Representations of $SL(2, \mathbb{R})$

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Plan of the presentation

1. Lie algebra $sl(2,\mathbb{C})$ and $sl(2,\mathbb{R})$.
2. Complex representations of $sl(2,\mathbb{C})$.
3. Representations of $sl(2,\mathbb{R})$.
4. Lie group $SL(2,\mathbb{R})$.
5. Unitary representations of $SL(2,\mathbb{R})$.

First we will start from the complex representations of $sl(2,\mathbb{C})$, because it is a complexification of $sl(2,\mathbb{R})$. Also we first want to outline the theory on the level of Lie algebras in order to obtain the Lie group representations by the exponential map.

Lie algebra $sl(2,\mathbb{C})$

It consists of 2 by 2 traceless matrices equipped with the scalar product

$$\langle X|Y\rangle = \mathrm{Tr}(XY), \qquad X, Y \in sl(2,\mathbb{C}). \tag{1}$$

We have the orthogonal basis of $sl(2,\mathbb{C})$ consisting of Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2}$$

and the commutation relation

$$\left[\frac{\mathrm{i}\sigma_i}{2}, \frac{\mathrm{i}\sigma_j}{2}\right] = -\epsilon_{ijk}\frac{\mathrm{i}\sigma_k}{2}. \tag{3}$$

Lie algebra $sl(2, \mathbb{C})$

It is convenient to introduce a triplet of operators A_-, A_+, N, which we will call **the standard triplet**, satisfying commutation relations

$$[N, A_\pm] = \pm A_\pm, \quad [A_+, A_-] = 2N. \tag{4}$$

Example of such a triplet:

$$A_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad N = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}. \tag{5}$$

It can be constructed as another basis of $sl(2, \mathbb{C})$ by transformation

$$A_\pm = \frac{1}{2}(\sigma_1 \pm \mathrm{i}\sigma_2), \quad N = \frac{1}{2}\sigma_3 \tag{6}$$

Representations of $sl(2,\mathbb{C})$

Let us introduce the Casimir operator

$$C = \frac{1}{4}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2), \tag{7}$$

in terms of the triplet A_-, A_+, N

$$C = \frac{1}{2}(A_+A_- + A_-A_+) + N^2. \tag{8}$$

The Casimir operator commutes with the representation.
Let us define a specific representation

$$A_-^l = \partial_w + lw^{-1}, \quad A_+^l = -w^2\partial_w + lw, \quad N^l = w\partial_w, \tag{9}$$

where $l \in \mathbb{C}$. It satisfies the $sl(2,\mathbb{C})$ commutation relations. The Casimir operator reads

$$C^l = l(l+1). \tag{10}$$

Representations of $sl(2,\mathbb{C})$

Let us define the space of monomials

$$\mathcal{W}^\eta := \{w^k \mid k \in \mathbb{Z} + \eta\}. \tag{11}$$

We have the representation

$$sl(2,\mathbb{C}) \ni X \mapsto X^{l,\eta} \in L(\mathcal{W}^\eta), \tag{12}$$

parametrized by $l, \eta \in \mathbb{C}$. The action of $X^{l,\eta}$:

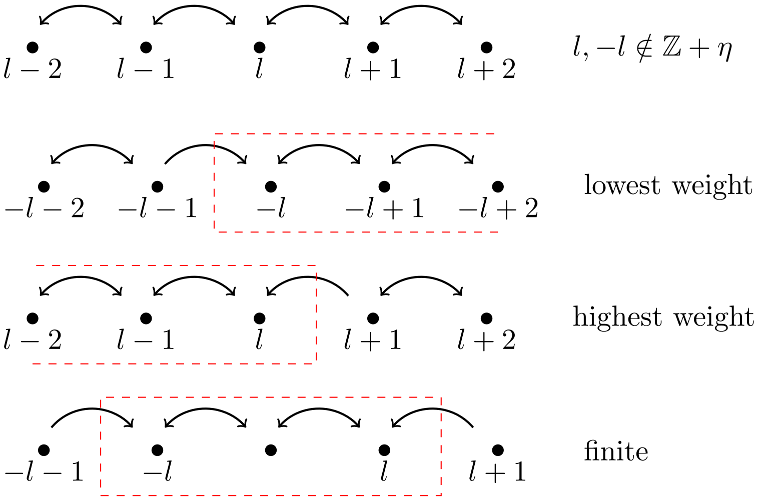
$$\begin{aligned} A_-^l w^k &= (k+l)w^{k-1}, \\ A_+^l w^k &= (-k+l)w^{k+1}, \\ N^l w^k &= kw^k. \end{aligned} \tag{13}$$

We have $X^{l,\eta} = X^{l,\eta+n}$ for $n \in \mathbb{Z}$.

Representations of $sl(2, \mathbb{C})$

1. The representation $X^{l,\eta}$ is irreducible if and only if $l, -l \notin \mathbb{Z} + \eta$.
2. Consider $X^{l,-l}$. Then it has an invariant subspace $\{w^k \,|\, k = -l, -l+1, \dots\}$.
   Let us define $X^{l,\mathrm{lw}}$ as $X^{l,-l}$ restricted to this subspace. It is irreducible if $l \neq 0, \frac{1}{2}, 1, \dots$
3. Consider $X^{l,l}$. Then it has an invariant subspace $\{w^k \,|\, k = \dots, l-1, l\}$.
   Let us define $X^{l,\mathrm{hw}}$ as $X^{l,l}$ restricted to this subspace. It is irreducible if $l \neq 0, \frac{1}{2}, 1, \dots$
4. If $l = 0, \frac{1}{2}, 1, \dots$, then $X^{l,-l} = X^{l,l}$ and we have an invariant subspace $\{w^k \,|\, k = -l, \dots, l\}$.
   Let us define X^{fin} as $X^{l,-l} = X^{l,l}$ restricted to this subspace.

Representations of $sl(2, \mathbb{C})$



Equivalence of representations $sl(2,\mathbb{C})$

For $l, -l \notin \mathbb{Z}+\eta$ let us introduce diagonal linear map $W_l : X^{l,\eta} \to X^{l,\eta}$.

$$W_l\, w^k := \frac{\Gamma(-l+k)}{\Gamma(l+1+k)} w^k. \tag{14}$$

By direct calculation we can check that

$$W_l^{-1} X^l\, W_l = X^{-1-l}, \qquad X \in sl(2,\mathbb{C}). \tag{15}$$

This shows the equivalence of representations $X^{l,\eta}$ and $X^{-l-1,\eta}$ for $l, -l \notin \mathbb{Z}+\eta$. This condition is given by the behavior of the Γ function. Consistently the Casimir operator is invariant with respect to transformation $l \mapsto -1-l$:

$$C^l = l(l+1) = C^{-1-l}. \tag{16}$$

Alternative forms of representation

Let us compute $X^{l,-} := w^l X^l w^{-l}$. We obtain

$$\begin{aligned} A_-^{l,-} &= \partial_w, \\ N^{l,-} &= w\partial_w - l, \\ A_+^{l,-} &= -w^2\partial_w + 2lw. \end{aligned} \tag{17}$$

With this representation in the case of $X^{l,\mathrm{lw}}$, the representation space is $\{w^k | k = 0, 1, \ldots\}$. For the representation $X^{l,\mathrm{fin}}$, the representation space is $\{w^k | k = 0, 1, \ldots, 2l\}$.

Alternative forms of representation

Another form of representations is $X^{l,+} := w^{-l} X^l w^l$. We obtain

$$
\begin{aligned}
A_-^{l,+} &= \partial_w + 2l, \\
N^{l,+} &= w\partial_w + l, \\
A_+^{l,+} &= -w^2 \partial_w.
\end{aligned}
\tag{18}
$$

With this representation in the case of $X^{l,\mathrm{lw}}$, the representation space is $\{w^k | k = 0, -1, \ldots\}$. For the representation $X^{l,\mathrm{fin}}$, the representation space is $\{w^k | k = 0, -1, \ldots, -2l\}$.

$SL(2,\mathbb{R})$ group

It consists of 2 by 2 matrices over $\mathbb{R}$ whose determinant is equal to 1. Its center is $\{\mathbb{1}, -\mathbb{1}\} \cong \mathbb{Z}_2$. We have the group $PSL(2,\mathbb{R}) := SL(2,\mathbb{R})/\mathbb{Z}_2$ with the neutral element as a center.
This group is isomorphic to homographies preserving $\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$.
The important decomposition (valid also for $SL(2,\mathbb{C})$)

$$\begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix}, \tag{19}$$

where

$$t = h_{12}, \quad s_1 = \frac{h_{11} - 1}{h_{12}}, \quad s_2 = \frac{h_{22} - 1}{h_{12}}. \tag{20}$$

If $h_{12} = 0$ the decomposition is invalid. We will treat this decomposition as a tool to heuristically construct group representations.

Representations of $SL(2,\mathbb{R})$

Alternative form of decomposition

$$\begin{pmatrix} 1 & 0 \\ s_2 & 1 \end{pmatrix} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ s_1 & 1 \end{pmatrix} = e^{s_2 A_-} \, e^{tA_+} \, e^{s_1 A_-}. \tag{21}$$

To find representations of the group we need to exponentiate A^l_+, N^l, A^l_-, obtaining

$$\begin{aligned} e^{tA^l_-} f(w) &= (1+tw^{-1})^l f(w+t), \\ e^{tN^l} f(w) &= f(e^t w), \\ e^{tA^l_+} f(w) &= (1+tw)^l f\left(\frac{w}{1+wt}\right). \end{aligned} \tag{22}$$

Thus we have the representation by homographies

$$h^l f(w) = (h_{11}+h_{12}w^{-1})^l (h_{21}w+h_{22})^l f\left(\frac{h_{11}w+h_{12}}{h_{12}w+h_{22}}\right) \tag{23}$$

Representations of $SL(2,\mathbb{R})$

By using operators $A_+^{l,-}, N^{l,-}, A_-^{l,-}$ we have

$$\begin{aligned} e^{tA_-^{l,-}} f(w) &= f(w+t), \\ e^{tN^{l,-}} f(w) &= e^{-lt} f(e^t w), \\ e^{tA_+^{l,-}} f(w) &= (1+tw)^l f\left(\frac{w}{1+wt}\right). \end{aligned} \tag{24}$$

The representation

$$h^{l,-} f(w) = (h_{21}w + h_{22})^{2l} f\left(\frac{h_{11}w + h_{12}}{h_{12}w + h_{22}}\right) \tag{25}$$

These formulas can suffer from possible multivaluedness of power functions. This is solved by going to the representations of the universal cover $\widetilde{SL}(2,\mathbb{R})$ or by other tricks showed in the following slides.

Lie algebra $sl(2,\mathbb{R})$

Lie algebra $sl(2,\mathbb{R})$ consists of matrices of the form

$$X = \begin{pmatrix} X_{11} & X_{12} \\ X_{21} & -X_{11} \end{pmatrix}. \tag{26}$$

It has a scalar product

$$\mathrm{Tr}(XY) = 2X_{11}Y_{11} + X_{12}Y_{21} + X_{21}Y_{12}. \tag{27}$$

Let us consider representation $X^{\bullet}$ of $sl(2,\mathbb{R})$. It can be extended by complex linearity to representation of $sl(2,\mathbb{C})$. Therefore representations of $sl(2,\mathbb{R})$ are the same as complex representations of $sl(2,\mathbb{C})$.

Representations of $sl(2,\mathbb{R})$

The main parameter describing representations of $sl(2,\mathbb{C})$ was l and η. This parametrization is standard for the theory of $su(2)$. In the case of $sl(2,\mathbb{R})$ the popular in the literature parametrization is given by parameter m, defined by

$$m = -2l - 1, \qquad l = -\frac{m+1}{2}. \tag{28}$$

The representation X^l with l expressed by m, will be denoted X^m. In our new notation we have

$$\begin{aligned} A_-^m w^k &= \Big(k - \frac{m+1}{2}\Big) w^{k-1}, \\ N^m w^k &= k w^k, \\ A_+^m w^k &= \Big(-k - \frac{m+1}{2}\Big) w^{k+1}. \end{aligned} \tag{29}$$

The Casimir operator is given by $C^m = \frac{m^2}{4} - \frac{1}{4}$.

Representations of $sl(2,\mathbb{R})$

1. The representation preserving $\mathcal{W}^\eta$ is dented $X^{m,\eta}$. It is irreducible if and only if $m, -m \notin 2\mathbb{Z}+1+2\eta$.
2. Consider $X^{m,\frac{m+1}{2}}$. Then it has an invariant subspace $\{w^k \mid k = \frac{m+1}{2}, \frac{m+3}{2}, \ldots\}$.
   Let us define $X^{m,\mathrm{lw}}$ as $X^{m,\frac{m+1}{2}}$ restricted to this subspace. It is irreducible if $m \neq -1, -2, -3, \ldots$.
3. Consider $X^{m,-\frac{m+1}{2}}$. Then it has an invariant subspace $\{w^k \mid k = \ldots, -\frac{m+3}{2}, -\frac{m+1}{2}\}$.
   Let us define $X^{m,\mathrm{hw}}$ as $X^{m,-\frac{m+1}{2}}$ restricted to this subspace. It is irreducible if $m \neq -1, -2, -3, \ldots$.
4. If $m = -1, -2, -3, \ldots$, then $X^{m,\frac{m+1}{2}} = X^{m,-\frac{m+1}{2}}$ and we have invariant subspace $\{w^k \mid k = \frac{m+1}{2}, \ldots, -\frac{m+1}{2}\}$.

$X^{m,\mathrm{lw}}$ and $X^{m,\mathrm{hw}}$ give representations of $PSL(2,\mathbb{R})$ for $m \in 2\mathbb{Z}+1$ and of $SL(2,\mathbb{R})$ for $m \in \mathbb{Z}$.

Unitarity of representations of $SL(2,\mathbb{R})$

Let $\eta \in \mathbb{R}$.

1. Suppose that we equip the space $\mathcal{W}^\eta$ with the sesquilinear scalar product $(f|g)$ where $f, g \in \mathcal{W}$, defined on the canonical basis by

$$(w^k|w^{k'}) = \delta_{k,k'}. \tag{30}$$

   Then

$$(X^{-\overline{m},\eta} f|g) + (f|X^{m,\eta} g) = 0. \tag{31}$$

2. Suppose we have another sesquilinear product, $(f|g)_m$, where $f, g \in \mathcal{W}$, defined on the canonical basis by

$$(w^k|w^{k'})_m = \delta_{k,k'} \frac{\Gamma(\frac{m}{2} + \frac{1}{2} + k)}{\Gamma(-\frac{m}{2} + \frac{1}{2} + k)}. \tag{32}$$

   Then

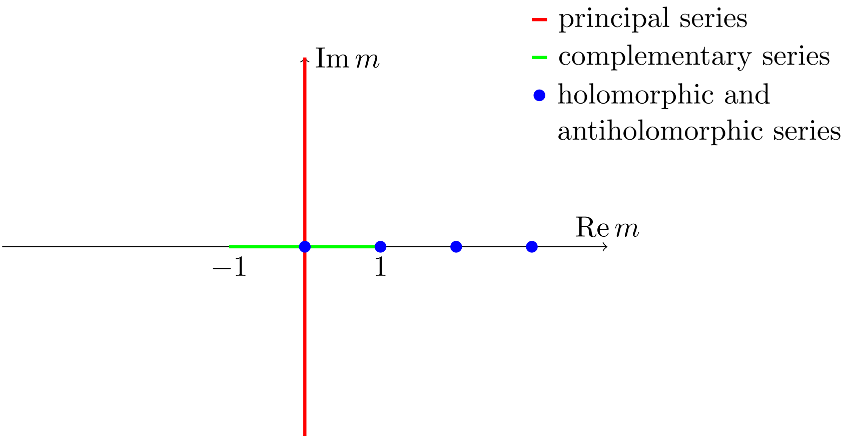
$$(X^{\overline{m},\eta} f|g)_m + (f|X^{m,\eta} g)_m = 0. \tag{33}$$

Unitary representations of $SL(2,\mathbb{R})$

We have positive scalar product in the following cases:

1. **The principal series**: $m = \mathrm{i}\mu \in \mathrm{i}\mathbb{R}$, $0 \le \eta < 1$. Then $X^{\mathrm{i}\mu,\eta}$ is unitary in the canonical scalar product $(w^k|w^k) = 1$.
2. **The complementary series**: $-1 < m < 1$, $-\frac{1-|m|}{2} < \eta < \frac{1-|m|}{2}$. Then $X^{m,\eta}$ is unitary in the scalar product
$$(w^k|w^k)_m = \frac{\Gamma(\frac{m}{2}+\frac{1}{2}+k)}{\Gamma(-\frac{m}{2}+\frac{1}{2}+k)}. \tag{34}$$
3. **The holomorphic or lowest weight series**: $m > -1$, $X^{m,\mathrm{lw}}$. It is unitary in the scalar product
$$(w^k|w^k)_m = \frac{k!}{\Gamma(k+m+1)}. \tag{35}$$
4. **The antiholomorphic or highest weight series**: $m > -1$, $X^{m,\mathrm{hw}}$.

Unitary representations of $SL(2, \mathbb{R})$



For general $m > -1$ the holomorphic and anthiholomorphic series gives the representations of $\widetilde{SL}(2, \mathbb{R})$.

Representations by homographies

The group $SL(2,\mathbb{R})$ acts on the $\mathbb{R}$. Let us integrate the flows generated by $A_-^{m,-}$ and $N^{m,-}$

$$\begin{aligned} e^{tA_-^{m,-}} f(x) &= f(x+t), \\ e^{tN^{m,-}} f(x) &= e^{\frac{m+1}{2}} f(e^t w). \end{aligned} \tag{36}$$

The flow generated by $A_+^{m,-}$ for $\eta = 0$ and $\eta = \frac{1}{2}$

$$\begin{aligned} e^{tA_+^{m,-}} f(x) &= |1+tx|^{-m-1} f\left(\frac{x}{1+xt}\right), \quad \eta = 0, \\ e^{tA_+^{m,-}} f(x) &= \operatorname{sgn}(1+tx)|1+tx|^{-m-1} f\left(\frac{x}{1+xt}\right), \quad \eta = \frac{1}{2}. \end{aligned} \tag{37}$$

Representations by homographies

This yields representation of $PSL(2,\mathbb{R})$ for $\eta = 0$ and of $SL(2,\mathbb{R})$ for $\eta = \frac{1}{2}$

$$h^{m,0}f(x) = |h_{12}x + h_{22}|^{-m-1} f\left(\frac{h_{11}x + h_{21}}{h_{12}x + h_{22}}\right),$$
$$h^{m,\frac{1}{2}}f(x) = \operatorname{sgn}(h_{12}x + h_{22})|h_{12}x + h_{22}|^{-m-1} f\left(\frac{h_{11}x + h_{21}}{h_{12}x + h_{22}}\right). \tag{38}$$

For arbitrary η we obtain representations of universal cover $\widetilde{SL}(2,\mathbb{R})$. The center of $\widetilde{SL}(2,\mathbb{R})$ is $\{\mathbb{1}_n \mid n \in \mathbb{Z}\}$. For $\tilde{h} \in \widetilde{SL}(2,\mathbb{R})$ corresponding to matrix $h \in SL(2,\mathbb{R})$ we can find n such that

$$\tilde{h} \in \mathrm{Ell}_{n-\frac{1}{2}} \cup \mathrm{Hyp}_n \cup \mathrm{Par}_n \cup \{\mathbb{1}_n\} \cup \mathrm{Ell}_{n+\frac{1}{2}}. \tag{39}$$

This means that $\tilde{h}$ can be accessed from $\mathbb{1}_n$ by one-parameter path.

Representations of $\widetilde{SL}(2,\mathbb{R})$

For $\tilde{h}$ belonging to n^{th} sector (as in previous slide) we have

$$\tilde{h}^{m,\eta} = e^{\mathrm{i}n2\pi\eta} \times$$

$$\begin{cases} |h_{22}+h_{12}x|^{-m-1} f\left(\frac{h_{11}x+h_{21}}{h_{12}x+h_{22}}\right), & (-1)^n(h_{22}+h_{12}x) > 0; \\ e^{-\mathrm{i}2\pi\eta}|h_{22}+h_{12}x|^{-m-1} f\left(\frac{h_{11}x+h_{21}}{h_{12}x+h_{22}}\right), & (-1)^n(h_{22}+h_{12}x) < 0, \\ & (-1)^n h_{12} < 0; \\ e^{\mathrm{i}2\pi\eta}|h_{22}+h_{12}x|^{-m-1} f\left(\frac{h_{11}x+h_{21}}{h_{12}x+h_{22}}\right), & (-1)^n(h_{22}+h_{12}x) < 0, \\ & (-1)^n h_{12} > 0; \end{cases}$$

Principal series representations

Consider spaces $L^2(\mathbb{R})$ and $L^2(\mathbb{S})$ and a unitary map $U : L^2(\mathbb{R}) \to L^2(\mathbb{S})$

$$Uf(u) = \frac{2^{\frac{1}{2}}}{|u+1|} f\Big(\mathrm{i}\frac{u-1}{u+1}\Big), \tag{40}$$

$$U^{-1}p(x) = \Big(\frac{2}{x^2+1}\Big)^{\frac{1}{2}} p\Big(\frac{\mathrm{i}+x}{\mathrm{i}-x}\Big). \tag{41}$$

Orthogonal basis on $L^2(\mathbb{S})$: $(u^k|u^{k'}) = \delta_{k,k'}2\pi$, $k, k' \in \mathbb{Z}+\eta$. Corresponding basis on $L^2(\mathbb{R})$

$$U^{-1}u^k = \Big(\frac{2}{x^2+1}\Big)^{\frac{1}{2}} \Big(\frac{\mathrm{i}+x}{\mathrm{i}-x}\Big)^k. \tag{42}$$

We have $(h^{-\overline{m},\overline{\eta}} f | h^{m,\eta} g) = (f|g)$. This gives the conditions for m and η for this series.

Complementary series representations

Let us introduce the scalar product on functions on $\mathbb{R}$

$$(f|g)_m = \frac{1}{2\Gamma(m)} \int\int \overline{f(x)} |x-y|^{m-1} g(y) \mathrm{d}x\, \mathrm{d}y, \quad 0 < m < 1;$$

$$(f|g)_0 = \int\int \overline{f(x)} g(x) \mathrm{d}x,$$

$$(f|g)_m =$$

$$-\frac{1}{2\Gamma(m)} \int\int \big(\overline{f(x)} - \overline{f(y)}\big) |x-y|^{m-1} \big(g(x) - g(y)\big) \mathrm{d}x\, \mathrm{d}y,$$

$$-1 < m < 0.$$

We have the equivalence of representations for m and $-m$.
By similar procedure as in the previous slide, we arrive at the basis in $(\cdot|\cdot)_m$

$$w^k = \Big(2\pi \cos\big(\frac{\pi}{2} m\big)\Big)^{-\frac{1}{2}} \Big(\frac{2}{x^2+1}\Big)^{\frac{1+m}{2}} \Big(\frac{\mathrm{i}+x}{\mathrm{i}-x}\Big)^k. \qquad (43)$$

Holomorphic series representations

Consider the upper complex half-plane:
$\mathbb{C}_+ := \{z = x + \mathrm{i}y \in \mathbb{C} : y \geq 0\}$, where $x = \operatorname{Re} z$ and $y = \operatorname{Im} z$.
For F, G analytic on $\mathbb{C}_+$, we define the scalar product

$$(F|G)_m = \frac{1}{\Gamma(m)} \int_{\mathbb{C}_+} y^{m-1} \overline{F(z)} G(z) \mathrm{d}^2 z, \qquad 0 < m;$$

$$(F|G)_0 = \int_{\mathbb{R}} \overline{F(x)} G(x) \mathrm{d}x,$$

$$(F|G)_m = \frac{1}{\Gamma(m)} \int_{\mathbb{C}_+} y^{m-1} \big(\overline{F(z)} G(z) - \overline{F(x)} G(x)\big) \mathrm{d}^2 z, \qquad -1 < m < 0.$$

Similarly as in preceding frames we obtain basis in $(\cdot|\cdot)_m$

$$w^k = \pi^{-\frac{1}{2}} 2^m \frac{(\mathrm{i} + z)^k}{(\mathrm{i} - z)^{k+m+1}}. \tag{44}$$

Antiholomorphic series representations are equivalent. They are obtained by taking antiholomorphic functions on $\mathbb{C}_+$.

Thank you for your attention !