# THE FEYNMAN PROPAGATOR ON CURVED SPACETIMES

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in collaboration with DANIEL SIEMSSEN York University The (classical or quantum) charged field on a curved spacetime with metric tensor  $g_{\mu\nu}$  in the presence of an external electromagnetic potential  $A_{\mu}$  and an external scalar potential Y satisfies the Klein-Gordon equation

$$\left(|g|^{-\frac{1}{4}}(\mathrm{i}\partial_{\mu} + A_{\mu})|g|^{\frac{1}{2}}g^{\mu\nu}(\mathrm{i}\partial_{\nu} + A_{\nu})|g|^{-\frac{1}{4}} + Y\right)\psi = 0.$$

In other words, the field is annihlated by the Klein-Gordon operator

$$K := |g|^{-\frac{1}{4}} (\mathrm{i}\partial_{\mu} + A_{\mu}) |g|^{\frac{1}{2}} g^{\mu\nu} (\mathrm{i}\partial_{\nu} + A_{\nu}) |g|^{-\frac{1}{4}} + Y.$$

I will discuss various propagators or two-point functions associated with K.

My talk is divided into three parts:

- 1. Flat case.
- 2. Stationary case.
- 3. Generic case.

### Part I. Flat case.

Let me start with the Minkowski space  $\mathbb{R}^{1,3}$  without external potentials. The Klein-Gordon operator is  $K := -\Box + m^2$ . The following propagators and 2-point functions should belong to the standard knowledge of every student of Quantum Field Theory.

• the forward/backward propagator

$$G^{\vee/\wedge}(x,y) \coloneqq \frac{1}{(2\pi)^4} \int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2 + m^2 \pm \mathrm{i}0 \operatorname{sgn} p_0} \,\mathrm{d}p,$$

• the Feynman/anti-Feynman propagator

$$G^{\mathrm{F}/\overline{\mathrm{F}}}(x,y) \coloneqq \frac{1}{(2\pi)^4} \int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2 + m^2 \mp \mathrm{i}0} \,\mathrm{d}p,$$

• the Pauli–Jordan propagator

$$G^{\mathrm{PJ}}(x,y) \coloneqq \frac{\mathrm{i}}{(2\pi)^3} \int \mathrm{e}^{-\mathrm{i}(x-y)\cdot p} \operatorname{sgn}(p_0) \delta(p^2 + m^2) \,\mathrm{d}p,$$

• the positive/negative frequency 2-point function

$$G^{(\pm)}(x,y) \coloneqq \frac{1}{(2\pi)^3} \int \mathrm{e}^{-\mathrm{i}(x-y)\cdot p} \theta(\pm p_0) \delta(p^2 + m^2) \,\mathrm{d}p.$$

From the point of view of operator theory,  $G^{\vee/\wedge}, G^{F/\overline{F}}$  are inverses of the Klein Gordon operator

$$KGf = GKf = f,$$

and  $G^{(\pm)}, G^{\mathrm{PJ}}$  are its bisolutions

$$KGf = GKf = 0.$$

The propagators express important quantities of QFT:

• the commutation relations

$$[\psi(x),\psi^*(y)] = -\mathrm{i}G^{\mathrm{PJ}}(x,y),$$

• the vacuum expectation of products of fields

$$(\Omega \mid \psi(x)\psi^*(y)\Omega) = G^{(+)}(x,y),$$
  
$$(\Omega \mid \psi^*(x)\psi(y)\Omega) = G^{(-)}(x,y),$$

• the vacuum expectation of time ordered products of fields

$$\left( \Omega \, \middle| \, \mathrm{T} \big( \psi(x)\psi^*(y) \big) \Omega \big) = -\mathrm{i} G^{\mathrm{F}}(x,y), \\ \left( \Omega \, \middle| \, \mathrm{T} \big( \psi^*(x)\psi(y) \big) \Omega \big) = -\mathrm{i} G^{\mathrm{F}}(x,y).$$

Note the identities satisfied by the propagators:

$$G^{\rm PJ} = G^{\vee} - G^{\wedge} \tag{1}$$

$$= iG^{(+)} - iG^{(-)},$$
 (2)

$$G^{\mathrm{F}} - G^{\mathrm{F}} = \mathrm{i}G^{(+)} + \mathrm{i}G^{(-)}, \qquad (3)$$

$$G^{\mathrm{F}} + G^{\mathrm{F}} = G^{\vee} + G^{\wedge}.$$
 (4)

The following facts are easy to see:

(1) the Klein-Gordon operator  $K = -\Box + m^2$  is essentially selfadjoint on  $C_c^{\infty}(\mathbb{R}^{1,3})$ ,

(2) For  $s > \frac{1}{2}$ , in the sense  $\langle t \rangle^{-s} L^2(\mathbb{R}^{1,3}) \rightarrow \langle t \rangle^s L^2(\mathbb{R}^{1,3})$ , the Feynman propagator is the boundary value of the resolvent of the Klein-Gordon operator:

$$\operatorname{s-lim}_{\epsilon \searrow 0} (K \mp i\epsilon)^{-1} = G^{F/\overline{F}}.$$

Here  $\langle t \rangle$  denotes the so-called "Japanese bracket"

$$\langle t \rangle := \sqrt{1 + t^2}.$$

#### Part I. Stationary case.

To simplify the exposition we will assume that  $g_{00} = 1$  and that there are no time/space cross terms. (These restrictions can be removed).

The spacetime is  $M = \mathbb{R} \times \Sigma$ .

$$\begin{split} K &= -(\mathrm{i}\partial_t + V)^2 + L, \\ L &= |g|^{-\frac{1}{4}}(\mathrm{i}\partial_i + A_i)|g|^{\frac{1}{2}}g^{ij}(\mathrm{i}\partial_j + A_j)|g|^{-\frac{1}{4}} + Y, \\ \text{where } g^{ij}, \, V, \, \vec{A} \text{ and } Y \text{ do not depend on } t. \end{split}$$

It is natural to apply the evolution approach and to rewrite the Klein-Gordon equation Ku = 0 as a 1st order equation for the Cauchy data

$$\left(\partial_t + \mathrm{i}B\right) \begin{bmatrix} u_1(t)\\u_2(t)\end{bmatrix} = 0,$$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} := \begin{bmatrix} u(t) \\ \mathrm{i}\partial_t u(t) - V u(t) \end{bmatrix}, \quad B := \begin{bmatrix} V & \mathbb{1} \\ L & V \end{bmatrix}.$$

We obtain the evolution

$$R(t,s) := e^{-i(t-s)B}$$

Note that B preserves the charge form

$$(u|Qv) = (u_1|v_2) + (u_2|v_1)$$

given by the matrix

$$Q := \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}.$$

If we assume that the Hamiltonian is positive, that is

$$H := BQ = \begin{bmatrix} L & V \\ V & \mathbb{1} \end{bmatrix} > 0,$$

there is also a natural Hilbert space, called the dynamical space on which B is self-adjoint, and which becomes naturally the 1-particle space after quantization. It has the scalar product  $(u|v)_{dyn} := (|B|^{-\frac{1}{2}}u|H|B|^{-\frac{1}{2}}v).$  We define the propagators in the evolution approach:

Pauli-Jordan bisolution $E^{\mathrm{PJ}}(t,s) \coloneqq R(t,s),$ forward inverse $E^{\vee}(t,s) \coloneqq \theta(t-s)R(t,s),$ backward inverse $E^{\wedge}(t,s) \coloneqq -\theta(s-t)R(t,s),$ pos./neg. freq. bisolution $E^{(\pm)}(t,s) \coloneqq R(t-s)\Pi^{(\mp)},$ Feynman/anti-Feynman inverse $E^{\mathrm{F}/\mathrm{F}}(t,s) \coloneqq \theta(t-s)R(t-s)\Pi^{(\mp)}$  $-\theta(s-t)R(t-s)\Pi^{(\pm)},$ Here  $\theta$  is the Heavyside function and $\Pi^{(\pm)} \coloneqq \mathbb{1}_{\mathbb{R}^{\pm}}(B).$ 

They act on functions 
$$t \mapsto w(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix}$$
 as follows:  
 $(E^{\bullet}w)(t) \coloneqq \int E^{\bullet}(t,s)w(s) \,\mathrm{d}s, \quad \bullet = \mathrm{PJ}, \lor, \land, (\pm), \mathrm{F}/\overline{\mathrm{F}}.$ 

We obtain also the propagators in the spacetime approach:

$$G^{\bullet} := -\mathrm{i} E_{12}^{\bullet}, \quad \bullet = \mathrm{PJ}, \vee, \wedge, \mathrm{F}/\overline{\mathrm{F}},$$
$$G^{(\pm)} := E_{12}^{(\pm)}, \quad E^{\bullet} = \begin{bmatrix} E_{11}^{\bullet} & E_{12}^{\bullet} \\ E_{21}^{\bullet} & E_{22}^{\bullet} \end{bmatrix}.$$

Essentially everything that was described in the flat case remains valid in the stationary case.

#### Part III. Generic case.

As is well-known, for every globally hyperbolic spacetime one can define the classical propagators  $G^{\wedge}, G^{\vee}, G^{\text{PJ}}$ .

To discuss the other propagators, which we call non-classical we will additionally assume that the Klein-Gordon operator is asymptotically stationary in the past and future.

We can partly show, and partly we conjecture, that the theory of propagators in the asymptotically stationary case is very similar to what was described for stationary spacetimes, even if it is much more difficult technically. We can rewrite the Klein-Gordon equation Ku = 0 as a 1st order equation for the Cauchy data

$$\begin{pmatrix} \partial_t + iB(t) \end{pmatrix} \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = 0,$$

$$\begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} := \begin{bmatrix} u(t) \\ i\partial_t u(t) - W(t)u(t) \end{bmatrix},$$

$$B(t) := \begin{bmatrix} W(t) & 1 \\ L(t) & \overline{W}(t) \end{bmatrix},$$

$$W(t) := V(t) + \frac{i}{4} |g|(t)^{-1} \partial_t |g|(t).$$

We make some technical assumptions, among other things, the operator L(t) should define equivalent norm for various times. The Hilbertizable space of Cauchy data

$$\mathcal{W} = L(t)^{\frac{1}{4}}L^2(\Sigma) \oplus L(t)^{-\frac{1}{4}}L^2(\Sigma)$$

equipped with the charge form given by the matrix

$$Q := \begin{bmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{bmatrix}.$$

becomes a Krein space. We define the pseudounitary evolution R(t,s) generated by  $B(t), \, {\rm that}$  is

$$\partial_t R(t, t') = -iB(t)R(t, t'),$$
  

$$R(t, t) = \mathbb{1}.$$

We assume that the Hamiltonian is positive, at least in the far future and past:

$$H(\pm\infty) := B(\pm\infty)Q \ge 0.$$

We introduce the in- and out-positive/negative frequency projection:

$$\begin{split} \Pi^{(+)}_{\pm} &:= \mathbb{1}_{\mathbb{R}^+}(B(\pm\infty)), \\ \Pi^{(-)}_{\pm} &:= \mathbb{1}_{\mathbb{R}^-}(B(\pm\infty)). \end{split}$$

They lead to the in-vacuum and the out-vacuum:

$$(\Omega_{\pm} | \psi(x)\psi^*(y)\Omega_{\pm}) = G_{\pm}^{(+)}(x,y),$$
  
$$(\Omega_{\pm} | \psi^*(x)\psi(y)\Omega_{\pm}) = G_{\pm}^{(-)}(x,y).$$

The following lemma uses the structure of Krein spaces: Lemma. For any s

$$\lim_{t \to -\infty} R(s,t) \operatorname{Ran} \Pi_{-}^{(+)}, \quad \lim_{t \to +\infty} R(s,t) \operatorname{Ran} \Pi_{+}^{(-)}$$

is a pair of complementary subspaces.

With help of the above lemma we first define the Feynman and anti-Feynman propagator  $E^{F/\overline{F}}$  in the evolution approach: in words, it describes particles travelling forward in time and antiparticles travelling backward in time. In the usual way, this leads to the Feynman and anti-Feynman propagator  $G^{F/\overline{F}}$  in the spacetime approach.

In a somewhat different setting, the construction of  $G^{\rm F}$  was given by A.Vasy et al and by Gérard-Wrochna. But it seems that the naturalness and simplicity of the above construction was realized only recently.

Here is the physical meaning of the Feynman propagator: it is the expectation value of the time-ordered product of fields between the in-vacuum and the out-vacuum:

$$G^{\mathrm{F}}(x,y) = \frac{\left(\Omega_{+} | \mathrm{T}\left(\hat{\phi}(x)\hat{\phi}(y)\right)\Omega_{-}\right)}{\left(\Omega_{+} | \Omega_{-}\right)}$$

The identities satisfied by the propagators in the generic case differ from the stationary case:

$$\begin{aligned} G^{\rm PJ} &= G^{\vee} - G^{\wedge} & (1)' \\ &= {\rm i}G^{(+)} - {\rm i}G^{(-)}, & (2)' \\ G^{\rm F} - G^{\rm F} &= {\rm i}G^{(+)}_{\pm} + {\rm i}G^{(-)}_{\pm} + \text{ smooth and "small"}, & (3)' \\ G^{\rm F} + G^{\rm F} &= G^{\vee} + G^{\wedge} + \text{ smooth and "small"}. & (4)' \end{aligned}$$

Thus on asymptotically stationary spacetimes we have two natural vacuum states and a single natural Feynman propagator. They are not defined locally—they depend globally on the whole spacetime. However, their singularities, and even more, the semiclassical expansion around the diagonal, are given by the local data.

Conjecture. On a large class of asymptotically stationary spacetimes (1) the operator K is essentially self-adjoint on  $C_c^{\infty}(M)$ , (2) in the sense  $\langle t \rangle^{-s} L^2(M) \to \langle t \rangle^s L^2(M)$ , where  $s > \frac{1}{2}$ ,  $s-\lim_{\epsilon \searrow 0} (K - i\epsilon)^{-1} = G^F$ . In a recent paper of A. Vasy this conjecture is proven for asymptotically Minkowskian spaces. It is also true if the spatial dimension is zero (when the Klein-Gordon operator reduces to the 1-dimensional Schrödinger operator). It is true as well on a large class of cosmological spacetimes. Presumably, one can prove it on symmetric spacetimes.

Surprisingly, we have not found a trace of this question in the older mathematical literature. Many respected mathematicians and mathematical physicists react with disgust to this question, claiming that it is completely non-physical.

However, in the physical literature there are many papers that take the self-adjointness of the Klein-Gordon operator for granted. The method of computing the Feynman propagator with external fields and possibly on curved spacetimes based on the identity

$$\lim_{\epsilon \searrow 0} \frac{1}{K - i\epsilon} = i \int_{0}^{\infty} e^{-itK} dt \qquad (*)$$

has even a name:

the Fock-Schwinger or Schwinger-DeWitt method.

Of course, without the self-adjointness of K, (\*) does not make sense.

Perhaps some of the experts in the audience may be surprised that the so-called Hadamard condition has not been mentioned in my talk so far.

In words, a two-point function satisfies the Hadamard condition if it is a positive definite bisolution of the Klein-Gordon equation whose wave front set is the same as in the flat case. The state defined by such a two-point function is called a Hadamard state.

Note that there are many Hadamard states. In particular, the inand out states, which we discussed, are Hadamard, as proven by Gérard and Wrochna. To my understanding, one can divide researchers interested in QFT on curved spacetimes into two categories.

- 1. The Feynmanists work with a global spacetime and use the distinguished in- and out states and the distinguished Feynman propagator. This is probably common among phenomenologically minded researchers.
- 2. The Hadamardists usually look at spacetimes locally and say that the reference state can be arbitrary as long as it satisfies the Hadamard condition. Most researchers in the mathematical QFT community belong to this category.

There is no contradiction between the Feynmanist and Hadamardist philosophy. Nevertheless, the emphasis of both approaches is quite different. My presentation tries to be a mathematical exposition of the Feynmanist approach.

I am actually not very fond of the Hadamard condition, which mathematically minded QFT-people like so much, because it is very abstract and hides the fact that one can say much more about the 2-point function of physically natural vacuum states than just the location of their wave front set.

### THANK YOU FOR YOUR ATTENTION

(This is the end of the main part of my slides. Note that I have some additional slides with "appendices", which normally I do not have time to cover in a talk.)

### Appendix I. Evolution in Hilbertizable spaces

Let  $\ensuremath{\mathcal{W}}$  be a Banach space. We say that a two-parameter family of bounded operators

$$\mathbb{R} \times \mathbb{R} \ni (t,s) \mapsto R(t,s) \in B(\mathcal{W}) \qquad (*)$$

is a strongly continuous evolution family on  ${\cal W}$  if for all r,s,t, we have the identities

$$R(t,t)=\mathbb{1}, \quad R(t,s)R(s,r)=R(t,r).$$

and the map (\*) is strongly continuous.

If R(t,s) = R(t-s,0) for all t,s, we say that the evolution is autonomous. Setting  $R(t) \coloneqq R(t,0)$ , we obtain a strongly continuous one-parameter group. As is well known, we can then write  $R(t) = e^{-itB}$ , where -iB is a certain unique, densely defined, closed operator called the generator of R(t).

If  $\mathcal{W}$  is a Hilbert space, then B is self-adjoint if and only if R is unitary.

Let  $\mathcal{W}$  be a topological vector space. We say that it is Hilbertizable if it has a topology of a Hilbert space for some scalar product  $(\cdot | \cdot)_{\bullet}$  on  $\mathcal{W}$ .

Let  $(\cdot | \cdot)_1$ ,  $(\cdot | \cdot)_2$  be two scalar products compatible with a Hilbertizable space  $\mathcal{W}$ . Then there exist constants  $0 < c \leq C$  such that

 $c(w \,|\, w)_1 \leq (w \,|\, w)_2 \leq C(w \,|\, w)_1.$ 

Let  $\{B(t)\}_{t\in\mathbb{R}}$  be a family of densely defined, closed operators on a Hilbertizable space  $\mathcal{W}$ . Let  $\mathcal{V}$  be another Hilbertizable space densely and continuously embedded in  $\mathcal{W}$ . The following theorem, due essentially to Kato, gives sufficient conditions for the existence of a (non-autononomous) evolution generated by  $\{B(t)\}_{t\in\mathbb{R}}$ 

**Theorem.** Suppose that the following conditions are satisfied:

- (a)  $\mathcal{V} \subset \text{Dom } B(t)$  so that  $B(t) \in B(\mathcal{V}, \mathcal{W})$  and  $t \mapsto B(t) \in B(\mathcal{V}, \mathcal{W})$  is norm-continuous.
- (b) For every t, scalar products  $(\cdot | \cdot)_{\mathcal{W},t}$  and  $(\cdot | \cdot)_{\mathcal{V},t}$  compatible with  $\mathcal{W}$  resp.  $\mathcal{V}$  have been chosen.
- (c) B(t) is self-adjoint in the sense of  $\mathcal{W}_t$  and the part  $\tilde{B}(t)$  of B(t) in  $\mathcal{V}_t$  is self-adjoint in the sense of  $\mathcal{V}_t$ .

(d) For  $C \in L^1_{\text{loc}}$  and all s, t

$$\|v\|_{\mathcal{W},s} \leq \|v\|_{\mathcal{W},t} \exp\left|\int_{s}^{t} C(r) \,\mathrm{d}r\right|, \\ \|w\|_{\mathcal{V},s} \leq \|w\|_{\mathcal{V},t} \exp\left|\int_{s}^{t} C(r) \,\mathrm{d}r\right|.$$

Then there exists a unique family of bounded operators  $\{R(t,s)\}_{s,t}$  on  $\mathcal{W}$ , preserving  $\mathcal{V}$ , called the evolution generated by B(t), such that:

(i) It is an evolution on  ${\cal W}$  and  ${\cal V}$ ,

(ii) For all  $v \in \mathcal{V}$  and s, t,

$$\label{eq:relation} \begin{split} \mathrm{i}\partial_t R(t,s)v &= B(t)R(t,s)v,\\ -\mathrm{i}\partial_s R(t,s)v &= R(t,s)B(s)v, \end{split}$$

where the derivatives are in the strong topology of  $\mathcal{W}$ .

#### Appendix II. Krein spaces

Suppose that a (complex) Hilbertizable space  $\mathcal{W}$  is equipped with a non-degenerate Hermitian form Q, sometimes called a charge form

$$\mathcal{W}\times\mathcal{W}\ni(v,w)\mapsto(v|Qw)=\overline{(w|Qv)}\in\mathbb{C}.$$

Note that often one starts from a real space with a symplectic form  $\omega$ . Then a charge form appears naturally as the complexification of  $i\omega$ .

An operator  $S_{\bullet}$  on  $(\mathcal{W}, Q)$  will be called an admissible involution if  $S_{\bullet}^2 = \mathbb{1}$  and there exists a scalar product  $(\cdot|\cdot)_{\bullet}$  compatible with the structure of  $\mathcal{W}$  such that

$$(v \mid Qw) = (v \mid S_{\bullet}w)_{\bullet}.$$

 $(\mathcal{W},Q)$  is called a Krein space if it possesses an admissible involution.

Every admissible involution  $S_{\bullet}$  defines a pair of projections

the positive projection 
$$\Pi_{\bullet}^{(+)} := \frac{1}{2}(\mathbb{1} + S_{\bullet}),$$
  
the negative projection  $\Pi_{\bullet}^{(-)} := \frac{1}{2}(\mathbb{1} - S_{\bullet}).$ 

**Theorem.** Let  $S_1$ ,  $S_2$  be a pair of admissible involutions on a Krein space  $(\mathcal{W}, Q)$ . Then we have two direct sum decompositions:

$$\mathcal{W} = \operatorname{Ran} \Pi_1^{(+)} \oplus \operatorname{Ran} \Pi_2^{(-)}$$
$$= \operatorname{Ran} \Pi_1^{(-)} \oplus \operatorname{Ran} \Pi_2^{(+)}$$

Let us sketch the proof. Set  $K \coloneqq S_2S_1$ . Then K is positive with respect to  $(\cdot | \cdot)_1$  and  $(\cdot | \cdot)_2$ . Hence we can define  $c \coloneqq \Pi_1^{(+)} \frac{1-K}{1+K} \Pi_1^{(-)}$ . Then the projections corresponding to the above direct sum decompositions are

$$\Lambda_{12}^{(+)} = \begin{bmatrix} \mathbb{1} & c \\ 0 & 0 \end{bmatrix}, \qquad \Lambda_{21}^{(-)} = \begin{bmatrix} 0 & -c \\ 0 & \mathbb{1} \end{bmatrix};$$
$$\Lambda_{12}^{(-)} = \begin{bmatrix} 0 & 0 \\ c^* & \mathbb{1} \end{bmatrix}, \qquad \Lambda_{21}^{(+)} = \begin{bmatrix} \mathbb{1} & 0 \\ -c^* & 0 \end{bmatrix}$$
where we use the direct sum Ran  $\Pi_1^{(+)} \oplus \operatorname{Ran} \Pi_1^{(-)}$ .

## Appendix III. Construction of the Feynman propagator.

Let  $(\mathcal{W}, Q)$  be a Krein space. A bounded invertible operator R on  $\mathcal{W}$  will be called pseudounitary (or symplectic) if

$$(Rv \,|\, QRw) = (v \,|\, Qw).$$

In the following theorem we describe generators of pseudounitary evolutions.

**Proposition.** Suppose that B is an operator on  $\mathcal{W}$  with domain containing a densely and continuously embedded Hilbertizable space  $\mathcal{V}$ . We assume that B is a generator of a group on  $\mathcal{W}$ , its part  $\tilde{B}$  in  $\mathcal{V}$  is a generator of a group on  $\mathcal{V}$ , and

$$(Bv \mid Qw) = \overline{(Bw \mid Qv)}, \quad v, w \in \mathcal{V}. \quad (*)$$

Then  $e^{-itB}$ ,  $t \in \mathbb{R}$ , is pseudounitary on  $(\mathcal{W}, Q)$ .

An operator B satisfying the above conditions is called a pseudounitary generator. The quadratic form (\*) is called the Hamiltonian quadratic form. Let B be a densely defined operator on  $\mathcal{W}$ . We say that it is stable if there exists an admissible involution  $S_{\bullet}$  such that B is self-adjoint for  $(\cdot|\cdot)_{\bullet}$ , Ker  $B = \{0\}$  and

$$S_{\bullet} = \operatorname{sgn}(B).$$

Every stable operator is a pseudounitary generator, and its Hamiltonian form is positive:

$$\left(v \mid BS_{\bullet}v\right) = \left(Bv \mid Qv\right) \ge 0.$$

Let  $\mathbb{R} \ni t \mapsto B(t) \in B(\mathcal{V}, \mathcal{W})$  satisfy the assumptions of the theorem about almost unitary evolutions. Assume also that B(t) is infinitesimally symplectic on  $(\mathcal{W}, Q)$  for all t. Then the evolution R(t, s) is symplectic.

Assume in addition that  $B(\pm\infty):= \operatorname*{s-lim}_{t\to\pm\infty} B(t)$  exist and are stable. Set

$$\Pi_{\pm}^{(+)} := \mathbb{1}_{\mathbb{R}^+} (B(\pm\infty)), \quad \Pi_{\pm}^{(-)} := \mathbb{1}_{\mathbb{R}^-} (B(\pm\infty)),$$
$$S_{\pm} := \operatorname{sgn} (B(\pm\infty)) = \Pi_{\pm}^{(+)} - \Pi_{\pm}^{(-)}.$$

which as we know are admissible involutions.

Then we can introduce the in/out positive frequency bisolutions  $E_{\pm}^{(+)}$ , and the in/out negative frequency bisolutions  $E_{\pm}^{(-)}$ .

$$E_{\pm}^{(+)}(t,s) := \lim_{\tau \to \pm \infty} R(t,\tau) \Pi_{\pm}^{(+)} R(\tau,s),$$
$$E_{\pm}^{(-)}(t,s) := \lim_{\tau \to \pm \infty} R(t,\tau) \Pi_{\pm}^{(-)} R(\tau,s).$$

Lemma. For any s

$$\lim_{t \to -\infty} R(s,t) \operatorname{Ran} \Pi_{-}^{(+)}, \quad \lim_{t \to +\infty} R(s,t) \operatorname{Ran} \Pi_{+}^{(-)},$$
$$\lim_{t \to -\infty} R(s,t) \operatorname{Ran} \Pi_{-}^{(-)}, \quad \lim_{t \to +\infty} R(s,t) \operatorname{Ran} \Pi_{+}^{(+)}$$

are two pairs of complementary subspaces.

With help of the above lemma we define two pairs of projections onto these subspaces:

$$\begin{array}{ll} \Lambda_{+-}^{(+)}(s), & \Lambda_{+-}^{(-)}(s), \\ \Lambda_{+-}^{(+)}(s), & \Lambda_{-+}^{(-)}(s). \end{array}$$

Now we can define the Feynman and anti-Feynman propagators in the evolution approach

$$E^{\mathbf{F}}(t,s) \coloneqq \theta(t-s)R(t,s)\Lambda_{+-}^{(+)}(s) - \theta(s-t)R(t,s)\Lambda_{+-}^{(-)}(s),$$
  
$$E^{\overline{\mathbf{F}}}(t,s) \coloneqq \theta(t-s)R(t,s)\Lambda_{-+}^{(-)}(s) - \theta(s-t)R(t,s)\Lambda_{-+}^{(+)}(s).$$

Set 
$$S_{\pm}(t) := \lim_{\tau \to \pm \infty} R(t,\tau) S_{\pm} R(\tau,t).$$
  
 $\Upsilon(t) \coloneqq \frac{1}{4} (2 + S_{-}(t) S_{+}(t) + S_{+}(t) S_{-}(t)).$ 

We have the identities

$$(E^{\mathrm{F}} - E^{\overline{\mathrm{F}}})(t,s) - \frac{1}{2} (E_{+}^{(+)} + E_{+}^{(-)} + E_{-}^{(+)} + E_{-}^{(-)})(t,s)$$
  
=  $\frac{1}{8} R(t,s) \Upsilon(s)^{-1} [S_{+}(s) - S_{-}(s), [S_{+}(s), S_{-}(s)]],$   
 $(E^{\mathrm{F}} + E^{\overline{\mathrm{F}}} - E^{\vee} - E^{\wedge})(t,s)$   
=  $\frac{1}{4} R(t,s) \Upsilon(s)^{-1} [S_{-}(s), S_{+}(s)].$ 

These identities simplify in some important situations. Suppose that for any (and hence for all) t

$$S_{-}(t)S_{+}(t) = S_{+}(t)S_{-}(t).$$

Then

$$E^{\rm F} + E^{\rm \overline{F}} = E^{\vee} + E^{\wedge},$$
  

$$E^{\rm F} - E^{\rm \overline{F}} = \frac{1}{2} \left( E^{(+)}_{+} + E^{(-)}_{+} + E^{(+)}_{-} + E^{(-)}_{-} \right).$$

If the evolution is autonomous, then  $E_{\pm}^{(+)}$  ,  $E_{\pm}^{(-)}$  collapse to two bisolutions

$$E_{+}^{(+)} = E_{-}^{(+)} \rightleftharpoons E^{(+)},$$
$$E_{+}^{(-)} = E_{-}^{(-)} \rightleftharpoons E^{(-)}.$$

We then have

$$E^{\overline{F}} + E^{\overline{F}} = E^{\vee} + E^{\wedge},$$
$$E^{\overline{F}} - E^{\overline{F}} = E^{(+)} + E^{(-)}.$$

Appendix IV. Attempt to define a distinguished self-adjoint extension of the Klein-Gordon operator

We will try to construct the resolvent of K. For  $z \in \mathbb{C}$  with  $\operatorname{Im} z \geq 0$ , let us perturb the generator B(t) by considering

$$B^{z}(t) := \begin{bmatrix} W(t) & \mathbb{1} \\ L(t) - z & \overline{W}(t) \end{bmatrix} = B(t) - \begin{bmatrix} 0 & 0 \\ z & 0 \end{bmatrix}$$

Suppose that the evolution family for  $B^{z}(t)$  is denoted  $R^{z}(t,s)$ .

For small  $\operatorname{Im} z$  we could expect that the following subspaces are complementary:

 $\lim_{t \to -\infty} R^{z}(s,t) \operatorname{Ran} \mathbb{1}_{\mathbb{R}^{+}}(B(t)), \quad \lim_{t \to +\infty} R^{z}(s,t) \operatorname{Ran} \mathbb{1}_{\mathbb{R}^{-}}(B(t))$ We define the corresponding pair of complementary projections:  $\Lambda_{-+}^{(+)z}(s), \quad \Lambda_{+-}^{(-)z}(s).$ Define the operator  $E^{\mathrm{F},z}$  by its integral kernel  $E^{\mathrm{F},z}(t,s)$  $\coloneqq \theta(t-s)R^{z}(t,s)\Lambda_{+-}^{(-)z}(s) - \theta(s-t)R^{z}(t,s)\Lambda_{-+}^{(+)z}(s).$  Set  $G^{\mathrm{F},z}(t,s) := E_{12}^{\mathrm{F},z}(t,s)$ , as an operator on  $L^2(M) = L^2(\mathbb{R}, L^2(\Sigma))$ . One can check that  $G^z$  satisfies the resolvent identity:

$$G^{\mathrm{F},z}(t,s) - G^{\mathrm{F},w}(t,s) = (z-w) \int G^{\mathrm{F},z}(t,\tau) G^{\mathrm{F},w}(\tau,s) \mathrm{d}\tau.$$

We hope that

$$G^{\mathrm{F},z} = \frac{1}{z - K^{\mathrm{sa}}},$$

where  $K^{sa}$  is a distinguished self-adjoint realization of the Klein-Gordon operator K. Unfortunately, it is not clear whether  $G^{F,z}$  is bounded.