical construction of the space $\mathcal{Z}$ and of a Langevin-Schrödinger dynamics $\mathrm{e}^{\mathrm{i}tZ}$ on $\mathcal{Z}$, which apart from (1.5) satisfies the condition

$$\mathrm{e}^{-\mathrm{i}t\Upsilon} = I_{\mathcal{K}}^* \mathrm{e}^{-\mathrm{i}tZ} I_{\mathcal{K}}. \tag{1.6}$$

Thus $\mathrm{e}^{-\mathrm{i}tZ}$ is a dilation of the contractive semigroup $\mathrm{e}^{-\mathrm{i}t\Upsilon}$ and $\mathrm{e}^{-\mathrm{i}tZ} \cdot \mathrm{e}^{\mathrm{i}tZ}$ is a dilation of the quantum Markov semigroup e^{tL}.

In this construction, at least formally, Z can be written in the form of a Pauli-Fierz operator

$$Z = \frac{1}{2}(\Upsilon + \Upsilon^*) + \mathrm{d}\Gamma(Z_{\mathrm{R}}) + \frac{1}{\sqrt{2\pi}} a(|1\rangle \otimes \nu) + \frac{1}{\sqrt{2\pi}} a^*(|1\rangle \otimes \nu). \tag{1.7}$$

The interaction that appears in (1.7) is quite singular and difficult mathematically. It is an example of a so-called *quantum white noise* [6].

Equation (1.5) suggests that quantum Langevin dynamics have perhaps more physical content than being just a mathematical device, and could be used as effective dynamics describing a small system interacting with environment. In fact, physicists (see e.g. [18]) often use such quantum Langevin dynamics to describe the interaction of a small system with an environment, e.g. with several heat baths.

Quantum Langevin dynamics are also often used to describe processes involving "continuous quantum measurements" [7]. One can then introduce observables describing "measurements performed in a given interval of time". Observables corresponding to measurements in non-overlapping time intervals commute, which can be a reasonable assumption in some idealized situations.

Note that the generator of a Langevin-Schrödinger dynamics is necessarily unbounded from below. This is often put forward as an argument against physical relevance of quantum Langevin dynamics. This argument is actually not justified, since unbounded from below generators of dynamics appear naturally in physics, especially in positive temperatures. We have seen such a situation when we discussed (1.3), since semi-Liouvilleans are typically unbounded from below. (See also a remark at the end of Subsect. 1.2).

1.4. Extended weak coupling limit. In [1], it was proposed by Accardi et al. that one could extend the idea of the weak coupling limit from the reduced dynamics to the dynamics on the whole system, and as a result one can obtain a justification of using quantum Langevin dynamics to describe quantum systems. They called their version of the weak coupling limit the *stochastic limit*. In our opinion, this name is not the best chosen, since the reduced weak coupling limit is just as "stochastic" as the extended one. Therefore we will use the name *extended weak coupling limit*.

The reduced weak coupling limit in the form considered by Davies has a rather clean mathematical formulation. Therefore, it was quickly appreciated by the mathematical physics community. The extended weak coupling limit is inevitably somewhat more complicated, in particular since it involves constructions that are, to a certain extent, arbitrary. Nevertheless, we believe that the idea of the extended weak coupling limit is valuable and sheds light on models used in physics, especially in quantum optics and quantum measurement theory. In our paper we would like to state and prove a new version of the extended weak coupling limit.

We start again from a dynamics generated by a "Pauli-Fierz operator" (1.3). As we discussed above, the reduced weak coupling limit leads to a quantum Markov semigroup with the generator given in a Lindblad form involving the operators Υ and ν. Given these data, we have a canonical construction of a quantum Langevin-Schrödinger dynamics $\mathrm{e}^{-\mathrm{i}tZ}$ acting on the "asymptotic space" $\mathcal{Z}$ such that (1.5) and (1.6) are satisfied. We also construct an appropriate identification operator $\Gamma(J_\lambda)$, which is a partial isometry mapping the physical space $\mathcal{H}$ into the asymptotic space $\mathcal{Z}$. Its main role is to scale the physical energy. There is some arbitrariness in the construction of the identification operator, since the frequencies away from the Bohr frequencies (differences of eigenvalues of K) do not matter in the limit $\lambda \searrow 0$. Finally, one needs what we call the "renormalizing operator" Z_{ren}, which takes care of the trivial part of the dynamics involving the eigenvalues of K. The main result of our paper can be stated as

$$\mathrm{s}^* - \lim_{\lambda\searrow 0} \mathrm{e}^{\mathrm{i}\lambda^{-2}tZ_{\mathrm{ren}}}\Gamma(J_\lambda)\mathrm{e}^{-\mathrm{i}\lambda^{-2}tH_\lambda}\Gamma(J_\lambda)^* = \mathrm{e}^{-\mathrm{i}tZ}, \tag{1.8}$$

where $\mathrm{s}^* - \lim$ denotes the strong* limit. Thus $\mathrm{e}^{-\mathrm{i}t(Z+\lambda^{-2}Z_{\mathrm{ren}})}$ can be viewed as the effective dynamics in the limit of $\lambda \searrow 0$.

Note that in the Heisenberg picture we obtain for any $B \in \mathcal{B}(\mathcal{Z})$,

$$\begin{aligned}\mathrm{s}^* - \lim_{\lambda\searrow 0} \mathrm{e}^{\mathrm{i}tZ_{\mathrm{ren}}}\Gamma(J_\lambda^*)\mathrm{e}^{-\mathrm{i}\lambda^{-2}tH_\lambda}\Gamma(J_\lambda)B\Gamma(J_\lambda^*)\mathrm{e}^{\mathrm{i}t\lambda^{-2}H_\lambda}\Gamma(J)^*\mathrm{e}^{-\mathrm{i}tZ_{\mathrm{ren}}}\\ = \mathrm{e}^{-\mathrm{i}tZ}B\mathrm{e}^{\mathrm{i}tZ}.\end{aligned}$$

Replacing B with $S \otimes 1$, pretending J_λ is unitary (which is justified, see e.g. Remark 4.5 or expression (6.33)), taking the conditional expectation $I_{\mathcal{K}}^* \cdot I_{\mathcal{K}}$ of both sides and using (1.5) we retrieve (1.4) – the reduced weak coupling limit.

One can also choose B of the form $1 \otimes A$ such that the strong limit $\Gamma(J_\lambda)B\Gamma(J_\lambda^*)$ as $\lambda \searrow 0$, exists. In that case, one can study fluctuations of reservoir quantities, see Theorem 5.7.

We can summarize the results of our paper in the following diagram (w.c.l. stands for weak coupling limit):

physical dynamics	extended w.c.l. $\longrightarrow$	quantum Langevin dynamics
$\downarrow$ reduction		$\downarrow$ reduction $\uparrow$ dilation
reduced physical dynamics	reduced w.c.l. $\longrightarrow$	quantum Markov semigroup + specific decomposition of Lindblad generator

1.5. Comparison with previous results. As mentioned already, we are surely not the first to come up with the concept of the extended weak coupling limit. Although the original idea is attributed to Spohn [29], the field was pioneered by Accardi et al. in [1] and a long list of works on the subject can be found in the book [3]. Recently, an interesting generalization has been made by [20].

On the heuristic level, the ideas of the extended weak coupling limit have been expressed by some physicists, e.g. by Gardiner and Collett, see [17] and Sect. 2.5 of [7].

The same idea was also applied to the low-density limit in [28] and [4], see also [5]. (The "reduced low density limit "has been put on rigorous footing in [15].)

Most previous results we are aware of have the following form: For a Hilbert space $\mathcal{R}$, let $\Phi(f) \in \Gamma_{\rm s}(\mathcal{R})$ be the exponential vector for the 1-particle vector $f \in \mathcal{R}$:

$$\Phi(f) = \exp\left(a^*(f)\right)\Omega. \tag{1.9}$$

Let $u, v \in \mathcal{K}$, $f, g \in \mathcal{H}_{\rm R}$, $s_1 < t_1, s_2 < t_2 \in \mathbb{R}$ and put $W_t^\lambda := {\rm e}^{{\rm i}\lambda^{-2}tH_0}{\rm e}^{-{\rm i}\lambda^{-2}tH_\lambda}$. Then, with all symbols having the same meaning as in the introduction above,

$$\left\langle u \otimes \Phi\left(\lambda \int_{s_1/\lambda^2}^{t_1/\lambda^2} {\rm e}^{-{\rm i}uH_{\rm R}} f {\rm d}u\right) \middle| (W_t^\lambda)^*(S\otimes 1)W_t^\lambda\, v \otimes \Phi\left(\lambda \int_{s_2/\lambda^2}^{t_2/\lambda^2} {\rm e}^{-{\rm i}uH_{\rm R}} g {\rm d}u\right)\right\rangle$$
$$\underset{\lambda\to 0}{\to} \left\langle u \otimes \Phi(1_{[s_1,t_1]} \otimes f) \middle| W_t^*(S\otimes 1)W_t\, v \otimes \Phi(1_{[s_2,t_2]} \otimes g)\right\rangle, \tag{1.10}$$

where W_t is the solution of an appropriate Langevin Schrödinger differential equation on the space $\mathcal{K}\otimes\Gamma_{\rm s}(L^2(\mathbb{R})\otimes\mathcal{H}_{\rm R})$ and $1_{[\cdot,\cdot]}$ is the indicator function of the interval $[\cdot,\cdot]$.

Note that both our approach and (1.10) express essentially the same physical idea. The scaling that we use to define J_λ is implicit in (1.10). The main advantages of our approach with respect to the previous works are

1) The asymptotic space $\mathcal{K}\otimes\Gamma_{\rm s}(L^2(\mathbb{R})\otimes\mathcal{H}_{\rm R})$ considered in (1.10) is much larger than the asymptotic space that we use (which is introduced in Subsect. 4.3). One can argue that our choice is more natural and "tailor-made" for the problem at hand – it closely resembles the original physical space without introducing unnecessary degrees of freedom.
2) We prove convergence in the $*$-strong sense, instead of (as outlined above) convergence of matrix elements of a class of rescaled coherent vectors. This is mathematically cleaner and more flexible.

3) Our approach allows to consider also limits of certain reservoir observables, see in particular Theorem 5.7.
4) We highlight the clear connection between the work of Davies [10] and Dümcke [14], and extended weak coupling limits. The latter follows rather easily from the results in [10] and [14].

A less important point of difference is the following: In the early works on the weak coupling limit, quasifree reservoirs were fermionic. If one chooses bosonic reservoirs, as we do, one has to control the unboundedness of the interaction term (since the bosonic creation and annihilation operators are unbounded). Although this is not difficult, see Theorem 4.1, we know of no place in the literature on the weak coupling limit where this difficulty is addressed. Of course, it is possible (and easy) to describe a version of our result where the Hamiltonian H_{R} is fermionic.

From the physical point of view, our results justify a lot of the manipulations one does with quantum Langevin dynamics (this is discussed in detail in [13]). In particular, Theorem 5.7 allows to identify fluctuations of reservoir number operators with limits of reservoir observables. These reservoir number operators (more specifically: their fluctuations) are heavily studied objects, see e.g. [7,8,27].

1.6. Outline. In Sect. 3, we construct a Langevin-Schrödinger dynamics associated with a specific decomposition of a Lindblad generator. In the first subsection of Sect. 4 we introduce the class of our physical models considered in our paper — Pauli-Fierz operators. In the remaining subsections of Sect. 4 we describe how to connect the setup of the physical model with that of the corresponding quantum Langevin dynamics. Our results are listed in Sect. 5 and their proofs are postponed to Sect. 6.

2. Preliminaries and Notations

We will use the formalism of second quantization, following the conventions adopted in [11].

For a Hilbert space $\mathcal{R}$ and $n \in \mathbb{N}$, we recall the projector Sym^n, which projects elements of the tensor power $\otimes^n \mathcal{R}$ onto symmetric tensors. Its range will be denoted $\Gamma_{\mathrm{s}}^n(\mathcal{R})$ – it is the n-particle subspace of the bosonic Fock space over $\mathcal{R}$. The symmetric (bosonic) second quantization of $\mathcal{R}$ is hence defined as

$$\Gamma_{\mathrm{s}}(\mathcal{R}) = \mathop{\oplus}_{n=0}^{\infty} \Gamma_{\mathrm{s}}^n(\mathcal{R}). \tag{2.1}$$

Note that we use the convention that $\otimes$ and $\oplus$ denote the tensor product and the direct sum in the category of Hilbert spaces. Sometimes we will use their algebraic counterparts. If $\mathcal{D}_1$ is a subspace of a Hilbert space $\mathcal{R}$, then

$$\overset{\mathrm{al}}{\Gamma}{}_{\mathrm{s}}^{n}(\mathcal{D}_1) = (\overset{\mathrm{al}}{\otimes}{}^{n} \mathcal{D}_1) \cap \Gamma_{\mathrm{s}}^n(\mathcal{R}), \tag{2.2}$$

where $\overset{\mathrm{al}}{\otimes}$ denotes the algebraic tensor product. We will often need

$$\overset{\mathrm{al}}{\Gamma}{}_{\mathrm{s}}(\mathcal{D}_1) = \mathrm{Span}\left\{\psi \mid \psi \in \overset{\mathrm{al}}{\Gamma}{}_{\mathrm{s}}^{n}(\mathcal{D}_1), n \in \mathbb{N}\right\}. \tag{2.3}$$

For $R \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{R})$, we heavily use the generalized creation and annihilation operators $a(R)$ and $a^*(R)$, as defined in [11]. Actually, we need even a slightly more general definition which is given now.

Assume that $\mathcal{D}_1$ is a dense subspace of the Hilbert space $\mathcal{R}$ and $R^* : \mathcal{K} \overset{\mathrm{al}}{\otimes} \mathcal{D}_1 \to \mathcal{K}$ is an unbounded operator. Let R stand for the adjoint of R^* in the sense of quadratic forms. (Note that the adjoint in the sense of forms is different from the adjoint in the sense of operators.) Define for all $n \in \mathbb{N}$,

$$a(R)\psi := \sqrt{n} \left(R^* \otimes \mathrm{Sym}^{n-1} \right) \psi, \qquad \psi \in \mathcal{K} \overset{\mathrm{al}}{\otimes} \overset{\mathrm{al}}{\Gamma}{}_{\mathrm{s}}^{n}(\mathcal{D}_1); \tag{2.4}$$

$a(R)$ is well defined as an unbounded operator and it defines a quadratic form on $\mathcal{K} \overset{\mathrm{al}}{\otimes} \overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{D}_1)$. Denote by $a^*(R)$ its adjoint in the sense of quadratic forms.

We write Ω for the vacuum vector in $\Gamma_{\mathrm{s}}(\mathcal{R})$:

$$\Omega = 1 \oplus 0 \oplus 0 \otimes 0 \oplus \dots . \tag{2.5}$$

s$-$lim will denote the strong limit. We say that the operators $A_{\lambda \in \mathbb{R}^+} \in \mathcal{B}(\mathcal{R})$ converge $*$-strongly to $A \in \mathcal{B}(\mathcal{R})$ (notation: $\mathrm{s}^* - \lim_{\lambda \downarrow 0} A_\lambda = A$) if

$$\mathrm{s} - \lim_{\lambda \downarrow 0} A_\lambda = A \qquad \text{and} \qquad \mathrm{s} - \lim_{\lambda \downarrow 0} A_\lambda^* = A^*. \tag{2.6}$$

If A is an operator, we will write

$$\Re A := \frac{1}{2}(A + A^*), \quad \Im A := \frac{1}{2\mathrm{i}}(A - A^*).$$

Our typical Hilbert space will be the tensor product of two Hilbert spaces. We will usually write A, B for $A \otimes 1$ and $1 \otimes B$.

3. Dilations

3.1. Unitary dilation of a contractive semigroup. Let $\mathcal{K}$ be a Hilbert space and let the family $\Theta_{t \in \mathbb{R}^+}$ be a contractive semigroup on $\mathcal{K}$:

$$\Theta_t \Theta_s = \Theta_{t+s}, \qquad \|\Theta_t\| \leq 1, \qquad t, s \in \mathbb{R}^+. \tag{3.1}$$

Definition 3.1. *We say that* $(\mathcal{Z}, I_{\mathcal{K}}, U_{t \in \mathbb{R}})$ *is a unitary dilation of* $\Theta_{t \in \mathbb{R}^+}$ *if*

1) $\mathcal{Z}$ *is a Hilbert space and* $U_{t \in \mathbb{R}} \in \mathcal{B}(\mathcal{Z})$ *is a unitary one-parameter group;*
2) $\mathcal{K} \subset \mathcal{Z}$ *and* $I_{\mathcal{K}}$ *is the embedding of* $\mathcal{K}$ *into* $\mathcal{Z}$*;*
3) *for all* $t \in \mathbb{R}^+$,

$$I_{\mathcal{K}}^* U_t I_{\mathcal{K}} = \Theta_t. \tag{3.2}$$

Assume that $\mathcal{K}$ is finite-dimensional and the semigroup Θ_t continuous. Then there exists a dissipative operator $-\mathrm{i}\Upsilon \in \mathcal{B}(\mathcal{K})$,

$$-\mathrm{i}\Upsilon + \mathrm{i}\Upsilon^* \leq 0, \tag{3.3}$$

such that $\Theta_t = \mathrm{e}^{-\mathrm{i}t\Upsilon}$.

3.2. Quantum Langevin dynamics. Let the family $\Lambda_{t\in\mathbb{R}^+}$ be a semigroup on $\mathcal{B}(\mathcal{K})$:

$$\Lambda_t\Lambda_s = \Lambda_{t+s}, \qquad t, s \in \mathbb{R}^+. \tag{3.4}$$

Definition 3.2. *We say that $(\mathcal{Z}, I_{\mathcal{K}}, U_{t\in\mathbb{R}})$ is a Langevin-Schrödinger dynamics for $\Lambda_{t\in\mathbb{R}^+}$ if*

1) *$\tilde{\mathcal{Z}}_{\mathrm{R}}$ is a Hilbert space and $U_{t\in\mathbb{R}}$ is a one-parameter unitary group on $\mathcal{Z} := \mathcal{K}\otimes\tilde{\mathcal{Z}}_{\mathrm{R}}$;*
2) *Ω is a normalized vector in $\tilde{\mathcal{Z}}_{\mathrm{R}}$ and $I_{\mathcal{K}}(\psi) := \psi\otimes\Omega \in \mathcal{Z}$ is the corresponding embedding of $\mathcal{K}$ into $\mathcal{Z}$;*
3) *for all $t \in \mathbb{R}^+$ and all $S \in \mathcal{B}(\mathcal{K})$,*

$$I_{\mathcal{K}}^* U_{-t} S\otimes 1 U_t I_{\mathcal{K}} = \Lambda_t(S). \tag{3.5}$$

The Heisenberg dynamics $\mathrm{e}^{\mathrm{i}tZ} \cdot \mathrm{e}^{-\mathrm{i}tZ}$ corresponding to a Langevin-Schrödinger dynamics will be called a *quantum Langevin dynamics*.

Definition 3.3. *We say that $\Lambda_{t\in\mathbb{R}_+}$ is a quantum Markov semigroup iff it is a semigroup on $\mathcal{B}(\mathcal{K})$ such that for any $t \in \mathbb{R}_+$ the map Λ_t is completely positive and preserves the unity.*

Clearly, if a semigroup Λ_t admits a Langevin-Schrödinger dynamics in the sense of Definition 3.2, then it is a quantum Markov semigroup.

Again, assume that $\mathcal{K}$ is finite dimensional. Assume that Λ_t is a continuous quantum Markov semigroup, so that we can define its generator L and we have $\Lambda_t = \mathrm{e}^{tL}$. Recall that then there exists a dissipative operator Υ on $\mathcal{K}$, another finite dimensional Hilbert space $\mathfrak{h}$ and an operator $\nu \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h})$, satisfying the condition

$$-\mathrm{i}\Upsilon + \mathrm{i}\Upsilon^* = -\nu^*\nu, \tag{3.6}$$

such that

$$L(S) = -\mathrm{i}(\Upsilon S - S\Upsilon^*) + \nu^* S\nu, \qquad S \in \mathcal{B}(\mathcal{K}). \tag{3.7}$$

Remark 3.1. If we choose an orthonormal basis $b_1, \ldots, b_d$ in $\mathfrak{h}$, then ν can be represented by a family of operators $\nu_1, \ldots, \nu_d \in \mathcal{B}(\mathcal{K})$, and then (3.7) can be rewritten as

$$L(S) = -\mathrm{i}(\Upsilon S - S\Upsilon^*) + \sum_{j=1}^{d} \nu_j^* S\nu_j, \qquad S \in \mathcal{B}(\mathcal{K}). \tag{3.8}$$

3.3. Construction of a Langevin-Schrödinger dynamics. Let $\mathcal{K}$, $\mathfrak{h}$ be finite dimensional Hilbert spaces, $\Re\Upsilon$ a self-adjoint operator on $\mathcal{K}$ and ν an operator from $\mathcal{K}$ to $\mathcal{K} \otimes \mathfrak{h}$. Setting $\Im\Upsilon := \nu^*\nu$ we obtain a dissipative operator $\Upsilon := \Re\Upsilon + \mathrm{i}\Im\Upsilon$ on $\mathcal{K}$.

Given the data $(\mathcal{K}, \Re\Upsilon, \mathfrak{h}, \nu)$ as above, we will construct a dilation for $\mathrm{e}^{\mathrm{i}t\Upsilon}$, which at the same time is a Langevin-Schrd̈inger dynamics for e^{tL}.

Introduce the operator Z_{R} on $\mathcal{Z}_{\mathrm{R}} := L^2(\mathbb{R}) \otimes \mathfrak{h} \cong L^2(\mathbb{R}, \mathfrak{h})$ as the operator of multiplication by the variable $x \in \mathbb{R}$:

$$(Z_{\mathrm{R}} f)(x) := x f(x).$$

Put

$$\mathcal{Z} = \mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_{\mathrm{R}}). \tag{3.9}$$

We define an unbounded linear functional on $L^2(\mathbb{R})$ with domain $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, denoted $\langle 1|$, by the obvious prescription

$$\langle 1| f = \int_{\mathbb{R}} f.$$

By $|1\rangle$, we denote the adjoint of $\langle 1|$ in the sense of forms.

We will also use the quadratic form from $\mathcal{K}$ to $\mathcal{K} \overset{\text{al}}{\otimes} \left(L^2(\mathbb{R}, \mathfrak{h}) \cap L^1(\mathbb{R}, \mathfrak{h})\right)$:

$$|1\rangle \otimes v. \tag{3.10}$$

Consider

$$\mathcal{D} := \mathcal{K} \overset{\text{al}}{\otimes} \overset{\text{al}}{\Gamma}_{\text{s}} \left(\text{Dom}(Z_{\text{R}})\right), \tag{3.11}$$

which is a dense subspace of $\mathcal{Z}$. As outlined in Sect. 2, using the fact that $L^2(\mathbb{R}, \mathfrak{h}) \cap L^1(\mathbb{R}, \mathfrak{h}) \subset \text{Dom}(Z_{\text{R}})$, we can define the quadratic forms $a(|1\rangle \otimes v)$ and $a^*(|1\rangle \otimes v)$ on $\mathcal{D}$. Hence, also the following expressions are quadratic forms on $\mathcal{D}$:

$$Z^+ = \Upsilon + (2\pi)^{-\frac{1}{2}} a(|1\rangle \otimes v) + (2\pi)^{-\frac{1}{2}} a^*(|1\rangle \otimes v) + \text{d}\Gamma(Z_{\text{R}}), \tag{3.12}$$

$$Z^- = \Upsilon^* + (2\pi)^{-\frac{1}{2}} a(|1\rangle \otimes v) + (2\pi)^{-\frac{1}{2}} a^*(|1\rangle \otimes v) + \text{d}\Gamma(Z_{\text{R}}). \tag{3.13}$$

It will be convenient to choose a family $b_{j \in \mathcal{J}} \in \mathfrak{h}$ and $C_{j \in \mathcal{J}} \in \mathcal{B}(\mathcal{K})$ indexed by a finite index set $\mathcal{J}$ such that

$$v = \sum_{j \in \mathcal{J}} C_j \otimes |b_j\rangle. \tag{3.14}$$

This can always be done, of course in many ways. Define, analogously to (2.4),

$$a(\text{e}^{\text{i}tZ_{\text{R}}}|1\rangle \otimes b_j), \qquad a^*(\text{e}^{\text{i}tZ_{\text{R}}}|1\rangle \otimes b_j), \tag{3.15}$$

as quadratic forms on $\mathcal{D}$. Note the equality

$$a(|1\rangle \otimes v) = \sum_{j \in \mathcal{J}} C_j^* \otimes a(|1\rangle \otimes b_j). \tag{3.16}$$

For $a \leq b$, let $\triangle_n[a, b] \subset \mathbb{R}^n$ be the simplex

$$\triangle_n[a, b] := \{(t_1, \ldots, t_n) \; : \; a < t_1 < \cdots < t_n < b\}. \tag{3.17}$$

Set

$$\begin{aligned} C_j^+ &= C_j, \\ C_j^- &= C_j^*. \end{aligned} \tag{3.18}$$

Now we combine these objects into something that is a priori a quadratic form, but turns out to be a bounded operator. For $t \geq 0$ we define

$$\begin{aligned} U_t &:= \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(Z_\mathrm{R})} \sum_{n=0}^{\infty} \int_{\triangle_n[0,t]} \mathrm{d}t_n \cdots \mathrm{d}t_1 \sum_{j_1,\dots,j_n \in \mathcal{J}} \sum_{\epsilon_1,\dots,\epsilon_n \in \{+,-\}} \\ &\quad \times (-\mathrm{i})^n (2\pi)^{-\frac{n}{2}} \mathrm{e}^{-\mathrm{i}(t-t_n)\Upsilon} C_{j_n}^{\epsilon_n} \mathrm{e}^{-\mathrm{i}(t_n - t_{n-1})\Upsilon} \cdots C_{j_1}^{\epsilon_1} \mathrm{e}^{-\mathrm{i}(t_1 - 0)\Upsilon} \\ &\quad \times \prod_{p=1,\dots,n:\ \epsilon_p = +} a^*(\mathrm{e}^{\mathrm{i}t_p Z_R}|1\rangle \otimes b_{j_p}) \prod_{p'=1,\dots,n:\ \epsilon_{p'} = -} a(\mathrm{e}^{\mathrm{i}t_{p'} Z_R}|1\rangle \otimes b_{j_{p'}}); \\ U_{-t} &:= U_t^*. \end{aligned} \tag{3.19}$$

(In the above expression $\prod_{p=1,\dots,n:\ \epsilon_p=+}$ should be understood as the product over these indices $p = 1, \dots, n$ that in addition satisfy the condition $\epsilon_p = +$.)

Finally, let $I_\mathcal{K}$ be the embedding of $\mathcal{K} \cong \mathcal{K} \otimes \Omega$ into $\mathcal{K} \otimes \Gamma_\mathrm{s}(\mathcal{Z}_\mathrm{R})$.

Theorem 3.2. *Let $Z^\pm$ be as defined in* (3.12) *and U_t as defined in* (3.19).

1) *The one-parameter family of quadratic forms U_t extends to a strongly continuous unitary group on $\mathcal{Z}$ and does not depend on the decomposition* (3.14).
2) *For $\psi, \psi' \in \mathcal{D}$, the function $\mathbb{R} \ni t \mapsto \langle\psi|U_t\psi'\rangle$ is differentiable away from $t = 0$, its derivative $t \mapsto \frac{\mathrm{d}}{\mathrm{d}t}\langle\psi|U_t\psi'\rangle$ is continuous away from 0 and at $t = 0$ it has the left and the right limit equal respectively to*

$$-\mathrm{i}\langle\psi|Z^+\psi'\rangle = \lim_{t\downarrow 0} t^{-1}\langle\psi|(U_t - 1)\psi'\rangle, \tag{3.20}$$

$$-\mathrm{i}\langle\psi|Z^-\psi'\rangle = \lim_{t\uparrow 0} t^{-1}\langle\psi|(U_t - 1)\psi'\rangle. \tag{3.21}$$

3) *The triple $(\mathcal{Z}, U_t, I_\mathcal{K})$ is a unitary dilation of the semigroup $\mathrm{e}^{-\mathrm{i}t\Upsilon}$ on $\mathcal{K}$:*

$$I_\mathcal{K}^* U_t I_\mathcal{K} = \mathrm{e}^{-\mathrm{i}t\Upsilon}. \tag{3.22}$$

4) *The triple $(\mathcal{Z}, U_t, 1_\mathcal{K})$ is a Langevin-Schrödinger dynamics for the semigroup e^{tL} on $\mathcal{B}(\mathcal{K})$:*

$$I_\mathcal{K}^* U_{-t}(S \otimes 1)U_t I_\mathcal{K} = \mathrm{e}^{tL}(S), \qquad S \in \mathcal{B}(\mathcal{K}). \tag{3.23}$$

We will say that U_t constructed in the above theorem is the Langevin-Schrödinger dynamics given by the data $(\mathcal{K}, \Re\Upsilon, \mathfrak{h}, \nu)$. Note that U_t can be written as $\mathrm{e}^{-\mathrm{i}tZ}$ for a uniquely defined self-adjoint operator Z on $\mathcal{Z}$. Clearly, $\mathcal{D}$ is not contained in the domain of Z and the quadratic forms Z^+ and Z^- are not generated by the operator Z (in fact, they are even not self-adjoint). On an appropriate domain, Z has the formal expression

$$Z = \Re\Upsilon + (2\pi)^{-\frac{1}{2}} a(|1\rangle \otimes \nu) + (2\pi)^{-\frac{1}{2}} a^*(|1\rangle \otimes \nu) + \mathrm{d}\Gamma(Z_\mathrm{R}), \tag{3.24}$$

which is the obvious "self-adjoint compromise" between Z^- and Z^+. This expression is formal since one needs a suitable regularization to give it a precise meaning. Such a regularization, under an additional assumption on the commutativity of the small system operators, is discussed e.g. in [9]. See also [21,31].

3.4. Alternative form of Langevin-Schrödinger equations. Proofs of Theorem 3.2 are contained in the literature, see e.g. [25]. In any case, this theorem involves well defined formulas and its proof follows by straightforward computations, which we leave to the reader. Nevertheless, we would like to mention a slightly different (though equivalent) form of Langevin-Schrödinger dynamics, which is closer to those usually appearing in the literature.

Let $\mathcal{G}$ denote the normalized Fourier transform on $L^2(\mathbb{R})$:

$$\mathcal{G}f(s) := (2\pi)^{-\frac{1}{2}} \int f(x)\mathrm{e}^{-\mathrm{i}sx}\mathrm{d}x.$$

We can treat it as a unitary operator on $\mathcal{Z}_\mathrm{R}$. We second quantize $\mathcal{G}$, obtaining an operator $\Gamma(\mathcal{G})$, which can be treated as an operator on $\mathcal{Z}$. Set

$$\hat{Z}_\mathrm{R} := \mathcal{G} Z_\mathrm{R} \mathcal{G}^*, \quad \hat{U}_t := \Gamma(\mathcal{G}) U_t \Gamma(\mathcal{G})^*. \tag{3.25}$$

Note that

$$\mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(\hat{Z}_\mathrm{R})} = \Gamma\left(\exp\left(-t\tfrac{\mathrm{d}}{\mathrm{d}s}\right)\right).$$

Then for $t \geq 0$ the formula (3.19) transforms into

$$\begin{aligned}\hat{U}_t = \Gamma\left(\exp\left(-t\tfrac{\mathrm{d}}{\mathrm{d}s}\right)\right) \sum_{n=0}^{\infty} \int_{\triangle_n[0,t]} \mathrm{d}t_n \cdots \mathrm{d}t_1 \sum_{j_1,\dots,j_n\in\mathcal{J}} \sum_{\epsilon_1,\dots,\epsilon_n\in\{+,-\}} (-\mathrm{i})^n \\ \times \mathrm{e}^{-\mathrm{i}(t-t_n)\Upsilon} C_{j_n}^{\epsilon_n} \mathrm{e}^{-\mathrm{i}(t_n-t_{n-1})\Upsilon} \cdots C_{j_1}^{\epsilon_1} \mathrm{e}^{-\mathrm{i}(t_1-0)\Upsilon} \\ \times \prod_{k=1,\dots,n:\ \epsilon_k=+} a^*(\delta_{t_k}\otimes b_{j_k}) \prod_{k'=1,\dots,n:\ \epsilon_{k'}=-} a(\delta_{t_{k'}}\otimes b_{j_{k'}}),\end{aligned} \tag{3.26}$$

where δ_t denotes the deltafunction at $t \in \mathbb{R}$, and (3.26) should be understood as a quadratic form between appropriate dense spaces. Equation (3.26) is sometimes referred to in the literature as the representation by *integral kernels*. It was introduced by Maassen [25]. See also [30,31,6,19] and Sect. VI, 3.2 of [26]. Differentiating (3.26) with respect to time we obtain (at least formally)

$$\begin{aligned}\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\hat{Z}_\mathrm{R})}\hat{U}_t = \left(\Upsilon + a^*(\delta_t\otimes\nu)\right)\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\hat{Z}_\mathrm{R})}\hat{U}_t \\ + \sum_{j\in\mathcal{J}} \nu_j^* \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(\hat{Z}_\mathrm{R})}\hat{U}_t a(\delta_t\otimes b_j),\end{aligned} \tag{3.27}$$

which essentially coincides with what is known in the literature under the name of the stochastic (or Langevin) Schrödinger equation.

4. The Pauli-Fierz Operator

4.1. Definitions and assumptions. Let $\mathcal{H} = \mathcal{K} \otimes \Gamma_\mathrm{s}(\mathcal{H}_\mathrm{R})$, where $\mathcal{K}$, $\mathcal{H}_\mathrm{R}$ are Hilbert spaces. We assume that $\mathcal{K}$ is finite-dimensional. Fix a self-adjoint operator H_R on $\mathcal{H}_\mathrm{R}$ and a self-adjoint operator K on $\mathcal{K}$. The operator H_0 on $\mathcal{H}$ given as

$$H_0 = K + \mathrm{d}\Gamma(H_\mathrm{R})$$

will be called the *free Pauli-Fierz operator*. We choose a $V \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathcal{H}_{\mathrm{R}})$ and we recall the generalized creation and annihilation operators $a(V)$ and $a^*(V)$ introduced in Sect. 2.

Theorem 4.1. *Set* $H^I(t) := \mathrm{e}^{-\mathrm{i}tH_0}(a^*(V) + a(V))\mathrm{e}^{\mathrm{i}tH_0}$. *Then*

$$W_{\lambda,t}\psi = \sum_{n=0}^{\infty} \int_{\triangle_n[0,t]} \mathrm{d}t_n \cdots \mathrm{d}t_1 \mathrm{e}^{\mathrm{i}tH_0}(\mathrm{i}\lambda)^n H^I(t_n)\cdots H^I(t_1)\psi \tag{4.1}$$

is well defined for all $\psi \in \mathcal{K} \overset{\mathrm{al}}{\otimes} \overset{\mathrm{al}}{\Gamma}_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$. $W_{\lambda,t}$ *extends to a 1-parameter unitary group on* $\mathcal{K} \otimes \Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$ *with a self-adjoint generator* H_λ. *The finite particle space* $\mathcal{K} \overset{\mathrm{al}}{\otimes} \overset{\mathrm{al}}{\Gamma}(\mathcal{H}_{\mathrm{R}})$ *belongs to the domain of* H_λ *and on* $\mathcal{K} \overset{\mathrm{al}}{\otimes} \overset{\mathrm{al}}{\Gamma}(\mathcal{H}_{\mathrm{R}})$*,*

$$H_\lambda = H_0 + \lambda \left(a(V) + a^*(V)\right). \tag{4.2}$$

H_λ will be called the *full Pauli-Fierz operator*.

We write

$$K = \sum_{k \in \mathrm{sp}(K)} k 1_{\mathcal{K}_k}, \tag{4.3}$$

where k, $1_{\mathcal{K}_k}$, are the eigenvalues and the spectral projections of K. We collect all Bohr frequencies in a set $\mathcal{F}$:

$$\mathcal{F} := \left\{\omega \in \mathbb{R} \middle| \omega = k - k' \text{ for some } k, k' \in \mathrm{sp}K\right\}. \tag{4.4}$$

We again denote by $I_{\mathcal{K}}$ the embedding of $\mathcal{K} = \mathcal{K} \otimes \Omega$ into $\mathcal{H}$, where $\Omega \in \Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})$ is the vacuum vector.

We now list the assumptions that we will need in our construction.

Assumption 4.2. *For any* $\omega \in \mathcal{F}$ *there exists a Hilbert space* $\mathfrak{h}_\omega$ *and an open set* $I_\omega \subset \mathbb{R}$ *with* $\omega \in I_\omega$ *and an identification*

$$\mathrm{Ran} 1_{I_\omega}(H_{\mathrm{R}}) \simeq L^2(I_\omega) \otimes \mathfrak{h}_\omega,$$

such that H_{R} *is the multiplication by the variable* $x \in I_\omega$. *We assume that* I_ω *are disjoint for distinct* $\omega \in \mathcal{F}$ *and we set* $I := \cup_{\omega \in \mathcal{F}} I_\omega$. *Thus if*

$$f \simeq \int_I^{\oplus} f(x)\mathrm{d}x \in \mathrm{Ran} 1_I(H_{\mathrm{R}}), \tag{4.5}$$

then

$$(H_{\mathrm{R}} f)(x) = x f(x),$$

for almost all x.

Assumption 4.3. *For any* $\omega \in \mathcal{F}$*, there exists a measurable function*

$$I_\omega \ni x \mapsto v(x) \in B(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}_\omega)$$

such that for $u \in \mathcal{K}$ for almost all $x \in I$ we have

$$(Vu)(x) = v(x)u.$$

Moreover, we assume that v is continuous in $\mathcal{F}$, so that for $\omega \in \mathcal{F}$ we can unambiguously define $v(\omega) \in \mathcal{B}(\mathcal{K}, \mathcal{K} \otimes \mathfrak{h}_\omega)$.

Assumption 4.4. *For all $S \in \mathcal{B}(\mathcal{K})$,*

$$\int_{\mathbb{R}^+} \mathrm{d}t \, \| V^* S \otimes 1 \, \mathrm{e}^{-\mathrm{i}tH_0} V \| < \infty. \tag{4.6}$$

4.2. Asymptotic reduced dynamics. Let

$$\mathfrak{h} := \underset{\omega \in \mathcal{F}}{\oplus} \mathfrak{h}_\omega. \tag{4.7}$$

We define the map $\nu_\omega : \mathcal{K} \to \mathcal{K} \otimes \mathfrak{h}_\omega$,

$$\nu_\omega := \sqrt{2\pi} \sum_{\substack{k, k' \in \mathrm{sp}K, \\ \omega = k - k'}} 1_{\mathcal{K}_k} v(\omega) 1_{\mathcal{K}_{k'}},$$

where $v(\omega)$ is well-defined by Assumption 4.3. We also define $\nu : \mathcal{K} \to \mathcal{K} \otimes \mathfrak{h}$

$$\nu := \sum_{\omega \in \mathcal{F}} \nu_\omega.$$

Under Assumption 4.4, we can define

$$\Upsilon := -\sum_{k \in \mathrm{sp}K} \mathrm{i} \int_0^\infty 1_{\mathcal{K}_k} V^* \mathrm{e}^{-\mathrm{i}t(K + H_\mathrm{R} - k)} V \, 1_{\mathcal{K}_k} \mathrm{d}t \tag{4.8}$$

$$= -\mathrm{i} \sum_{\omega \in \mathcal{F}} \sum_{k - k' = \omega} \int_0^\infty 1_{\mathcal{K}_k} V^* 1_{\mathcal{K}_{k'}} \mathrm{e}^{-\mathrm{i}t(H_\mathrm{R} - \omega)} V \, 1_{\mathcal{K}_k} \mathrm{d}t. \tag{4.9}$$

Remark that $-\mathrm{i}\Upsilon$ is a dissipative operator and hence it generates a contractive semigroup on $\mathcal{K}$. Note that

$$\begin{aligned} \mathrm{i}\Upsilon - \mathrm{i}\Upsilon^* &= \sum_{\omega \in \mathcal{F}} \sum_{k - k' = \omega} \int_{-\infty}^\infty 1_{\mathcal{K}_k} V^* 1_{\mathcal{K}_{k'}} \mathrm{e}^{-\mathrm{i}t(H_\mathrm{R} - \omega)} V \, 1_{\mathcal{K}_k} \mathrm{d}t \\ &= 2\pi \sum_{\omega \in \mathcal{F}} \sum_{k - k' = \omega} 1_{\mathcal{K}_k} v^*(\omega) 1_{\mathcal{K}_{k'}} v(\omega) \, 1_{\mathcal{K}_k} \quad = \quad \nu^* \nu, \end{aligned}$$

and thus Υ and ν satisfy the condition (3.6). Therefore,

$$L(S) = -\mathrm{i}(\Upsilon S - S\Upsilon^*) + \nu^* S \nu, \qquad S \in \mathcal{B}(\mathcal{K}), \tag{4.10}$$

is the generator of a quantum Markov semigroup.

4.3. Asymptotic space and dynamics. We introduce the asymptotic space and the asymptotic dynamics that we will use in our paper. The asymptotic reservoir one-particle spaces are

$$\mathcal{Z}_{\mathrm{R}_\omega} := L^2(\mathbb{R}, \mathfrak{h}_\omega), \tag{4.11}$$

$$\mathcal{Z}_{\mathrm{R}} := \bigoplus_{\omega\in\mathcal{F}} \mathcal{Z}_{\mathrm{R}_\omega} = L^2(\mathbb{R}, \mathfrak{h}). \tag{4.12}$$

For $\omega \in \mathcal{F}$, we have the orthogonal projections

$$1_{\mathrm{R}_\omega} : \mathcal{Z}_{\mathrm{R}} \to \mathcal{Z}_{\mathrm{R}_\omega}.$$

Let Z_{R} be the operator of multiplication by the variable in $\mathbb{R}$ on $\mathcal{Z}_{\mathrm{R}}$.

Clearly, we can construct from $(\mathcal{Z}, I_{\mathcal{K}}, \nu, \Re\Upsilon)$ the Langevin-Schrödinger dynamics of Theorem 3.2. We denote it by U_t and its generator by Z.

Finally, we define a renormalizing Hamiltonian Z_{ren} on $\mathcal{Z}$:

$$Z_{\mathrm{ren}} := K + \mathrm{d}\Gamma\left(\bigoplus_{\omega\in\mathcal{F}} \omega 1_{\mathrm{R}_\omega}\right). \tag{4.13}$$

4.4. Scaling. For $\lambda > 0$, we define the family of partial isometries $J_{\lambda,\omega} : \mathcal{Z}_{\mathrm{R}_\omega} = L^2(\mathbb{R}, \mathfrak{h}_\omega) \to L^2(I_\omega, \mathfrak{h}_\omega)$, which on $g_\omega \in \mathcal{Z}_{\mathrm{R}_\omega}$ act as

$$(J_{\lambda,\omega} g_\omega)(y) = \begin{cases} \frac{1}{\lambda} g_\omega(\frac{y-\omega}{\lambda^2}), & \text{if } y \in I_\omega; \\ 0, & \text{if } y \in \mathbb{R}\backslash I_\omega. \end{cases} \tag{4.14}$$

Since $L^2(I_\omega, \mathfrak{h}_\omega) \subset \mathcal{H}_{\mathrm{R}}$, $J_{\lambda,\omega}$ can be viewed as a map from $\mathcal{Z}_{\mathrm{R},\omega}$ to $\mathcal{H}_{\mathrm{R}}$. We have

$$J^*_{\lambda,\omega} J_{\lambda,\omega} = 1_{\lambda^{-2}(I_\omega-\omega)}(Z_{\mathrm{R}}) 1_{\mathrm{R}_\omega} \xrightarrow[\lambda\downarrow 0]{\text{strongly}} = 1_{\mathrm{R}_\omega} \qquad J_{\lambda,\omega} J^*_{\lambda,\omega} = 1_{I_\omega}(H_{\mathrm{R}}). \tag{4.15}$$

We set $J_\lambda : \mathcal{Z}_{\mathrm{R}} \to \mathcal{H}_{\mathrm{R}}$ defined for $g = (g_\omega)_{\omega\in\mathcal{F}}$ by

$$J_\lambda g := \sum_{\omega\in\mathcal{F}} J_{\lambda,\omega} g_\omega.$$

Note that

$$J_\lambda J^*_\lambda = 1_I(H_{\mathrm{R}}).$$

In what follows, we will mainly need the second quantized $\Gamma(J_\lambda)$, which will also be used to denote the operator

$$1 \otimes \Gamma(J_\lambda) \in \mathcal{B}(\mathcal{Z}, \mathcal{H}).$$

Remark 4.5. In the definition of J_λ there is a lot of freedom. What matters is what happens near the Bohr frequencies. In fact, essentially the only requirement on J_λ is that Lemma 6.5 holds and that both $J^*_\lambda J_\lambda$ and $J_\lambda J^*_\lambda$ converge strongly to 1. The form of J_λ also reflects that different frequencies "do not see each other" in the weak coupling limit (see e.g. [16,2] for an explicit discussion).

The following fact is immediate:

Proposition 4.6. *We have*

$$s^* - \lim_{\lambda\downarrow 0} \mathrm{e}^{\mathrm{i}\lambda^{-2} t Z_{\mathrm{ren}}} \Gamma(J^*_\lambda) \mathrm{e}^{-\mathrm{i}\lambda^{-2} t H_0} \Gamma(J_\lambda) = \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(Z_{\mathrm{R}})}.$$

5. Results

The full dynamics in the interaction picture will be denoted by

$$T_\lambda(t, t_0) = \mathrm{e}^{\mathrm{i}tH_0}\mathrm{e}^{-\mathrm{i}(t-t_0)H_\lambda}\mathrm{e}^{-\mathrm{i}t_0H_0}. \tag{5.1}$$

We start with two versions of older results by Davies about the reduced weak coupling limit. However, in most presentations of this subject contained in the literature the perturbation is assumed to be bounded. This is not the case in Theorem 5.1.

Theorem 5.1. *Assume Assumptions 4.2, 4.3, 4.4. Let* $T \leq \infty$.

1) *Let* Υ *be as defined in* (4.8). *Then*

$$\lim_{\lambda\downarrow 0} I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t_0) I_{\mathcal{K}} = \mathrm{e}^{-\mathrm{i}(t-t_0)\Upsilon} \tag{5.2}$$

uniformly for $T \geq t \geq t_0 \geq -T$.

2) *Let* L *be as defined in* (4.10). *Then*

$$\lim_{\lambda\downarrow 0} I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t_0)\ S\otimes 1\ T_\lambda(\lambda^{-2}t, \lambda^{-2}t_0)^* I_{\mathcal{K}} = \mathrm{e}^{(t-t_0)L}(S) \tag{5.3}$$

uniformly for $T \geq t \geq t_0 \geq -T$.

We will prove Theorem 5.1 1) in Subsect. 6.3 – it is an important step of the proof of our main result. Theorem 5.1 2) can be proven by similar arguments, or, which is easier in our framework, it follows immediately from Theorem 5.7.

The following result is a version of a result of Dümcke [14]. Apart from its intrinsic interest, we will need it as an important step in the proof of our main result.

Theorem 5.2. *Assume Assumptions 4.2, 4.3, 4.4 and let* $T < \infty$, $\ell \in \mathbb{N}$ *and* $S_1, \dots, S_\ell \in \mathcal{B}(\mathcal{K})$. *Then*

$$\begin{aligned}&\lim_{\lambda\downarrow 0} I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t_\ell) S_\ell \cdots S_2 T_\lambda(\lambda^{-2}t_2, \lambda^{-2}t_1) S_1 T_\lambda(\lambda^{-2}t_1, \lambda^{-2}t_0) I_{\mathcal{K}}\\ &\quad = \mathrm{e}^{-\mathrm{i}(t-t_\ell)\Upsilon} S_\ell \dots S_2 \mathrm{e}^{-\mathrm{i}(t_2-t_1)\Upsilon} S_1 \mathrm{e}^{-\mathrm{i}(t_1-t_0)\Upsilon}\end{aligned} \tag{5.4}$$

uniformly for ordered times $T \geq t \geq t_\ell \geq \dots \geq t_1 \geq t_0 \geq -T$.

Clearly, Theorem 5.1 1 is a special case of Theorem 5.2, corresponding to $\ell = 0$, or all $S_i = 1$.

Remark 5.3. Strictly speaking, Theorems 5.1 1) and 5.2 are somewhat different from the results in [10] and [14]. In our setup, the latter are consequences of Theorem 5.7 and Theorem 3.2. (see Remark 5.8).

Note that the above results did not involve any dilations, nor the identification operator $\Gamma(J_\lambda)$.

Our main result describes the extended weak coupling limit for Pauli-Fierz operators and reads

Theorem 5.4. *Assume Assumptions 4.2, 4.3, 4.4. Let U_t be the Langevin-Schrödinger dynamics constructed from $(\mathcal{Z}, I_{\mathcal{K}}, \nu, \Re\Upsilon)$ with $\mathcal{Z}, \nu, \Upsilon$ defined in Sect. 4.3. Let also $Z_{\rm ren}$ be as defined in Sect. 4.3. Then,*

$$\mathrm{s}^* - \lim_{\lambda\downarrow 0} \Gamma(J_\lambda^*) T_\lambda(\lambda^{-2}t, \lambda^{-2}t_0)\Gamma(J_\lambda) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(Z_{\rm R})} U_{t-t_0} \mathrm{e}^{-\mathrm{i}t_0\mathrm{d}\Gamma(Z_{\rm R})}, \tag{5.5}$$

$$\mathrm{s}^* - \lim_{\lambda\downarrow 0} \mathrm{e}^{\mathrm{i}\lambda^{-2}tZ_{\rm ren}} \Gamma(J_\lambda^*) \mathrm{e}^{-\mathrm{i}\lambda^{-2}tH_\lambda}\Gamma(J_\lambda) = U_t. \tag{5.6}$$

Remark 5.5. Weaker versions of Theorem 5.1 and Theorem 5.2 follow immediately from Theorem 5.4. They are weaker because the uniformity in time is lacking.

Remark that on $\mathrm{Dom} Z_{\rm ren}$,

$$[Z_{\rm ren}, U_t] = 0, \tag{5.7}$$

as can be checked from the explicit expression for U_t. The generator $Z_{\rm ren}$ could be considered as the free (i.e. $\mathcal{K}$ and R are decoupled) Hamiltonian in the weak coupling limit and hence (5.7) expresses the conservation of the 'decoupled' energy. In the reduced weak coupling limit we have an analogous situation: the generator of the limiting quantum Markov semigroup L commutes with the generator of the free evolution $\mathrm{i}[K, \cdot]$.

A consequence of Theorem 5.4 is now given. Its advantage is that it does not involve explicitly the operators J_λ.

Recall the notation in Assumption 4.2. Let $I \ni x \mapsto g(x) \in \mathcal{B}(\mathfrak{h}(x))$ be a measurable function such that $\sup_{x\in I} \|g(x)\| < 1$ and $x \mapsto g(x)$ is continuous in a neighbourhood of $\mathcal{F}$. Remark that this requirement makes sense because of Assumption 4.3. Define the contractive multiplication operator $G \in \mathcal{B}(\mathcal{H}_{\rm R})$ as,

$$(Gf)(x) = g(x)f(x), \tag{5.8}$$

and remark that $\Gamma(G)$ is also a contractive operator on $\Gamma_{\rm s}(\mathcal{H}_{\rm R})$. Let $\mathcal{C}$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$, generated by

$$S \otimes 1, \quad \text{and} \quad 1 \otimes \Gamma(G), \tag{5.9}$$

with $S \in \mathcal{B}(\mathcal{K})$ and G as defined above. Let $\mathcal{C}_{\rm as}$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{Z})$ generated by

$$S \otimes 1 \quad \text{and} \quad 1 \otimes \Gamma(1 \otimes p) \tag{5.10}$$

with $S \in \mathcal{B}(\mathcal{K})$ and $p \in \bigoplus_{\omega\in\mathcal{F}} \mathcal{B}(\mathfrak{h}_\omega)$.

Proposition 5.6. *There exists a unique $*$-homomorphism $\Theta : \mathcal{C} \to \mathcal{C}_{\rm as}$ such that*

$$\Theta(S \otimes \Gamma(G)) = S \otimes \Gamma\left(1 \otimes \left(\bigoplus_{\omega\in\mathcal{F}} g(\omega)\right)\right), \qquad S \in \mathcal{B}(\mathcal{K}), \tag{5.11}$$

where G and g are related by (5.8). We have

$$\Theta(A) = \mathrm{s}^* - \lim_{\lambda\searrow 0} \Gamma(J_\lambda^*) A \Gamma(J_\lambda), \quad A \in \mathcal{C}. \tag{5.12}$$

Theorem 5.7. *Assume Assumptions 4.2 4.3, 4.4 and let the family U_t be as in Theorem 5.4. For any $\ell \in \mathbb{N}$, any $A_1, \ldots, A_\ell \in \mathcal{C}$ and (not necessarily ordered) times $t_0, t_1, \ldots, t_\ell, t \in \mathbb{R}$,*

$$\lim_{\lambda\searrow 0} I_\mathcal{K}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t_\ell) A_n \ldots A_2 T_\lambda(\lambda^{-2}t_2, \lambda^{-2}t_1) A_1 T_\lambda(\lambda^{-2}t_1, \lambda^{-2}t_0) I_\mathcal{K}$$

$$= I_\mathcal{K}^* U_{(t-t_\ell)} \Theta(A_n) \ldots \Theta(A_2) U_{(t_2-t_1)} \Theta(A_1) U_{(t_1-t_0)} I_\mathcal{K}. \tag{5.13}$$

Remark 5.8. The results in [10] and [14] correspond to Theorem 5.7 where $A_1 \ldots, A_\ell$ are elements of $\mathcal{B}(\mathcal{K})$ and hence $\Theta(A_{1,\ldots,\ell}) = A_{1,\ldots,\ell}$.

6. Proofs

6.1. Proof of Theorem 4.1. Existence of the physical dynamics. We will prove a somewhat stronger theorem. Let P_n be the orthogonal projector on $\mathcal{K} \otimes \Gamma_\mathrm{s}^n(\mathcal{H}_\mathrm{R})$ and let the dense subspace $\mathcal{D}_1$ of $\mathcal{H}$ be defined as

$$\psi \in \mathcal{D}_1 \Leftrightarrow \text{ there exists } C \text{ such that for } n = 0, 1, 2 \ldots \text{ we have } \|P_n\psi\| \leq \frac{C^n}{\sqrt{n!}}.$$

Theorem 6.1. *For $\psi \in \mathcal{D}_1$, the series (4.1) defining $W_{\lambda,t}\psi$ is absolutely convergent, belongs to $\mathcal{D}_1$, is continuous wrt $t \in \mathbb{R}$ and we have $W_{\lambda,t}W_{\lambda,s}\psi = W_{\lambda,t+s}\psi$, $\|W_{\lambda,t}\psi\|^2 = \|\psi\|^2$. Therefore, $W_{\lambda,t}$ extends uniquely to a strongly continuous unitary group on $\mathcal{H}$. By Stone's theorem, it has a self-adjoint generator H_λ, and by a theorem of Nelson, $\mathcal{D}_1$ is a core for H_λ.*

Proof. It is enough to assume that $\lambda = 1$, so that we will write W_t. We can also assume that $t \geq 0$. Let $\psi \in \mathcal{D}_1$ with $\|P_m\psi\| \leq \frac{C^m}{\sqrt{m!}}$. Note that we have

$$P_n W_t \psi = \sum_{q=0}^{\infty} \int_{\triangle_n[0,t]} \sum_{m=\max\{n-q,0\}}^{\infty} \mathrm{d}t_m \cdots \mathrm{d}t_1$$

$$\times \mathrm{e}^{\mathrm{i}tH_0} \mathrm{i}^n P_n H^I(t_q) \cdots H^I(t_1) P_m \psi.$$

Note also that

$$\|H^I(t_q) \cdots H^I(t_1) P_m\| \leq \begin{cases} 2^q \|V\|^q \frac{\sqrt{(m+q)!}}{\sqrt{m!}}, & m \geq n-q, \\ 0, & m < n-q. \end{cases}$$

Therefore,

$$\begin{aligned}\|P_n W_t \psi\| &\leq \sum_{q=0}^{\infty} \sum_{m=\max\{n-q,0\}}^{\infty} \frac{(2t\|V\|)^q}{q!} \frac{\sqrt{(m+q)!}}{\sqrt{m!}} \frac{C^m}{\sqrt{m!}} \\ &\leq \sum_{p=n}^{\infty} \sum_{q=0}^{p} \frac{1}{\sqrt{p!}} \frac{(2t\|V\|)^q C^{p-q} p!}{q!(p-q)!} \\ &\leq \sum_{p=n}^{\infty} \frac{1}{\sqrt{p!}} (2t\|V\| + C)^p \\ &\leq \frac{(2t\|V\| + C)^n}{\sqrt{n!}} \sum_{r=0}^{\infty} \frac{(2t\|V\| + C)^r}{\sqrt{r!}} \\ &= \frac{(2t\|V\| + C)^n}{\sqrt{n!}} C_1.\end{aligned}$$

This proves the absolute convergence of the series and the fact that it belongs to $\mathcal{D}_1$. The rest of claims is now straightforward. □

6.2. Decomposition of interaction.

Lemma 6.2. *There is a finite index set $\mathcal{J}$ and families $D_{j\in\mathcal{J}} \in \mathcal{B}(\mathcal{K})$ and $\phi_{j\in\mathcal{J}} \in \mathcal{H}_\mathrm{R}$ such that*

$$V = \sum_{j\in\mathcal{J}} D_j \otimes |\phi_j\rangle, \tag{6.1}$$

and such that the function

$$h(t) := \sum_{j,j'\in\mathcal{J}} |\langle \phi_{j'}|\mathrm{e}^{-\mathrm{i}tH_\mathrm{R}} \phi_j\rangle| \tag{6.2}$$

is integrable:

$$\|h\|_1 := \int_{\mathbb{R}} \mathrm{d}t\, h(t) < +\infty. \tag{6.3}$$

Moreover, we can choose this decomposition so that all $\phi_{j\in\mathcal{J}}$ are continuous in $\mathcal{F}$ and for all $j \in \mathcal{J}$ there is at most one $\omega \in \mathcal{F}$ such that $\phi_j(\omega) \neq 0$.

Proof. Let $\{w_p\}_{p\in\mathcal{P}}$ be an orthonormal basis of eigenvectors of K, so that $Kw_p = k_p w_p$. For each $p \in \mathcal{P}$, there exists a family $\{\phi_{p,m}\}_{m\in\mathcal{M}}$ in $\mathcal{H}_\mathrm{R}$ such that

$$Vw_p = \sum_{m\in\mathcal{M}} w_m \otimes \phi_{m,p}.$$

We set $S = |w_m\rangle\langle w_m|$. Now

$$\langle w_p|V^* S \mathrm{e}^{\mathrm{i}tH_0} V w_p\rangle = \langle \phi_{p,m}|\mathrm{e}^{\mathrm{i}tH_\mathrm{R}} \phi_{p,m}\rangle \mathrm{e}^{\mathrm{i}tk_m}$$

is integrable by Assumption 4.4.

Then we choose a partition of unity $\chi_\omega \in C_c^\infty(\mathbb{R})$ together with $\chi_\infty \in C^\infty(\mathbb{R})$ such that $\chi_\omega = 1$ on a neighborhood of ω, $\chi_\omega = 0$ on a neighborhood of $\mathcal{F}\backslash\{\omega\}$ and $\chi_\infty = 0$ on a neighborhood of $\mathcal{F}$ and $\sum_m \chi_m + \chi_\infty = 1$. We set $\phi_{m,p,\omega} := \chi_\omega(H_{\mathrm{R}})\phi_{m,p}$, $\phi_{m,p,\infty} := \chi_\infty(H_{\mathrm{R}})\phi_{m,p}$, $D_{m,p,\omega} = D_{m,p,\infty} := |w_m\rangle\langle w_p|$. Hence, the index set $\mathcal{J}$ is chosen as $\mathcal{M}\times\mathcal{P}\times(\mathcal{F}\cup\{\infty\})$ and elementary properties of the Fourier transform imply the integrability of (6.2). □

If for a given $j \in \mathcal{J}$ and $\omega \in \mathcal{F}$, we have $\phi_j(\omega) \neq 0$, then this ω will be referred to as $\omega(j)$. If for a given j, there is no $\omega \in \mathcal{F}$ such that $\phi_j(\omega) \neq 0$, then $\omega(j)$ is chosen arbitrarily. For further reference let us record the identity

$$v(\omega) = \sum_{j\in\mathcal{J}\,:\,\omega(j)=\omega} D_j \otimes |\phi_j(\omega)\rangle, \quad \omega \in \mathcal{F}. \tag{6.4}$$

6.3. Proof of Theorems 5.1. Reduced weak coupling limit. For operators $A_1, \dots, A_p$ we will write

$$\prod_{i=1}^{p} A_i := A_p \cdots A_1.$$

We will also write $D(t) := \mathrm{e}^{-\mathrm{i}tK} D \mathrm{e}^{\mathrm{i}tK}$.

Define

$$\begin{aligned} &G_\lambda(t,t_0)\\ &\quad:= \sum_{n=0}^{\infty} (\mathrm{i}\lambda)^{-2n} \int_{\triangle_{2n}[t_0,t]} \mathrm{d}t_1 \dots \mathrm{d}t_{2n}\\ &\qquad\times \prod_{i=1}^{n} I_{\mathcal{K}}^* H^{\mathrm{I}}(\lambda^{-2}t_{2i}) H^{\mathrm{I}}(\lambda^{-2}t_{2i-1}) I_{\mathcal{K}}\\ &\quad= \sum_{n=0}^{\infty} (\mathrm{i}\lambda)^{-2n} \sum_{j_1,\dots,j_{2n}\in\mathcal{J}} \int_{\triangle_{2n}[t_0,t]} \mathrm{d}t_1 \dots \mathrm{d}t_{2n}\\ &\qquad\times \prod_{p=1}^{n} D_{j_{2p}}^*(\lambda^{-2}t_{2p}) D_{j_{2p-1}}(\lambda^{-2}t_{2p-1}) \langle \phi_{j_{2p}} | \mathrm{e}^{\mathrm{i}\lambda^{-2}(t_{2p}-t_{2p-1})H_{\mathrm{R}}} \phi_{j_{2p-1}} \rangle. \end{aligned}$$

Lemma 6.3. *For all $T \leq \infty$,*

$$\lim_{\lambda\downarrow 0} \sup_{0\leq t_0\leq t\leq T} \| I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t_0) I_{\mathcal{K}} - G_\lambda(t,t_0) \| = 0. \tag{6.5}$$

Proof. Set

$$\begin{aligned} D_j^+ &= D_j,\\ D_j^- &= D_j^*. \end{aligned} \tag{6.6}$$

For $n = 0, 1, 2, \dots$, let $\mathrm{Pair}(2n)$ denote the set of pairings of $\{1, \dots, 2n\}$. That means, $\sigma \in \mathrm{Pair}(2n)$ iff it is a permutation $\sigma \in S_{2n}$ satisfying $\sigma(2p-1) < \sigma(2p)$, $p = 1, \dots, n$, and $\sigma(2p-1) < \sigma(2p+1)$, $p = 1, \dots, n-1$. We will write $\epsilon(p) = +$ for even p

and $\epsilon(p) = -$ for odd p. One can visualize the above definitions as follows: $\sigma(2p-1)$ corresponds to the p^{th} creator in the order of increasing time and $\sigma(2p)$ corresponds to the annihilator paired with this creator by the Wick theorem.

Using first the Dyson expansion and then the Wick theorem we obtain

$$\begin{aligned}
&I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t_0) I_{\mathcal{K}} \\
&= \sum_{n=0}^{\infty} \sum_{\sigma \in \mathrm{Pair}(2n)} \sum_{j_1,\ldots,j_{2n} \in I} (\mathrm{i}\lambda)^{-2n} \int_{\triangle_{2n}[t_0,t]} \mathrm{d}t_{2n} \cdots \mathrm{d}t_1 \\
&\quad \times \prod_{i=1}^{2n} D_{j_i}^{\epsilon(\sigma(i))}(\lambda^{-2}t_i) \\
&\quad \times \prod_{p=1}^{n} \langle \phi_{j_{\sigma(2p)}} | \mathrm{e}^{\mathrm{i}\lambda^{-2}(t_{\sigma(2p)} - t_{\sigma(2p-1)})H_{\mathrm{R}}} \phi_{j_{\sigma(2p-1)}} \rangle \\
&=: \sum_{n=0}^{+\infty} C_n.
\end{aligned} \tag{6.7}$$

Assume for simplicity $t_0 = 0$. Abbreviating $\|D\| := \sup_{j \in \mathcal{J}} \|D_j\|$, we obtain a uniform estimate

$$\begin{aligned}
&\|C_n\| \\
&\leq (\|D\|\lambda^{-1})^{2n} \sum_{\pi \in \mathrm{Pair}(2n)} \int_{\triangle_{2n}[0,t]} \mathrm{d}t_1 \ldots \mathrm{d}t_{2n} \\
&\quad \times \prod_{p=1}^{n} h(\lambda^{-2}(t_{\pi(2p)} - t_{\pi(2p-1)})) \\
&= \frac{(\|D\|\lambda^{-1})^{2n}}{2^n n!} \sum_{\pi \in S_{2n}} \int_{\triangle_{2n}[0,t]} \mathrm{d}t_1 \ldots \mathrm{d}t_{2n} \\
&\quad \times \prod_{p=1}^{n} h(\lambda^{-2}|t_{\pi(2p)} - t_{\pi(2p-1)}|) \\
&= \frac{(\|D\|\lambda^{-1})^{2n}}{2^n n!(2n)!} \sum_{\pi \in S_{2n}} \int_{[0,t]^{2n}} \mathrm{d}t_1 \ldots \mathrm{d}t_{2n} \\
&\quad \times \prod_{p=1}^{n} h(\lambda^{-2}|t_{\pi(2p)} - t_{\pi(2p-1)}|) \\
&\leq \frac{(\|D\|\lambda^{-1})^{2n} t^n}{2^n n!} \left(\int_{-t}^{t} \mathrm{d}s\, h(\lambda^{-2}|s|) \right)^n \\
&\leq \frac{(\|D\|)^{2n}}{2^n n!} t^n \|h\|_1^n.
\end{aligned} \tag{6.8}$$

First we used that each pairing can be represented by $2^n n!$ permutations. Then we allowed to permute $t_1, \ldots, t_{2n}$. The last inequality has been obtained by a change of integration variables. The bound (6.8) shows that the series (6.7) is absolutely convergent. We will exploit this now since we estimate the series term by term.

Given a pairing σ, the term in the sum (6.7) is estimated by

$$\lambda^{-2n}\int_{\triangle_{2n}([0,t])}\mathrm{d}t_1\ldots\mathrm{d}t_{2n}\prod_{i=1}^{n}h(\lambda^{-2}(t_{\pi(2i)}-t_{\pi(2i-1)})). \tag{6.9}$$

We are going to show that (6.9) does not vanish only for the time consecutive pairing: that is for the pairing given by the identity permutation (also called "nonnested, noncrossing pairings" for obvious reasons). Assume there is i such that $\pi(2i)-\pi(2i-1)>1$ and let p be such that $s_1 := t_{\pi(2i-1)} < t_{\pi(p)} < t_{\pi(2i)} =: s_2$. Then

$$\begin{aligned}
&\lambda^{-2n}\int_{\triangle_{2n}([0,t])}\mathrm{d}t_1\ldots\mathrm{d}t_{2n}\prod_{i=1}^{n}h(\lambda^{-2}(s_2-s_1))\\
&\leq \lambda^{-2}t^{n-2}(2\|h\|_1)^{n-1}\int_0^t\mathrm{d}t_{\pi(p)}\int_0^{t_{\pi(p)}}\mathrm{d}s_1\int_{t_{\pi(p)}}^t\mathrm{d}s_2\,h(\lambda^{-2}(s_2-s_1))\\
&\leq t^{n-2}(2\|h\|_1)^{n-1}\int_0^t\mathrm{d}t_{\pi(p)}\int_0^{t_{\pi(p)}}\mathrm{d}s_1\int_{\lambda^{-2}(t_{\pi(p)}-s_1)}^{\infty}\mathrm{d}u\,h(u).
\end{aligned} \tag{6.10}$$

The last line vanishes uniformly in $0\leq t\leq T$ by the dominated convergence theorem, since the expression is dominated by $t^2\|h\|_1$, and the du-integral vanishes as $\lambda\downarrow 0$ whenever $s_1 < t_{\pi(p)}$. This ends the proof since $G_\lambda(t,t_0)$ is the sum of all terms with time-consecutive pairings. □

Now notice that $G_\lambda(t,t_0)$ can be written in the form familiar from the weak coupling limit for Friedrichs Hamiltonians. In fact, if we consider the Hilbert space $\mathcal{K}\oplus(\mathcal{K}\otimes\mathcal{H}_\mathrm{R})$ with the Friedrichs-type Hamiltonian

$$\tilde{H}_\lambda := \begin{bmatrix} K & \lambda V\\ \lambda V^* & K+H_\mathrm{R}\end{bmatrix},$$

then we can write

$$G_\lambda(t,t_0) = \mathrm{e}^{\mathrm{i}\lambda^{-2}tK}I_\mathcal{K}^*\mathrm{e}^{-\mathrm{i}\lambda^{-2}t\tilde{H}_\lambda}I_\mathcal{K}\mathrm{e}^{-\mathrm{i}\lambda^{-2}t_0K}.$$

Therefore, we can apply Theorem 2.1 in [10]. More precisely, define

$$\begin{aligned}
Q_{\lambda,s} &:= \lambda^{-2}\int_0^s\mathrm{d}u\,\mathrm{e}^{\mathrm{i}\lambda^{-2}uK}I_\mathcal{K}^*H^\mathrm{I}(\lambda^{-2}u)H^\mathrm{I}(0)I_\mathcal{K}\\
&= \int_0^{\lambda^{-2}s}V^*\mathrm{e}^{-\mathrm{i}u(K+H_\mathrm{R})}V\,\mathrm{d}u
\end{aligned} \tag{6.11}$$

and remark that by Assumption 4.4,

1) For all $\tau_1>0$, there is $c>0$ such that

$$0\leq s\leq\tau_1,\ \lambda\leq 1 \Rightarrow \|Q_{\lambda,s}\|\leq c. \tag{6.12}$$

2) For all $0 < \tau_0 \leq \tau_1 < \infty$,

$$\lim_{\lambda \downarrow 0} \sup_{\tau_0 \leq s \leq \tau_1} \| Q_{\lambda,s} - Q \| = 0, \tag{6.13}$$

with

$$Q := \int_0^{+\infty} V^* \mathrm{e}^{-\mathrm{i}u(K+H_{\mathrm{R}})} V \mathrm{d}u < \infty. \tag{6.14}$$

The aforementioned theorem by Davies allows us to conclude from 1) and 2) above, and the fact that $\Upsilon = \sum_{k \in \mathrm{sp} K} 1_{\mathcal{K}_k} K 1_{\mathcal{K}_k}$, that for all $T < \infty$,

$$\lim_{\lambda \downarrow 0} \sup_{t_0 \leq t \leq T} \| G_\lambda(t, t_0) - \mathrm{e}^{-\mathrm{i}(t-t_0)\Upsilon} \| = 0. \tag{6.15}$$

6.4. Proof of Theorem 5.2. Weak coupling limit for correlations. We follow very closely the strategy of Dümcke in [14]. The case $\ell = 0$ has been already proven. For notational reasons, we restrict ourselves to the case $\ell = 1$. Higher ℓ are proven in exactly the same way.

The theorem for the case $\ell = 1$ follows immediately from Theorem 5.1 and the following lemma:

Lemma 6.4. *For all $T \leq \infty$ and $S \in \mathcal{B}(\mathcal{K})$,*

$$\begin{aligned} \lim_{\lambda \downarrow 0} \sup_{-T \leq t_0 < t' < t < T} & \| I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t')\ S \otimes 1\ T_\lambda(\lambda^{-2}t', t_0) I_{\mathcal{K}} \\ & - I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t') I_{\mathcal{K}} S I_{\mathcal{K}}^* T_\lambda(t', t_0) I_{\mathcal{K}} \| = 0. \end{aligned} \tag{6.16}$$

Proof. Using first the Dyson expansion and then the Wick theorem we obtain

$$\begin{aligned} & I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t')\ S \otimes 1\ T_\lambda(\lambda^{-2}t', t_0) I_{\mathcal{K}} \qquad (6.17) \\ & \quad - I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t') I_{\mathcal{K}} \\ & = \sum_{n=0}^{\infty} \sum_{\sigma \in \mathrm{Pair}(2n)} \sum_{j_1, \ldots, j_{2n} \in \mathcal{J}} (\mathrm{i}\lambda)^{-2n} \qquad (6.18) \\ & \quad \times \sum_{p=0}^{n} \int_{\triangle_p[t_0, t'] \times \triangle_{2n-p}[t', t]} \mathrm{d}t_{2n} \cdots \mathrm{d}t_1 \\ & \quad \times \prod_{i=p+1}^{2n} D_{j_i}^{\epsilon(\sigma(i))}(\lambda^{-2}t_i) S \prod_{i'=1}^{p} D_{j_{i'}}^{\epsilon(\sigma(i'))}(\lambda^{-2}t_{i'}) \\ & \quad \times \prod_{p=1}^{n} \left\langle \phi_{j_{\sigma(2p)}} | \mathrm{e}^{\mathrm{i}\lambda^{-2}(t_{\sigma(2p)} - t_{\sigma(2p-1)}) H_{\mathrm{R}}} \phi_{j_{\sigma(2p-1)}} \right\rangle . \\ & =: \sum_{n=0}^{+\infty} C_n(S). \end{aligned}$$

Assume for simplicity $t_0 = 0$. Exactly as in (6.8), we prove that

$$\|C_n(S)\| \le \|S\| \frac{(\|D\|)^{2n}}{2^n n!} t^n \|h\|_1^n, \qquad (6.19)$$

so we can again estimate the series (6.18) term by term.

The term in the sum (6.18) corresponding to the pairing σ is estimated by

$$\left(\frac{\|D\|}{\lambda}\right)^{2n} \|S\| \int_{\triangle_p[0,t']\times\triangle_{2n-p}[t',t]} \mathrm{d}t_1 \ldots \mathrm{d}t_{2n} \prod_{i=1}^{n} h(\lambda^{-2}(t_{\pi(2i)} - t_{\pi(2i-1)})). \qquad (6.20)$$

We are going to show all such terms with a pairing crossing t' vanish in the limit $\lambda \searrow 0$.

Assume there is a i such that

$$s_1 := t_{\pi(2i-1)} < t' < t_{\pi(2i)} =: s_2. \qquad (6.21)$$

Then

$$\begin{aligned}
&\lambda^{-2n} \int_{\triangle_p[0,t']\times\triangle_{n-p}[t',t]} \mathrm{d}t_1 \ldots \mathrm{d}t_{2n} \prod_{i=1}^{n} h(\lambda^{-2}(s_2 - s_1)) \\
&\le \lambda^{-2}(2t\|h\|_1)^{n-1} \int_0^{t'} \mathrm{d}s_1 \int_{t'}^{t} \mathrm{d}s_2\, h(\lambda^{-2}(s_2 - s_1)) \\
&\le (2t\|h\|_1)^{n-1} \int_0^{t'} \mathrm{d}s_1 \int_{\lambda^{-2}(t'-s_1)}^{+\infty} \mathrm{d}u\, h(u).
\end{aligned}$$

To prove that this term vanishes uniformly in t', we have to show

$$\lim_{\lambda\downarrow 0} \sup_{0\le t'\le t} \int_0^{t'} \mathrm{d}s_1 \int_{\lambda^{-2}(t'-s_1)}^{+\infty} \mathrm{d}u\, h(u) = 0. \qquad (6.22)$$

This follows since for each $s_1 < t'$, the integral over u vanishes as $\lambda \downarrow 0$ and the whole expression is bounded by $t'\|h\|_1$.

Since we have established that no pairing crosses the t' point, the problem factorizes and (6.16) is true. □

6.5. Convergence of annihilation operators.

Lemma 6.5. *For $\psi \in \mathcal{D}$ and $j \in \mathcal{J}$,*

$$\lim_{\lambda\downarrow 0} \lambda^{-1} a(J_\lambda^* \mathrm{e}^{\mathrm{i}\lambda^{-2}t(H_\mathrm{R}-\omega(j))}\phi_j)\psi \qquad (6.23)$$

$$= a\left(\mathrm{e}^{\mathrm{i}tZ_\mathrm{R}}|1\rangle \otimes \phi_j(\omega(j))\right)\psi \qquad (6.24)$$

uniformly in $t \in \mathbb{R}$.

Proof. We use

$$\lambda^{-1}J_\lambda^* e^{i\lambda^{-2}t(H_R-\omega(j))}\phi_j(x) = \bigoplus_{\omega\in\mathcal{F}} \lambda^{-1}J_{\lambda,\omega}^* e^{i\lambda^{-2}t(H_R-\omega(j))}\phi_j(x). \tag{6.25}$$

Now

$$\begin{aligned}
&\lambda^{-1}J_{\lambda,\omega}^* e^{i\lambda^{-2}t(H_R-\omega(j))}\phi_j(x)\\
&= e^{it(x+\lambda^{-2}(\omega-\omega(j)))}\left(1_{I_\omega}(H_R)\phi_j\right)(\omega+\lambda^2 x)\\
&\underset{\lambda\searrow 0}{\longrightarrow} \begin{cases}\phi_j(\omega(j)), & \omega=\omega(j);\\ 0, & \omega\neq\omega(j).\end{cases}
\end{aligned}$$

□

6.6. Resummation formula. The following lemma gives a convenient expression for the full dynamics in terms of the reduced dynamics.

Lemma 6.6.

$$\begin{aligned}
&T_\lambda(t,t_0)\\
&= \sum_{m=0}^{\infty}\int_{\triangle_m[t_0,t]} dt_1\dots dt_m \sum_{\epsilon_1,\dots,\epsilon_p\in\{+,-\}}\ \sum_{j_1,\dots,j_m\in\mathcal{J}} (-i)^m\lambda^m\\
&\times I_{\mathcal{K}}^* T_\lambda(t,t_m)D_{j_m}^{\epsilon_m}(t_m)T_\lambda(t_m,t_{m-1})\cdots D_{j_1}^{\epsilon_1}(t_1)T_\lambda(t_1,t_0)I_{\mathcal{K}}\otimes 1_{\Gamma_s(\mathcal{H}_R)}\\
&\times \prod_{i=1,\dots,m\,:\,\epsilon_i=+} a^*(e^{it_iH_R}\phi_{j_i}) \prod_{i'=1,\dots,m\,:\,\epsilon_{i'}=-} a(e^{it_{i'}H_R}\phi_{j_{i'}}).
\end{aligned} \tag{6.26}$$

Proof. $\widetilde{\mathrm{Pair}}(n)$ will denote the set of all pairings *inside* the set $\{1,\dots,n\}$. That means, $\sigma\in\widetilde{\mathrm{Pair}}(n)$ iff there is $p=0,1,\dots,[n/2]$ such that σ is an injection of $\{1,\dots,2p\}$ into $\{1,\dots,n\}$ satisfying $\sigma(2i-1)<\sigma(2i+1)$, $i=1,\dots,p-1$ and $\sigma(2i-1)<\sigma(2i)$, $i=1,\dots,p$. For $\sigma\in\widetilde{\mathrm{Pair}}(n)$, let $\mathrm{Ran}\sigma$ denote the image of σ. We say that a sequence $\epsilon_1,\dots\epsilon_n$ of $\{+,-\}$ is *compatible* with $\sigma\in\widetilde{\mathrm{Pair}}(n)$ iff

$$\epsilon_{\sigma(2i-1)}=-,\quad \epsilon_{\sigma(2i)}=+,\quad i=1,\dots,p.$$

Applying the Dyson expansion and then the Wick theorem we obtain that the left-hand side of (6.26) equals

$$\begin{aligned}
&\sum_{n=0}^{\infty}\int_{\triangle_n[t_0,t]} dt_1\cdots dt_n \sum_{j_1,\dots,j_n\in\mathcal{J}}\ \sum_{\epsilon_1,\dots,\epsilon_n\in\{+,-\}}\\
&\times\prod_{r=1}^{n}(-i)\lambda D_{j_r}^{\epsilon_r}(t_r)\\
&\times a^{\epsilon_n}(e^{it_nH_R}\phi_{j_n})\cdots a^{\epsilon_1}(e^{it_1H_R}\phi_{j_1})
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \int_{\triangle_n} \mathrm{d}t_1 \cdots \mathrm{d}t_n \sum_{j_1,\dots,j_n \in \mathcal{J}} \sum_{\sigma \in \widetilde{\mathrm{Pair}(n)}} \sum_{\substack{\epsilon_1,\dots,\epsilon_n \in \{+,-\} \\ \text{compatible with } \sigma}}$$

$$\times \prod_{r=1}^{n} (-\mathrm{i})\lambda D_{j_r}^{\epsilon_r}(t_r)$$

$$\times \prod_{i \in \{1,\dots,n\} \setminus \mathrm{Ran}\sigma \,:\, \epsilon_i = +} a^*(e^{\mathrm{i}t_i H_{\mathrm{R}}} \phi_{j_i}) \prod_{i' \in \{1,\dots,n\} \setminus \mathrm{Ran}\sigma \,:\, \epsilon_{i'} = -} a(e^{\mathrm{i}t_{i'} H_{\mathrm{R}}} \phi_{j_{i'}})$$

$$\times \prod_{q=1}^{p} \left\langle \phi_{j_{\sigma(2q)}} | \mathrm{e}^{\mathrm{i}(t_{\sigma(2q)} - t_{\sigma(2q-1)}) H_{\mathrm{R}}} \phi_{j_{\sigma(2q-1)}} \right\rangle. \tag{6.27}$$

Applying the Wick theorem and the Dyson expansion backwards we see that the right-hand side of (6.27) equals the right-hand side of (6.26). □

6.7. Proof of Theorem 5.4. Set

$$D_{j,\omega} := \sum_{e-e'=\omega} 1_{\mathcal{K}_e} D_j 1_{\mathcal{K}_{e'}}.$$

Recall the operators ν_ω and ν defined in Subsect. 4.2. They can be expressed in terms of $D_{j,\omega}$ by

$$\nu_\omega = \sqrt{2\pi} \sum_{j \in \mathcal{J} \,:\, \omega(j) = \omega} D_{j,\omega} \otimes |\phi_j(\omega)\rangle,$$

$$\nu = \sqrt{2\pi} \sum_{j \in \mathcal{J}} D_{j,\omega(j)} \otimes |\phi_j(\omega(j))\rangle.$$

We use first the resummation formula (6.26), and then we replace $D_j^{\epsilon}(t)$ with

$$\sum_{\omega \in \mathcal{F}} D_{j,\omega}^{\epsilon} \mathrm{e}^{-\mathrm{i}\epsilon\omega t}.$$

We compute in terms of a quadratic form on $\mathcal{D}$:

$$\Gamma(J_\lambda^*) T_\lambda(\lambda^{-2}t, \lambda^{-2}t_0) \Gamma(J_\lambda)$$

$$= \sum_{m=0}^{\infty} \int_{\triangle_m[t_0,t]} \mathrm{d}t_1 \dots \mathrm{d}t_m \sum_{j_1,\dots,j_m} \sum_{\epsilon_1,\dots,\epsilon_m \in \{+,-\}} (\mathrm{i}\lambda)^{-m}$$

$$\times I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t_m) D_{j_m}^{\epsilon_m}(\lambda^{-2}t_m) T_\lambda(\lambda^{-2}t_m, \lambda^{-2}t_{m-1}) \cdots$$

$$\times \cdots D_{j_1}^{\epsilon_1}(\lambda^{-2}t_1) T_\lambda(\lambda^{-2}t_1, \lambda^{-2}t_0) I_{\mathcal{K}} \otimes 1_{\Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})}$$

$$\times \prod_{i=1,\dots,m \,:\, \epsilon_i = +} a^*(J_\lambda^* \mathrm{e}^{\mathrm{i}\lambda^{-2}t_i H_{\mathrm{R}}} \phi_{j_i})$$

$$
\begin{aligned}
&\times \Gamma(J_\lambda^* J_\lambda) \prod_{i'=1,\ldots,m\,:\,\epsilon_i=-} a(J_\lambda^* \mathrm{e}^{\mathrm{i}\lambda^{-2}t_{i'}H_{\mathrm{R}}}\phi_{j_{i'}}) \\
&= \sum_{m=0}^{\infty} \int_{\triangle_m[t_0,t]} \mathrm{d}t_1 \ldots \mathrm{d}t_m \sum_{j_1,\ldots,j_m} \sum_{\epsilon_1,\ldots,\epsilon_m\in\{+,-\}} \sum_{\omega_1,\ldots,\omega_m\in\mathcal{F}} (\mathrm{i}\lambda)^{-m} \\
&\times \prod_{p=1}^{m} \mathrm{e}^{\mathrm{i}(\omega_p-\omega(j_p))\lambda^{-2}t} \\
&\times I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t_m) D_{j_m,\omega_m}^{\epsilon_m} T_\lambda(\lambda^{-2}t_m, \lambda^{-2}t_{m-1}) \cdots \\
&\times \cdots D_{j_1,\omega_1}^{\epsilon_1} T_\lambda(\lambda^{-2}t_1, \lambda^{-2}t_0) I_{\mathcal{K}} \otimes 1_{\Gamma_{\mathrm{s}}(\mathcal{H}_{\mathrm{R}})} \\
&\times \prod_{i=1,\ldots,m\,:\,\epsilon_i=+} a^*(J_\lambda^* \mathrm{e}^{\mathrm{i}\lambda^{-2}t_i(H_{\mathrm{R}}-\omega(j_i))}\phi_{j_i}) \\
&\times \Gamma(J_\lambda^* J_\lambda) \prod_{i'=1,\ldots,m\,:\,\epsilon_i=-} a(J_\lambda^* \mathrm{e}^{\mathrm{i}\lambda^{-2}t_{i'}(H_{\mathrm{R}}-\omega(j_{i'}))}\phi_{j_{i'}}).
\end{aligned}
\tag{6.28}
$$

Now by Theorem 5.2, we have a uniform limit

$$
\begin{aligned}
&\lim_{\lambda\searrow 0} I_{\mathcal{K}}^* T_\lambda(\lambda^{-2}t, \lambda^{-2}t_m) D_{j_m,\omega_m}^{\epsilon_m} T_\lambda(\lambda^{-2}t_m, \lambda^{-2}t_{m-1}) \cdots \\
&\quad \times \cdots D_{j_1,\omega_1}^{\epsilon_1} T_\lambda(\lambda^{-2}t_1, \lambda^{-2}t_0) I_{\mathcal{K}} \\
&= \mathrm{e}^{-\mathrm{i}(t-t_m)\Upsilon} D_{j_m,\omega_m}^{\epsilon_m} \mathrm{e}^{-\mathrm{i}(t_m-t_{m-1})\Upsilon} \ldots \\
&\quad \times \cdots D_{j_1,\omega_1}^{\epsilon_1} \mathrm{e}^{\mathrm{i}(t_1-t_0)\Upsilon}.
\end{aligned}
\tag{6.29}
$$

By Lemma 6.5, for $\psi, \psi' \in \mathcal{D}$ we have the uniform limits

$$
\begin{aligned}
&\lim_{\lambda\searrow 0} \prod_{i'=1,\ldots,m\,:\,\epsilon_{i'}=-} \lambda^{-1} a(J_\lambda^* \mathrm{e}^{\mathrm{i}\lambda^{-2}t_{i'}(H_{\mathrm{R}}-\omega(j_{i'}))}\phi_{j_{i'}})\psi' \\
&= \prod_{i'=1,\ldots,m\,:\,\epsilon_{i'}=-} a\left(\mathrm{e}^{\mathrm{i}t_{i'}Z_{\mathrm{R}}}|1\rangle \otimes \phi_{j_{i'}}(\omega(j_{i'}))\right)\psi', \\
&\lim_{\lambda\searrow 0} \prod_{i=1,\ldots,m\,:\,\epsilon_i=+} \lambda^{-1} a(J_\lambda^* \mathrm{e}^{\mathrm{i}\lambda^{-2}t_i(H_{\mathrm{R}}-\omega(j_i))}\phi_{j_i})\psi \\
&= \prod_{i=1,\ldots,m\,:\,\epsilon_i=+} a\left(\mathrm{e}^{\mathrm{i}t_i Z_{\mathrm{R}}}|1\rangle \otimes \phi_{j_i}(\omega(j_i))\right)\psi.
\end{aligned}
$$

Clearly, $\mathrm{s}-\lim_{\lambda\searrow 0}\Gamma(J_\lambda^* J_\lambda) = 1$. Thus, (6.28), as a quadratic form on $\mathcal{D}$, up to an error of the order $o(\lambda^0)$ equals

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \int_{\triangle_m[t_0,t]} \mathrm{d}t_1 \ldots \mathrm{d}t_m \sum_{\epsilon_1,\ldots,\epsilon_m\in\{+,-\}} \sum_{\omega_1,\ldots,\omega_m\in\mathcal{F}} (-\mathrm{i})^m \\
&\times \prod_{p=1}^{m} \mathrm{e}^{\mathrm{i}(\omega_p-\omega(j_p))\lambda^{-2}t}
\end{aligned}
$$

$$\times e^{-i(t-t_m)\Upsilon} D^{\epsilon_m}_{j_m,\omega_m} e^{-i(t_m-t_{m-1})\Upsilon} \dots$$
$$\times \cdots D^{\epsilon_1}_{j_1,\omega_1} e^{-i(t_1-t_0)\Upsilon} \otimes 1_{\Gamma_s(\mathcal{H}_R)}$$
$$\times \prod_{i=1,\dots,m\,:\,\epsilon_i=+} a\left(e^{it_i Z_R}|1\rangle \otimes \phi_{j_i}(\omega(j_i))\right)$$
$$\times \prod_{i'=1,\dots,m\,:\,\epsilon_{i'}=-} a\left(e^{it_{i'} Z_R}|1\rangle \otimes \phi_{j_{i'}}(\omega(j_{i'}))\right).$$

By the Riemann-Lebesgue Lemma, in the limit $\lambda \searrow 0$, all the terms with $\omega(j_p) \neq \omega_p$ for some p disappear, and we obtain

$$\sum_{m=0}^{\infty} \int_{\triangle_m[t_0,t]} dt_1 \dots dt_m \sum_{\epsilon_1,\dots,\epsilon_m \in\{+,-\}} (-i)^m$$
$$\times e^{-i(t-t_m)\Upsilon} D^{\epsilon_m}_{j_m,\omega_m} e^{-i(t_m-t_{m-1})\Upsilon} \dots$$
$$\times \cdots D^{\epsilon_1}_{j_1,\omega_1} e^{-i(t_1-t_0)\Upsilon} \otimes 1_{\Gamma_s(\mathcal{H}_R)}$$
$$\times \prod_{i=1,\dots,m\,:\,\epsilon_i=+} a\left(e^{it_i Z_R}|1\rangle \otimes \phi_{j_i}(\omega(j_i))\right)$$
$$\times \prod_{i'=1,\dots,m\,:\,\epsilon_{i'}=-} a\left(e^{it_{i'} Z_R}|1\rangle \otimes \phi_{j_{i'}}(\omega(j_{i'}))\right)$$
$$= e^{it d\Gamma(Z_R)} U_{t-t_0} e^{-it_0 d\Gamma(Z_R)}.$$

Thus we obtained, for $\psi, \psi' \in \mathcal{D}$,

$$\lim_{\lambda\downarrow 0} \langle \psi | \Gamma(J_\lambda^*) T_\lambda(\lambda^{-2}t, \lambda^{-2}t_0) \Gamma(J_\lambda)\psi' \rangle \tag{6.30}$$
$$= \langle \psi | e^{it d\Gamma(Z_R)} U_{t-t_0} e^{-it_0 d\Gamma(Z_R)} \psi' \rangle. \tag{6.31}$$

By density, (6.30) can be extended to the weak limit on the whole space. But the weak convergence of contractions to a unitary operator implies the strong* convergence. This yields (5.5).

Note that

$$\Gamma(J_\lambda^*) e^{-it\lambda^{-2}t H_0} (1 - \Gamma\left(J_\lambda J_\lambda^*\right)) = 0.$$

Therefore,

$$e^{i\lambda^{-2}t Z_{ren}} \Gamma(J_\lambda^*) e^{-i\lambda^{-2}t H_\lambda} \Gamma(J_\lambda)$$
$$= e^{i\lambda^{-2}t Z_{ren}} \Gamma(J_\lambda^*) e^{-i\lambda^{-2}t H_0} \Gamma(J_\lambda)$$
$$\times \Gamma(J_\lambda^*) e^{i\lambda^{-2}t H_0} e^{-i\lambda^{-2}t H_\lambda} \Gamma(J_\lambda). \tag{6.32}$$

The equality (5.6) now follows from (4.16) and (5.5) since the strong limit of a product of uniformly bounded operators is the product of limits.

6.8. Proof of Theorem 5.7. Remark that

$$(1-\Gamma(J_\lambda J_\lambda^*))\mathrm{e}^{\mathrm{i}\lambda^{-2}tH_0}\mathrm{e}^{-\mathrm{i}\lambda^{-2}tH_\lambda}\Gamma(J_\lambda)\xrightarrow[\lambda\searrow 0]{\text{strongly}} 0. \tag{6.33}$$

This follows since for $\psi\in\mathcal{Z}$

$$\begin{aligned}&\|(1-\Gamma(J_\lambda J_\lambda^*))\mathrm{e}^{\mathrm{i}\lambda^{-2}tH_0}\mathrm{e}^{-\mathrm{i}\lambda^{-2}tH_\lambda}\Gamma(J_\lambda)\psi\|^2\\&=\|\mathrm{e}^{\mathrm{i}\lambda^{-2}tH_0}\mathrm{e}^{-\mathrm{i}\lambda^{-2}tH_\lambda}\Gamma(J_\lambda)\psi\|^2-\|\Gamma(J_\lambda J_\lambda^*)\mathrm{e}^{\mathrm{i}\lambda^{-2}tH_0}\mathrm{e}^{-\mathrm{i}\lambda^{-2}tH_\lambda}\Gamma(J_\lambda)\psi\|^2\\&=\|\Gamma(J_\lambda)\psi\|^2-\|\Gamma(J_\lambda^*)\mathrm{e}^{\mathrm{i}\lambda^{-2}tH_0}\mathrm{e}^{-\mathrm{i}\lambda^{-2}tH_\lambda}\Gamma(J_\lambda)\psi\|^2\xrightarrow[\lambda\searrow 0]{}0,\end{aligned}$$

where we used $\mathrm{s-}\lim_{\lambda\searrow 0} J_\lambda^* J_\lambda = 1$, Theorem 5.4 and the fact that $\Gamma(J_\lambda)$ is a partial isometry.

Theorem 5.7 is now proven by using (5.5), (5.12), (6.33) and the fact that for multiplication operators G as in the text preceding Theorem 5.7 we have

$$[\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(H_{\mathrm{R}})},\Gamma(G)]=0\qquad[\Gamma(J_\lambda J_\lambda^*),\Gamma(G)]=0.$$

Acknowledgements. The research of J. D. was partly supported by the Postdoctoral Training Program HPRN-CT-2002-0277 and the Polish KBN grants SPUB127 and 2 P03A 027 25. Part of the work was done when both authors visited the Erwin Schrödinger Institute (J. D. as a Senior Research Fellow), as well as during a visit of J. D. at K. U. Leuven supported by a grant of the ESF. W. D. R. is a Postdoctoral Fellow supported by FWO-Flanders.

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Communicated by H.-T. Yau