SCATTERING THEORY IN NONRELATIVISTIC QUANTUM FIELD THEORY

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BASIC ABSTRACT SCATTERING THEORY

We are given two self-adjoint operators H_0 and $H = H_0 + V$.

The Møller (or wave) operators (if they exist) are defined as

$$S^{\pm} := \text{s-}\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}tH_0}\,.$$

They satisfy $S^{\pm} H_0 = H S^{\pm}$ and are isometric.

Scattering operator

Scattering operator is introduced as

$$S = S^{+*} S^{-}.$$

It satisfies $H_0 S = S H_0$.
If $\mathrm{Ran} S^{+} = \mathrm{Ran} S^{-}$, then it is unitary.

Alternative scattering operator

In the old literature, sometimes one can find a scattering operator of a different kind

$$\tilde{S} = S^+ S^{-*},$$

which satisfies $\tilde{S} H = H \tilde{S}$. Both scattering operators are closely related:

$$\tilde{S}^* = S^- S S^{-*}.$$

Standard interpretation of quantum mechanics

Let $\rho \geq 0$, $\mathrm{Tr}\rho = 1$, be the density matrix representing a state prepared at time t_-.

Let $A = A^*$ represent an observable measured at time t_+.

Expectation of the measurement equals

$$\mathrm{Tr}\, A\, \mathrm{e}^{\mathrm{i}(-t_+ + t_-)H}\, \rho\, \mathrm{e}^{\mathrm{i}(-t_- + t_+)H}\, .$$

Physical interpretation of scattering operator

At time t_- the state $\mathrm{e}^{-\mathrm{i}t_- H_0}\,\rho\,\mathrm{e}^{\mathrm{i}t_- H_0}$ is prepared.

The experimentalist measures at time t_+ the observable $\mathrm{e}^{\mathrm{i}t_+ H_0}\,A\,\mathrm{e}^{-\mathrm{i}t_+ H_0}$.

For $t_- \to -\infty$, $t_+ \to \infty$, the expectation of the measurement converges to

$$\mathrm{Tr}\, A S \rho S^*.$$

Scattering cross-sections in abstract setting I

Assume that the observable A commutes with H_0.

Let P be a projection commuting with H_0. Assume that the experimentalist prepares a state ρ such that $P\rho P = \rho$ (but he does not control ρ more closely).

Let $\sigma_1 < \sigma_2$ be numbers such that

$$\sigma_1 P \leq PSAS^* P \leq \sigma_2 P.$$

(We choose P small enough so that $\sigma_2 - \sigma_1$ is small).

Scattering cross-sections in abstract setting II

Then

$$\begin{aligned}
\sigma_1 \quad &\leq \quad \liminf_{t_- \to -\infty,\ t_+ \to \infty} \mathrm{Tr}\ A\, \mathrm{e}^{\mathrm{i}(t_+ - t_-)H} \rho\, \mathrm{e}^{-\mathrm{i}(t_+ - t_-)H} \\
&\leq \quad \limsup_{t_- \to -\infty,\ t_+ \to \infty} \mathrm{Tr}\ A\, \mathrm{e}^{\mathrm{i}(t_+ - t_-)H} \rho\, \mathrm{e}^{-\mathrm{i}(t_+ - t_-)H} \ \leq \ \sigma_2.
\end{aligned}$$

Thus for any $\epsilon > 0$, there exists T such that for $t_- \leq -T, \quad T \leq t_+$, the expectation value of the measurement lies between $\sigma_1 - \epsilon$ and $\sigma_2 + \epsilon$.

Problem with eigenvalues

It is easy to see that if the standard Møller operators exist and $H_0\Psi = E\Psi$, then $H\Psi = E\Psi$.

In practice, the standard formalism of scattering theory is usually applied to Hamiltonians H_0 which have only absolutely continuous spectrum.

In quantum field theory, typically, both H_0 and H have ground states, and these ground states are different. Thus, standard scattering theory is not applicable. Instead, one can sometimes try other approaches.

Abelian Møller operators

Abelian Møller operators are defined as

$$S^{\pm}_{\mathrm{Ab}} := \mathrm{s-}\lim_{\epsilon \searrow 0} 2\epsilon \int_0^{\infty} \mathrm{e}^{-2\epsilon t}\, \mathrm{e}^{\pm \mathrm{i} t H}\, \mathrm{e}^{\mp \mathrm{i} t H_0}\, \mathrm{d}t.$$

They satisfy $S^{\pm}_{\mathrm{Ab}} H_0 = H S^{\pm}_{\mathrm{Ab}}$, but do not have to be isometric.

If the standard Møller operator exists, then so do the Abelian Møller operators, and they coincide.

Adiabatic Møller operators

Switch on the interaction adiabatically:

$$U_\epsilon(0) = 1, \quad \frac{\mathrm{d}}{\mathrm{d}t} U_\epsilon(t) = \mathrm{i} U_\epsilon(t)(H_0 + \mathrm{e}^{-\epsilon|t|}\, V).$$

One can introduce the adiabatic Møller operators

$$S_{\mathrm{ad}}^{\pm} := \mathrm{w}-\lim_{\epsilon \searrow 0} \lim_{t \to \pm\infty} U_\epsilon(t)\, \mathrm{e}^{-\mathrm{i}tH_0}\,.$$

One expects that $S_{\mathrm{Ab}}^{\pm} = S_{\mathrm{ad}}^{\pm} = S_{\mathrm{ur}}^{\pm}$. (Subscript ur stands for unrenormalized)

Renormalization of Møller operators

Suppose that the vacuum amplitude operators $Z^\pm := S_{\mathrm{ur}}^{\pm *} S_{\mathrm{ur}}^\pm$ has a trivial kernel. Then we can define the renormalized Møller operators

$$S_{\mathrm{rn}}^\pm := S_{\mathrm{ur}}^\pm (Z^\pm)^{-1/2}.$$

They also satisfy $S_{\mathrm{rn}}^\pm H_0 = H S_{\mathrm{rn}}^\pm$ and are isometric.

If $\mathrm{Ran} S_{\mathrm{rn}}^+ = \mathrm{Ran} S_{\mathrm{rn}}^-$, then the renormalized scattering operator

$$S_{\mathrm{rn}} = S_{\mathrm{rn}}^{+*} S_{\mathrm{rn}}^-$$

is unitary and $H_0 S_{\mathrm{rn}} = S_{\mathrm{rn}} H_0$.

Dyson series
for unrenormalized Møller operators

Set $V(t) = \mathrm{e}^{\mathrm{i}tH_0}\, V\, \mathrm{e}^{-\mathrm{i}tH_0}$. Expanding in formal power series we obtain

$$\begin{aligned}
S_{\mathrm{Ab}}^{+} &= \lim_{\epsilon\searrow 0} \sum_{n=0}^{\infty} \int_{\infty>t_n>\cdots>t_1>0} \\
&\quad \mathrm{i}^n\, \mathrm{e}^{-\epsilon t_n}\, V(t_n)\cdots V(t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1, \\
S_{\mathrm{ad}}^{+} &= \lim_{\epsilon\searrow 0} \sum_{n=0}^{\infty} \int_{\infty>t_n>\cdots>t_1>0} \\
&\quad \mathrm{i}^n\, \mathrm{e}^{-\epsilon(t_n+\cdots+t_1)}\, V(t_n)\cdots V(t_1)\mathrm{d}t_n\cdots\mathrm{d}t_1.
\end{aligned}$$

Dyson series for unrenormalized scattering operators

For $S_{\rm ur} := S_{\rm ur}^{+*} S_{\rm ur}^{-}$, after performing the $\epsilon \searrow 0$ limit we get

$$S_{\rm ur}^{+} = \sum_{n=0}^{\infty} \int_{\infty > t_n > \cdots > t_1 > -\infty} \mathrm{i}^n V(t_n) \cdots V(t_1) \mathrm{d}t_n \cdots \mathrm{d}t_1.$$

After expanding each term in Feynman diagrams, this formal expansion is the usual starting point for analysis of scattering amplitudes in quantum field theory.

In our lectures, after a brief outline of scattering for Schrödinger operators, (the best known example of scattering theory), we will discuss scattering for QFT with localized interactions. We will see that it is quite different from the Schrödinger case.

Physical and asymptotic spaces I

Our starting point will be a physical Hilbert space $\mathcal{H}$ and Hamiltonian H.

Our aim will be to guess the asymptotic Hamiltonians $H^{\pm \mathrm{as}}$ and Hilbert spaces $\mathcal{H}^{\pm \mathrm{as}}$ as well as Møller operators $S^{\pm} : \mathcal{H}^{\pm \mathrm{as}} \to \mathcal{H}$, which should be isometric (preferably unitary), and intertwine the asymptotic and physical Hamiltonians, i.e. $HS^{\pm} = S^{\pm} H^{\pm \mathrm{as}}$.

Of course, these conditions do not determine asymptotic spaces, Hamiltonians and Møller operators completely. One needs to use physical intuition to give a natural definitions.

Physical and asymptotic spaces II

One way to define Møller operators is to introduce a natural identification operators $J^{\pm} : \mathcal{H}^{\pm \mathrm{as}} \to \mathcal{H}$ such that

$$S^{\pm} := \mathrm{s-}\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH} J^{\pm} \mathrm{e}^{-\mathrm{i}tH^{\pm \mathrm{as}}}$$

The usual scattering operator $S = S^{+*}S^{-}$ maps $\mathcal{H}^{-\mathrm{as}}$ into $\mathcal{H}^{+\mathrm{as}}$.

The alternative scattering operator $\tilde{S} = S^{+}S^{-*}$ acts on the physical space $\mathcal{H}$.

Art of scattering theory

There is no single set-up of scattering theory. Special set-ups are used e.g. for

1. many-body Schrödinger operators: Enss, Sigal-Soffer, Graf, D.
2. local relativistic QFT: Haag-Ruelle,
3. classical waves.

They will not be discussed here.

SCATTERING THEORY FOR 2-BODY SCHRÖDINGER OPERATORS

Assume that

$$\begin{aligned} H_0 &= -\Delta, \\ H &= -\Delta + V(x). \end{aligned}$$

We say that the potential $V(x)$ is short range if

$$|V(x)| \leq C(1+|x|)^{-1-\mu}, \quad \mu > 0.$$

Then one can show that $S^{\pm} := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}tH_0}$ exist and their ranges are $\mathrm{Ran}1_{\mathrm{c}}(H)$.

The T-matrix

Introduce the T-operator

$$S = 1 + \mathrm{i}T.$$

Let ξ be the momentum variable. Let $\hat{\xi} = \xi|\xi|^{-1}$ be the angular variable. In the momentum representation, the T-operator has the distributional kernel

$$T(\xi_+, \xi_-) \;\; = \;\; \delta(|\xi_+| - |\xi_-|)T(|\xi_+|, \hat{\xi}_+, \hat{\xi}_-).$$

The scattering cross-section at the energy $\lambda^2/2$, incoming angle $\hat{\xi}_-$ and outgoing angle $\hat{\xi}_+$ is defined as

$$\sigma(\lambda, \hat{\xi}_+, \hat{\xi}_-) := |T(\lambda, \hat{\xi}_+, \hat{\xi}_-)|^2.$$

Measuring scattering cross-sections I

Suppose that we prepare a state concentrated around the momentum around ξ_- and measure the probability of finding the particle of momentum around ξ_+, where the energies are the same: $|\xi_-|^2/2 = |\xi_+|^2/2$. If the scattering amplitude is well behaved (sufficiently continuous) then the probability of the measurement is proportional to $\sigma(|\xi_+|, \hat{\xi}_+, \hat{\xi}_-)$.

Measuring scattering cross-sections II

Let us make it more precise. Let D denote the momentum operator. Suppose that we want to measure the observable $a(D)$.

Fix the incoming angle $\eta_- \in S^{d-1}$. Let us assume that $\hat{\xi}_- \mapsto T(|\xi_+|, \hat{\xi}_+, \hat{\xi}_-)$ is continuous at $\hat{\xi}_- = \hat{\eta}_-$, uniformly for $\xi_+ \in \operatorname{supp} a$. Prepare a state whose density matrix has the form

$$\rho(\xi_-, \xi'_-) = \rho_{\mathrm{en}}(|\xi_-|, |\xi'_-|)\rho_{\mathrm{an}}(\hat{\xi}_-, \hat{\xi}'_-).$$

Measuring scattering cross-sections III

Then for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\rho_{\mathrm{an}}(\hat{\xi}_-, \hat{\xi}'_-)$ is supported in the set

$$|\hat{\xi}_- - \hat{\eta}_-| \leq \delta, \quad |\hat{\xi}'_- - \hat{\eta}_-| \leq \delta,$$

then the expectation value of the measurement differs from

$$\int a(\xi_+)\sigma(|\xi_+|, \hat{\xi}_+, \hat{\eta}_-)\rho_{\mathrm{en}}(|\xi_+|, |\xi_+|)|\xi_+|^{d-1}\mathrm{d}\xi_+ \times \int \rho_{\mathrm{an}}(\hat{\xi}_-, \hat{\xi}'_-)\mathrm{d}\hat{\xi}_-\mathrm{d}\hat{\xi}'_-.$$

by at most ϵ.

Long-range potentials

Suppose that the potential satisfies $V = V_\mathrm{l} + V_\mathrm{s}$ where V_s is short-range and

$$|\partial_x^\alpha V_\mathrm{l}| \leq C_\alpha (1 + |x|)^{-|\alpha|-\mu}, \quad \mu > 0.$$

We then say that the potential is long range.

It includes the physically relevant Coulomb potential $V(x) = z|x|^{-1}$. One can show that for such potentials standard Møller operators do not exit. This is one of manifestations of the infra-red problem.

Cross-sections for long-range potentials

In many quantum mechanics textbooks one approximates long-range potentials by a sequence of short-range potentials, e.g., the Coulomb potential by the Yukawa potentials $V_\mu = z\,\mathrm{e}^{-\mu|x|}\,|x|^{-1}$. For short-range potentials one can construct Møller and scattering operators, which leads to scattering cross-sections

$$\sigma_\mu(\lambda, \hat{\xi}_1, \hat{\xi}_2).$$

Then one shows that there exist

$$\lim_{\mu \searrow 0} \sigma_\mu(\lambda, \hat{\xi}_1, \hat{\xi}_2),$$

which is interpreted as the scattering cross-section for V.

Modified Møller operators

There exist better approaches to long-range scattering. One can define modified Møller operators for long-range potentials. For instance, for an appropriate

$$S(t,\xi) = \frac{t\xi^2}{2} + \text{corrections}$$

there exists

$$S_{\mathrm{lr}}^{\pm} := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\,\mathrm{e}^{-\mathrm{i}S(t,D)}\,.$$

It is isometric, $S_{\mathrm{lr}}^{\pm} H_0 = H S_{\mathrm{lr}}^{\pm}$ and $\mathrm{Ran} S_{\mathrm{lr}}^{\pm} = \mathrm{Ran} 1_{\mathrm{c}}(H)$.

Freedom of definition

However, in general there is no canonical choice of $S_{\mathrm{lr}}^{\pm}$. If we have two modified Møller operator $S_{\mathrm{lr},1}^{\pm}$ and $S_{\mathrm{lr},2}^{\pm}$, then there exists a phase $\psi^{\pm}$ such that

$$S_{\mathrm{lr},1}^{\pm} = S_{\mathrm{lr},2}^{\pm}\, \mathrm{e}^{\mathrm{i}\psi^{\pm}(D)}\ .$$

This arbitrariness disappears in scattering cross-sections, which are canonically defined.

Asymptotic momenta

For long-range potentials, there exists a self-adjoint operator $D^{\pm}$ such that, for any $g \in C_c(\mathbb{R}^d)$,

$$g(D^{\pm}) = \mathrm{s}-\lim_{t\to\infty} \mathrm{e}^{\mathrm{i}tH}\, g(D)\, \mathrm{e}^{-\mathrm{i}tH}\, 1_{\mathrm{c}}(H).$$

Unlike modified Møller operators, asymptotic momenta are canonically defined. Modified Møller operators can be introduced as isometric operators satisfying

$$g(D^{\pm}) = S_{\mathrm{lr}}^{\pm} g(D) S_{\mathrm{lr}}^{\pm *}.$$

SECOND QUANTIZATION

1-particle Hilbert space: $\mathcal{Z}$.

Symmetrization/antisymmetrization projections

$$
\begin{aligned}
\Theta_{\mathrm{s}} &:= \frac{1}{n!}\sum_{\sigma\in S_n}\Theta(\sigma),\\
\Theta_{\mathrm{a}} &:= \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{sgn}\sigma\Theta(\sigma).
\end{aligned}
$$

n-particle bosonic/fermionic space: $\otimes_{\mathrm{s/a}}^n\mathcal{Z} := \Theta_{\mathrm{s/a}}\otimes^n\mathcal{Z}$.

Bosonic/fermionic Fock space: $\Gamma_{\mathrm{s/a}}(\mathcal{Z}) := \mathop{\oplus}\limits_{n=0}^{\infty}\otimes_{\mathrm{s/a}}^n\mathcal{Z}$.

Vacuum vector: $\Omega = 1 \in \otimes_{\mathrm{s/a}}^0\mathcal{Z} = \mathbb{C}$.

Creation and annihilation operators

For $z \in \mathcal{Z}$ we define the creation operator

$$a^*(z)\Psi := \sqrt{n+1}z \otimes_{\mathrm{s/a}} \Psi, \quad \Psi \in \otimes_{\mathrm{s/a}}^n \mathcal{Z},$$

and the annihilation operator $a(z) := (a^*(z))^*$.

Traditional notation: identify $\mathcal{Z}$ with $L^2(\Xi)$ for some measure space $(\Xi, \mathrm{d}\xi)$. If z equals a function $\Xi \ni \xi \mapsto z(\xi)$, then

$$a^*(z) = \int z(\xi) a_\xi^* \mathrm{d}\xi, \quad a(z) = \int \overline{z}(\xi) a_\xi \mathrm{d}\xi.$$

Field and Weyl operators

For $f \in \mathcal{Z}$ we introduce field operators

$$\phi(f) := \frac{1}{\sqrt{2}}(a^*(f) + a(f)),$$

and Weyl operators

$$W(f) := \mathrm{e}^{\mathrm{i}\phi(f)} .$$

For later reference note that

$$(\Omega | W(f)\Omega) = \mathrm{e}^{-\|f\|^2/4} .$$

Wick quantization

Let $b \in B\left(\otimes_{\mathrm{s/a}}^{n}\mathcal{Z}, \otimes_{\mathrm{s/a}}^{m}\mathcal{Z}\right)$ with the integral kernel $b(\xi_1, \cdots \xi_m, \xi_n', \cdots, \xi_1')$. The Wick quantization of the polynomial b is the operator

$$
\begin{aligned}
B &= \int b(\xi_1, \cdots \xi_m, \xi_n', \cdots, \xi_1') \\
&\quad a^*(\xi_1) \cdots a^*(\xi_m) a(\xi_n') \cdots a(\xi_1') \mathrm{d}\xi_1, \cdots \xi_n \mathrm{d}\xi_1' \cdots \mathrm{d}\xi_m'.
\end{aligned}
$$

For $\Phi \in \otimes_{\mathrm{s/a}}^{k+m}\mathcal{Z}$, $\Psi \in \otimes_{\mathrm{s/a}}^{k+n}\mathcal{Z}$, it is defined by

$$
(\Phi | B\Psi) = \frac{\sqrt{(n+k)!(m+k)!}}{k!} (\Phi | b \otimes 1_{\mathcal{Z}}^{\otimes k} \Psi).
$$

Second quantization

For an operator q on $\mathcal{Z}$ we define the operator $\Gamma(q)$ on $\Gamma_{\mathrm{s/a}}(\mathcal{Z})$ by

$$\Gamma(q)\Big|_{\otimes_{\mathrm{s/a}}^n \mathcal{Z}} = q \otimes \cdots \otimes q.$$

Similarly, for an operator h we define the operator $\mathrm{d}\Gamma(h)$ by

$$\mathrm{d}\Gamma(h)\Big|_{\otimes_{\mathrm{s/a}}^n \mathcal{Z}} = h \otimes 1^{(n-1)\otimes} + \cdots 1^{(n-1)\otimes} \otimes h.$$

Traditional notation: If h is the multiplication operator by $h(\xi)$, then $\mathrm{d}\Gamma(h) = \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi$.

Note the identity $\Gamma(\mathrm{e}^{\mathrm{i}th}) = \mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}$.

NONRELATIVISTIC QED

Free photons

Let $L^2_{\mathrm{tr}}(\mathbb{R}^3, \mathbb{R}^3)$ describe divergenceless (transversal) square integrable vector fields on $\mathbb{R}^3$. Free photons are described by the Hilbert space $\mathcal{H}_{\mathrm{ph}} := \Gamma_{\mathrm{s}}(L^2_{\mathrm{tr}}(\mathbb{R}^3, \mathbb{R}^3))$ and the Hamiltonian

$$H_{\mathrm{ph}} \;=\; \sum_s \int a_s^*(\xi)|\xi|a_s(\xi)\mathrm{d}\xi.$$

where $e_s(\xi) \cdot \xi = 0$, $e_s(\xi) \cdot e_{s'}(\xi) = \delta_{s,s'}$ are two polarization vectors.

Vector potential

The vector potential is the operator given by

$$A(x) = \sum_s (2\pi)^{-3} \int e_s(\xi) a_s^*(\xi) \frac{\mathrm{e}^{\mathrm{i}x\xi}}{\sqrt{2|\xi|}} \mathrm{d}\xi + \mathrm{hc}$$

Actually, we will need to replace it by the smeared vector potential

$$A_\rho(x) = \sum_s (2\pi)^{-3} \int \rho(\xi) e_s(\xi) a_s^*(\xi) \frac{\mathrm{e}^{\mathrm{i}x\xi}}{\sqrt{2|\xi|}} \mathrm{d}\xi + \mathrm{hc}$$

where $\rho \in C_\mathrm{c}(\mathbb{R}^3)$ is a cutoff equal 1 for $|\xi| < \Lambda$. (In what follows we drop the subscript ρ).

N-body matter Hamiltonians

n identical particles of charge z, mass m,

Bose/Fermi statistics, in a vector potential $A(x)$ and in a scalar potential $V(x)$, are described by the Hilbert space $\Gamma^n_{\mathrm{s/a}}(L^2(\mathbb{R}^d))$ and the Hamiltonian

$$
\begin{aligned}
H^n \;&=\; \sum_{i=1}^{n}\left(\frac{1}{2m}(D_i - zA(x_i))^2 + zV(x_i)\right) \\
&\quad + \sum_{1\le i<j\le n} \frac{z^2}{4\pi}|x_i - x_j|^{-1}.
\end{aligned}
$$

2nd-quantized matter Hamiltonians

Matter can be described in the 2nd quantized formalism, with the Hilbert space $\mathcal{H} = \Gamma_{\mathrm{s/a}}(L^2(\mathbb{R}^d))$ and the Hamiltonian

$$\begin{aligned} H &= \bigoplus_{n=0}^{\infty} H^n \\ &= \int b^*(x) \left(\frac{1}{2}(D - zA(x))^2 + zV(x) \right) b(x) \mathrm{d}x \\ &\quad + \frac{1}{2} \frac{z^2}{4\pi} \int \int b^*(x) b^*(y) |x-y|^{-1} b(y) b(x) \mathrm{d}x \mathrm{d}y. \end{aligned}$$

Matter interacting with photons

Suppose we have a number of species of particles. The total Hilbert space is $\bigotimes_j \mathcal{H}_j \otimes \mathcal{H}_{\mathrm{ph}}$ and the total Hamiltonian is

$$\begin{aligned} H &= \sum_j \int b_j^*(x) \left(\frac{1}{2m_j} (D - z_j A(x))^2 + z_j V(x) \right) b_j(x) \mathrm{d}x \\ &\quad + \frac{1}{2} \sum_{j,k} \frac{z_j z_k}{4\pi} \int \int b_j^*(x) b_k^*(y) |x-y|^{-1} b_k(y) b_j(x) \mathrm{d}x \mathrm{d}y \\ &\quad + \sum_s \int a_s^*(\xi) |\xi| a_s(\xi) \mathrm{d}\xi. \end{aligned}$$

Single particle interacting with photons

A single particle interacting with radiation is described by a Hilbert space $L^2(\mathbb{R}^d) \otimes \mathcal{H}_{\mathrm{ph}}$ and the Hamiltonian (sometimes called the Pauli-Fierz Hamiltonian)

$$\begin{aligned} H &= \frac{1}{2m}\Big((D - zA(x))^2 + zV(x)\Big) \\ &\quad + \sum_s \int a_s^*(\xi)|\xi|a_s(\xi)\mathrm{d}\xi. \end{aligned}$$

It is an example of a Hamiltonian where a small system (a particle) interacts with a large quantum environment (photons).

SCATTERING FOR HAMILTONIANS OF QUANTUM FIELD THEORY

Typical Hamiltonians of QFT have (at least formally) the form

$$
\begin{aligned}
H_\lambda \;:=\;& \int h(\xi) a^*(\xi) a(\xi) \mathrm{d}\xi \\
+\;& \lambda \int \sum_{n,m} v_{n,m}(\xi_1, \cdots \xi_m, \xi_n', \cdots, \xi_1') \\
& a^*(\xi_1) \cdots a^*(\xi_m) a(\xi_n') \cdots a(\xi_1') \mathrm{d}\xi_1, \cdots \xi_m \mathrm{d}\xi_1' \cdots \mathrm{d}\xi_n'
\end{aligned}
$$

where e.g. $h(\xi) = \sqrt{\xi^2 + m^2}$ describes the 1-particle energy. The polynomials should be even in fermionic variables.

Localized interactions

Assume that $v_{n,m}(\xi_1, \cdots \xi_m, \xi'_n, \cdots, \xi'_1)$ are smooth and decay fast in all directions. This is a simplifying assumption, which is not satisfied in most interesting theories. Nevertheless, there are physically relevant examples, where this assumption is fulfilled, besides we can use it as an introductory step before studying more relevant translation invariant systems.

Self-adjointness

We do not worry too much about the self-adjointness of H_λ. If we do not know how to do otherwise, we work with formal power series.

In fact, in the case of fermions there is no problem, since the perturbation is bounded. In the case of bosons, the self-adjointness is OK if the perturbation is of degree 1 or 2 but small enough. Otherwise it can be proven only under special assumptions (e.g. for $P(\phi)_2$ interactions).

Literature

Scattering operator was used in QFT from the very beginning. It was present already in the work of Schwinger, Tomonaga, Feynman and Dyson

Much of mathematical literature about scattering in QFT is old and often not very satisfactory. Let us mention

1. K.O. Friedrichs: "Perturbations of spectra in Hilbert spaces" 1965
2. K. Hepp: "La theorie de la renormalisation" 1969
3. А. С. Шварц: "Математические основы квантовой теории поля " 1975

Interactions that do not polarize the vacuum

Suppose that $v_{n,0} = v_{0,n} = 0$. Then Ω is an eigenvector of both H_0 and H. Then standard wave operators exist, at least formally.

Unfortunately, physically realistic Hamiltonians often polarize the vacuum.

Ground states for localized interactions

One can show, at least formally, that H_λ possesses a ground state $H_\lambda \Omega_\lambda = E_\lambda \Omega_\lambda$,

$$
\begin{aligned}
\Omega_\lambda &= \sum_{n=0}^{\infty} \lambda^n \Omega_n, \\
E_\lambda &= \sum_{n=0}^{\infty} \lambda^n E_n.
\end{aligned}
$$

Møller operators for localized interactions

Unrenormalized Møller operators exist, at least as formal power series

$$
\begin{aligned}
S_{\mathrm{ur}}^{\pm} &= \mathrm{s-}\lim_{\epsilon \searrow 0} 2\epsilon \int_0^\infty \mathrm{e}^{-2\epsilon t}\, \mathrm{e}^{\pm \mathrm{i} t H}\, \mathrm{e}^{\mp \mathrm{i} t (H_0 - E)}\, \mathrm{d}t \\
&= \sum_{n=0}^{\infty} \lambda^n S_{\mathrm{ur},n}^{\pm}.
\end{aligned}
$$

$Z = S_{\mathrm{ur}}^{-*} S_{\mathrm{ur}}^{-} = S_{\mathrm{ur}}^{+*} S_{\mathrm{ur}}^{+}$ is proportional to identity and equals $Z = |(\Omega_\lambda|\Omega)|^2$. The renormalized Møller operators $S_{\mathrm{rn}}^{\pm} := S_{\mathrm{ur}}^{\pm} Z^{-1/2}$ are formally unitary and so is the renormalized scattering operator $S_{\mathrm{rn}} := S_{\mathrm{rn}}^{+*} S_{\mathrm{rn}}^{-}$.

Asymptotic fields for localized interactions

Lehman-Symanzik-Zimmermann introduce an alternative approach based on asymptotic fields

$$\begin{aligned} a^{\pm}_{\lambda}(f) &:= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\, a(\mathrm{e}^{-\mathrm{i}th}\, f)\, \mathrm{e}^{-\mathrm{i}tH}, \\ a^{*\pm}_{\lambda}(f) &:= \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\, a^{*}(\mathrm{e}^{-\mathrm{i}th}\, f)\, \mathrm{e}^{-\mathrm{i}tH}, \end{aligned}$$

(at least as formal power series). They satisfy the usual CCR. Asymptotic annihilation operators kill the perturbed ground state

$$a^{\pm}_{\lambda}(f)\Omega_{\lambda} = 0$$

Møller operator from asymptotic fields

The (renormalized) Møller operators can be defined with help of asymptotic fields

$$S^{\pm}_{\mathrm{rn},\lambda} a^*(f_1) \cdots a^*(f_n)\Omega \;=\; a^{*\pm}_{\lambda}(f_1) \cdots a^{*\pm}_{\lambda}(f_n)\Omega_\lambda$$

They are formally unitary and intertwine the CCR:

$$\begin{aligned} S^{\pm}_{\mathrm{rn},\lambda} a^*(f) &= a^{*\pm}_{\lambda}(f) S^{\pm}_{\mathrm{rn},\lambda}, \\ S^{\pm}_{\mathrm{rn},\lambda} a(f) &= a^{\pm}_{\lambda}(f) S^{\pm}_{\mathrm{rn},\lambda}. \end{aligned}$$

Note that there is no need for renormalization.

Scattering operator from asymptotic fields

The (renormalized) scattering operators can be defined with help of asymptotic fields, even skipping Møller operators, as the unique (up to a phase factor) unitary operators satisfying

$$
\begin{aligned}
\tilde{S}_{\mathrm{rn},\lambda} a_\lambda^{*-}(f) &= a_\lambda^{*+}(f) \tilde{S}_{\mathrm{rn},\lambda}, \\
\tilde{S}_{\mathrm{rn},\lambda} a_\lambda^{-}(f) &= a_\lambda^{+}(f) \tilde{S}_{\mathrm{rn},\lambda}.
\end{aligned}
$$

Translation-invariant interactions

Basic Hamiltonians of QFT have a translation-invariant interaction, and their scattering theory (even just formal) is more complicated. On the level of the interactions this is expressed by a delta function:

$$\begin{aligned} & v_{n,m}(\xi_1, \cdots \xi_m, \xi'_n, \cdots, \xi'_1) \\ = \; & \tilde{v}_{n,m}(\xi_1, \cdots \xi_m, \xi'_n, \cdots, \xi'_1) \\ & \delta(\xi_1 + \cdots + \xi_m - \xi'_n - \cdots - \xi'_1), \end{aligned}$$

SCATTERING THEORY OF VAN HOVE HAMILTONIANS

Let $\xi \mapsto h(\xi) \in [0, \infty[$ describe the energy and $\xi \mapsto z(\xi)$ the interaction. Van Hove Hamiltonian is a self-adjoint operator formally defined as

$$H = \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi + \int \overline{z}(\xi) a_\xi \mathrm{d}\xi + \int z(\xi) a_\xi^* \mathrm{d}\xi.$$

To avoid the ultraviolet problem we will always assume

$$\int_{h \geq 1} |z(\xi)|^2 \mathrm{d}\xi < \infty.$$

Van Hove Hamiltonian
Infrared case A

Let

$$\int_{h<1} \frac{|z(\xi)|^2}{h(\xi)^2} \mathrm{d}\xi \quad < \quad \infty.$$

Introduce the dressing operator

$$U := \exp\left(-a^*(\frac{z}{h}) + a(\frac{z}{h})\right).$$

and the ground state energy

$$E := -\int \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi.$$

Van Hove Hamiltonian
Infrared case A continued

Let

$$H_0 \quad = \quad \int h(\xi) a_\xi^* a_\xi \mathrm{d}\xi.$$

In Case A, the operator H is well defined and, up to a constant, is unitarily equivalent to H_0:

$$H - E = U H_0 U^*$$

Therefore H has the spectrum $[E, \infty[$ and

$$\Psi = \exp\left(-\int \frac{|z(\xi)|^2}{2h(\xi)^2} \mathrm{d}\xi \right) \exp\left(\int a^*(\xi) \frac{z(\xi)}{h(\xi)} \mathrm{d}\xi \right) \Omega.$$

is its unique ground state.

Van Hove Hamiltonian
Infrared case B

Let

$$\begin{aligned}\int_{h<1}\frac{|z(\xi)|^2}{h(\xi)}\mathrm{d}\xi &< \infty;\\ \int_{h<1}\frac{|z(\xi)|^2}{h(\xi)^2}\mathrm{d}\xi &= \infty.\end{aligned}$$

Then H is well defined, has the spectrum $[E,\infty[$, but has no bound states.

Van Hove Hamiltonian
Infrared case C

Let

$$\begin{aligned}
\int_{h<1} |z(\xi)|^2 \mathrm{d}\xi &< \infty; \\
\int_{h<1} \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi &= \infty.
\end{aligned}$$

Then H is well defined, but $\mathrm{sp}H =]-\infty, \infty[$.

Unrenormalized Møller operators for Van Hove Hamiltonians

Assume that h has an absolutely continuous spectrum (as an operator on $L^2(\Xi)$) and Case A or B:

$$\int \frac{|z(\xi)|^2}{h(\xi)} \mathrm{d}\xi \;\;<\;\; \infty.$$

Then there exists

$$S_{\mathrm{ur}}^{\pm} \;\;:=\;\; \mathrm{s}-\lim_{\epsilon\searrow 0} \epsilon \int_0^\infty \mathrm{e}^{-\epsilon t}\, \mathrm{e}^{\mathrm{i}tH}\, \mathrm{e}^{-\mathrm{i}t(H_0+E)}\, \mathrm{d}\xi.$$

We have $S_{\mathrm{ur}}^{\pm} = UZ$, where

$$Z = \exp\left(- \int \frac{|z(\xi)|^2}{h^2(\xi)} \mathrm{d}\xi \right).$$

Renormalized Møller and scattering operators for Van Hove Hamiltonians

In Case A, the vacuum renormalization constant is nonzero and we can renormalize $S_{\mathrm{ur}}^{\pm}$, obtaining the dressing operator:

$$S_{\mathrm{rn}}^{\pm} := S_{\mathrm{ur}}^{\pm} Z^{-1/2} = U.$$

The scattering operator is (unfortunately) trivial:

$$S = S_{\mathrm{rn}}^{+*} S_{\mathrm{rn}}^{-} = 1.$$

Asymptotic fields for Van Hove Hamiltonians

It is easy to see that in Case A, B and C, for $f \in \mathrm{Dom} h^{-1}$, there exist asymptotic fields:

$$a^{\pm}(f) := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\, a(\mathrm{e}^{-\mathrm{i}th}\, f)\, \mathrm{e}^{-\mathrm{i}tH} = a(f) + (f|h^{-1}z),$$

$$a^{*\pm}(f) := \lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\, a^*(\mathrm{e}^{-\mathrm{i}th}\, f)\, \mathrm{e}^{-\mathrm{i}tH} = a^*(f) + (z|h^{-1}f).$$

This allows us to compute that the scattering operator ($\tilde{S} = 1$) even in Case B and C. In Case A the asymptotic representation of the CCR is Fock but in Case B and C it is not.

SPECTRAL PROPERTIES OF PAULI-FIERZ HAMILTONIANS

Let $\mathcal{K}$ be a Hilbert space with a self-adjoint operator K describing the small system. Typical example of K is a Schrödinger operator. Usually, we will assume that K has discrete eigenvalues, which is the case if $K = -\Delta + V(x)$ with $\lim_{|x|\to\infty} V(x) = \infty$. The full Hilbert space will be $\mathcal{H} := \mathcal{K} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))$.

Generalized spin-boson or Pauli-Fierz Hamiltonians

We will discuss at length a class of Hamiltonians, which is often used in physics and mathematics literature to ilustrate basic properties of a small system interacting with bosonic fields.

Let $\xi \mapsto v(\xi) \in B(\mathcal{K})$.
We take, e.g. $h(\xi) := \sqrt{\xi^2 + m^2}$, $m \geq 0$.
Set $H := H_0 + V$ where

$$\begin{aligned} H_0 &= K \otimes 1 + 1 \otimes \int h(\xi) a^*(\xi) a(\xi) \mathrm{d}\xi, \\ V &= \int v(\xi) \otimes a^*(\xi) \mathrm{d}\xi + \mathrm{hc}. \end{aligned}$$

Spectrum of Pauli-Fierz Hamiltonians

Theorem D.-Gérard Assume that $(K + \mathrm{i})^{-1}$ is compact and

$$\int (1 + h(\xi)^{-1})\|v(\xi)\|^2 \mathrm{d}\xi < \infty.$$

Then H is self-adjoint and bounded from below.

If $E := \inf \mathrm{sp} H$, then $\mathrm{sp}_{\mathrm{ess}} H = [E + m, \infty[$.

Ground state of Pauli-Fierz Hamiltonians

Theorem Bach-Fröhlich-Sigal, Arai-Hirokawa, Gérard.
If in addition

$$\int (1 + h(\xi)^{-2})\|v(\xi)\|^2 \mathrm{d}\xi < \infty,$$

then H has a ground state (the infimum of its spectrum is an eigenvalue).

Embedded point spectrum of Pauli-Fierz Hamiltonians

One does not expect that H has point spectrum embedded in its continuous spectrum. In fact, one can often prove for a small nonzero coupling constant that the spectrum of $H_\lambda := H + \lambda V$ in $]E+m, \infty[$ is purely absolutely continuous, e.g. Bach-Fröhlich-Sigal-Soffer.

In particular, if $m = 0$, this means that the only eigenvalue of H_λ is at the bottom of its spectrum. It often can be proven to be nondegenerate.

SCATTERING THEORY OF PAULI-FIERZ HAMILTONIANS I

In the case of Pauli-Fierz Hamiltonians the usual formalism of scattering in QFT does not apply, because of the presence of the small system.

It is convenient to use a version of the LSZ formalism and start with asymptotic fields.

I will follow the formalism of D-Gerard.
Fröhlich-Griesemer-Schlein use a slightly different setup.

Set $\mathcal{Z}_1 := \mathrm{Dom} h^{-1/2} \subset L^2(\mathbb{R}^d)$.

Basic theorem

Theorem D.-Gérard. Let for f from a dense subspace

$$\int_0^\infty \left\| \int \mathrm{e}^{\mathrm{i}th(\xi)} f(\xi)v(\xi)\mathrm{d}\xi + \mathrm{hc} \right\| \mathrm{d}t < \infty.$$

1. for $f \in \mathcal{Z}_1$ there exists

$$W^{\pm}(f) := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH} 1 \otimes W(\mathrm{e}^{-\mathrm{i}th} f)\, \mathrm{e}^{-\mathrm{i}tH};$$

2. $W^{\pm}(f_1)W^{\pm}(f_2) = \mathrm{e}^{-\mathrm{i}\mathrm{Im}(f_1|f_2)} W^{\pm}(f_1 + f_2)$, $f_1, f_2 \in \mathcal{Z}_1$;
3. $\mathbb{R} \ni t \mapsto W^{\pm}(tf)$ is strongly continuous;
4. $\mathrm{e}^{\mathrm{i}tH} W^{\pm}(f)\, \mathrm{e}^{-\mathrm{i}tH} = W^{\pm}(\mathrm{e}^{\mathrm{i}th} f)$;
5. if $H\Psi = E\Psi$, then $(\Psi|W^{\pm}(f)\Psi) = \mathrm{e}^{-\|f\|^2/4} \|\Psi\|^2$.

Asymptotic fields for Pauli-Fierz Hamiltonians

We introduce asymptotic fileds

$$\phi^{\pm}(f) := \frac{\mathrm{d}}{\mathrm{i}\mathrm{d}t} W^{\pm}(tf)\Big|_{t=0}$$

and asymptotic creation/annihilation operators

$$\begin{aligned} a^{*\pm}(f) &:= \frac{1}{\sqrt{2}}(\phi(f) + \mathrm{i}\phi(\mathrm{i}f)), \\ a^{\pm}(f) &:= \frac{1}{\sqrt{2}}(\phi(f) - \mathrm{i}\phi(\mathrm{i}f)). \end{aligned}$$

Asymptotic vacua for Pauli-Fierz Hamiltonians

Two equivalent definitions:

$$\begin{aligned}\mathcal{K}_0^{\pm} &:= \left\{\Psi \ : \ (\Psi|W^{\pm}(f)\Psi) = \mathrm{e}^{-\|f\|^2/4}\,\|\Psi\|^2\right\} \\ &= \left\{\Psi \ : \ a^{\pm}(f)\Psi = 0\right\}.\end{aligned}$$

The last item of the previous theorem can be reformulated as

$$\mathcal{H}_{\mathrm{p}}(H) \subset \mathcal{K}_0^{\pm},$$

where $\mathcal{H}_{\mathrm{p}}(H)$ denotes the span of eigenvectors of H.

Asymptotic Fock representation

Define

$$\mathcal{H}^{\pm}_{[0]} := \mathrm{Span}^{\mathrm{cl}} \left\{ W^{\pm}(f)\Psi \ : \ \Psi \in \mathcal{K}^{\pm}_0, \ f \in \mathcal{Z}_1 \right\}.$$

Then $\mathcal{H}^{\pm}_{[0]}$ is the smallest space containing the asymptotic vacua and invariant wrt asymptotic creation operators.

Asymptotic completeness for massive Pauli-Fierz Hamiltonians

Theorem Assume that $m > 0$. Then

1. Hoegh-Kroehn, D.-Gérard. $\mathcal{H}_{[0]}^{\pm} = \mathcal{H}$, in other words, the asymptotic representations of the CCR are Fock.
2. D.-Gérard. $\mathcal{K}_0^{\pm} = \mathcal{H}_{\mathrm{p}}(H)$, in other words, all the asymptotic vacua are linear combinations of eigenvectors.

Conjectures about asymptotic completeness for massless Pauli-Fierz Hamiltonians

Conjectures. D.-Gérard. Assume that $h(\xi)=|\xi|$ and

$$\int(1+h(\xi)^{-2})\|v(\xi)\|^2\mathrm{d}\xi<\infty.$$

Then

1. $\mathcal{H}_{[0]}^{\pm}=\mathcal{H}$,
2. $\mathcal{K}_0^{\pm}=\mathcal{H}_{\mathrm{p}}(H)$.

Conjecture is true if $\dim\mathcal{K}=1$ (i.e. for van Hove Hamiltonians). It is also true if $v(\xi)=0$ for $|\xi|<\epsilon$, $\epsilon>0$, (as remarked by Fröhlich-Griesemer-Schlein).

Asymptotic Hamiltonian for the asymptotic Fock sector

The operator $K_0^{\pm} := H\Big|_{\mathcal{K}_0^{\pm}}$ describes the energies of asymptotic vacua (bound state energies, if asymptotic completeness is true).

Define the asymptotic space $\mathcal{H}_0^{\pm\mathrm{as}} := \mathcal{K}_0^{\pm} \otimes \Gamma_{\mathrm{s}}(L^2(\mathbb{R}^d))$ and the asymptotic Hamiltonian

$$H_0^{\pm\mathrm{as}} \;:=\; K_0^{\pm} \otimes 1 + 1 \otimes \int h(\xi) a^*(\xi) a(\xi) \mathrm{d}\xi.$$

Møller operators for the asymptotic Fock sector

There exists a unitary operator

$$S_0^{\pm} : \mathcal{H}_0^{\pm\mathrm{as}} \longrightarrow \mathcal{H}_{[0]}^{\pm} \subset \mathcal{H}$$

called the Møller operator (for the asymptotic Fock sector) such that

$$\begin{aligned} & S_0^{\pm}\Psi \otimes a^*(f_1)\cdots a^*(f_n)\ \Omega \\ = \ & a^{*\pm}(f_1)\cdots a^{*\pm}(f_n)\ \Psi, \quad \Psi \in \mathcal{K}_0^{\pm}. \end{aligned}$$

Intertwining properties of Møller operators

We have

$$
\begin{aligned}
S_0^{\pm} 1 \otimes a^*(f) &= a^{*\pm}(f) S_0^{\pm}, \\
S_0^{\pm} 1 \otimes a(f) &= a^{\pm}(f) S_0^{\pm}, \\
S_0^{\pm} H_0^{\pm\mathrm{as}} &= H S_0^{\pm}.
\end{aligned}
$$

Scattering operators for the asymptotic Fock sector

Define

$$S_{00} = S_0^{+*} S_0^{-}.$$

It satisfies $S_{00} H_0^{-\mathrm{as}} = H_0^{+\mathrm{as}} S_{00}$.

If $\mathcal{H}_{[0]}^{+} = \mathcal{H}_{[0]}^{-}$, then S_{00} is unitary on $\mathcal{H}_0^{+\mathrm{as}} = \mathcal{H}_0^{-\mathrm{as}}$.

Relaxation to the ground state I

In practice, one often expects (and sometimes one can prove) that H only absolutely continuous spectrum except for a unique ground state Ψ_{gr}. Thus

$$\mathrm{w}-\lim_{|t|\to\infty} \mathrm{e}^{\mathrm{i}tH} = |\Psi_{\mathrm{gr}})(\Psi_{\mathrm{gr}}|.$$

If in addition asymptotic completeness holds, then the asymptotic space is $\mathcal{H}_0^{\pm\mathrm{as}} = \Gamma_{\mathrm{s}}(\mathcal{Z})$.

Relaxation to the ground state II

Introduce the C^*-algebra

$$\mathfrak{A} := B(\mathcal{K}) \otimes \mathrm{CCR}(\mathcal{Z})$$

where $\mathrm{CCR}(\mathcal{Z}) = \mathrm{Span}^{\mathrm{cl}}\{W(f) \ : \ f \in \mathcal{Z}\}$.

Theorem Assume asymptotic completeness and the absence of bound states except for a unique ground state. Let $A \in \mathfrak{A}$. Then

$$\mathrm{w-}\lim_{|t|\to\infty} \mathrm{e}^{\mathrm{i}tH} A \, \mathrm{e}^{-\mathrm{i}tH} = |\Psi_{\mathrm{gr}})(\Psi_{\mathrm{gr}}| \ \ (\Psi_{\mathrm{gr}}|A\Psi_{\mathrm{gr}}).$$

REPRESENTATIONS OF THE CCR

Let $\mathcal{Y}$ be a real vector space equipped with an antisymmetric form ω. (Usually we assume that ω is symplectic, i.e. is nondegenerate). Let $U(\mathcal{H})$ denote the set of unitary operators on a Hilbert space $\mathcal{H}$. We say that

$$\mathcal{Y} \ni y \mapsto W^{\pi}(y) \in U(\mathcal{H})$$

is a representation of the CCR over $\mathcal{Y}$ in $\mathcal{H}$ if

$$W^{\pi}(y_1)W^{\pi}(y_2) = \mathrm{e}^{-\frac{\mathrm{i}}{2}y_1\omega y_2}\,W^{\pi}(y_1+y_2), \qquad y_1, y_2 \in \mathcal{Y}.$$

Regular representations of the CCR

Let $\mathcal{Y} \ni y \mapsto W^\pi(y)$ be a representation of the CCR. Clearly,

$$\mathbb{R} \ni t \mapsto W^\pi(ty) \in U(\mathcal{H})$$

is a 1-parameter group. We say that a representation of the CCR) is regular if this group is strongly continuous for each $y \in \mathcal{Y}$.

Field operators

Assume that $y \mapsto W^\pi(y)$ is a regular representation of the CCR.

$$\phi^\pi(y) := -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} W^\pi(ty)\Big|_{t=0}.$$

$\phi^\pi(y)$ will be called the field operator corresponding to $y \in \mathcal{Y}$. We have Heisenberg canonical commutation relation

$$[\phi^\pi(y_1), \phi^\pi(y_2)] = \mathrm{i} y_1 \omega y_2.$$

Creation/annihilation operators

Let $\mathcal{Z}$ be a complex vector space with a scalar product $(\cdot|\cdot)$. It has a symplectic form $\mathrm{Im}(\cdot|\cdot)$ Suppose that

$$\mathcal{Z} \ni f \mapsto W^\pi(f) \in U(\mathcal{H})$$

is a regular representation of the CCR. For $f \in \mathcal{Z}$ we introduce creation/annihilation operators

$$a^{\pi*}(f) := \frac{1}{\sqrt{2}}(\phi^\pi(f) + \mathrm{i}\phi^\pi(\mathrm{i}f)), \qquad a^\pi(f) := \frac{1}{\sqrt{2}}(\phi^\pi(f) - \mathrm{i}\phi^\pi(\mathrm{i}f)).$$

They satisfy the usual relations

$$\begin{aligned} &[a^\pi(f_1), a^\pi(f_2)] = 0, \quad [a^{\pi*}(f_1), a^{\pi*}(f_2)] = 0, \\ &[a^\pi(f_1), a^{\pi*}(f_2)] = (f_1|f_2). \end{aligned}$$

Fock representation of the CCR

Consider the creation/annihilation operators acting on the Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$. Then $\phi(f) := \frac{1}{\sqrt{2}}(a^*(f) + a(f))$ are self-adjoint and

$$\mathcal{Z} \ni f \mapsto \exp \mathrm{i}\phi(f)$$

is a regular representation of the CCR called the Fock representation. The vacuum Ω is characterized by either of the following equivalent equations:

$$\begin{aligned} a(f)\Omega &= 0, \qquad f \in \mathcal{Z}; \\ (\Omega| \, \mathrm{e}^{\mathrm{i}\phi(f)} \, \Omega) &= \mathrm{e}^{-\frac{1}{4}(f|f)}, \quad f \in \mathcal{Z}. \end{aligned}$$

Coherent representation of the CCR I

Let g be an antilinear functional on $\mathcal{Z}$ (not necessarily bounded), that is $g \in \mathcal{Z}^*$. Then

$$\mathcal{Z} \ni f \mapsto W_g(f) := W(f)\, \mathrm{e}^{\mathrm{i}\mathrm{Re}(g|f)} \in U(\Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}}))$$

is a regular representation of the CCR called the $[g]$-coherent representation. The corresponding creation/annihilation operators are

$$\begin{aligned} a_g(f) &= a(f) + \frac{1}{\sqrt{2}}(f|g), \\ a_g^*(f) &= a^*(f) + \frac{1}{\sqrt{2}}(g|f). \end{aligned}$$

Coherent representation of the CCR II

The vector Ω is characterized by either of the following equations:

$$
\begin{aligned}
a_g(f)\Omega &= \frac{1}{\sqrt{2}}(f|g)\Omega, \\
(\Omega|W_g(f)\Omega) &= \mathrm{e}^{-\frac{1}{4}(f|f)+\mathrm{i}\mathrm{Re}(f|g)}.
\end{aligned}
$$

The representation $f \mapsto W_g(f)$ is unitarily equivalent to the Fock representation iff g is a bounded functional $g \in \mathcal{Z}^{\mathrm{cpl}}$. More generally, W_{g_1} is equivalent to W_{g_2} iff $g_1 - g_2 \in \mathcal{Z}^{\mathrm{cpl}}$.

Coherent sectors of a CCR representation I

Suppose that

$$\mathcal{Z} \ni f \mapsto W^{\pi}(f) \in U(\mathcal{H})$$

is a representation of the CCR (e.g. obtained by asymptotic limits, so that $\pi = \pm$). Let g be be an antilinear functional on $\mathcal{Z}$. How can we find all subrepresentations of W^{π} equivalent to a multiple of the $[g]$-coherent representation?

Coherent sectors of a CCR representation II

Define

$$\begin{aligned}
\mathcal{K}_g^\pi &:= \{\Psi \in \mathcal{H} \ :\ a^\pi(f)\Psi = \sqrt{2}(g|f)\Psi\} \\
&= \{\Psi \in \mathcal{H} \ :\ (\Psi|W^\pi(f)\Psi) = \|\Psi\|^2 \, \mathrm{e}^{-\frac{1}{4}(f|f)+\mathrm{iRe}(f|g)}\},
\end{aligned}$$

$$\begin{aligned}
\mathcal{H}_{[g]}^\pi &:= \mathrm{Span}^{\mathrm{cl}} \left\{ a^{\pi *}(f_1)\cdots a^{\pi *}(f_1)\Psi \ :\ \Psi \in \mathcal{K}_g^\pi,\ f_i \in \mathcal{Z} \right\} \\
&= \mathrm{Span}^{\mathrm{cl}} \left\{ W^\pi(f)\Psi \ :\ \Psi \in \mathcal{K}_g^\pi,\ f \in \mathcal{Z} \right\}.
\end{aligned}$$

Coherent sectors of a CCR representation III

We define an isometric operator $S_g^\pi : \mathcal{K}_g^\pi \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}}) \to \mathcal{H}$ by

$$
\begin{aligned}
& S_g^\pi \ \Psi \otimes a_g^*(f_1) \cdots a_g^*(f_n)\Omega \\
= \quad & \qquad a^{\pi *}(f_1) \cdots a^{\pi *}(f_n)\Psi,
\end{aligned}
$$

$$
\begin{aligned}
& S_g^\pi \ \Psi \otimes W_g(f)\Omega \\
= \quad & \qquad W^\pi(f)\Psi.
\end{aligned}
$$

Coherent sectors of a CCR representation IV

Theorem.

1. $\mathcal{H}^{\pi}_{[g]}$ is an invariant subspace for W^{π}.
2. $S^{\pi}_{g} : \mathcal{K}^{\pi}_{g} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}}) \to \mathcal{H}^{\pi}_{[g]}$ is unitary.
3. $S^{\pi}_{g}\ 1 \otimes W_{g}(f) = W^{\pi}(f)\ S^{\pi}_{g}$.
4. If U is unitary such that $U 1 \otimes W_{g}(f) = W^{\pi}(f)\ U$, then $\mathrm{Ran} U \subset \mathcal{H}^{\pi}_{[g]}$.

Thus on

$$\bigoplus_{[g]\in\mathcal{Z}^{*}/\mathcal{Z}^{\mathrm{cpl}}} \mathcal{H}^{\pi}_{[g]} \subset \mathcal{H}$$

the representation W^{π} is well understood – it is of the coherent type.

Covariant CCR representations

Let h be a self-adjoint operator on $\mathcal{Z}^{\rm cpl}$ and H a self-adjoint operator on $\mathcal{H}$. We say that (W^π, h, H) is a covariant representation of the CCR iff

$$\mathrm{e}^{\mathrm{i}tH}\, W^\pi(f)\, \mathrm{e}^{-\mathrm{i}tH} = W^\pi(\mathrm{e}^{\mathrm{i}th} f),\ \ f \in \mathcal{Z}.$$

Example. Fock representation, $(W, h, \mathrm{d}\Gamma(h))$:

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma(h)}\, W(f)\, \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma(h)} = W(\mathrm{e}^{\mathrm{i}th} f).$$

Covariant coherent CCR representations

Let $g \in h^{-1}\mathcal{Z}^{\mathrm{cpl}}$. Set $z = \frac{1}{\sqrt{2}}hg$. Introduce the van Hove Hamiltonian

$$\mathrm{d}\Gamma_g(h) := \mathrm{d}\Gamma(h) + a^*(z) + a(z) + (z|h^{-1}z).$$

Then $(W^g, h, \mathrm{d}\Gamma_g(h))$ is covariant:

$$\mathrm{e}^{\mathrm{i}t\mathrm{d}\Gamma_g(h)}\, W_g(f)\, \mathrm{e}^{-\mathrm{i}t\mathrm{d}\Gamma_g(h)} = W_g(\mathrm{e}^{\mathrm{i}th}\, f).$$

This is obvious for $g \in \mathcal{Z}^{\mathrm{cpl}}$, because then

$$\begin{aligned}
\mathrm{d}\Gamma_g(h) &= W(\mathrm{i}g)\mathrm{d}\Gamma(h)W(-\mathrm{i}g),\\
W_g(f) &= W(\mathrm{i}g)W(f)W(-\mathrm{i}g).
\end{aligned}$$

Restricting covariant representation to a Fock sector

Suppose that $\mathcal{Z} \ni f \mapsto W^\pi(f) \in U(\mathcal{H})$ is a representation of the CCR covariant for h, H:

$$\mathrm{e}^{\mathrm{i}tH} W^\pi(f)\, \mathrm{e}^{-\mathrm{i}tH} = W^\pi(\mathrm{e}^{\mathrm{i}th} f).$$

It is easy to restrict it to the Fock sector:

Theorem. $\mathcal{K}_0^\pi$ and $\mathcal{H}_{[0]}^\pi$ are $\mathrm{e}^{\mathrm{i}tH}$-invariant. Let $K_0^\pi := H\Big|_{\mathcal{K}_0^\pi}$ and on $\mathcal{K}_0^\pi \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$ set

$$H_0^\pi = K_0^\pi \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h).$$

Then $HS_0^\pi = S_0^\pi H_0^\pi$.

Restricting covariant representation to a coherent sector I

Theorem. Let $g \in h^{-1/2}\mathcal{Z}$. Then $\mathcal{H}^{\pi}_{[g]}$ is $\mathrm{e}^{\mathrm{i}tH}$-invariant and there exists a unique operator K^{π}_{g} on $\mathcal{K}^{\pi}_{g}$ such that if on $\mathcal{K}^{\pi}_{g} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$ we set

$$H^{\pi}_{g} := K^{\pi}_{g} \otimes 1 + 1 \otimes \mathrm{d}\Gamma_{g}(h),$$

then $HS^{\pi}_{g} = S^{\pi}_{g} H^{\pi}_{g}$.

Restricting covariant representation to a coherent sector II

Thus restricted to $\mathcal{H}^{\pi}_{[g]}$, the covariant representation (W^{π}, h, H) is unitarily equivalent to

$$\big(1 \otimes W^{g}, 1 \otimes h, K^{\pi}_{g} \otimes 1 + 1 \otimes \mathrm{d}\Gamma_{g}(h)\big)\,.$$

In particular, if $g \notin \mathcal{Z}^{\mathrm{cpl}}$, then the Hamiltonian does not have a ground state inside this sector. Nevertheless, inside this sector, we have good control on the dynamics!

SCATTERING THEORY OF PAULI-FIERZ HAMILTONIANS II

Below we reformulate the basic theorem.

Theorem Under the same assumptions as before

1. for $f \in \mathcal{Z}_1$ there exists
$$W^{\pm}(f) := \mathrm{s-}\lim_{t\to\pm\infty} \mathrm{e}^{\mathrm{i}tH}\, 1\otimes W(\mathrm{e}^{-\mathrm{i}th}\, f)\,\mathrm{e}^{-\mathrm{i}tH};$$
2. $\mathcal{Z}_1 \ni f \mapsto W^{\pm}(f)$ are representations of the CCR.
3. These representations are regular.
4. $(W^{\pm}, h, H)$ are covariant.
5. The Fock sector of $W^{\pm}$ contains all eigenvectors of H.

Asymptotic g-coherent subspace

Let $g \in \mathcal{Z}^*$. Then one can define

$$\mathcal{K}_g^{\pm} \quad := \quad \{\Psi \in \mathcal{H} \ : \ (\Psi|W^{\pm}(f)\Psi) = \|\Psi\|^2 \, \mathrm{e}^{-\frac{1}{4}(f|f)+\mathrm{i}\mathrm{Re}(f|g)}\},$$

$$\mathcal{H}_{[g]}^{\pm} \quad := \quad \mathrm{Span}^{\mathrm{cl}} \left\{ W^{\pm}(f)\Psi \ : \ \Psi \in \mathcal{K}_g^{\pi}, \ f \in \mathcal{Z} \right\},$$

as well as the asymptotic Hilbert spaces

$$\mathcal{H}_g^{\pm\mathrm{as}} := \mathcal{K}_g^{\pm} \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}^{\mathrm{cpl}})$$

asymptotic Hamiltonians

$$H_g^{\pm\mathrm{as}} := K_g^{\pm} \otimes 1 + 1 \otimes \mathrm{d}\Gamma(h).$$

g-coherent Møller operators I

The Møller operators $S_g^\pm : \mathcal{H}_g^{\pm\mathrm{as}} \to \mathcal{H}_{[g]}^\pm \subset \mathcal{H}$ intertwine field operators and the Hamiltonians:

$$
\begin{aligned}
S_g^\pm 1 \otimes a_g^*(f) &= a^{*\pm}(f) S_g^\pm, \\
S_g^\pm 1 \otimes a_g(f) &= a^\pm(f) S_g^\pm, \\
S_g^\pm H_g^{\pm\mathrm{as}} &= H S_g^\pm.
\end{aligned}
$$

One can define scattering operator between sectors g_1 and g_2:

$$
S_{g_2,g_1} := S_{g_2}^{+*} S_{g_1}^-.
$$

g-coherent Møller operators II

Define the g-coherent identifier $J_g^{\pm} : \mathcal{H}_g^{\pm \mathrm{as}} \to \mathcal{H}$ by

$$J_g^{\pm} \; \Psi \otimes W_g(f)\Omega = 1 \otimes W(f) \; \Psi.$$

Then we can introduce Møller operators using this identifier:

$$S_g^{\pm} = \mathrm{s-}\lim_{t \to \pm\infty} \mathrm{e}^{\mathrm{i}tH} \, J_g^{\pm} \, \mathrm{e}^{-\mathrm{i}tH_g^{\pm \mathrm{as}}} \; .$$

Incoming/outgoing coherent subspaces

In the physical space we can distinguish the space where asymptotic CCR are coherent:

$$\mathcal{H}^{\pm}_{[\mathrm{coh}]} \quad := \quad \bigoplus_{g \in \mathcal{Z}^* / \mathcal{Z}^{\mathrm{cpl}}} \mathcal{H}^{\pm}_{[g]} \subset \mathcal{H}.$$

We also introduce the corresponding asymptotic spaces

$$\mathcal{H}^{\pm \mathrm{as}}_{\mathrm{coh}} \quad := \quad \bigoplus_{g \in \mathcal{Z}^* / \mathcal{Z}^{\mathrm{cpl}}} \mathcal{H}^{\pm \mathrm{as}}_{g}.$$

Coherent Møller and scattering operators

We have the Møller operators $S_{\mathrm{coh}}^{\pm} : \mathcal{H}_{\mathrm{coh}}^{\pm\mathrm{as}} \to \mathcal{H}_{[\mathrm{coh}]}^{\pm}$

$$S_{\mathrm{coh}}^{\pm} \quad := \quad \bigoplus_{g \in \mathcal{Z}^*/\mathcal{Z}^{\mathrm{cpl}}} S_g^{\pm}.$$

intertwining the asymptotic and the physical Hamiltonian

$$S_{\mathrm{coh}}^{\pm} H_{\mathrm{coh}}^{\pm\mathrm{as}} = H S_{\mathrm{coh}}^{\pm}.$$

Finally, we have an object that is perhaps the most interesting physically: the coherent scattering operator $S_{\mathrm{coh}} : \mathcal{H}_{\mathrm{coh}}^{-\mathrm{as}} \to \mathcal{H}_{\mathrm{coh}}^{+\mathrm{as}}$

$$S_{\mathrm{coh}} := S_{\mathrm{coh}}^{+*} S_{\mathrm{coh}}^{-}.$$

Soft bosons I

Assume that all asymptotic fields are g-coherent for some unbounded g. Typically one can expect that all the unboundedness of g is concentrated at the zero energy, that is for any $\epsilon > 0$, $\|1_{[\epsilon,\infty[}(h)g\| < \infty$. By modifying g we can assume that $1_{[\epsilon,\infty[}(h)g = 0$. The one-particle space can be split as $\mathcal{Z} = \mathcal{Z}_{\leq\epsilon} \oplus \mathcal{Z}_{>\epsilon}$, where

$$\mathcal{Z}_{\leq\epsilon} := 1_{[0,\epsilon]}(h)\mathcal{Z}, \quad \mathcal{Z}_{>\epsilon} := 1_{]\epsilon,\infty[}(h)\mathcal{Z}.$$

Then the Fock space splits as

$$\Gamma_{\mathrm{s}}(\mathcal{Z}) \simeq \Gamma_{\mathrm{s}}(\mathcal{Z}_{\leq\epsilon}) \otimes \Gamma_{\mathrm{s}}(\mathcal{Z}_{>\epsilon}),$$

and the vacuum splits as $\Omega = \Omega_{\leq\epsilon} \otimes \Omega_{>\epsilon}$.

Soft bosons II

The projection

$$P^{\pm}_{\leq\epsilon} := S^{\pm}_{g}\ 1\otimes|\Omega_{>\epsilon})(\Omega_{>\epsilon}|\ S^{\pm *}_{g}$$

projects onto the particle with a cloud of soft bosons of frequency less than ϵ. It is canonically defined and can serve as a substitute of the ground state. In case of the infrared problem

$$\bigcap_{\epsilon>0} \mathrm{Ran} P^{\pm}_{\leq\epsilon} = \{0\}.$$

If the infra-red problem is absent, then

$$\bigcap_{\epsilon>0} \mathrm{Ran} P^{\pm}_{\leq\epsilon} = \mathbb{C}\Psi_{\mathrm{gr}}.$$

CONCLUSION

There exists a flexible mathematical formalism to describe scattering theory for second-quantized Hamiltonians with localized interactions. It can often describe quite difficult situations, involving e.g. an infrared catastrophe. Its key ingredient is the concept of representations of the CCR or CAR.

The situation is much more difficult for translation-invariant Hamiltonians. Rigorous results are very limited (many-body Schrödinger operators Enss, Sigal, Soffer, Graf, D., Haag-Ruelle theory, Compton scattering at weak coupling and small energy Fröhlich-Griesemer-Schlein).