Unitary representations of $\widetilde{SL}(2, \mathbb{R})$

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Plan

1. Schroedinger operators with inverse square potential
$$-\frac{\partial^2}{\partial x^2} + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2}$$
2. Universal cover of $SL(2,\mathbb{R})$
3. Integrating $sl(2,\mathbb{R})$ representations to $\widetilde{SL}(2,\mathbb{R})$ representations
4. Lowest and highest weight representations of $\widetilde{SL}(2,\mathbb{R})$ on $L^2[0,\infty[$
5. Principal and complementary representations of $\widetilde{SL}(2,\mathbb{R})$ on $L^2[0,\infty[\oplus L^2[0,\infty[$.

Consider a formal differential expression:

$$L_{m^2} = -\frac{\partial^2}{\partial x^2} + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2}.$$

When does it define

- a **self-adjoint** operator?
- a **closed** operator?

We have corresponding **minimal and maximal operators** on domains:

$$\mathrm{Dom}(L_{m^2}^{max}) = \left\{ f \in L^2[0,\infty[\,:\, L_{m^2} f \in L^2[0,\infty[\right\}$$
$$\mathrm{Dom}(L_{m^2}^{min}) = (C_c^\infty]0,\infty[)^{cl}.$$

Properties

- homogeneous of degree -2
- $L_{m^2}^{min} \subset L_{m^2}^{max}$
- $\left(L_{m^2}^{min}\right)^* = L_{\overline{m}^2}^{max} \implies L_{m^2}^{min}$ is **Hermitian** for $m^2 \in \mathbb{R}$.

Notice that formally

$$L_{m^2}\, x^{\frac{1}{2}+m} = 0$$

Suppose:

- $\operatorname{Re} m > -1$: $\overline{x^{\frac{1}{2}+m}}\, x^{\frac{1}{2}+m} = x^{1+2\operatorname{Re} m}$
- ξ - smooth cutoff function with compact support

then

$$x^{\frac{1}{2}+m}\xi \in \operatorname{Dom}(L_{m^2}^{max}).$$

Definition

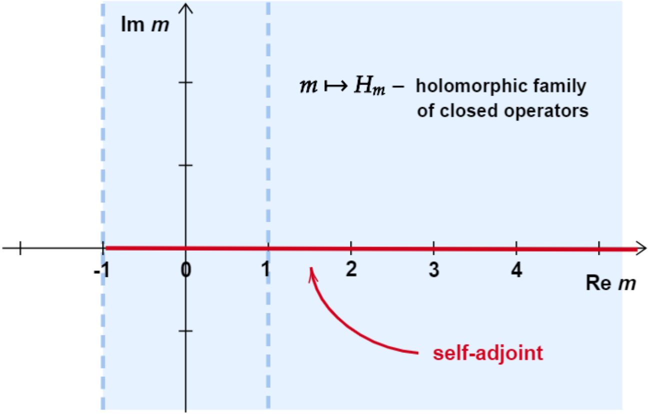
For $\operatorname{Re} m > -1$, define operator H_m as a restriction of $L_{m^2}^{max}$ to domain:

$$\operatorname{Dom}(L_{m^2}^{min}) \cup \mathbb{C}x^{\frac{1}{2}+m}\xi.$$

- $\operatorname{sp} H_m = [0, \infty[$
- $H_m^* = H_{\overline{m}} \implies H_m$ is **self-adjoint** for $m \in \mathbb{R}$
- $m \mapsto H_m$ is a 1-parameter holomorphic family of closed operators.

Schroedinger operator with $1/x^2$ potential

1. $1 < \operatorname{Re} m$:
$$H_m = L_{m^2}^{min} = L_{m^2}^{max}$$
2. $-1 < \operatorname{Re} m < 1$:
$$L_{m^2}^{min} \subsetneq H_m \subsetneq L_{m^2}^{max}$$
and the codimension of the domain is 1.



Schroedinger operator with $1/x^2$ potential

H_m generates a holomorphic 1-parameter semigroup:

$$e^{-\frac{t}{2}H_m}(x,y) = \sqrt{\frac{2}{\pi t}} I_m\Big(\frac{xy}{t}\Big) e^{-\frac{x^2+y^2}{2t}}, \qquad \operatorname{Re} t > 0$$

$$e^{\pm i\frac{t}{2}H_m}(x,y) = e^{\pm i\frac{\pi}{2}(m+1)} \sqrt{\frac{2}{\pi t}} \mathcal{I}_m\Big(\frac{xy}{t}\Big) e^{\mp i\frac{x^2+y^2}{2t}}, \qquad \pm \operatorname{Im} t \geqslant 0$$

Kernel of the resolvent:

$$\frac{1}{(H_m + k^2)}(x,y) = \frac{1}{k} \begin{cases} I_m(kx)K_m(ky) & 0 < x < y \\ I_m(ky)K_m(kx) & 0 < y < x \end{cases}, \qquad \operatorname{Re} k > 0$$

Universal cover of $SL(2,\mathbb{R})$

$SL(2,\mathbb{R})$ has subgroups:

$$SO(2) = \left\{ u_\phi := \begin{bmatrix} \cos\phi & \sin\phi \\ -\sin\phi & \cos\phi \end{bmatrix} \ : \ \phi \in]-\pi,\pi] \right\}$$

$$AN(2,\mathbb{R}) = \left\{ t := \begin{bmatrix} a & 0 \\ n & \frac{1}{a} \end{bmatrix} \ : \ n \in \mathbb{R}, \ a > 0 \right\}$$

KAN decomposition:

$$SL(2,\mathbb{R}) = SO(2) \times AN(2,\mathbb{R}).$$

Lets define a group:

$$\widetilde{SL}(2,\mathbb{R}) := \mathbb{R} \times AN(2,\mathbb{R})$$

and maps:

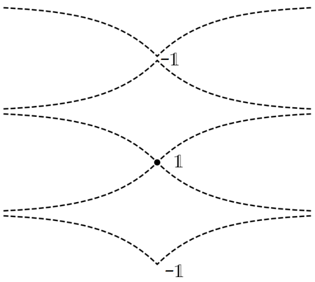
$$\tau : \widetilde{SL}(2,\mathbb{R}) \ni (\phi, t) \longmapsto u_\phi t \in SL(2,\mathbb{R})$$

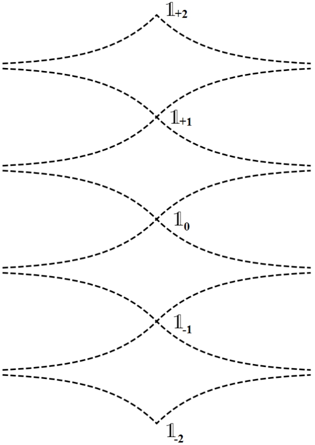
$$\phi : \widetilde{SL}(2,\mathbb{R}) \ni (\phi, t) \longmapsto \phi \in \mathbb{R}$$

$\widetilde{SL}(2,\mathbb{R})$ is the universal cover of $SL(2,\mathbb{R})$:

- $\tau : \widetilde{SL}(2,\mathbb{R}) \to SL(2,\mathbb{R})$ is a surjective homomorphism
- $\widetilde{SL}(2,\mathbb{R})$ is a simply connected Lie group.

Universal cover of $SL(2, \mathbb{R})$





Center of $\widetilde{SL}(2, \mathbb{R})$:

$$\mathbb{1}_n := (\pi n, (-1)^n \mathbb{1}) \in \mathbb{R} \times AN(2, \mathbb{R})$$

$\mathbb{1}_n$ cover $\mathbb{1}$ and $-\mathbb{1}$:

$$\widetilde{SL}(2, \mathbb{R}) \ni \mathbb{1}_n \stackrel{\tau}{\longmapsto} (-1)^n \mathbb{1} \in SL(2, \mathbb{R}).$$

G - Lie group with Lie algebra $\mathfrak{g}$

$\widetilde{G}$ - universal cover of G

Exponential maps:

$$\exp : \mathfrak{g} \ni X \longmapsto e^X \in G$$
$$\widetilde{\exp} : \mathfrak{g} \ni X \longmapsto \widetilde{\exp} X \in \widetilde{G}$$

Theorem

Let

$$\pi : \mathfrak{g} \longrightarrow gl(V)$$

be a representation of the Lie algebra g. If G is ***compact****, there exists exactly one representation of* $\widetilde{G}$

$$\widetilde{\pi} : \widetilde{G} \longrightarrow GL(V)$$

such that

$$\widetilde{\pi}(\widetilde{\exp} X) = e^{\pi(X)}.$$

Representations of $\widetilde{SL}(2,\mathbb{R})$

We will define a "representation" of $sl(2,\mathbb{R})$:

$$sl(2,\mathbb{R}) \ni X \stackrel{\pi}{\longmapsto} \pi(X) - \textbf{unbounded} \text{ operator on a Hilbert space V}$$

which integrates to a representation of $\widetilde{SL}(2,\mathbb{R})$:

$$\widetilde{\pi} : \widetilde{SL}(2,\mathbb{R}) \ni \tilde{h} \longmapsto \widetilde{\pi}(\tilde{h}) \in B(V)$$

i.e.

$$\begin{array}{ccc} sl(2,\mathbb{R}) \ni X & \xrightarrow{\widetilde{\exp}} & \widetilde{\exp}X \in \widetilde{SL}(2,\mathbb{R}) \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ \pi(X) & \xrightarrow{\exp} & \boxed{e^{\pi(X)} = \widetilde{\pi}\left(\widetilde{\exp}X\right)} \end{array}$$

<u>Problems:</u>

- $\pi(X)$ are unbounded operators
- $SL(2,\mathbb{R})$ is not compact $\implies$ π doesn't have to integrate to a representations of $\widetilde{SL}(2,\mathbb{R})$.

$sl(2,\mathbb{R})$ standard triplet

Matrices

$$A_+, A_-, N \in sl(2,\mathbb{R})$$

are called a **standard triplet** if

$$[A_+, A_-] = 2N, \qquad [N, A_\pm] = \pm A_\pm.$$

Introduce standard triplet in $sl(2,\mathbb{R})$:

$$A_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad A_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad N = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

Then,

$$e^{s_+ A_+} = \begin{bmatrix} 1 & s_+ \\ 0 & 1 \end{bmatrix}, \quad e^{s_- A_-} = \begin{bmatrix} 1 & 0 \\ s_- & 1 \end{bmatrix}, \quad e^{tN} = \begin{bmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{bmatrix}.$$

Any $\tilde{h} \in \widetilde{SL}(2,\mathbb{R})$ can be written using

$$e^{s_+ A_+}, e^{s_- A_-}, e^{tN}, \ \mathbb{1}_n.$$

Therefore, representation $\widetilde{\pi}(\tilde{h})$ for $\tilde{h} \in \widetilde{SL}(2,\mathbb{R})$ can be written using

$$e^{s_+ \pi(A_+)}, e^{s_- \pi(A_-)}, e^{t\pi(N)}, \ \widetilde{\pi}(\mathbb{1}_n).$$

$SL(2,\mathbb{R})$ representation on $L^2[0,\infty[$

1. Define operators on $L^2[0,\infty[$:

$$A := -\frac{i}{2}(x\partial_x + \partial_x x) \qquad \text{dilation operator}$$

$$K := x^2$$

$$H_m := -\partial_x^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2} \qquad \text{Schroedinger operator}$$

2. Define representation π of $sl(2,\mathbb{R})$ on $L^2[0,\infty[$

$$\left.\begin{array}{lcl} A_+ & \overset{\pi}{\longmapsto} & \frac{i}{2}H_m \\ A_- & \longmapsto & -\frac{i}{2}K \\ N & \longmapsto & -\frac{i}{2}A \end{array}\right\} \text{formally satisfy the commutation relations of } sl(2,\mathbb{R})$$

- the Casimir operator $C = \frac{1}{2}(A_+A_- + A_-A_+) + N^2$

$$\pi(C) = \frac{1}{4}m^2 - \frac{1}{4}$$

- $\pi(g)$ are antihermitian $\implies$ $e^{\pi(g)}$ are unitary and bounded.

$SL(2,\mathbb{R})$ representation on $L^2[0,\infty[$

Kernels of the three operators:

1. $\pi(A_+) = \frac{i}{2} H_m$

$$e^{\alpha\pi(A_+)}(x,y) = e^{\frac{i}{2}\pi(m+1)}\sqrt{\frac{2}{\pi|\alpha|}}\mathcal{I}_m\left(\frac{xy}{|\alpha|}\right)e^{-i\frac{x^2+y^2}{2\alpha}}, \quad \operatorname{Im}\alpha \geqslant 0$$

2. $\pi(A_-) = -\frac{i}{2} K$

$$e^{-\frac{i}{2}\beta K}f(x) = e^{-\frac{i}{2}\beta x^2}f(x)$$
$$e^{\beta\pi(A_-)}(x,y) = e^{-\frac{i}{2}\beta x^2}\delta(x-y)$$

3. $\pi(N) = -\frac{i}{2} A$

$$e^{itA}f(x) = e^{t/2}f(e^t x)$$
$$e^{\gamma\pi(N)}(x,y) = e^{-\frac{\gamma}{4}}\delta\left(e^{-\frac{\gamma}{2}}x - y\right).$$

$$SL(2,\mathbb{R}) = SO(2) \times AN(2,\mathbb{R}) =]-\pi,\pi] \times AN(2,\mathbb{R})$$

We have a map

$$\phi : SL(2,\mathbb{R}) \longrightarrow]-\pi,\pi].$$

Using ϕ we can partition $SL(2,\mathbb{R})$ into sectors:

$$\begin{aligned}
Y_- &:= \{h \,:\, \phi(h) \in \,]-\pi, 0[\,\} &&= \{h \,:\, h_{12} < 0\} && \\
Z_0 &:= \{h \,:\, \phi(h) = 0\} &&= \{h \,:\, h_{12} = 0, h_{11} > 0\} &&= AN(2,\mathbb{R}) \\
Y_+ &:= \{h \,:\, \phi(h) \in \,]0, \pi[\,\} &&= \{h \,:\, h_{12} > 0\} &&= -Y_- \\
Z_1 &:= \{h \,:\, \phi(h) = \pi\} &&= \{h \,:\, h_{12} = 0, h_{11} < 0\} &&= -AN(2,\mathbb{R})
\end{aligned}$$

$SL(2,\mathbb{R})$ is a disjoint sum of sectors:

$$SL(2,\mathbb{R}) = Y_- \sqcup Z_0 \sqcup Y_+ \sqcup Z_1.$$

$\widetilde{SL}(2,\mathbb{R})$ representation on $L^2[0,\infty[$

$$\widetilde{SL}(2,\mathbb{R}) = \mathbb{R} \times AN(2,\mathbb{R})$$

We have a map

$$\widetilde{\phi} : \widetilde{SL}(2,\mathbb{R}) \longrightarrow \mathbb{R}.$$

Using $\widetilde{\phi}$ we can partition $\widetilde{SL}(2,\mathbb{R})$ into sectors:

$$\tilde{Y}_{n+\frac{1}{2}} := \left\{ \tilde{h} \in \widetilde{SL}(2,\mathbb{R}) \ : \ \phi(\tilde{h}) \in]n\pi, (n+1)\pi[\right\}$$
$$\tilde{Z}_n := \left\{ \tilde{h} \in \widetilde{SL}(2,\mathbb{R}) \ : \ \phi(\tilde{h}) = n\pi \right\}$$

- $\mathbb{1}_n \in \tilde{Z}_n$
- $\widetilde{SL}(2,\mathbb{R})$ is a disjoint sum of sectors:

$$\widetilde{SL}(2,\mathbb{R}) = \bigsqcup_{n\in\mathbb{Z}} \tilde{Y}_{n+\frac{1}{2}} \ \sqcup \ \tilde{Z}_n.$$

- we can identify

$$SL(2,\mathbb{R}) \supset Y_- \cup Z_0 \cup Y_+ \quad \cong \quad \tilde{Y}_{-\frac{1}{2}} \cup \tilde{Z}_0 \cup \tilde{Y}_{\frac{1}{2}} \subset \widetilde{SL}(2,\mathbb{R}).$$

$\widetilde{SL}(2,\mathbb{R})$ representation on $L^2[0,\infty[$

For $h \in Z_0 = AN(2,\mathbb{R})$

$$h = \begin{bmatrix} a & 0 \\ n & \frac{1}{a} \end{bmatrix} = e^{\frac{n}{a}A_-} e^{2\log(a)N}$$

so the representation is

$$\widetilde{\pi}(h) = e^{\frac{n}{a}\left(-\frac{i}{2}K\right)} e^{2\log(a)\left(-\frac{i}{2}A\right)}.$$

For $h \in Y_+ \sqcup Y_-$ $(h_{12} \neq 0)$

$$h = e^{\frac{h_{22}-1}{h_{12}}A_-} e^{h_{12}A_+} e^{\frac{h_{11}-1}{h_{12}}A_-}.$$

So the representation is given by

$$\widetilde{\pi}(h) = e^{\frac{h_{22}-1}{h_{12}}\left(-\frac{i}{2}K\right)} e^{h_{12}\left(\frac{i}{2}H_m\right)} e^{\frac{h_{11}-1}{h_{12}}\left(-\frac{i}{2}K\right)}.$$

Kernel of this operator equals

$$\widetilde{\pi}(h)(x,y) = e^{i\frac{\pi}{2}(m+1)\mathrm{sgn}(h_{12})}\sqrt{\frac{2}{\pi|h_{12}|}}\,\mathcal{I}_m\left(\frac{xy}{|h_{12}|}\right) e^{-\frac{i}{2h_{12}}(h_{11}x^2+h_{22}y^2)}.$$

$\widetilde{SL}(2,\mathbb{R})$ representation on $L^2[0,\infty[$

$\boxed{\mathbb{1} \in SL(2,\mathbb{R})}$

$$\mathbb{1} = \lim_{\epsilon \searrow 0} \begin{bmatrix} 1 & \pm\epsilon \\ 0 & 1 \end{bmatrix}$$

$$\mathbb{1} = \widetilde{\pi}(\mathbb{1}) = s - \lim_{\epsilon \searrow 0} e^{\pm \frac{i}{2}\epsilon H_m}$$

Therefore, kernels of these operators

$$e^{\pm i \frac{\pi}{2}(m+1)} \sqrt{\frac{2}{\pi\epsilon}} \mathcal{I}_m\left(\frac{xy}{\epsilon}\right) e^{\mp \frac{i}{2\epsilon}(x^2+y^2)} \xrightarrow{\epsilon \searrow 0} \delta(x-y) = \widetilde{\pi}(\mathbb{1})(x,y)$$

$\boxed{\mathbb{1}_{\pm 1} \in \widetilde{SL}(2,\mathbb{R})}$

$$\mathbb{1}_{\pm 1} = \lim_{\epsilon \searrow 0} \begin{bmatrix} -1 & \mp\epsilon \\ 0 & -1 \end{bmatrix}$$

These matrices correspond to operators with kernels:

$$e^{\mp i \frac{\pi}{2}(m+1)} \sqrt{\frac{2}{\pi\epsilon}} \mathcal{I}_m\left(\frac{xy}{\epsilon}\right) e^{\mp \frac{i}{2\epsilon}(x^2+y^2)}.$$

So the operators converge to

$$\widetilde{\pi}(\mathbb{1}_{\pm 1}) = e^{\pm i\pi(m+1)} \mathbb{1}.$$

$\boxed{\mathbb{1}_n \in \widetilde{SL}(2,\mathbb{R})}$

$$\mathbb{1}_n = (\pi n, (-1)^n \mathbb{1}) \in \mathbb{R} \times AN(2,\mathbb{R}), \quad n \in \mathbb{Z}$$

Center of $\widetilde{SL}(2,\mathbb{R})$:

$$Z(\widetilde{SL}(2,\mathbb{R})) = \{\mathbb{1}_n \ : \ n \in \mathbb{Z}\}.$$

Then, from Schur's lemma:

$$\widetilde{\pi}(\mathbb{1}_n) = e^{i\pi\ n\,(m+1)}\mathbb{1}.$$

In particular:

- $m = 1, 3, \ldots$: $\widetilde{\pi}$ is also a representation of $PSL(2,\mathbb{R})$ and $SL(2,\mathbb{R})$

$$\widetilde{\pi}(\mathbb{1}_n) = \mathbb{1}$$

- $m = 0, 2, \ldots$: $\widetilde{\pi}$ is also a representation of $SL(2,\mathbb{R})$

$$\widetilde{\pi}(\mathbb{1}_n) = (-1)^n \mathbb{1}.$$

We calculated $\widetilde{\pi}(\tilde{h})$ for all $\tilde{h} \in \widetilde{SL}(2,\mathbb{R})$.

Lets take a standard triplet in $sl(2,\mathbb{C})$:

$$\mathcal{A}_+, \mathcal{A}_-, \mathcal{N} \in sl(2,\mathbb{C}) = \mathbb{C}sl(2,\mathbb{R}).$$

Consider space of monomials:

$$W^m := \left\{ w^k \ : \ k \in \mathbb{Z} + m \right\}, \quad m \in \mathbb{C}.$$

Define representation $X^{l,m}$ of $sl(2,\mathbb{C})$ on W^m:

$$\left.\begin{array}{lcl} \mathcal{A}_+ & \longmapsto & \mathcal{A}^l_+ := -w^2\partial_w + lw \\ \mathcal{A}_- & \longmapsto & \mathcal{A}^l_- := \partial_w + lw^{-1} \\ \mathcal{N} & \longmapsto & \mathcal{N}^l := w\partial_w \end{array}\right\} \begin{array}{c}\text{satisfy the commutation} \\ \text{relations of } sl(2,\mathbb{C})\end{array}$$

Spectrum of $\mathcal{N}^l$:

$$\mathcal{N}^l w^k = k w^k \implies \boxed{\mathrm{sp}(\mathcal{N}^l) \subset \mathbb{Z} + m.}$$

Theorem

Suppose

- π *is an irreducible representation of* $sl(2,\mathbb{C})$
- $\pi(\mathcal{N})$ *has an eigenvector with eigenvalue* $m \in \mathbb{C}$.

Then, there exists $l \in \mathbb{C}$ *s.t. the representation* π *is equivalent to* $X^{l,m}$ *(or one of its subrepresentations).*

Example of a standard triplet in $sl(2,\mathbb{C})$:

$$\mathcal{A}_+ = \frac{1}{2}\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}, \qquad \mathcal{A}_- = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}, \qquad \mathcal{N} = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

Notice: $\boxed{(2i\mathcal{N})^2 = -\mathbb{1}.}$

Connection between this $sl(2,\mathbb{C})$ standard triplet and previous $sl(2,\mathbb{R})$ standard triplet:

$$\boxed{\mathcal{N} = \frac{i}{2}(A_- - A_+).}$$

Representation of $\mathbb{1}_n$

Fact 1

From Schur's lemma:

$$\widetilde{\pi}(\mathbb{1}_n) = e^{i\pi\, n\,(m+1)}\mathbb{1}.$$

Fact 2

For $X \in sl(2, \mathbb{R})$ such that $(2X)^2 = -\mathbb{1}$:

$$\mathbb{1}_n = \widetilde{\exp}(\pi n X)$$

The representation $\widetilde{\pi}$ is an exponent of representation π:

$$\widetilde{\pi}(\mathbb{1}_n) = e^{\pi n\, \pi(X)}.$$

From 1 and 2 X has to satisfy:

$$\mathrm{sp}(\pi(X)) \subset i(m + \mathbb{Z}), \qquad (2X)^2 = -\mathbb{1}$$

Therefore

$$X = i\mathcal{N}, \qquad \mathcal{N} = \frac{1}{2}\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}.$$

We can write $\mathcal{N}$ from $sl(2,\mathbb{C})$ standard triplet using A_+, A_- from $sl(2,\mathbb{R})$ standard triplet:

$$\mathcal{N} = \frac{i}{2}(A_- - A_+)$$

Then, the operator corresponding to $\mathcal{N}$:

$$\pi(\mathcal{N}) = \frac{1}{4}(K + H_m) = \frac{1}{4}\left(-\partial_x^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2} + x^2\right),$$

which is a Hamiltonian of a **harmonic oscillator + potential** $1/x^2$.

$\Longrightarrow$ $\pi(\mathcal{N})$ is an operator with spectrum bounded from below

$\Longrightarrow$ π is equivalent to $X^{l,lw}$.

Lowest weight representation of $\widetilde{SL}(2,\mathbb{R})$ on $L^2[0,\infty[$

Theorem

The representation (m,lw) of $\widetilde{SL}(2,\mathbb{R})$ *can be represented as bounded operators on* $L^2[0,\infty[$ *with kernels:*

$\tilde{h} \in \tilde{Y}_{n+\frac{1}{2}}$:

$$\tilde{h}^{m,lw}(x,y) = e^{i\pi(m+1)(n+\frac{1}{2})}\sqrt{\frac{2}{\pi|h_{12}|}}\mathcal{I}_m\left(\frac{xy}{|h_{12}|}\right)e^{-\frac{i}{2h_{12}}(h_{11}x^2+h_{22}y^2)}$$

$\tilde{h} \in \tilde{Z}_n$:

$$\tilde{h}^{m,lw}(x,y) = e^{i\pi n(m+1)-\log(|h_11|)/4}e^{-i\frac{h_{21}}{2h_{11}}x^2}\delta\left(|h_{11}|^{-1}x - y\right).$$

Its generators are

$$\begin{aligned}
\pi(A_+) &= \frac{i}{2}H_m &&= \frac{i}{2}\left(-\partial_x^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2}\right)\\
\pi(A_-) &= -\frac{i}{2}K &&= -\frac{i}{2}x^2\\
\pi(N) &= -\frac{i}{2}A &&= -\frac{1}{4}(x\partial_x + \partial_x x).
\end{aligned}$$

Lowest and highest weight representations of $\widetilde{SL}(2,\mathbb{R})$

Lowest weight representation generators:

$$
\begin{aligned}
A_+^{m,lw} &= \frac{i}{2}H_m &&= \frac{i}{2}\left(-\partial_x^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2}\right)\\
A_-^{m,lw} &= -\frac{i}{2}K &&= -\frac{i}{2}x^2\\
N^{m,lw} &= -\frac{i}{2}A &&= -\frac{1}{4}(x\partial_x + \partial_x x).
\end{aligned}
$$

$$
\mathcal{N}^{m,lw} = \frac{i}{2}\left(A_-^{m,lw} - A_+^{m,lw}\right) = \frac{1}{4}\left(-\partial_x^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2} + x^2\right)
$$

Highest weight representation generators:

$$
\begin{aligned}
A_+^{m,hw} &= -\frac{i}{2}H_m\\
A_-^{m,hw} &= \frac{i}{2}K\\
N^{m,hw} &= -\frac{i}{2}A.
\end{aligned}
$$

$$
\mathcal{N}^{m,hw} = \frac{i}{2}\left(A_-^{m,hw} - A_+^{m,hw}\right) = -\frac{1}{4}\left(-\partial_x^2 + \left(m^2 - \frac{1}{4}\right)\frac{1}{x^2} + x^2\right).
$$

Define a representation of $sl(2,\mathbb{R})$ on $L^2[0,\infty[\oplus L^2[0,\infty[)$:

$$\left.\begin{array}{lcl} A_+ & \longmapsto & -\frac{i}{2}H_m \oplus (-H_m) \\ A_- & \longmapsto & \frac{i}{2}K \oplus (-K) \\ N & \longmapsto & -\frac{1}{2}A \oplus A \end{array}\right\} \text{ formally satisfy the commutation relations of } sl(2,\mathbb{R})$$

Operator $H_m \oplus (-H_m)$

For $-1 < \mathrm{Re}\, m < 1$ we have operators

$$L_{m^2}^{min} \oplus (-L_{m^2}^{min}) \qquad L_{m^2}^{max} \oplus (-L_{m^2}^{max}).$$

$(f^+, f^-) \in \mathrm{Dom}\big(L_{m^2}^{max} \oplus (-L_{m^2}^{max})\big)$:

$$f^+(x) \sim \alpha_+^+ x^{\frac{1}{2}+m} + \alpha_-^+ x^{\frac{1}{2}-m}$$
$$f^-(x) \sim \alpha_+^- x^{\frac{1}{2}+m} + \alpha_-^- x^{\frac{1}{2}-m},$$

where $\alpha_+^+, \alpha_-^+, \alpha_+^-, \alpha_-^- \in \mathbb{C}$.

Definition

Define $H_m^{\omega_+,\omega_-}$ as a restriction of $L_{m^2}^{max} \oplus (-L_{m^2}^{max})$ to functions satisfying condition:

$$\alpha_+^+ \omega_+ = \alpha_+^-$$
$$\alpha_-^+ \omega_- = \alpha_-^-,$$

where $\omega_+, \omega_- \in \mathbb{C} \cup \{\infty\}$.

$H_m^{\omega_+,\omega_-}$, $A \oplus A$, $K \oplus (-K)$ generate **principal and complementary series.**