THE FEYNMAN PROPAGATOR ON A CURVED SPACETIME

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Quantum Field Theory on curved spacetimes may sound like a specialized subject. In reality it is not—Lorentzian manifolds are a natural playground for quantum fields.

I will restrict myself to the linear theory, which in the curved case is quite nontrivial and offers some interesting conceptual questions. It also leads to interesting problems in PDE's and the operator theory.

I will start with discussing QFT on the flat Minkowski space $\mathbb{R}^{1,3}$. The free charged field on the flat $\mathbb{R}^{1,3}$ satisfies the Klein-Gordon equation

$$(-\Box + m^2)\psi(x) = 0.$$

To study the field $\psi(x)$ one introduces various propagators or two-point functions.

- the forward/backward propagator

$$G^{\vee/\wedge}(x,y) \coloneqq \frac{1}{(2\pi)^4} \int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2 + m^2 \pm \mathrm{i}0 \operatorname{sgn} p_0} \, \mathrm{d}p,$$

- the Feynman/anti-Feynman propagator

$$G^{\mathrm{F}/\overline{\mathrm{F}}}(x,y) \coloneqq \frac{1}{(2\pi)^4} \int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2 + m^2 \mp \mathrm{i}0} \, \mathrm{d}p,$$

- the Pauli–Jordan propagator

$$G^{\mathrm{PJ}}(x,y) \coloneqq \frac{\mathrm{i}}{(2\pi)^3} \int \mathrm{e}^{-\mathrm{i}(x-y)\cdot p} \operatorname{sgn}(p_0) \delta(p^2 + m^2) \, \mathrm{d}p,$$

- the positive/negative frequency 2-point function

$$G^{(\pm)}(x,y) \coloneqq \frac{1}{(2\pi)^3} \int \mathrm{e}^{-\mathrm{i}(x-y)\cdot p} \theta(\pm p_0) \delta(p^2 + m^2) \, \mathrm{d}p.$$

Mathematically, $G^{\vee/\wedge}, G^{\mathrm{F}/\overline{\mathrm{F}}}$ are inverses of the Klein Gordon operator

$$(-\Box + m^2)Gf = G(-\Box + m^2)f = f,$$

and $G^{(\pm)}, G^{\mathrm{PJ}}$ are its bisolutions

$$(-\Box + m^2)Gf = G(-\Box + m^2)f = 0.$$

These propagators express various important quantities of Quantum Field Theory:

- the commutation relations

$$[\psi(x), \psi^*(y)] = -\mathrm{i}G^{\mathrm{PJ}}(x, y),$$

- the vacuum expectation of products of fields

$$(\Omega \mid \psi(x)\psi^*(y)\Omega) = G^{(+)}(x, y),$$
$$(\Omega \mid \psi^*(x)\psi(y)\Omega) = G^{(-)}(x, y),$$

- the vacuum expectation of time ordered products of fields

$$\big(\Omega \,\big|\, \mathrm{T}\big(\psi(x)\psi^*(y)\big)\Omega\big) = -\mathrm{i}G^{\mathrm{F}}(x, y).$$

Note the identities satisfied by the propagators:

$$
\begin{aligned}
G^{\mathrm{PJ}} &= G^{\vee} - G^{\wedge} && (1)\\
&= \mathrm{i}G^{(+)} - \mathrm{i}G^{(-)}, && (2)\\
G^{\mathrm{F}} - G^{\overline{\mathrm{F}}} &= \mathrm{i}G^{(+)} + \mathrm{i}G^{(-)}, && (3)\\
G^{\mathrm{F}} + G^{\overline{\mathrm{F}}} &= G^{\vee} + G^{\wedge}, && (4)\\
G^{\mathrm{F}} &= \mathrm{i}G^{(+)} + G^{\wedge} = \mathrm{i}G^{(-)} + G^{\vee}, && (5)\\
G^{\overline{\mathrm{F}}} &= -\mathrm{i}G^{(+)} + G^{\vee} = -\mathrm{i}G^{(-)} + G^{\wedge}. && (6)
\end{aligned}
$$

The following fact is easy to see:

(1) the Klein-Gordon operator $-\Box + m^2$ is essentially self-adjoint on $C_{\mathrm{c}}^\infty(\mathbb{R}^{1,3})$,

(2) For $s > \frac{1}{2}$, in the sense $\langle t\rangle^{-s} L^2(\mathbb{R}^{1,3}) \to \langle t\rangle^{s} L^2(\mathbb{R}^{1,3})$, the Feynman propagator is the boundary value of the resolvent of the Klein-Gordon operator:

$$\operatorname*{s-lim}_{\epsilon\searrow 0}(-\Box + m^2 - \mathrm{i}\epsilon)^{-1} = G^{\mathrm{F}}.$$

Here $\langle t\rangle$ denotes the so-called "Japanese bracket"

$$\langle t\rangle := \sqrt{1+t^2}.$$

Quantum field theory on the flat Minkowski space is very simple. More interesting, but still linear, is QFT on a curved spacetime in the presence of an external electromagnetic potential A and an external scalar potential Y, based on the (generalized) Klein-Gordon equation

$$\Big(|g|^{-\frac{1}{4}}(\mathrm{i}\partial_\mu + A_\mu)|g|^{\frac{1}{2}}g^{\mu\nu}(\mathrm{i}\partial_\nu + A_\nu)|g|^{-\frac{1}{4}} + Y\Big)\psi = 0.$$

All the constructions and relations that we described for the flat Minkowski space generalize easily to the stationary case.

The situation is more complicated in the generic, possibly non-stationary case.

It is well known that on an arbitrary globally hyperbolic spacetime the forward, backward and Pauli–Jordan propagators are still well-defined. We use the name classical propagators as the joint name for these three propagators. The identity (1) still holds. The Pauli–Jordan propagator is still responsible for the commutation relations of fields.

The other propagators, which we call non-classical, are more difficult. In the literature it is often claimed that it makes no sense to ask for distinguished non-classical propagators on generic spacetimes. We will argue that on a large class of spacetimes that are asymptotically stationary in the future and past there exist distinguished non-classical propagators.

It is rather obvious that the in/out positive/negative frequency bisolutions are distinguished. Let us denote them by $G_{\pm}^{(+)}$ and $G_{\pm}^{(-)}$. (The plus/minus in the parentheses corresponds to positive/negative frequencies; the plus/minus without parentheses corresponds to the future/past). The identity (2) now splits into two independent identities

$$G^{\mathrm{PJ}} = \mathrm{i}G_{\pm}^{(+)} - \mathrm{i}G_{\pm}^{(-)}.$$

We now have two distinguished vacuum states, the in–vacuum and the out–vacuum:

$$(\Omega_\pm \,|\, \psi(x)\psi^*(y)\Omega_\pm) = G_\pm^{(+)}(x,y),$$
$$(\Omega_\pm \,|\, \psi^*(x)\psi(y)\Omega_\pm) = G_\pm^{(-)}(x,y).$$

Note that the states Ω_+ and Ω_- satisfy the so-called Hadamard condition about the wave front set, as proven by Gerard and Wrochna.

It is less obvious that the Feynman propagator also possesses a natural generalization. It describes what in popular science books is expressed as particles travelling forward in time and antiparticles travelling backwards in time. After quantization the Feynman propagator satisfies

$$\frac{\left(\Omega_+ \,\middle|\, \mathrm{T}\big(\psi(x)\psi^*(y)\big)\Omega_-\right)}{(\Omega_+ \,|\, \Omega_-)} = -\mathrm{i}G^{\mathrm{F}}(x,y)$$

Note that the identities (3)–(6) no longer hold. (They are still true on the level of singularities of the respective functions).

The main goal of Quantum Field Theory is to compute scattering amplitudes. This is done by evaluating Feynman diagrams, where we put the Feynman propagator at the lines. Therefore, the Feynman propagator is a central object in QFT.

In what follows I will describe the construction of the Feynman propagator. The basic idea will be to introduce the time variable and rewrite the 2nd order Klein-Gordon equation into a 1st order evolution equation.

I will follow the Bourbaki style emphasizing general constructions.

Let $\mathcal{W}$ be a Banach space. We say that a two-parameter family of bounded operators

$$\mathbb{R} \times \mathbb{R} \ni (t, s) \mapsto R(t, s) \in B(\mathcal{W}) \qquad (*)$$

is a strongly continuous evolution family on $\mathcal{W}$ if for all r, s, t, we have the identities

$$R(t, t) = \mathbb{1}, \quad R(t, s)R(s, r) = R(t, r).$$

and the map $(*)$ is strongly continuous.

If $R(t,s) = R(t-s,0)$ for all t, s, we say that the evolution is autonomous. Setting $R(t) := R(t,0)$, we obtain a strongly continuous one-parameter group. As is well known, we can then write $R(t) = \mathrm{e}^{-\mathrm{i}tB}$, where $-\mathrm{i}B$ is a certain unique, densely defined, closed operator called the generator of $R(t)$.

If $\mathcal{W}$ is a Hilbert space, then B is self-adjoint if and only if R is unitary.

Unfortunately, the evolution of the Cauchy data on non-stationary curved spacetimes is non-autonomous. Besides, it is usually not unitary for any scalar product.

However, one can often assume that the evolution preserves a class of equivalent scalar products. To formalize this idea it is convenient to introduce the concept of Hilbertizable spaces.

Let $\mathcal{W}$ be a topological vector space. We say that it is Hilbertizable if it has a topology of a Hilbert space for some scalar product $(\cdot\,|\,\cdot)$ on $\mathcal{W}$.

Let $(\cdot\,|\,\cdot)_1$, $(\cdot\,|\,\cdot)_2$ be two scalar products compatible with a Hilbertizable space $\mathcal{W}$. Then there exist constants $0 < c \leq C$ such that

$$c(w\,|\,w)_1 \leq (w\,|\,w)_2 \leq C(w\,|\,w)_1.$$

Let $\{B(t)\}_{t\in\mathbb{R}}$ be a family of densely defined, closed operators on a Hilbertizable space $\mathcal{W}$. Let $\mathcal{V}$ be another Hilbertizable space densely and continuously embedded in $\mathcal{W}$. The following theorem, due essentially to Kato, gives sufficient conditions for the existence of a (non-autononomous) evolution generated by $\{B(t)\}_{t\in\mathbb{R}}$

Theorem. Suppose that the following conditions are satisfied:

(a) $\mathcal{V} \subset \operatorname{Dom} B(t)$ so that $B(t) \in B(\mathcal{V}, \mathcal{W})$ and $t \mapsto B(t) \in B(\mathcal{V}, \mathcal{W})$ is norm-continuous.

(b) For every t, scalar products $(\cdot \mid \cdot)_{\mathcal{W},t}$ and $(\cdot \mid \cdot)_{\mathcal{V},t}$ compatible with $\mathcal{W}$ resp. $\mathcal{V}$ have been chosen.

(c) $B(t)$ is self-adjoint in the sense of $\mathcal{W}_t$ and the part $\tilde{B}(t)$ of $B(t)$ in $\mathcal{V}_t$ is self-adjoint in the sense of $\mathcal{V}_t$.

(d) For $C \in L^1_{\mathrm{loc}}$ and all s, t

$$\|v\|_{\mathcal{W},s} \leq \|v\|_{\mathcal{W},t} \exp\left|\int_s^t C(r)\,\mathrm{d}r\right|,$$

$$\|w\|_{\mathcal{V},s} \leq \|w\|_{\mathcal{V},t} \exp\left|\int_s^t C(r)\,\mathrm{d}r\right|.$$

Then there exists a unique family of bounded operators $\{R(t,s)\}_{s,t}$ on $\mathcal{W}_{-\frac{1}{2}}$, preserving $\mathcal{V}$, called the evolution generated by $B(t)$, such that:

(i) It is an evolution on $\mathcal{W}$ and $\mathcal{V}$,

(ii) For all $v \in \mathcal{V}$ and s, t,

$$\begin{aligned} \mathrm{i}\partial_t R(t,s)v &= B(t)R(t,s)v, \\ -\mathrm{i}\partial_s R(t,s)v &= R(t,s)B(s)v, \end{aligned}$$

where the derivatives are in the strong topology of $\mathcal{W}$.

Consider the inhomogeneous evolution equation

$$\big(\partial_t + \mathrm{i}B(t)\big)v(t) = w(t). \qquad (**)$$

We will say that an operator $E^\bullet$ is a bisolution resp. an inverse or Green's operator of $(**)$ if

$$\begin{array}{rll} & \big(\partial_t + \mathrm{i}B(t)\big)E^\bullet w = 0 & E^\bullet\big(\partial_t + \mathrm{i}B(t)\big)v = 0, \\ \text{resp.} & \big(\partial_t + \mathrm{i}B(t)\big)E^\bullet w = w, & E^\bullet\big(\partial_t + \mathrm{i}B(t)\big)v = v. \end{array}$$

The most obvious bisolutions and inverses are defined by the following integral kernels:

the Pauli-Jordan bisolution $E^{\mathrm{PJ}}(t,s) := R(t,s)$,

the forward inverse $E^{\vee}(t,s) := \theta(t-s)R(t,s)$,

the backward inverse $E^{\wedge}(t,s) := -\theta(s-t)R(t,s)$.

They act on functions $t \mapsto w(t)$ as follows:

$$(E^{\bullet}w)(t) := \int E^{\bullet}(t,s)w(s)\,\mathrm{d}s, \quad \bullet = \mathrm{PJ}, \vee, \wedge,$$

Clearly,

$$E^{\mathrm{PJ}} = E^{\vee} - E^{\wedge}.$$

Let $S_\pm \in B(\mathcal{W})$ satisfy $S_\pm^2 = \mathbb{1}$. In other words, S_+ and S_- are two bounded involutions on $\mathcal{W}_0$,

Define

$$\Pi_\pm := \frac{1}{2}(\mathbb{1} + S_\pm), \quad \text{projection onto out/in-particles}$$

$$\Pi_\mp := \frac{1}{2}(\mathbb{1} - S_-), \quad \text{projection onto out/in-antiparticles.}$$

Clearly,

$$S_\pm = \Pi_\pm - \Pi_\mp.$$

Define $E_{\pm}^{(+)}$ and $E_{\pm}^{(-)}$ by the following integral kernels:

the in/out particle bisolution

$$E_{\pm}^{(+)}(t,s) := \operatorname*{s-lim}_{\tau\to\pm\infty} R(t,\tau)\Pi_{\pm}R(\tau,s),$$

the in/out antiparticle bisolution

$$E_{\pm}^{(-)}(t,s) := -\operatorname*{s-lim}_{\tau\to\pm\infty} R(t,\tau)\Pi_{\mp}R(\tau,s),$$

provided the above limits exist. Clearly,

$$E^{\mathrm{PJ}} = E_{\pm}^{(+)} - E_{\pm}^{(-)}.$$

$E_{\pm}^{(+)}$ and $E_{\pm}^{(-)}$ will describe the 2-point functions of the in- and out-vacuum.

Set

$$\begin{aligned}
S_\pm(t) &\coloneqq \operatorname*{s-lim}_{\tau\to\pm\infty} R(t,\tau)S_\pm R(\tau,t),\\
\Pi_\pm(t) &\coloneqq \operatorname*{s-lim}_{\tau\to\pm\infty} R(t,\tau)\Pi_\pm R(\tau,t),\\
\Pi_\mp(t) &\coloneqq \operatorname*{s-lim}_{\tau\to\pm\infty} R(t,\tau)\Pi_\mp R(\tau,t).
\end{aligned}$$

We say that the evolution satisfies asymptotic complementarity if for some, and hence for any t,

$$\begin{aligned}\mathcal{W} &= \operatorname{Ran}\Pi_{-}(t) \oplus \operatorname{Ran}\Pi_{\overline{+}}(t)\\ &= \operatorname{Ran}\Pi_{\overline{-}}(t) \oplus \operatorname{Ran}\Pi_{+}(t)\end{aligned}$$

Suppose that asymptotic complementarity holds. Then there exist two pairs of complementary projections corresponding to these pairs of subspaces

$$\begin{aligned}\Lambda_{-+}(t) + \Lambda_{\overline{+-}}(t) &= \mathbb{1},\\ \Lambda_{\overline{-+}}(t) + \Lambda_{+-}(t) &= \mathbb{1}.\end{aligned}$$

Define the operators $E^{-\overline{+}}$ and $E^{\overline{-}+}$ by their integral kernels

$$E^{-\overline{+}}(t,s) := \theta(t-s)R(t,s)\Lambda_{\overline{+-}}(s) - \theta(s-t)R(t,s)\Lambda_{-+}(s),$$
$$E^{\overline{-}+}(t,s) := \theta(t-s)R(t,s)\Lambda_{+-}(s) - \theta(s-t)R(t,s)\Lambda_{\overline{-+}}(s).$$

$E^{-\overline{+}}$ will be called the particle-in—antiparticle-out inverse and $E^{\overline{-}+}$ will be called the antiparticle-in—particle-out inverse. They are abstract versions of the Feynman and anti-Feynman inverse.

We have the identities

$$\big(E^{-\overline{\mp}} - E^{\overline{-}+}\big)(t,s) - \frac{1}{2}\big(E_{+}^{(+)} + E_{+}^{(-)} + E_{-}^{(+)} + E_{-}^{(-)}\big)(t,s)$$
$$= \frac{1}{8}R(t,s)\Upsilon(s)^{-1}\big[S_{+}(s) - S_{-}(s), [S_{+}(s), S_{-}(s)]\big].$$

$$\big(E^{-\overline{\mp}} + E^{\overline{-}+} - E^{\vee} - E^{\wedge}\big)(t,s)$$
$$=\frac{1}{4}R(t,s)\Upsilon(s)^{-1}[S_{-}(s), S_{+}(s)].$$

These identities simplify in some important situations. Suppose that for any (and hence for all) t

$$S_-(t)S_+(t) = S_+(t)S_-(t).$$

Then

$$\begin{aligned} E^{-\overline{+}} + E^{\overline{-}+} &= E^{\vee} + E^{\wedge}, \\ E^{-\overline{+}} - E^{\overline{-}+} &= \frac{1}{2}\big(E_+^{(+)} + E_+^{(-)} + E_-^{(+)} + E_-^{(-)}\big). \end{aligned}$$

If the evolution is autonomous, it is natural to assume that $S_+ = S_-$, requiring that it commutes with the generator B. Then $E_{\pm}^{(+)}$, $E_{\pm}^{(-)}$ collapse to two bisolutions

$$
\begin{aligned}
E_{+}^{(+)} &= E_{-}^{(+)} =: E^{(+)}, \\
E_{+}^{(-)} &= E_{-}^{(-)} =: E^{(-)}.
\end{aligned}
$$

We have

$$
\begin{aligned}
E^{-\overline{+}} + E^{\overline{-}+} &= E^{\vee} + E^{\wedge}, \\
E^{-\overline{+}} - E^{\overline{-}+} &= E^{(+)} + E^{(-)}.
\end{aligned}
$$

Suppose that a complex vector space $\mathcal{W}$ is equipped with a non-degenerate Hermitian form Q, sometimes called a charge form

$$\mathcal{W} \times \mathcal{W} \ni (v, w) \mapsto (v|Qw) \in \mathbb{C}.$$

If the dimension of $\mathcal{W}$ is infinite, we assume that $\mathcal{W}$ is Hilbertizable and Q is bounded.

A charge form appears naturally as the complexification of $\mathrm{i}\omega$, where ω is the symplectic form.

Abusing terminology, a bounded invertible operator R on $\mathcal{W}$ will be called symplectic if

$$(Rv \,|\, QRw) = (v \,|\, Qw).$$

An operator $S_\bullet$ on $(\mathcal{W}, Q)$ will be called an admissible involution if $S_\bullet^2 = \mathbb{1}$ and there exists a scalar product $(\cdot|\cdot)_\bullet$ compatible with the structure of $\mathcal{W}$ such that

$$(v \,|\, Qw) = (v \,|\, S_\bullet w)_\bullet.$$

$(\mathcal{W}, Q)$ is called a Krein space if it possesses an admissible involution.

$S_\bullet$ defines a pair of projections

$$\text{the positive projection} \quad \Pi_\bullet := \frac{1}{2}(\mathbb{1} + S_\bullet),$$
$$\text{the negative projection} \quad \Pi_{\overline{\bullet}} := \frac{1}{2}(\mathbb{1} - S_\bullet).$$

Theorem. Let S_1, S_2 be a pair of admissible involutions on a Krein space $(\mathcal{W}, Q)$. Then we have two direct sum decompositions:

$$\begin{aligned} \mathcal{W} &= \operatorname{Ran}\Pi_1 \oplus \operatorname{Ran}\Pi_{\overline{2}} \\ &= \operatorname{Ran}\Pi_{\overline{1}} \oplus \operatorname{Ran}\Pi_2. \end{aligned}$$

Let us sketch the proof. Set $K := S_2 S_1$. Then K is positive with respect to $(\cdot \mid \cdot)_1$ and $(\cdot \mid \cdot)_2$. Hence we can define $c := \Pi_1 \frac{\mathbb{1}-K}{\mathbb{1}+K} \Pi_{\overline{1}}$. Then the projections corresponding to the above direct sum decompositions are

$$
\Lambda_{12} = \begin{bmatrix} \mathbb{1} & c \\ 0 & 0 \end{bmatrix}, \qquad \Lambda_{\overline{21}} = \begin{bmatrix} 0 & -c \\ 0 & \mathbb{1} \end{bmatrix};
$$

$$
\Lambda_{\overline{12}} = \begin{bmatrix} 0 & 0 \\ c^* & \mathbb{1} \end{bmatrix}, \qquad \Lambda_{21} = \begin{bmatrix} \mathbb{1} & 0 \\ -c^* & 0 \end{bmatrix}.
$$

where we use the direct sum $\operatorname{Ran} \Pi_1 \oplus \operatorname{Ran} \Pi_{\overline{1}}$.

Proposition. Suppose that B is an operator on $\mathcal{W}$ with domain containing $\mathcal{V}$. We assume that B is a generator of a group on $\mathcal{W}$, its part $\tilde{B}$ in $\mathcal{V}$ is a generator of a group on $\mathcal{V}$, and

$$\big(Bv \mid Qw\big) = \overline{\big(Bw|Qv\big)}, \quad v, w \in \mathcal{V}.$$

Then $\mathrm{e}^{-\mathrm{i}tB}$, $t \in \mathbb{R}$, is symplectic on $(\mathcal{W}, Q)$.

An operator B is called a infinitesimally symplectic if the above conditions hold. The above quadratic form is called the energy or Hamiltonian quadratic form.

Let B be a densely defined operator on $\mathcal{W}$. We say that it is stable if there exists an admissible involution $S_\bullet$ such that B is self-adjoint for $(\cdot|\cdot)_\bullet$, $\operatorname{Ker} B = \{0\}$ and

$$S_\bullet = \operatorname{sgn}(B).$$

Note that every stable operator is infinitsimally symplectic, and its Hamiltonian form is positive:

$$\big(v \mid BS_\bullet v\big) = \big(Bv \mid Qv\big) \geq 0.$$

Let $\mathbb{R} \ni t \mapsto B(t) \in B(\mathcal{V}, \mathcal{W})$ satisfy the assumptions of the theorem about almost unitary evolutions. Assume also that $B(t)$ is infinitesimally symplectic on $(\mathcal{W}, Q)$ for all t. Then the evolution $R(t,s)$ is symplectic.

Assume in addition that $B_\pm := \underset{t\to\pm\infty}{\text{s-lim}}\, B(t)$ exist and are stable. Then we set

$$S_\pm := \operatorname{sgn}\left(B_\pm\right)$$

which as we know are admissible involutions. Then we can introduce the corresponding bisolutions $E_\pm^{(+)}$ and $E_\pm^{(-)}$.

We also know that asymptotic complementarity is true for S_+ and S_-, and hence the inverses $E^{-\overline{+}}$ and $E^{\overline{-}+}$ are well defined.

With this choice the bisolutions $E_{\pm}^{(+)}$ are called the in/out positive frequency bisolutions, the bisolutions $E^{\overline{\pm}}$ are called the in/out negative frequency bisolutions.

The inverses $E^{-\overline{+}}$ and $E^{\overline{-}+}$ are denoted E^{F}, resp. $E^{\overline{\mathrm{F}}}$, and called the Feynman, resp. the anti-Feynman inverse.

Let us go back to the Klein-Gordon equation on a globally hyperbolic, asymptotically stationary manifold M

$$K\psi = 0,$$

where

$$K := \big(|g|^{-\frac{1}{4}}(\mathrm{i}\partial_\mu + A_\mu)|g|^{\frac{1}{2}}g^{\mu\nu}(\mathrm{i}\partial_\nu + A_\nu)|g|^{-\frac{1}{4}} + Y\big)\psi$$

is the Klein-Gordon operator.

It is possible and helpful to introduce a time variable t, so that the spacetime is $M = \mathbb{R}\times\Sigma$. We can assume that there are no time-space cross terms so that the metric can be written as

$$-g_{00}(t,\vec{x})\,\mathrm{d}^2t + g_{ij}(t,\vec{x})\,\mathrm{d}x^i\,\mathrm{d}x^j.$$

By conformal rescaling we can assume that $g_{00} = 1$, so that, setting $V := A^0$, we have

$$\begin{aligned} K &= -(\mathrm{i}\partial_t + V)^2 + L, \\ L &= -|g|^{-\frac{1}{4}}(\mathrm{i}\partial_i + A_i)|g|^{\frac{1}{2}}g^{ij}(\mathrm{i}\partial_j + A_j)|g|^{-\frac{1}{4}} + Y. \end{aligned}$$

We rewrite the Klein-Gordon equation $Ku = 0$ as a 1st order equation for the Cauchy data

$$\Big(\partial_t + \mathrm{i}B(t)\Big)\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = 0,$$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} := \begin{bmatrix} u(t) \\ \mathrm{i}\partial_t u(t) - W(t)u(t) \end{bmatrix},$$

$$B(t) := \begin{bmatrix} W(t) & \mathbb{1} \\ L(t) & \overline{W}(t) \end{bmatrix},$$

$$W(t) := V(t) + \frac{\mathrm{i}}{4}|g|(t)^{-1}\partial_t |g|(t).$$

We make some technical assumptions, among other things, the operator $L(t)$ should define equivalent norm for various times. The Hilbertizable space of Cauchy data

$$\mathcal{W} = L(t)^{\frac{1}{4}} L^2(\Sigma) \oplus L(t)^{-\frac{1}{4}} L^2(\Sigma)$$

equipped with the charge form given by the matrix

$$Q := \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}.$$

becomes a Krein space. As described above, we define the symplectic dynamics $R(t, s)$ generated by $B(t)$.

Note that if

$$E = \begin{pmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{pmatrix}$$

is a bisolution/inverse of $\partial_t + \mathrm{i}B(t)$, then E_{12} is a bisolution/inverse of K.

We can now define both the classical propagators

$$G^{\mathrm{PJ}} := -\mathrm{i}E_{12}^{\mathrm{PJ}}, \quad G^{\vee} := -\mathrm{i}E_{12}^{\vee}, \quad G^{\wedge} := -\mathrm{i}E_{12}^{\wedge}$$

and the non-classical propagators

$$G^{(\pm)} := E_{12}^{(\pm)}, \quad G^{\mathrm{F}} := -\mathrm{i}E_{12}^{\mathrm{F}}, \quad G^{\overline{\mathrm{F}}} := -\mathrm{i}E_{12}^{\overline{\mathrm{F}}}.$$

They are inverses or bisolutions of K.

Thus on asymptotically stationary spacetimes we have two natural vacuum states and a single natural Feynman propagator. They are not defined locally—they depend globally on the whole spacetime. However, their singularities, and even more, the semiclassical expansion around the diagonal, are given by the local data.

Conjecture. On a large class of asymptotically stationary spacetimes
(1) the operator K is essentially self-adjoint on $C_{\mathrm{c}}^{\infty}(M)$,
(2) in the sense $\langle t\rangle^{-s}L^2(M) \to \langle t\rangle^{s}L^2(M)$, where $s > \frac{1}{2}$,

$$\operatorname*{s-lim}_{\epsilon\searrow 0}(K - \mathrm{i}\epsilon)^{-1} = G^{\mathrm{F}}.$$

In a recent paper of A. Vasy this conjecture is proven for asymptotically Minkowskian spaces. It is true if the spatial dimension is zero (when the Klein-Gordon operator reduces to the 1-dimensional Schrödinger operator). It is also true on a large class of cosmological spacetimes. Presumably, one can prove it on symmetric spacetimes.

Surprisingly, we have not found a trace of this question in the older mathematical literature. Many respected mathematicians and mathematical physicists react with disgust to this question, claiming that it is completely non-physical.

However, in the physical literature there are many papers that take the self-adjointness of the Klein-Gordon operator for granted. The method of computing the Feynman propagator with external fields and possibly on curved spacetimes based on the identity

$$\lim_{\epsilon \searrow 0} \frac{1}{K - \mathrm{i}\epsilon} = \mathrm{i} \int_0^\infty \mathrm{e}^{-\mathrm{i}tK} \, \mathrm{d}t \qquad (*)$$

has even a name:

the Fock–Schwinger or Schwinger–DeWitt method.

Of course, without the self-adjointness of K, $(*)$ does not make sense.

Let us try to construct a self-adjoint extension of K by constructing its resolvent. For $z \in \mathbb{C}$ with $\operatorname{Im} z \geq 0$, let us perturb the generator B by considering

$$B^z(t) := \begin{bmatrix} W(t) & \mathbb{1} \\ L(t) - z & \overline{W}(t) \end{bmatrix} = B(t) - \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix}.$$

Suppose that the evolution family for $B^z(t)$ is denoted $R^z(t,s)$. Set

$$S_{\pm}^z(t) \coloneqq \operatorname*{s-lim}_{\tau \to \pm\infty} R^z(t,\tau) S_{\pm} R^z(\tau,t),$$
$$\Pi_{\pm}^z(t) \coloneqq \operatorname*{s-lim}_{\tau \to \pm\infty} R^z(t,\tau) \Pi_{\pm} R^z(\tau,t),$$
$$\Pi_{\mp}^z(t) \coloneqq \operatorname*{s-lim}_{\tau \to \pm\infty} R^z(t,\tau) \Pi_{\mp} R^z(\tau,t).$$

For small z we should have asymptotic complementarity in the sense

$$\mathcal{W} = \operatorname{Ran}\Pi^z_-(t) \oplus \operatorname{Ran}\Pi^z_+(t),$$

We have the corresponding pair of projections:

$$\Lambda^z_{-+}(t) + \Lambda^z_{+-}(t) = \mathbb{1}.$$

Define the operator $E^{\mathrm{F},z}$ by its integral kernel

$$E^{\mathrm{F},z}(t,s)$$
$$:=\theta(t-s)R^z(t,s)\Lambda^z_{+-}(s)-\theta(s-t)R^z(t,s)\Lambda^z_{-+}(s).$$

Set $G^{\mathrm{F},z}(t,s) := E^{\mathrm{F},z}_{12}(t,s)$, as an operator on $L^2(\mathbb{R},L^2(\Sigma))$. One can check that G^z satisfies the resolvent identity:

$$G^{\mathrm{F},z}(t,s)-G^{\mathrm{F},w}(t,s)=(z-w)\int G^{\mathrm{F},z}(t,\tau)G^{\mathrm{F},w}(\tau,s)\mathrm{d}\tau.$$

One can expect that

$$G^{\mathrm{F},z}=\frac{1}{z-K^{\mathrm{sa}}},$$

where K^{sa} is a distinguished self-adjoint realization of the Klein-Gordon operator K. Unfortunately, it is not clear whether $G^{\mathrm{F},z}$ is bounded.