Introduction to Quantization–Problems-1

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Problem 0.1. *Compute the Fourier transforms of*

$$e^{x^2-xy-y^2}, \quad e^{i\alpha x^2}, \quad e^{ix+ixy}.$$

Problem 0.2. *Which limits exist? Compute them*

$$\lim_{R\to\infty}\int_{-R}^{R} e^{i\alpha x^2}dx, \tag{0.1}$$

$$\lim_{\epsilon\to 0}\int e^{i\alpha x^2-\epsilon x^2}dx, \tag{0.2}$$

$$\lim_{R\to\infty}\int_{0}^{R} xe^{i\alpha x^2}dx, \tag{0.3}$$

$$\lim_{\epsilon\to 0}\int_{0}^{\infty} xe^{i\alpha x^2-\epsilon x^2}dx. \tag{0.4}$$

Problem 0.3. *Let $\phi \in C_c^\infty(\mathbb{R})$ such that $\phi = 1$ on a neighborhood of 0. Show that the following limit exists and does not depend on the choice of ϕ:*

$$\lim_{R\to\infty}\int x^n e^{i\alpha x^2}\phi(x/R)dx. \tag{0.5}$$

Hint. Consider the operator $L := (1+2i\alpha x)^{-1}(1+\partial_x)$. Note that $Le^{i\alpha x^2} = e^{i\alpha x^2}$. Hence for any N we have $L^N e^{i\alpha x^2} = e^{i\alpha x^2}$. After inserting this operator, we can integrate by parts. Note that (0.5) is called the *oscillatory integral of $x^n e^{i\alpha x^2}$*.

Problem 0.4. *Find an operator U such that*

$$U\mathrm{Op}(a)U^{-1} = \mathrm{Op}(a_1),$$

where

1. $a_1(x,p) = a(-x,-p)$
2. $a_1(x,p) = a(-p,x)$
3. $a_1(x,p) = a(x+\alpha, p+\beta)$
4. $a_1(x,p) = a(x+p,p)$

Problem 0.5. *For which values of $t \in \mathbb{R}$ the following operators are positive:*

1. $\mathrm{Op}(p^2 + \omega^2 x^2 + t)$,
2. $\mathrm{Op}(x^2 p^2 + t)$.

Problem 0.6. *Consider $a, b \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^2)$.*

1. *Compute the star product $a \star b$ as a formal power series in $\hbar$ up to the term of the order $\hbar^4$.*
2. *Suppose that $\mathrm{supp}(a) \cap \mathrm{supp}(b) = \emptyset$. Show that as a formal power series in $\hbar$ we have $a \star b = 0$. Give an example of such a, b such that $\mathrm{Op}(a)\mathrm{Op}(b) \neq 0$. (One can show that $\mathrm{Op}(a)\mathrm{Op}(b) = O(\hbar^{\infty})$)*
3. *Let $b = 1$ on $\mathrm{supp}(a)$. Show that $a \star b - b \star a = 0$ as a formal power series.*

Problem 0.7. *Compute the Weyl, x,p- and p,x-symbols of*

1. $\hat{x}\hat{p}$,
2. $\hat{x}^2\hat{p}^2$,
3. *the orthogonal projection onto* $\mathrm{e}^{-\frac{1}{2t}x^2}$,
4. $\mathrm{e}^{\mathrm{i}\xi\hat{x}+\eta\hat{p}}$
5. $\mathrm{e}^{-\frac{1}{2}\hat{x}^2}\mathrm{e}^{-\frac{1}{2}\hat{p}^2}$.

Problem 0.8. *Express the following operator in terms of $\hat{x}$, $\hat{p}$:*

1. $\mathrm{Op}(x^3 p)$,
2. $\mathrm{Op}(x^2 p^2)$.