Introduction to Quantization–Problems 2

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Problem 0.1. *Find the $x,p-$, $p,x-$, Weyl, Wick, anti-Wick symbols of the following operators:*

1. $\hat{x}^2+\hat{p}^2$, $(\hat{x}^2+\hat{p}^2)^2$;
2. $\hat{x}\hat{p}$, $(\hat{x}\hat{p})^2$.

Problem 0.2. *Let $a\in C^\infty(\mathbb{R}^d\oplus\mathbb{R}^d)$ and $a(x,p)$ is everywhere nonzero. Using the semiclassical ($\hbar$-dependent) quantization $\mathrm{Op}_\hbar$, show that*

$$\mathrm{Op}_\hbar(a)\mathrm{Op}_\hbar(a^{-1})=\mathbb{1}+\mathrm{Op}_\hbar(r),$$

where r is of order 2 in $\hbar$. Compute the principal and subprincipal symbol of r.

Problem 0.3. *Let $a,b\in C_\mathrm{c}^\infty(\mathbb{R}^d\oplus\mathbb{R}^d)$ and $a=1$ on $\mathrm{supp}b$. Show that*

$$\mathrm{Op}_\hbar(a)\mathrm{Op}_\hbar(b)=\mathrm{Op}(b)+O(\hbar^\infty).$$

Problem 0.4. *Given $a\in C_\mathrm{c}^\infty(\mathbb{R}^d\oplus\mathbb{R}^d)$, compute b:*

1. $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{p}^2}\mathrm{Op}_\hbar(a)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{p}^2}=\mathrm{Op}_\hbar(b)$,
2. $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\hat{x}^2}\mathrm{Op}_\hbar(a)\mathrm{e}^{-\frac{\mathrm{i}}{\hbar}\hat{x}^2}=\mathrm{Op}_\hbar(b)$,
3. $\mathcal{F}_\hbar\mathrm{Op}_\hbar(a)\mathcal{F}_\hbar^{-1}=\mathrm{Op}_\hbar(b)$,
4. $\mathrm{e}^{\frac{\mathrm{i}t}{\hbar}(\hat{p}^2+\hat{x}^2)}\mathrm{Op}_\hbar(a)\mathrm{e}^{-\frac{\mathrm{i}t}{\hbar}(\hat{p}^2+\hat{x}^2)}=\mathrm{Op}_\hbar(b)$,

Problem 0.5. *Compute b:*

1. $\mathrm{Op}(b)=\hat{x}^2\hat{p}^2+\hat{p}^2\hat{x}^2$,
2. $\mathrm{Op}(b)=\mathrm{Op}(xp)^2-\mathrm{Op}(x^2p^2)$,

3. $\mathrm{Op}(x^2+p^2)^2 - \mathrm{Op}\big((x^2+p^2)^2\big)$.

Problem 0.6. *Compute the integral kernel of*

1. $\mathrm{Op}\Big(\frac{1}{x^2+p^2+1}\Big)$,
2. $\mathrm{Op}(\mathrm{e}^{-|x|-|p|})$.

Problem 0.7. *Suppose that the operator A on $L^2(\mathbb{R})$ has the integral kernel $A(x,y)$ and the symbol $a(x,p)$. What is the integral kernel and the symbol of*

$$[\hat{x}, A], \quad [\hat{p}, A]?$$

Problem 0.8. *Compute the Wick symbol of the operators $\hat{x}^n$ and $\hat{p}^n$. For which n it is equal to x^n and p^n?*

Problem 0.9. *Compute the anti-Wick quantization of x^n, p^n. For which n it is equal to $\hat{x}^n$ and $\hat{p}^n$?*

Problem 0.10. *Prove that if $f, x^n f \in L^1(\mathbb{R})$, then $\mathrm{e}^{\mathrm{i}t\Delta} f$ is n times differentiable.*

Problem 0.11. *Prove that if $f, f^{(n)} \in L^1$, then $|\hat{f}(\xi)| \le c\langle\xi\rangle^{-n}$.*

Problem 0.12. *What is the semiclassical prediction of the number of eigenvalues below the energy E for the following Hamiltonians on $L^2(\mathbb{R}^3)$:*

(1) $-\Delta + (\vec{x})^2$,

(2) $-\Delta - \frac{1}{|\vec{x}|}$,

(3) $-\Delta - \theta(1-|\vec{x}|)$.

Problem 0.13. *For what $\alpha, \beta \ge 0$ the operator* $\mathrm{Op}(\mathrm{e}^{-\alpha x^2 - \beta p^2})$

(1) *is proportional to a projection;*

(2) *is positive;*

(3) *is trace class;*

(4) *is self-adjoint;*

(5) *belongs to Ψ^0_{00};*

Problem 0.14. *Let $\alpha, \beta, \gamma \in \mathbb{R}$.*

(1) *Does $a(x,p) = \mathrm{e}^{\mathrm{i}\alpha x^2 + \mathrm{i}\beta xp + \mathrm{i}\gamma p^2}$ belong to S^0_{00}?*

(2) *Show that if $4\alpha\gamma = \beta^2$, then $\mathrm{Op}(a)$ is unitary.*

(3) *Show that $\mathrm{Op}(a)$ is always proportional to a unitary operator.*

Problem 0.15. *Let $a(x) \neq 0$ everywhere. Consider the expansion*

$$(1 + a\Delta)^{-1} \simeq \sum_{n=-\infty}^{-2} \mathrm{Op}(b_n),$$

where $b_n(x,\xi)$ is homogeneous in ξ of degree n. Compute b_2, b_3, b_4.

Problem 0.16. *Which of the following functions belong to $S^m(T^\#\mathbb{R})$?*

(1) $\sqrt{1+\xi^2}$,

(2) $\frac{\sqrt{1+x^2+\xi^2}}{\sqrt{1+x^2}}$,

(3) $(1+x^2+\xi^2)^{-\frac{1}{2}}$,

(4) $\mathrm{e}^{-\xi^2(1+x^2)}$,

(5) $\mathrm{e}^{\mathrm{i}\xi^2}$.

Problem 0.17. *Consider the following operators on $\mathbb{R}^2$*

(1) $-\mathrm{i}\partial_x - \partial_y^2$,

(2) $-\Big(\partial_x + \frac{\mathrm{i}y}{\sqrt{x^2+y^2+1}}\Big)^2 - \Big(\partial_y - \frac{\mathrm{i}x}{\sqrt{x^2+y^2+1}}\Big)^2$,

(3) $\partial_x\partial_y$,

(4) $\frac{1}{\sqrt{x^2+y^2+1}}(x\partial_x + y\partial_y)$,

(5) $(1+x^2)\partial_x^2 + (1+y^2)\partial_y^2 + x^2 + y^2$.

a *What are their principal symbols?*

b *What are their subprincipal symbols?*

c *Which ones are elliptic?*

d *If they are not elliptic, what are the subsets of $\mathbb{R}^2$ where they are elliptic?*

e *If they are not elliptic, what are the conical subsets of $\mathrm{T}^\#\mathbb{R}^2$ where they are elliptic?*

Problem 0.18. *What is the singular support and the wave front set of the following distributions on $\mathbb{R}^2$ with the coordinates denoted (x, y):*

(1) $\delta(x)$,

(2) $\mathrm{e}^{-\sqrt{x^2+y^2}}$,

(3) $|x+y|$

(4) $\theta(1-x^2-y^2)$,

(5) $\frac{1}{x-\mathrm{i}0} := \lim_{\epsilon\searrow 0}\frac{1}{x-\mathrm{i}\epsilon}$,

(6) $\frac{1}{(x-\mathrm{i}0)(y-\mathrm{i}0)}$.

(7) $\frac{1}{(x-\mathrm{i}0)}|y|$.