Geometry of the universal cover of $SL(2, \mathbb{R})$

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$\mathsf{SL}(2,\mathbb{R})$

Group: matrix multiplication

$$\mathsf{SL}(2,\mathbb{R}) = \{g \in \mathbb{M}_{2x2}(\mathbb{R}) : \det g = 1\}$$

Surface in $\mathbb{R}^4$

$$\det \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = 1 \qquad (g_{11}, g_{12}, g_{21}, g_{22}) \in \mathbb{R}^4$$

$$g_{11}g_{22} - g_{12}g_{21} = 1$$

$\mathrm{SL}(2,\mathbb{R})$ as a group

- If $g, h \in \mathrm{SL}(2,\mathbb{R})$, then $\det gh = \det g \det h = 1$, $gh \in \mathrm{SL}(2,\mathbb{R})$.
- If $g, h, k \in \mathrm{SL}(2,\mathbb{R})$ then $ghk = g(hk) = (gh)k$
- Neutral element of the group is $\mathbb{1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.
- Inverse element g^{-1}

$$\begin{aligned} \det\left(g^{-1}g\right) &= \det(\mathbb{1}) \\ \det\left(g^{-1}\right)\det(g) &= \det(\mathbb{1}) \\ \det\left(g^{-1}\right) &= 1. \end{aligned}$$

$SL(2,\mathbb{R})$ as a surface

The determinant condition can be also viewed as a definition of a 3-surface in $\mathbb{R}^4 \ni (g_{11}, g_{12}, g_{21}, g_{22})$.

$$\det g = 1 \iff g_{11}g_{22} - g_{12}g_{21} = 1.$$

Equation of this surface can be rewritten as

$$\left(\frac{g_{11}+g_{22}}{2}\right)^2 - \left(\frac{g_{11}-g_{22}}{2}\right)^2 - \left(\frac{g_{12}+g_{21}}{2}\right)^2 + \left(\frac{g_{12}-g_{21}}{2}\right)^2 = 1,$$

which is an equation of a 3-hyperbola in $\mathbb{R}^4$,
or a unit sphere in $\mathbb{R}^{2,2}$.

1-hyperbola

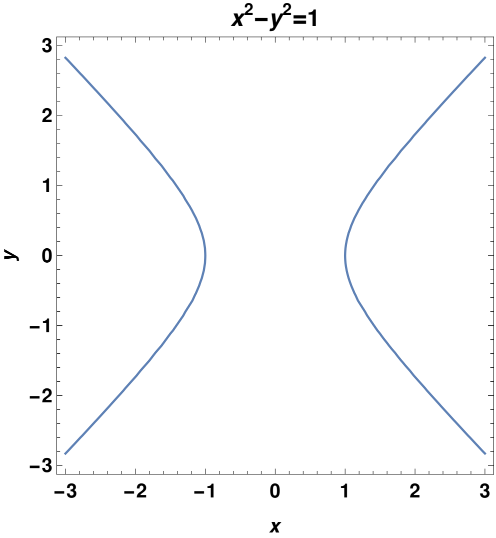


Figure: 1-hyperbola in $\mathbb{R}^2$

2-hyperbolas

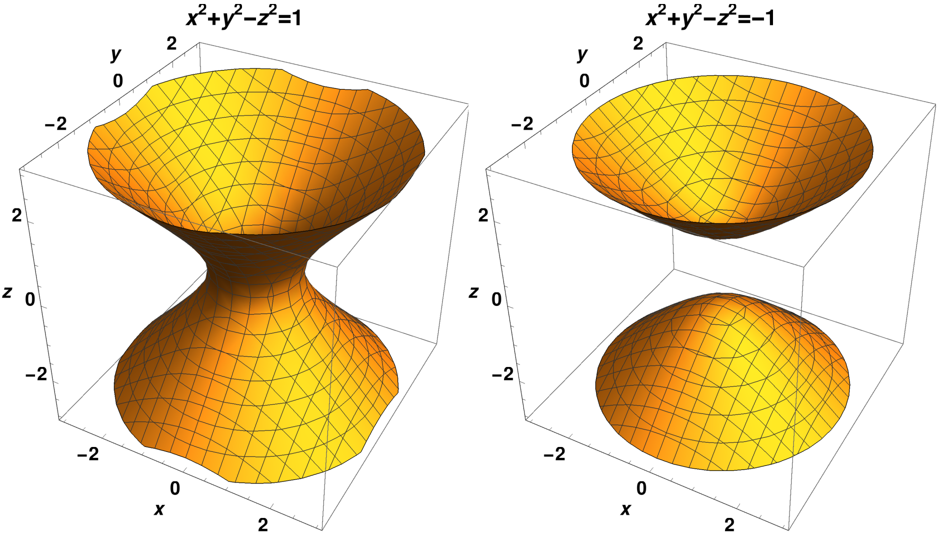
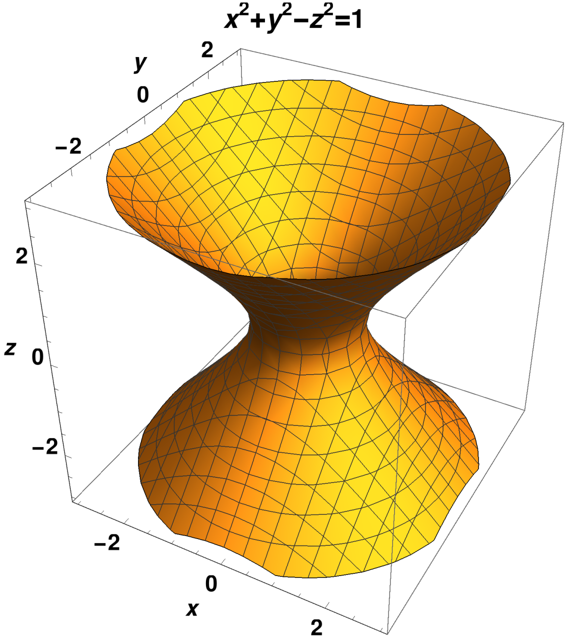
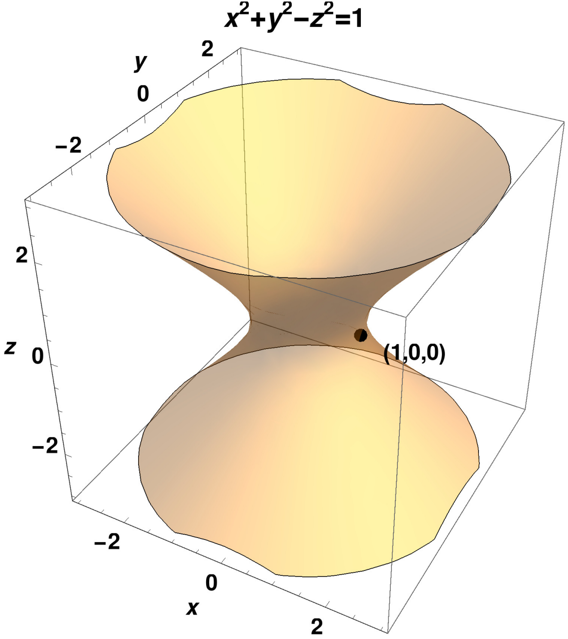


Figure: Two 2-hyperbolas in $\mathbb{R}^3$

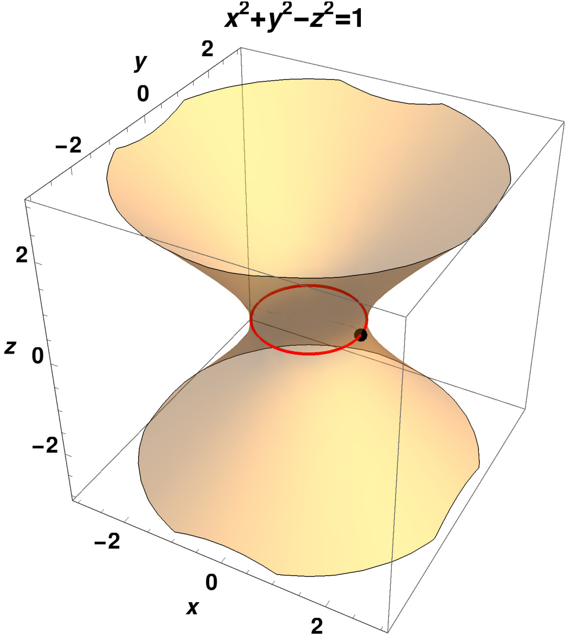
2-hyperbola



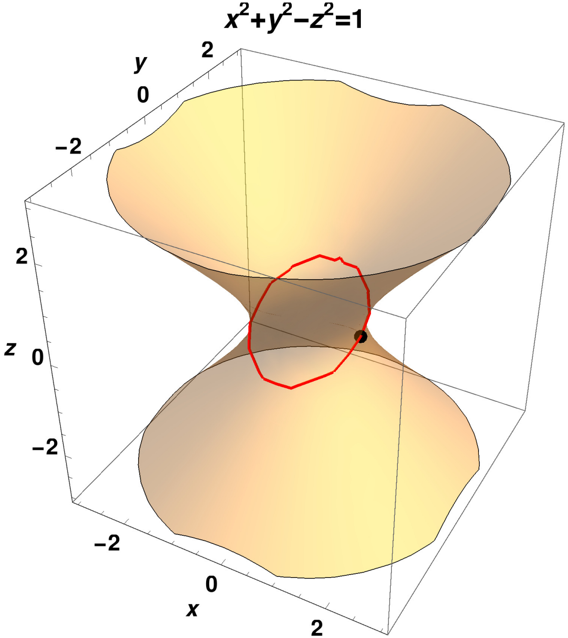
2-hyperbola



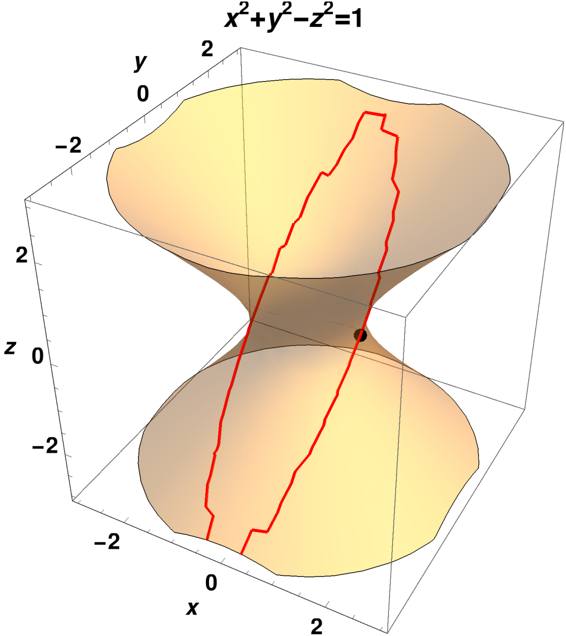
2-hyperbola geodesics: elliptic



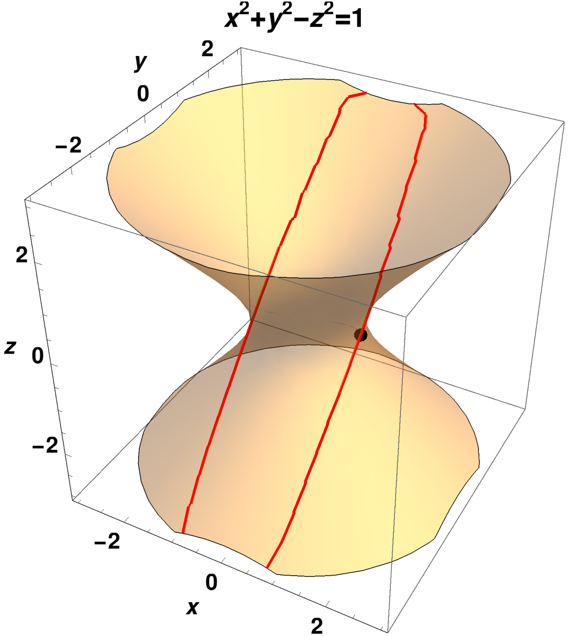
2-hyperbola geodesics: elliptic



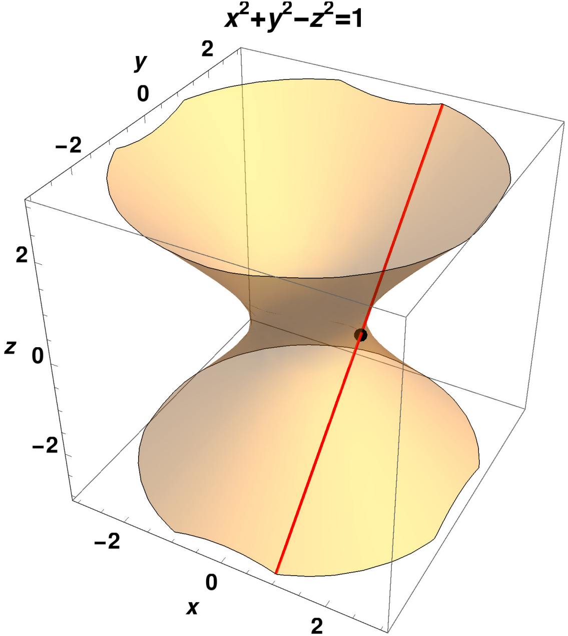
2-hyperbola geodesics: elliptic



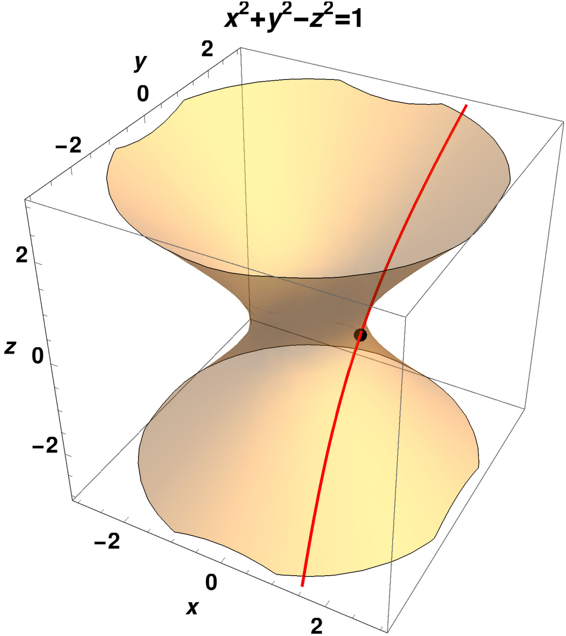
2-hyperbola geodesics: elliptic



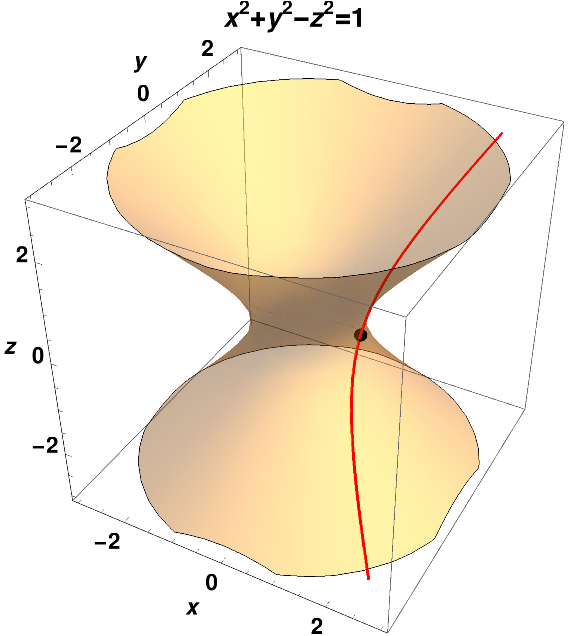
2-hyperbola geodesics: parabolic



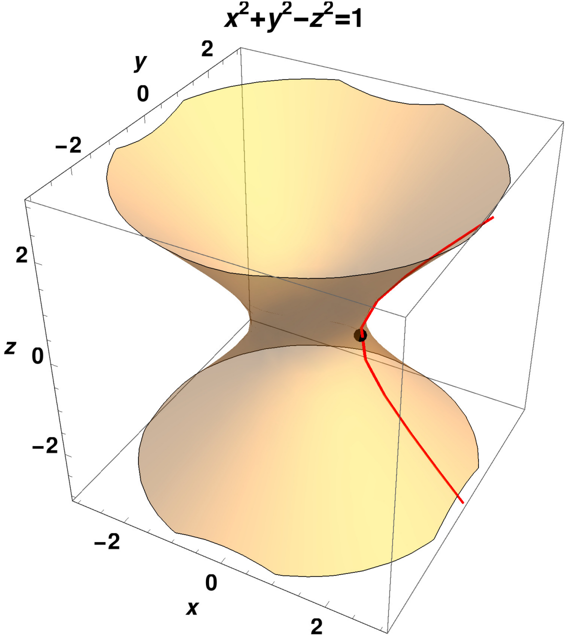
2-hyperbola geodesics: hyperbolic



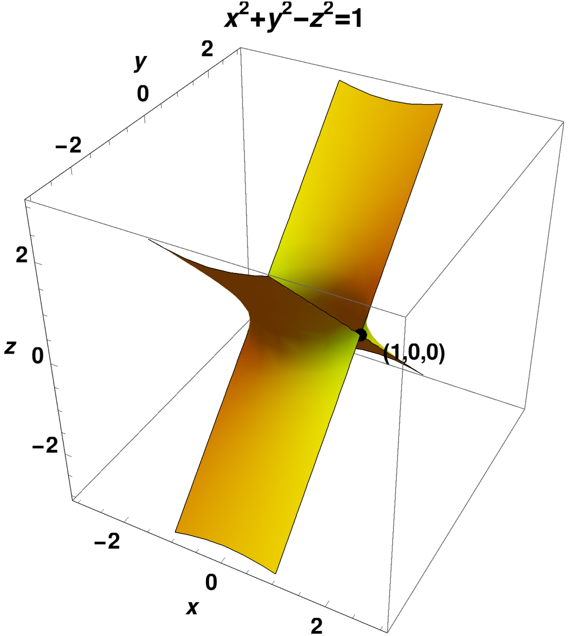
2-hyperbola geodesics: hyperbolic



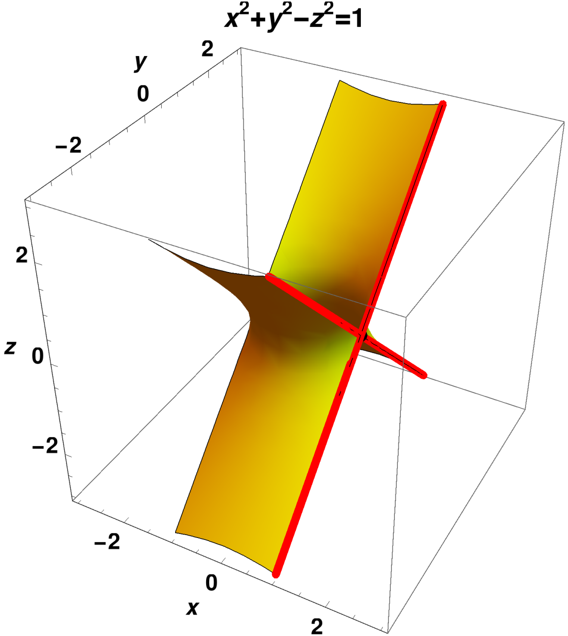
2-hyperbola geodesics: hyperbolic



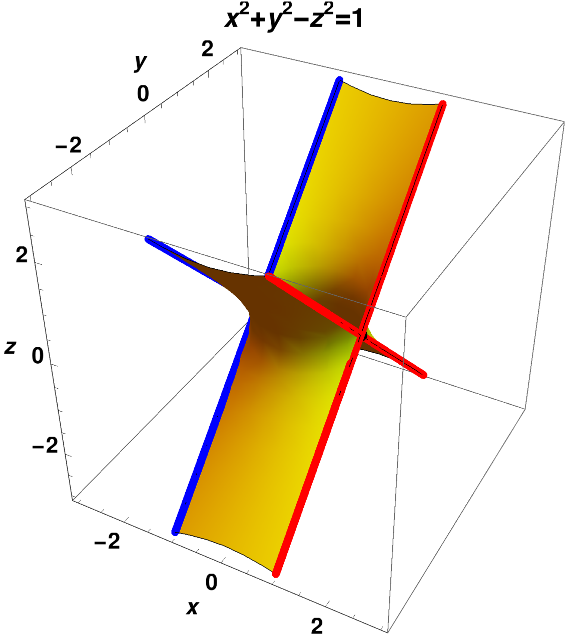
2-hyperbola elements: EII



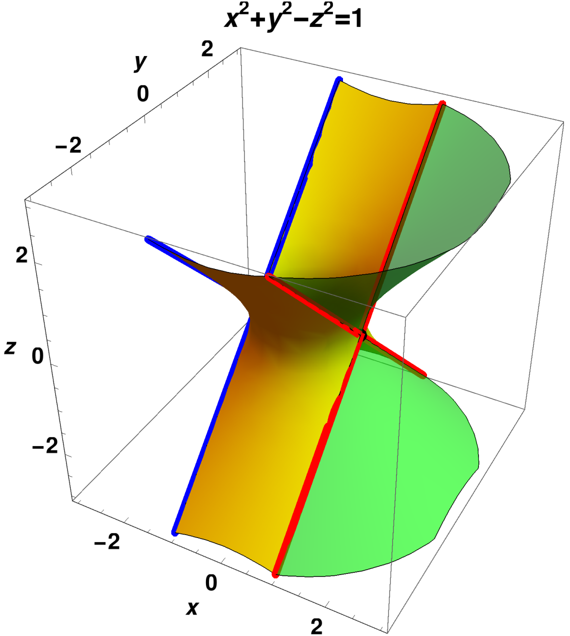
2-hyperbola elements: Ell, Par^{+}



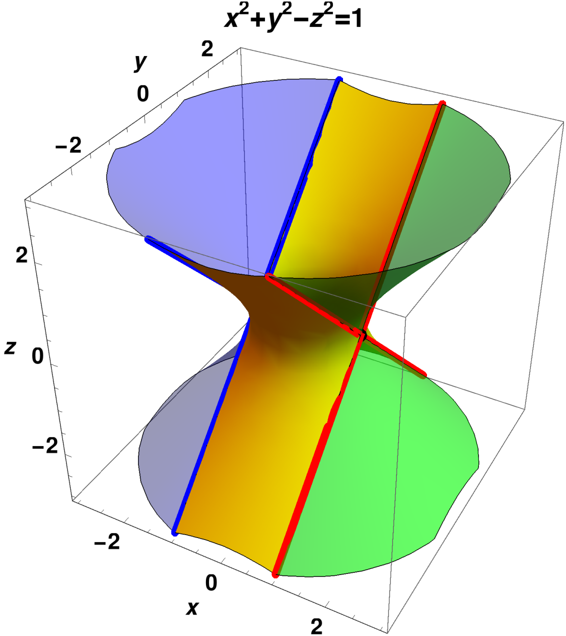
2-hyperbola elements: Ell, Par^+, Par^-



2-hyperbola elements: Ell, Par^+, Par^-, Hyp^+



2-hyperbola elements: Ell, Par^+, Par^-, Hyp^+, Hyp^-



$\mathfrak{sl}(2, \mathbb{R})$

$\mathfrak{sl}(2,\mathbb{R})$ - the Lie algebra of the $SL(2,\mathbb{R})$ group

Tangent space at the group identity

$$\mathfrak{sl}(2,\mathbb{R}) = T_{\mathbb{1}} SL(2,\mathbb{R})$$

$\mathfrak{sl}(2,\mathbb{R})$ - the Lie algebra of the $SL(2,\mathbb{R})$ group

$X \in \mathfrak{sl}(2,\mathbb{R})$ and $t \in \mathbb{R}$
$\exp(tX) \equiv 1 + tX + \frac{(tX)^2}{2} + \dots$

$$\det \exp(tX) = 1$$

$$\det \begin{bmatrix} 1 + tX_{11} & tX_{12} \\ tX_{21} & 1 + tX_{22} \end{bmatrix} + O(t^2) = 1$$

$$(1 + tX_{11})(1 + tX_{22}) + O(t^2) = 1 \qquad \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0}$$

$$X_{11} + X_{22} = 0$$

$\mathfrak{sl}(2, \mathbb{R})$ - the Lie algebra of the $SL(2, \mathbb{R})$ group

$$\mathfrak{sl}(2, \mathbb{R}) = \{X \in \mathbb{M}_{2x2}(\mathbb{R}) : \mathrm{Tr}\, X = 0\}$$

$$X = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & -X_{11} \end{bmatrix}$$

$$X_{11}, X_{12}, X_{21} \in \mathbb{R}$$

Scalar product in $\mathbb{M}_{2x2}(\mathbb{R})$

$$A, B \in \mathbb{M}_{2x2}(\mathbb{R}), \text{ then}$$

$$\langle A, B \rangle \equiv \tfrac{1}{2} \operatorname{Tr} AB$$

- symmetric

$$\langle A, B \rangle = \frac{1}{2} \operatorname{Tr} AB = \frac{1}{2} \operatorname{Tr} BA = \langle B, A \rangle$$

- non-degenerate

$$\begin{aligned} \forall_B \quad 0 &= \langle A, B \rangle \\ 0 &= \operatorname{Tr} AB \\ 0 &= a_{11} b_{11} + a_{12} b_{21} + a_{21} b_{12} + a_{22} b_{22} \\ &\implies A = 0 \end{aligned}$$

Basis of the $\mathfrak{sl}(2, \mathbb{R})$ algebra

The Pauli matrices

$$\mathrm{i}\sigma_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

all belong to the $\mathfrak{sl}(2, \mathbb{R})$ algebra.
They form an orthogonal basis of the algebra

$$\langle \mathrm{i}\sigma_2, \mathrm{i}\sigma_2 \rangle = -1, \quad \langle \sigma_1, \sigma_1 \rangle = 1, \quad \langle \sigma_3, \sigma_3 \rangle = 1$$

$$\langle \mathrm{i}\sigma_2, \sigma_1 \rangle = 0, \quad \langle \mathrm{i}\sigma_2, \sigma_3 \rangle = 0, \quad \langle \sigma_1, \sigma_3 \rangle = 0$$

Basis of the $\mathfrak{sl}(2, \mathbb{R})$ algebra

The previously introduced parametrisation of $X \in \mathfrak{sl}(2, \mathbb{R})$

$$X = X_0(X_{11}, X_{12}, X_{21}) = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & -X_{11} \end{bmatrix},$$

in the new basis

$$X = X(t, s, w) = t\,\mathrm{i}\sigma_2 + s\sigma_1 + w\sigma_3,$$

becomes

$$X_{11} = w, \quad X_{12} = s + t, \quad X_{21} = s - t$$

$$t = \frac{X_{12} - X_{21}}{2}, \quad s = \frac{X_{12} + X_{21}}{2}, \quad w = X_{11}.$$

Classification of elements of the $\mathfrak{sl}(2,\mathbb{R})$ algebra

$X \in \mathfrak{sl}(2,\mathbb{R})$

$$
\begin{aligned}
X = X(t,s,w) &= t\,\mathrm{i}\sigma_2 + s\sigma_1 + w\sigma_3 \\
\langle X, X\rangle &= \langle X(s,t,w), X(s,t,w)\rangle \\
&= -t^2 + s^2 + w^2.
\end{aligned}
$$

This scalar product has a signature $(-,+,+)$ - Minkowski space.

Classification of elements of the $\mathfrak{sl}(2,\mathbb{R})$ algebra

$$X \in \mathfrak{sl}(2,\mathbb{R})$$

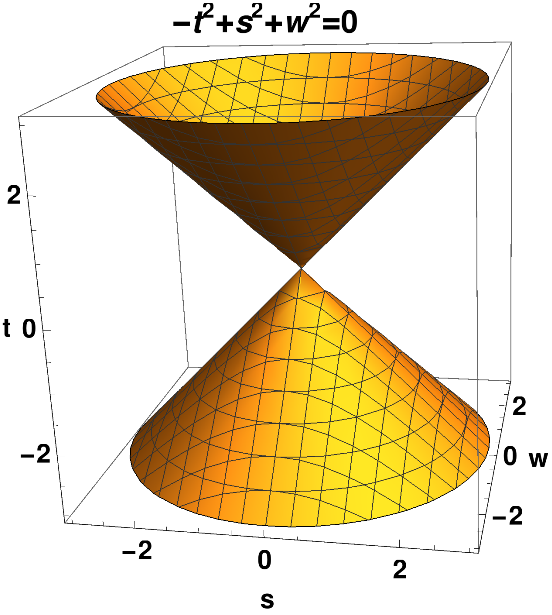
$$\begin{aligned} \text{par} &\equiv \{X \neq 0 : \langle X, X\rangle = 0\} \\ \text{ell} &\equiv \{X : \langle X, X\rangle < 0\} \\ \text{hyp} &\equiv \{X : \langle X, X\rangle > 0\} \end{aligned}$$

examples

$$A_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad A_+^2 = A_-^2 = 0 \qquad \operatorname{Tr} 0 = 0$$

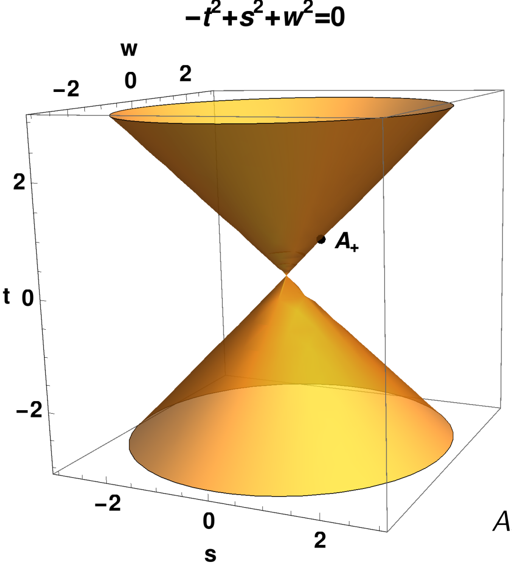
$$N = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix} \qquad N^2 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{4} \end{bmatrix} \qquad \operatorname{Tr} N^2 = \frac{1}{2}$$

Classification of elements of the algebra



Classification of elements of the algebra

$$A_+ = X(\tfrac{1}{2}, \tfrac{1}{2}, 0)$$

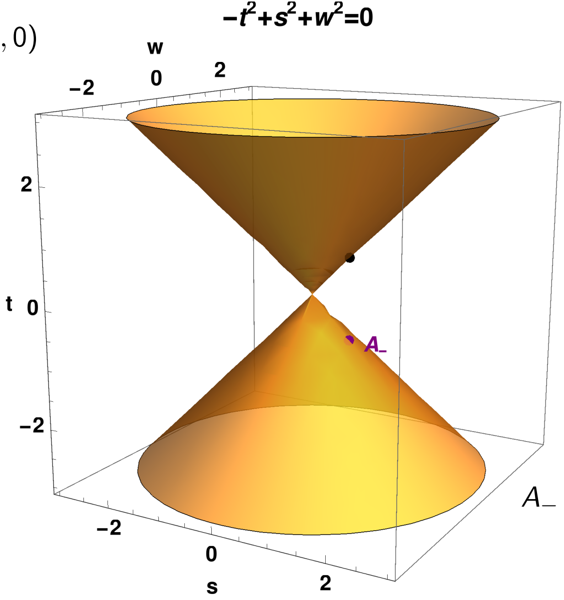


$$A_+ \in par$$

$$A_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Classification of elements of the algebra

$A_- = X(-\frac{1}{2}, \frac{1}{2}, 0)$

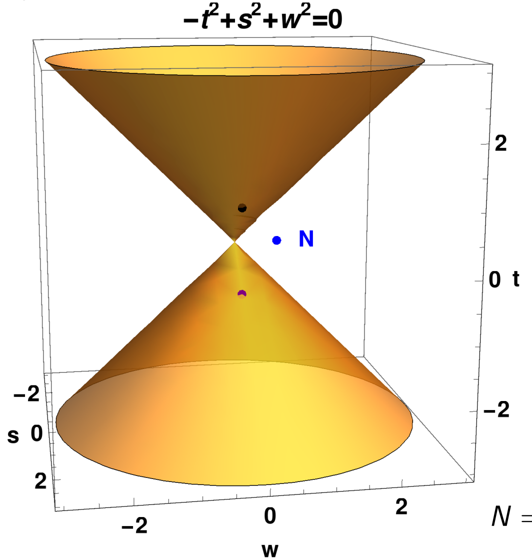


$A_- \in par$

$$A_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

Classification of elements of the algebra

$N = X(0,0,1)$



$N \in hyp$

$$N = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}$$

$\mathsf{SL}(2, \mathbb{R})$

Classification of elements of the $\mathrm{SL}(2,\mathbb{R})$ group

$g \in \mathrm{SL}(2,\mathbb{R})$

$$\begin{aligned} Par^+ &= \{g \neq \mathbb{1} : \mathrm{Tr}\, g = 2\} \\ Par^- &= \{g \neq -\mathbb{1} : \mathrm{Tr}\, g = -2\} \end{aligned}$$

$$Ell = \{g : -2 < \mathrm{Tr}\, g < 2\}$$

$$\begin{aligned} Hyp^+ &= \{g : \mathrm{Tr}\, g > 2\} \\ Hyp^- &= \{g : \mathrm{Tr}\, g < -2\} \end{aligned}$$

1 parameter groups in $\mathrm{SL}(2,\mathbb{R})$

$X \in \mathfrak{sl}(2,\mathbb{R})$
$\tau \in \mathbb{R}$

$$\exp(\tau X) \equiv \mathbb{1} + \tau X + \frac{\tau^2}{2!}X^2 + \ldots$$

1 parameter groups in $\mathrm{SL}(2,\mathbb{R})$: *par* elements

- If $X \in par$ then $X^2 = 0$

$\tau \in \mathbb{R}$

$$\exp(\tau X) = \mathbb{1} + \tau X$$
$$\mathrm{Tr}(\exp(\tau X)) = \mathrm{Tr}(\mathbb{1} + \tau X) = 2$$
$$\exp(\tau X) \in Par^{+}$$

1 parameter groups in $\mathrm{SL}(2,\mathbb{R})$: *hyp* elements

- If $X \in hyp$ then $X^2 = \lambda \mathbb{1}, \lambda > 0$

$\tau \in \mathbb{R}_+$

$$\begin{aligned}\exp(\tau X) &= \mathbb{1} + \tau X + \frac{(\tau X)^2}{2!} + \dots \\ &= \mathbb{1} + \tau X + \frac{(\tau\sqrt{\lambda})^2}{2!}\mathbb{1} + \dots \\ &= \cosh(\sqrt{\lambda}\tau)\mathbb{1} + \frac{\sinh(\sqrt{\lambda}\tau)}{\sqrt{\lambda}}X\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(\exp(\tau X)) &= \mathrm{Tr}\left(\cosh(\sqrt{\lambda}\tau)\mathbb{1} + \frac{\sinh(\sqrt{\lambda}\tau)}{\sqrt{\lambda}}X\right) \\ &= 2\cosh(\sqrt{\lambda}\tau) \\ & 2\cosh(\sqrt{\lambda}\tau) > 2\end{aligned}$$

$$\exp(\tau X) \in Hyp^+$$

1 parameter groups in $\mathrm{SL}(2,\mathbb{R})$: *ell* elements

- If $X \in$ *ell* then $X^2 = -\lambda\mathbb{1}, \lambda > 0$

$\tau \in \mathbb{R}_+$

$$\begin{aligned}\exp(\tau X) &= \mathbb{1} + \tau X + \frac{(\tau X)^2}{2!} + \dots \\ &= \mathbb{1} + \tau X + \frac{(\mathrm{i}\sqrt{\lambda}\tau)^2}{2!}\mathbb{1} + \dots \\ &= \cos\left(\sqrt{\lambda}\tau\right)\mathbb{1} + \frac{\sin\left(\sqrt{\lambda}\tau\right)}{\sqrt{\lambda}}X\end{aligned}$$

$$\begin{aligned}\mathrm{Tr}(\exp(\tau X)) &= \mathrm{Tr}\left(\cos\left(\sqrt{\lambda}\tau\right)\mathbb{1} + \frac{\sin\left(\sqrt{\lambda}\tau\right)}{\sqrt{\lambda}}X\right) \\ &= 2\cos\left(\sqrt{\lambda}\tau\right)\end{aligned}$$

$$-2 \leq 2\cos\left(\sqrt{\lambda}\tau\right) \leq 2$$

$$\exp(\tau X) \in \mathit{Ell} \cup \{\mathbb{1}, -\mathbb{1}\}$$

1 parameter groups in $SL(2,\mathbb{R})$: overview

$X \in \mathfrak{sl}(2,\mathbb{R})$ can be expressed in the basis of Pauli matrices

$$X = X(t,s,w) \equiv t\,\mathrm{i}\sigma_2 + s\sigma_1 + w\sigma_3$$

$$X^2(s,t,w) = \frac{-t^2+s^2+w^2}{2}\mathbb{1}$$

$$\lambda = -t^2+s^2+w^2$$

$$\exp(\tau X(t,s,w)) = \begin{cases} \mathbb{1} + \tau X(t,s,w), & \lambda = 0, \\ \cosh(\tau\sqrt{\lambda})\mathbb{1} + \frac{\sinh(\tau\sqrt{\lambda})}{\sqrt{\lambda}} X(t,s,w), & \lambda > 0, \\ \cos(\tau\sqrt{-\lambda})\mathbb{1} + \frac{\sin(\tau\sqrt{-\lambda})}{\sqrt{-\lambda}} X(t,s,w), & \lambda < 0. \end{cases}$$

1 parameter groups in $\mathrm{SL}(2,\mathbb{R})$

$(t, s, w) = (0, 1, 0)$, $X(0, 1, 0) = \sigma_1 \to \lambda = 1$

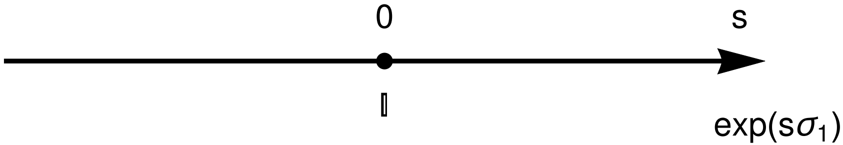
$$\exp(\tau X(0, 1, 0)) = \cosh\tau\mathbb{1} + \sinh\tau\sigma_1$$

examples $x \in \mathbb{R}_+$ and $\tau = \ln x$,

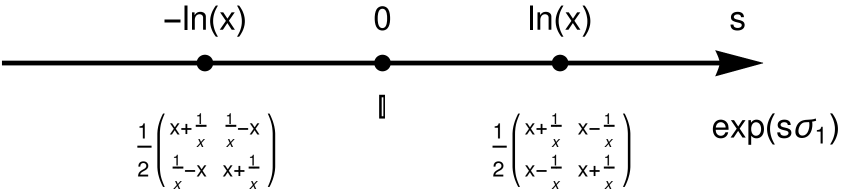
$$\exp(\ln x X(0, 1, 0)) = \frac{1}{2}\begin{bmatrix} x + \frac{1}{x} & x - \frac{1}{x} \\ x - \frac{1}{x} & x + \frac{1}{x} \end{bmatrix}.$$

The cases $(0, \alpha, \beta)$, are very similar.

1 parameter groups in $SL(2, \mathbb{R})$: (0,1,0) direction



1 parameter groups in $SL(2,\mathbb{R})$: (0,1,0) direction



1 parameter groups in $\mathrm{SL}(2,\mathbb{R})$

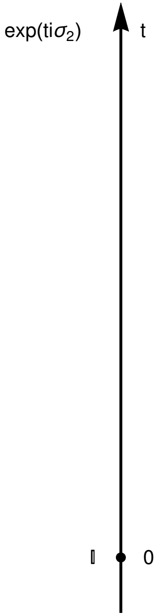
$(t,s,w) = (1,0,0)$, $X(1,0,0) = \mathrm{i}\sigma_2 \to \lambda = -1$

$$\exp(\tau X(1,0,0)) = \cos\tau \mathbb{1} + \sin\tau \mathrm{i}\sigma_2$$

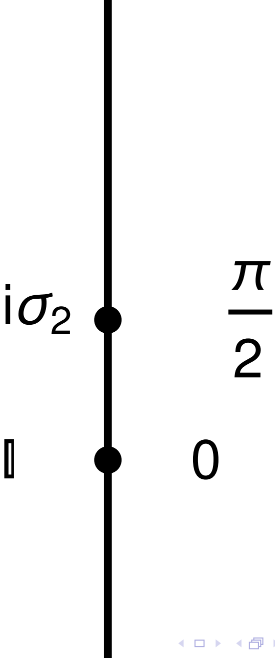
Periodic function, period $T = 2\pi$, depends on α in $X(\alpha,0,0)$.
examples:

$$\tau = \frac{\pi}{4} \to \frac{1}{\sqrt{2}}\mathbb{1} + \frac{1}{\sqrt{2}}\mathrm{i}\sigma_2$$
$$\tau = \frac{\pi}{2} \to \mathrm{i}\sigma_2$$
$$\tau = \pi \to -\mathbb{1}$$
$$k \in \mathbb{Z}$$
$$\tau = k\pi \to (-1)^k \mathbb{1}$$

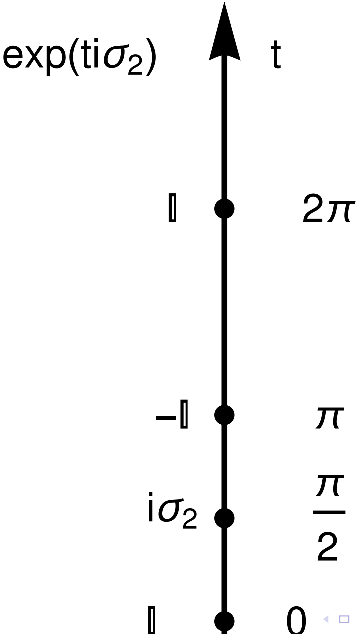
1 parameter groups in $SL(2,\mathbb{R})$: (1,0,0) direction



1 parameter groups in $SL(2, \mathbb{R})$: (1,0,0) direction



1 parameter groups in $SL(2, \mathbb{R})$: (1,0,0) direction



1 parameter groups in $SL(2,\mathbb{R})$

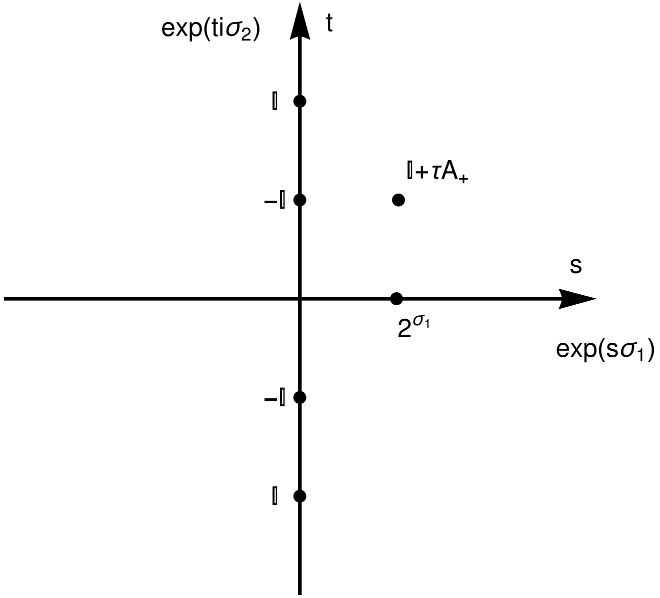
$(t, s, w) = (\alpha, \beta, 0)$,
$X(\alpha, \beta, 0) = \alpha i\sigma_2 + \beta\sigma_1$

$$\lambda = -\alpha^2 + \beta^2$$

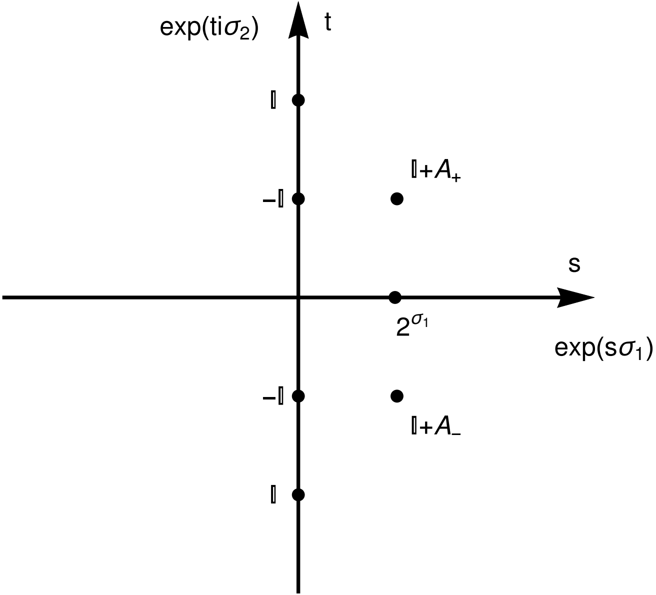
example

$$\exp\left(\tau X(\frac{1}{2}, \frac{1}{2}, 0)\right) = \mathbb{1} + \tau\frac{1}{2}\left(i\sigma_2 + \sigma_1\right) = \mathbb{1} + \tau A_+$$

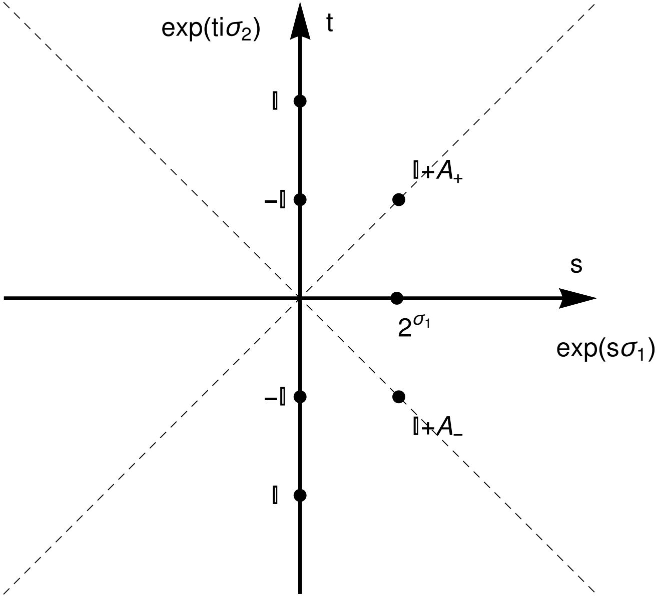
1 parameter groups in $SL(2,\mathbb{R})$



1 parameter groups in $SL(2, \mathbb{R})$



1 parameter groups in $SL(2,\mathbb{R})$



1 parameter groups in $\mathrm{SL}(2, \mathbb{R})$

$$(t, s, w) = (\sin\theta, \cos\theta, 0),$$
$$X(\sin\theta, \cos\theta, 0) = \sin\theta i\sigma_2 + \cos\theta\sigma_1$$
$\lambda = -\sin^2\theta + \cos^2\theta \quad \text{if} \quad \theta \in]\frac{\pi}{4}, \frac{3\pi}{4}[$ then $\lambda < 0$ and

$$\exp(\tau X(\sin\theta, \cos\theta, 0)) = \cos(\tau\sqrt{-\lambda})\mathbb{1} + \frac{\sin(\tau\sqrt{-\lambda})}{\sqrt{-\lambda}} X(\sin\theta, \cos\theta, 0)$$

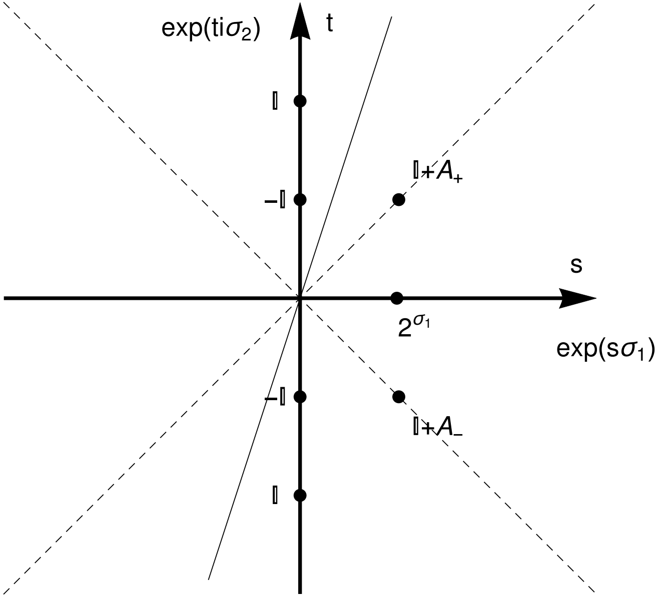
is also a periodic function.

Example

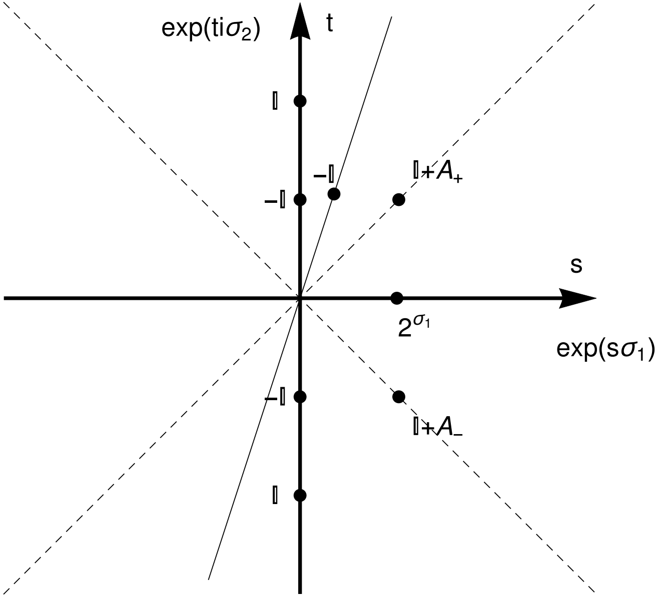
$\tau = \frac{\pi k}{\sqrt{-\lambda}}, k \in \mathbb{Z}$

$$\exp\left(\frac{\pi k}{\sqrt{-\lambda}} X(\alpha, \beta, 0)\right) = (-1)^k \mathbb{1}$$

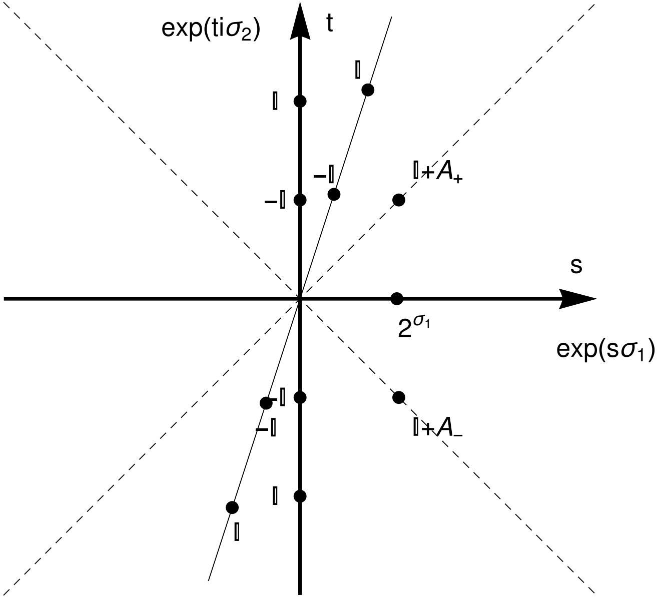
1 parameter groups in $SL(2,\mathbb{R})$: periodic EII



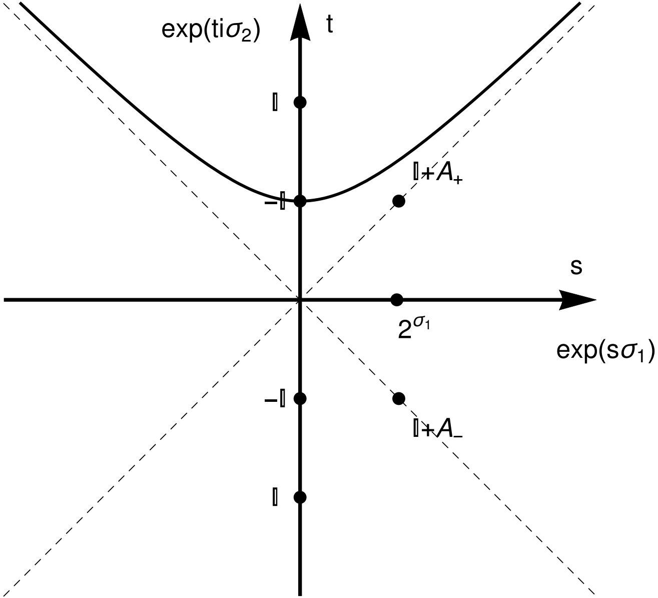
1 parameter groups in $SL(2,\mathbb{R})$: periodic Ell



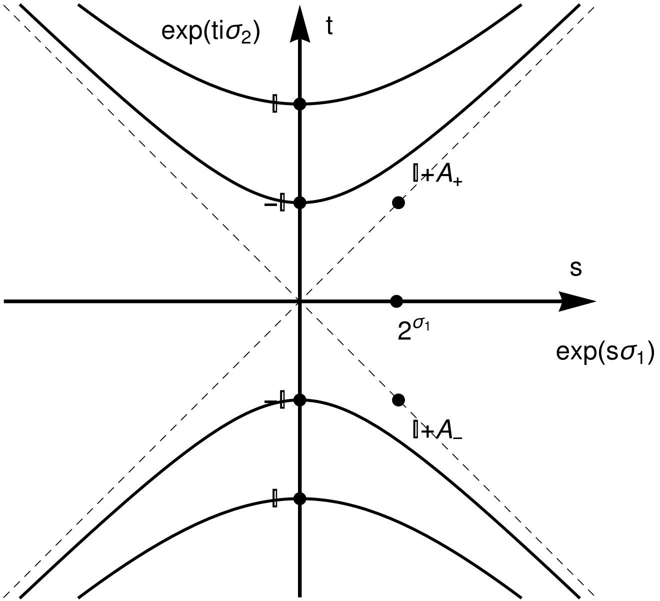
1 parameter groups in $SL(2,\mathbb{R})$: periodic EII



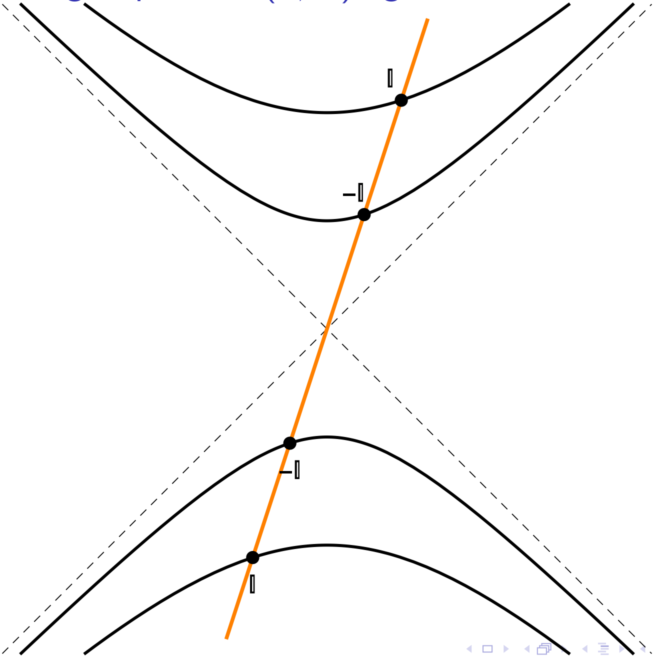
1 parameter groups in $SL(2,\mathbb{R})$: periodic Ell



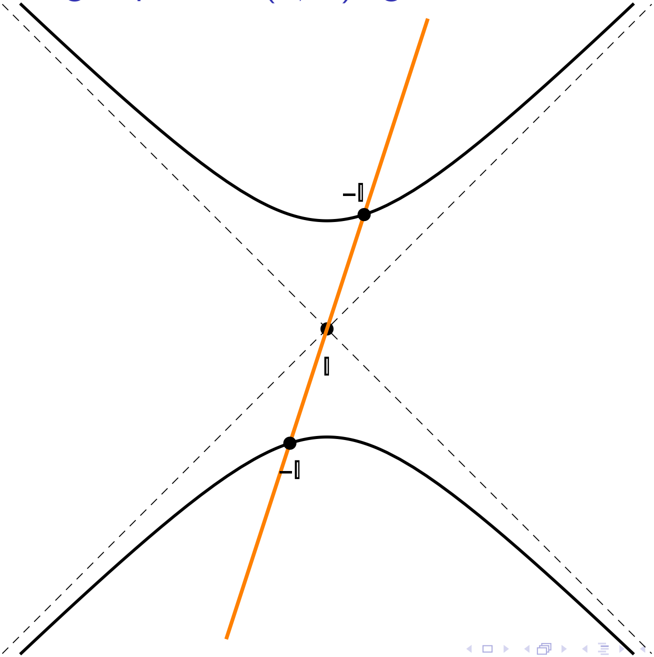
1 parameter groups in $SL(2,\mathbb{R})$: periodic EII



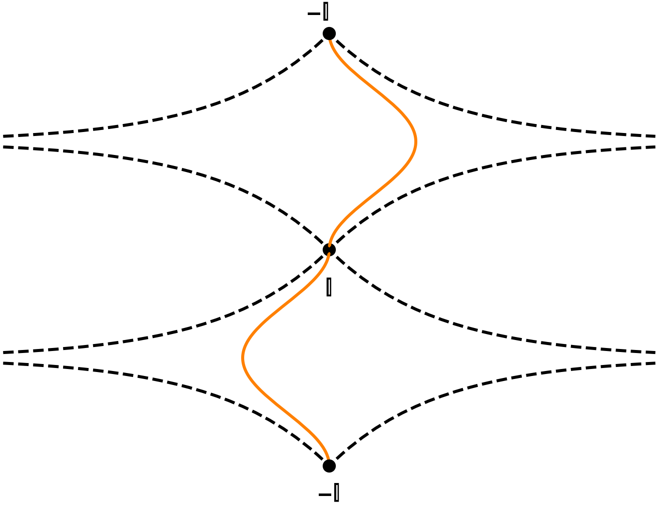
1 parameter groups in $SL(2,\mathbb{R})$: general view



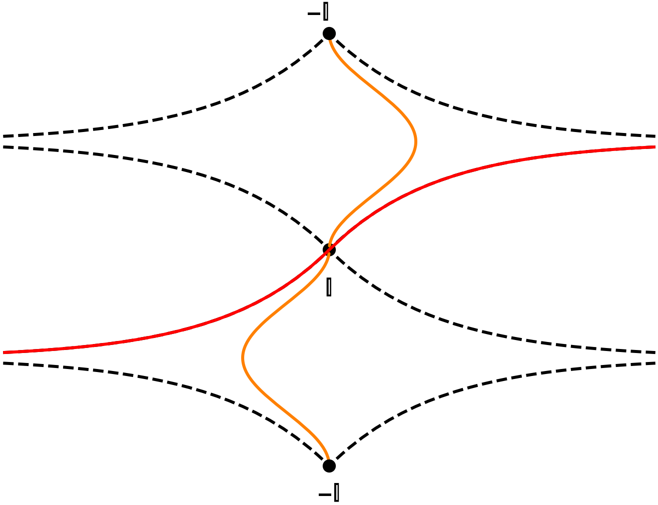
1 parameter groups in $SL(2, \mathbb{R})$: general view



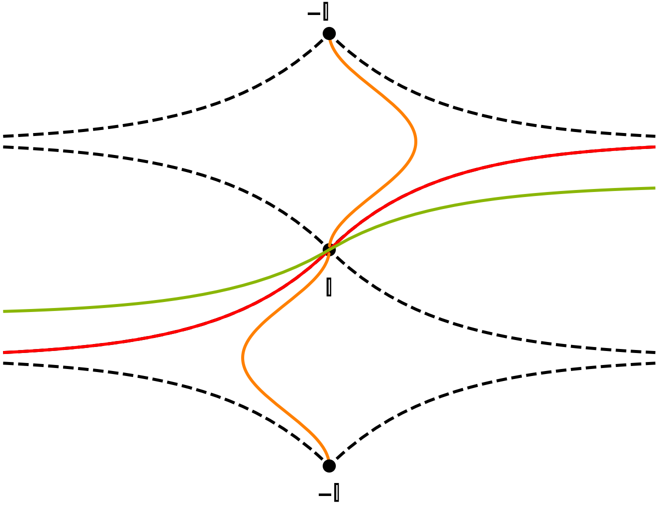
1 parameter groups in $SL(2,\mathbb{R})$: general view



1 parameter groups in $SL(2, \mathbb{R})$: general view



1 parameter groups in $SL(2, \mathbb{R})$: general view



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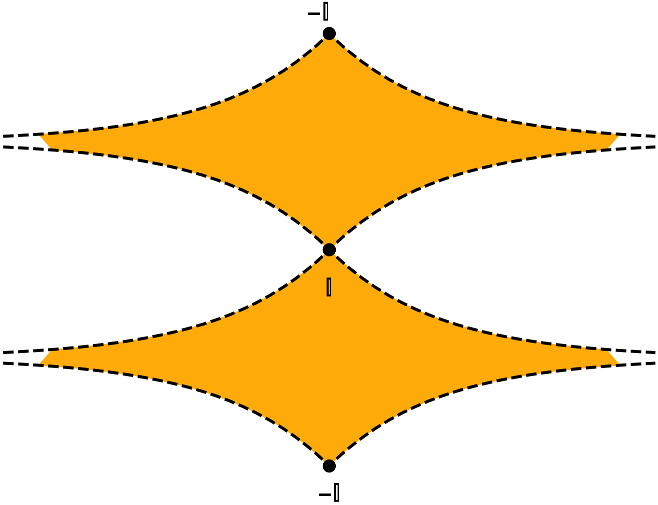

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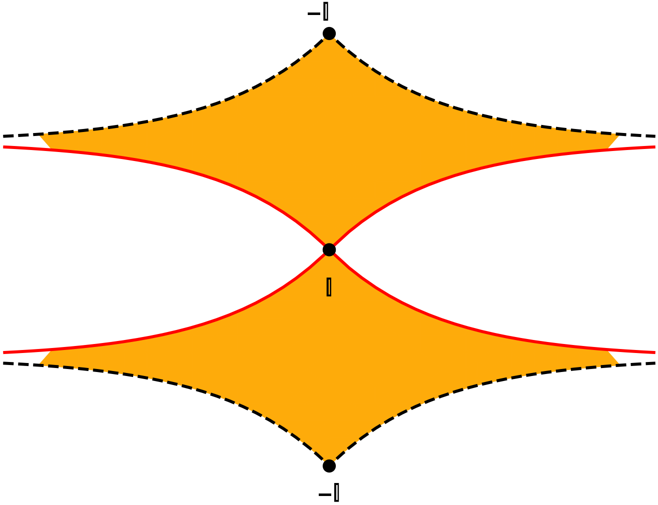
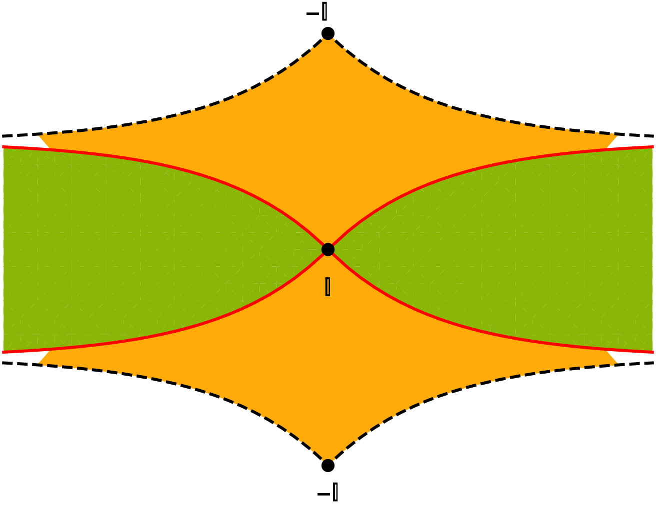
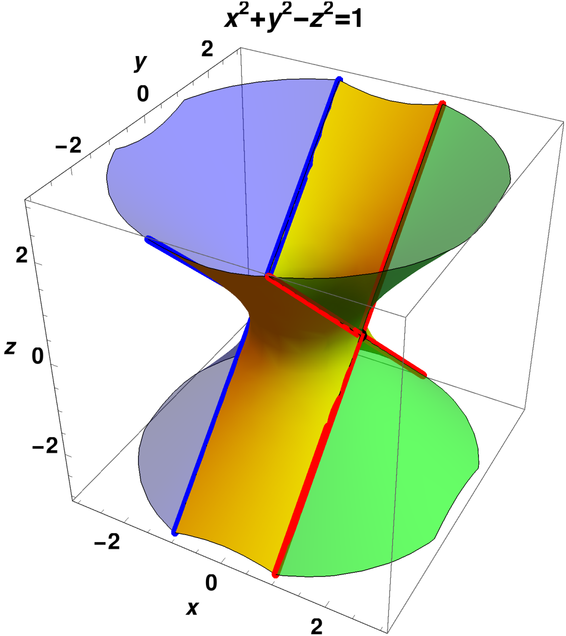


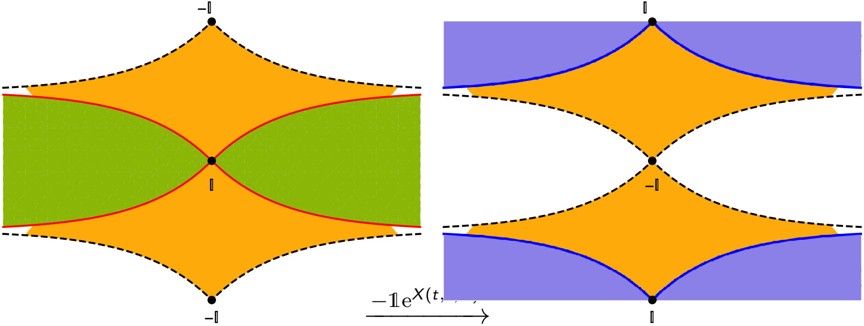
1 parameter groups in $SL(2, \mathbb{R})$



2-hyperbola elements: Ell, Par^+, Par^-, Hyp^+, Hyp^-



How to reach $-\mathbb{1}$ neighbourhood



The KAN decomposition of elements of the $\mathrm{SL}(2,\mathbb{R})$ group

The $\mathrm{SL}(2,\mathbb{R})$ group contains $SO(2)$ group which consists of matrices of the form

$$\mathrm{u}_\varphi \equiv \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix}.$$

The group $AN(2,\mathbb{R})$ of lower triangular matrices with determinant equal to one and positive entries on the diagonal have elements of the following form

$$t = \begin{bmatrix} a & 0 \\ n & \frac{1}{a} \end{bmatrix}, \quad a > 0.$$

The map

$$SO(2) \times AN(2,\mathbb{R}) \ni (u_\varphi, t) \mapsto tu \in SL(2,R)$$

is a continuous bijection.

The KAN decomposition of elements of the $\mathrm{SL}(2, \mathbb{R})$ group

If $g \in \mathrm{SL}(2, \mathbb{R})$, then

$$g = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} = \begin{bmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{bmatrix} \begin{bmatrix} a & 0 \\ n & \frac{1}{a} \end{bmatrix},$$

where $a > 0, \varphi \in]-\pi, \pi]$ and

$$a = \frac{1}{\sqrt{g_{12}^2 + g_{22}^2}}$$

$$\cos\varphi = \frac{g_{22}}{\sqrt{g_{12}^2 + g_{22}^2}} \qquad \sin\varphi = \frac{g_{12}}{\sqrt{g_{12}^2 + g_{22}^2}}$$

$$n = \frac{g_{11}g_{12} + g_{22}g_{21}}{\sqrt{g_{12}^2 + g_{22}^2}}$$

The KAN decomposition of elements of the $\mathrm{SL}(2,\mathbb{R})$ group

The KAN decomposition gives a map

$$\mathrm{SL}(2,\mathbb{R}) \ni g \mapsto \varphi(g) \in]-\pi,\pi].$$

example

$$\varphi\left(\mathrm{u}_{\varphi_1} t\right) = \varphi_1$$

This shows that $\mathrm{SL}(2,\mathbb{R})$ group is homotopic to a circle $]-\pi,\pi]$.

The KAN decomposition of elements of the $SL(2,\mathbb{R})$ group

Classification of $SL(2,\mathbb{R})$ elements with the KAN decomposition

$$Y^+ = \{g : \varphi(g) \in]0,\pi[\}$$

$$Y^- = \{g : \varphi(g) \in]-\pi,0[\} = -Y^+$$

$$AN^+ = \{g : \varphi(g) = 0\}$$

$$AN^- = \{g : \varphi(g) = \pi\}$$

Lemma

$AN \cdot AN = AN$
$AN \cdot Y^+ = Y^+$
$Y^+ \cdot AN = Y^+$
$Y^+ \cdot Y^+ \subset Y^+ \cup AN^- \cup Y^-$

$\widetilde{\mathsf{SL}}(2, \mathbb{R})$

$\widetilde{\mathrm{SL}}(2,\mathbb{R})$ - the universal covering group of $\mathrm{SL}(2,\mathbb{R})$

$$\widetilde{\mathrm{SL}}(2,\mathbb{R}) \equiv \mathbb{R} \times AN(2,\mathbb{R}),$$

Let $(\phi_i, t_i) \in \widetilde{\mathrm{SL}}(2,\mathbb{R})$ then

$$(\phi_1, t_1)(\phi_2, t_2) = (\phi, t)$$

where

$$\mathrm{u}_\phi t \equiv \mathrm{u}_{\phi_1} t_1 \mathrm{u}_{\phi_2} t_2,$$

and ϕ, with the use of **Lemma**, can be determined as follows

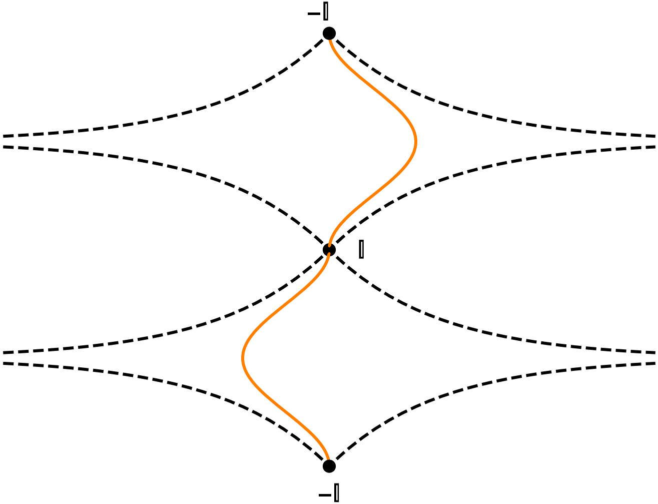
$$\phi_1 = n_1\pi, \quad \phi_2 = n_2\pi \implies \phi = (n_1+n_2)\pi$$

$$\phi_1 \in]n_1\pi, (n_1+1)\pi[\quad \phi_2 = n_2\pi \implies \phi \in](n_1+n_2)\pi, (n_1+n_2+1)\pi[$$

...

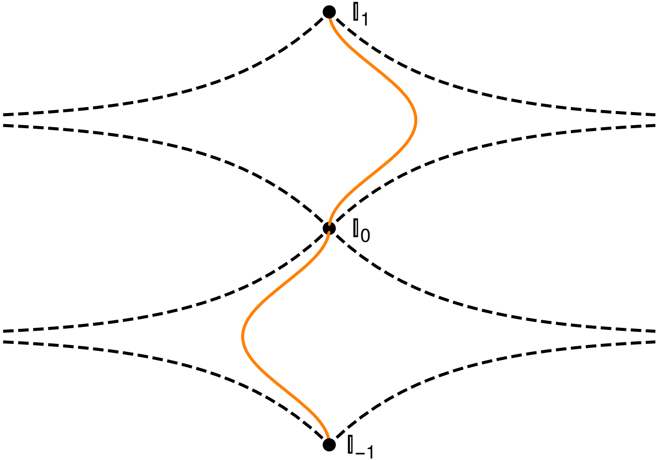
n_i will be used as subscripts for $\mathbb{1}$s from $\widetilde{\mathrm{SL}}(2,\mathbb{R})$.

1 parameter groups in $\widetilde{\mathrm{SL}}(2,\mathbb{R})$: general view



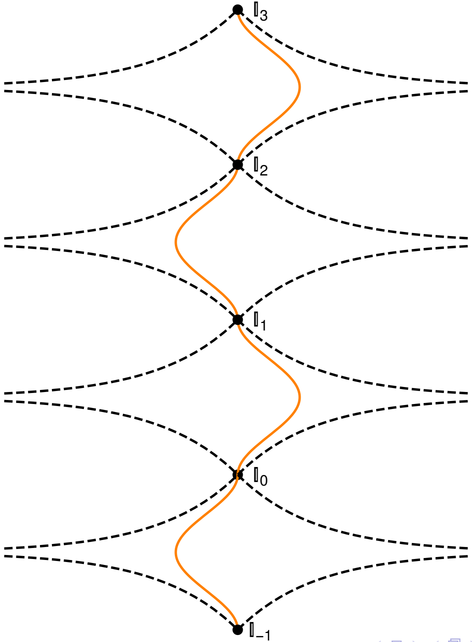
$\mathrm{SL}(2,\mathbb{R})$ situation

1 parameter groups in $\widetilde{SL}(2,\mathbb{R})$: general view

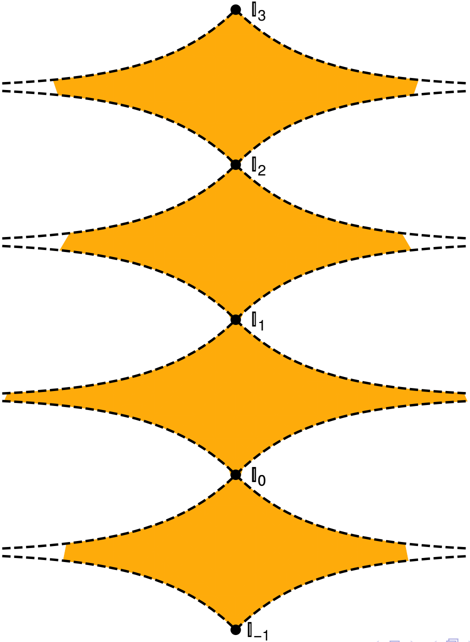


$\widetilde{SL}(2,\mathbb{R})$ situation

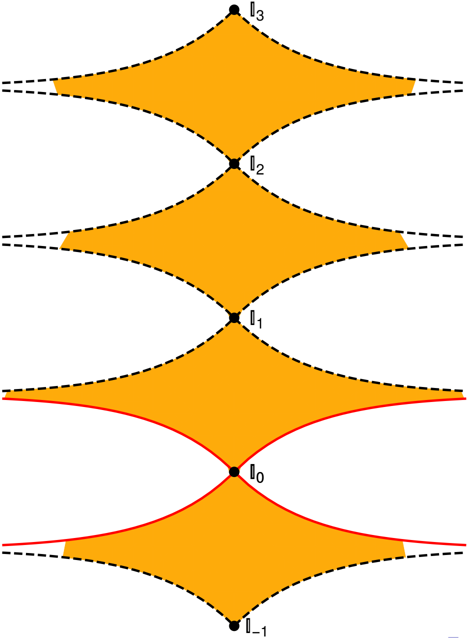
1 parameter groups in $\widetilde{SL}(2,\mathbb{R})$: general view



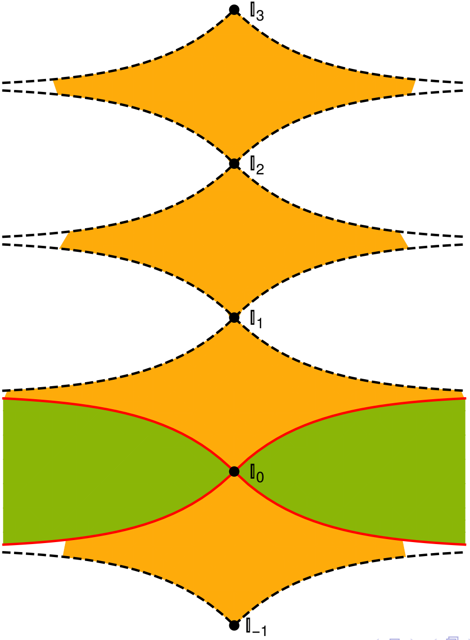
1 parameter groups in $\widetilde{SL}(2,\mathbb{R})$: general view



1 parameter groups in $\widetilde{\mathrm{SL}}(2, \mathbb{R})$: general view



1 parameter groups in $\widetilde{SL}(2,\mathbb{R})$: general view



1 parameter groups in $\widetilde{SL}(2,\mathbb{R})$: general view