HOMOGENEOUS SCHRÖDINGER OPERATORS AND THE LIE ALGEBRA $sl(2,\mathbb{R})$

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Homogeneous Schrödinger operators, called also Bessel operators, are given by

$$H_m = -\partial_x^2 + \big(-\frac{1}{4} + m^2 \big) \frac{1}{x^2}$$

with the boundary condition $\sim x^{m+\frac{1}{2}}$ near 0. They are very common in applications and have quite sophisticated properties.

I will describe one application: representations of the Lie group $SL(2, \mathbb{R})$ and its universal covering.

Instead of an introduction, let me describe one important situation where Bessel operators appear.

Consider the Laplacian in d dimensions in spherical coordinates:

$$\Delta_d = -\partial_r^2 - \frac{d-1}{r}\partial_r - \frac{1}{r^2}\Delta_{\mathbb{S}^{d-1}}.$$

On spherical harmonics of order ℓ it becomes

$$-\partial_r^2 - \frac{d-1}{r}\partial_r + \frac{\ell(\ell+d-2)}{r^2}.$$

For all d and ℓ it is simply transformed to the Bessel operator with $m := \ell + \frac{d}{2} - 1$:

$$r^{\frac{d}{2}-\frac{1}{2}}\Big(-\partial_r^2 - \frac{d-1}{r}\partial_r + \frac{\ell(\ell+d-2)}{r^2}\Big)r^{-\frac{d}{2}+\frac{1}{2}}.$$
$$= -\partial_r^2 + \Big(m^2 - \frac{1}{4}\Big)\frac{1}{r^2},$$

Thus the Bessel operator for integer and half-integer values of m describes the radial part of the Laplacian in all dimensions.

Let $\alpha \in \mathbb{C}$. Consider the formal expression

$$L_\alpha = -\partial_x^2 + \Big(-\frac{1}{4} + \alpha\Big)\frac{1}{x^2},$$

We would like to interpret it as

a closed (unbounded) operator on $L^2[0,\infty[$.

Two naive interpretations of L_α:

• The minimal operator $L_\alpha^{\min}$: We start from L_α on $C_{\rm c}^\infty]0,\infty[$, and then we take its closure.

• The maximal operator $L_\alpha^{\max}$: We consider the domain consisting of all $f \in L^2[0,\infty[$ such that $L_\alpha f \in L^2[0,\infty[$.

Clearly, $L_\alpha^{\min} \subset L_\alpha^{\max}$.

Let

$$A := \frac{1}{2\mathrm{i}}(x\partial_x + \partial_x x),$$

be the generator of dilations. Clearly,

$$(a^{\mathrm{i}A} f)(x) = \sqrt{a} f(ax), \quad a > 0.$$

We say that B is homogeneous of degree ν if

$$a^{\mathrm{i}A} B a^{-\mathrm{i}A} = a^{\nu} B.$$

Clearly, L_α is homogeneous of degree -2.

We will see that it is often natural to write $\alpha = m^2$, $m \in \mathbb{C}$.
Notice that

$$L_{m^2} x^{\frac{1}{2} \pm m} = 0.$$

Let ξ be a compactly supported cutoff equal 1 around 0.

Note that $x^{\frac{1}{2}+m}\xi$ belongs to $\mathrm{Dom} L_{m^2}^{\max}$ iff $-1 < \mathrm{Re}\, m$.

Therefore, if $|\mathrm{Re}\,(m)| < 1$, then

$$x^{\frac{1}{2}+m}\xi,\ x^{\frac{1}{2}-m}\xi \in \mathrm{Dom} L_{m^2}^{\max}.$$

Theorem 1. • *For* $1 \leq \operatorname{Re} m$, $L_{m^2}^{\min} = L_{m^2}^{\max}$.

- *For* $-1 < \operatorname{Re} m < 1$, $L_{m^2}^{\min} \subsetneq L_{m^2}^{\max}$, *and the codimension of their domains is* 2.
- $(L_{\alpha}^{\min})^* = L_{\overline{\alpha}}^{\max}$. *Hence, for* $\alpha \in \mathbb{R}$, $L_{\alpha}^{\min}$ *is Hermitian.*
- $L_{\alpha}^{\min}$ *and* $L_{\alpha}^{\max}$ *are homogeneous of degree* -2.

Let $\mathrm{Re}\,(m) > -1$. We define the operator H_m to be the restriction of $L_{m^2}^{\max}$ to

$$\mathrm{Dom} L_{m^2}^{\min} + \mathbb{C} x^{\frac{1}{2}+m} \xi.$$

Clearly,

- For $1 \leq \mathrm{Re}\, m$, $L_{m^2}^{\min} = H_m = L_{m^2}^{\max}$.
- For $-1 < \mathrm{Re}\, m < 1$, $L_{m^2}^{\min} \subsetneq H_m \subsetneq L_{m^2}^{\max}$ and the codimension of the domains is 1.

Theorem 2.

- $H_m^* = H_{\overline{m}}$. *Hence, for* $m \in]-1, \infty[$, H_m *is self-adjoint.*
- H_m *is homogeneous of degree* -2.
- $\mathrm{sp}\, H_m = [0, \infty[$.
- $\{\mathrm{Re}\, m > -1\} \ni m \mapsto H_m$ *is a holomorphic family of closed operators.*

For real m this theorem is classic. It was extended to complex m by L. Bruneau, J. D., V. Georgescu.

We can explicitly compute the integral kernels of various functions of H_m. For instance,

$$\mathrm{e}^{-\frac{t}{2}H_m}(x,y) = \sqrt{\frac{2}{\pi t}}\mathcal{I}_m\Big(\frac{xy}{t}\Big)\mathrm{e}^{-\frac{x^2+y^2}{2t}}, \qquad \mathrm{Re}\, t \geq 0;$$

$$\mathrm{e}^{\pm\frac{\mathrm{i}t}{2}H_m}(x,y) = \mathrm{e}^{\pm\mathrm{i}\frac{\pi}{2}(m+1)}\sqrt{\frac{2}{\pi t}}\mathcal{J}_m\Big(\frac{xy}{t}\Big)\mathrm{e}^{\frac{\mp\mathrm{i}x^2\mp\mathrm{i}y^2}{2t}}, \quad \pm\mathrm{Im}\, t \geq 0.$$

One can also compute the kernel of the resolvent, which we also write in two equivalent ways:

$$\frac{1}{(H_m+k^2)}(x,y)=\frac{1}{k}\begin{cases}\mathcal{I}_m(kx)\,\mathcal{K}_m(ky) & \text{if } 0<x<y,\\ \mathcal{I}_m(ky)\,\mathcal{K}_m(kx) & \text{if } 0<y<x,\end{cases}\qquad \operatorname{Re}k>0;$$

$$\frac{1}{(H_m-k^2)}(x,y)=\pm\frac{\mathrm{i}}{k}\begin{cases}\mathcal{J}_m(kx)\,\mathcal{H}_m^{\pm}(ky) & \text{if } 0<x<y,\\ \mathcal{J}_m(ky)\,\mathcal{H}_m^{\pm}(kx) & \text{if } 0<y<x,\end{cases}\qquad \pm\operatorname{Im}k>0.$$

Here we use various kinds of Bessel family functions for dimension 1:

$$\text{the } \textit{modified Bessel function} \quad \mathcal{I}_m(z) := \sqrt{\frac{\pi z}{2}} I_m(z),$$

$$\text{the } \textit{MacDonald function} \quad \mathcal{K}_m(z) := \sqrt{\frac{2z}{\pi}} K_m(z),$$

$$\text{the } \textit{Bessel function} \quad \mathcal{J}_m(z) := \sqrt{\frac{\pi z}{2}} J_m(z),$$

$$\text{the } \textit{Hankel function of the 1st kind} \quad \mathcal{H}^+_m(z) := \sqrt{\frac{\pi z}{2}} H^+_m(z),$$

$$\text{the } \textit{Hankel function of the 2nd kind} \quad \mathcal{H}^-_m(z) := \sqrt{\frac{\pi z}{2}} H^-_m(z).$$

Question 1. The RHS is well defined (and even analytic) for all m. The LHS is well defined only for $\mathrm{Re}\,(m) > -1$. Does the RHS define anything useful when the LHS is ill defined?

Set $\Xi_m(t) = \mathrm{e}^{\mathrm{i}\ln(2)t}\frac{\Gamma(\frac{m+1+\mathrm{i}t}{2})}{\Gamma(\frac{m+1-\mathrm{i}t}{2})}$. Using the operators

$$A := \frac{1}{2\mathrm{i}}(x\partial_x + \partial_x x), \qquad\qquad K := x^2,$$

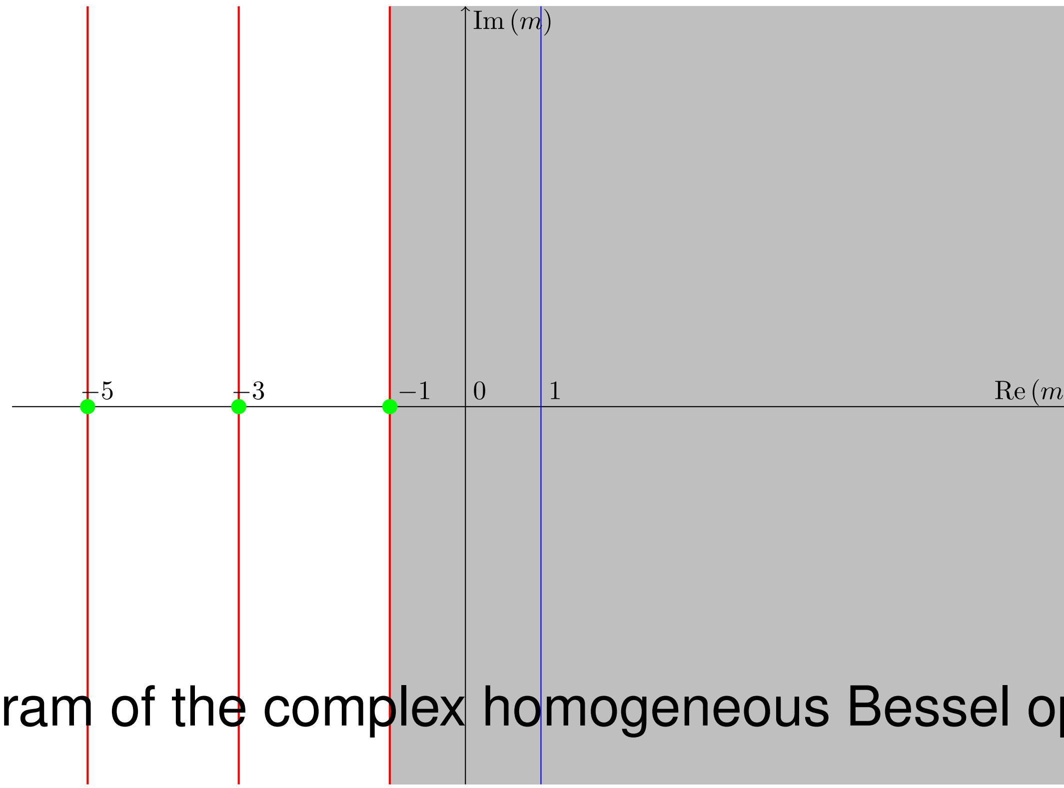
we have

$$H_m := \Xi_m^{-1}(A)K^{-1}\Xi_m(A).$$

$\Xi_m(A)$ is bounded and invertible for all m such that $\mathrm{Re}\, m \neq -1, -3, \dots$. Therefore, the RHS defines a closed operator for all such m.

$\Xi_m(A)$ is unitary for all $m \in \mathbb{R}$. Therefore, for $m \in \mathbb{R}$, the RHS is well-defined and self-adjoint.

Question 2. LHS is defined only in the grey region. RHS is defined everywhere except for the red lines. What is the meaning of the RHS when the LHS is ill defined?



Phase diagram of the complex homogeneous Bessel operator

Let ξ be a compactly supported cutoff equal 1 around 0. Let $n = 1, 2, \ldots$ and $-1 - 2n < \mathrm{Re}\,(m) \leq 1 - 2n$. Define $\mathcal{H}_m$ to be $L^2(\mathbb{R}_+)$ enlarged by adding the n-dimensional space

$$\mathrm{Span}(x^{\frac{1}{2}+m}, \ \ldots, \ x^{\frac{1}{2}+m+2n-4}, \ x^{\frac{1}{2}+m+2n-2}\xi).$$

H_m with the boundary condition $\sim x^{\frac{1}{2}+m}$ is a well-defined closed operator on $\mathcal{H}_m$ and the distributional kernel of $\mathrm{e}^{\frac{\mathrm{i}t}{2}H_m}$ is given by the usual formula.

This answers Question 1.

Actually, if $\mathrm{Re}\,(m) < 1-2n$, then $x^{\frac{1}{2}+m+2n-2}$ is square integrable near infinity. Therefore, it is not necessary to put ξ. For $-1-2n < \mathrm{Re}\,(m) < 1-2n$ (in the nth zone between vertical red lines)

$$\mathrm{Span}(x^{\frac{1}{2}+m}, \dots, x^{\frac{1}{2}+m+2n-4}, x^{\frac{1}{2}+m+2n-2})$$

is a distinguished subspace of $\mathcal{H}_m$ mapped by the operator H_m into itself. Therefore, the operator H_m induces an operator $\tilde{H}_m$ on

$$\mathcal{H}_m/\mathrm{Span}(x^{\frac{1}{2}+m}, \dots, x^{\frac{1}{2}+m+2n-4}, x^{\frac{1}{2}+m+2n-2}) \simeq L^2(\mathbb{R}_+).$$

This answers Question 2.

For $\mathrm{Re}\,(m) \leq -1$, $m \neq -1, -3, \dots$, the space $\mathcal{H}_m$ is equipped with a natural bilinear form

$$\langle f|g\rangle_m := \int_0^\infty x^{2m+1}\big(x^{-2m-1}f(x)g(x)\big)\mathrm{d}x.$$

(Here x^{2m+1} is an irregular distribution!) $\langle\cdot|\cdot\rangle_m$ is compatible with the usual scalar product:

$$\langle\overline{f}|g\rangle_m = (f|g), \quad f, g \in L^2(\mathbb{R}_+) \subset \mathcal{H}_m.$$

The operator H_m is self-transposed wrt $\langle\cdot|\cdot\rangle_m$:

$$\langle f|H_m g\rangle_m = \langle H_m f|g\rangle_m$$
$$= \int_0^\infty x^{2m+1}\big(\partial_x x^{-m-\frac{1}{2}}f(x)\big)\big(\partial_x x^{-m-\frac{1}{2}}g(x)\big)\mathrm{d}x.$$

One can also define realizations of

$$L_{m^2} = -\partial_x^2 + \big(-\frac{1}{4} + m^2\big)\frac{1}{x^2}$$

with mixed boundary conditions $\sim x^{\frac12+m} + \kappa x^{\frac12-m}$. They were studied by many people, including S. Richard and me, but I will not speak about them today, because they are not homogeneous.

Instead, I will describe another class of related interesting operators, which I learned from a paper by K. Andrzejewski. Similar operators (in a more complicated setup) can be also found in papers by A. Vasy.

Let us double the Hilbert space and consider

$$L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+).$$

The first component will be denoted by $f^\uparrow$ and the second by $f^\downarrow$. Let $\omega \in \mathbb{C}$. Consider the operator

$$H_m^\omega := -\partial_x^2 + \big(-\frac{1}{4} + m^2\big)\frac{1}{x^2} \oplus -\Big(-\partial_x^2 + \big(-\frac{1}{4} + m^2\big)\frac{1}{x^2}\Big)$$

with the boundary condition

$$f^\uparrow \sim a_+ x^{\frac{1}{2}+m} + a_- x^{\frac{1}{2}-m}, \quad f^\downarrow \sim a_+ \omega x^{\frac{1}{2}+m} + a_- \omega^{-1} x^{\frac{1}{2}-m}.$$

Then for $-1 < \mathrm{Re}\,(m) < 1$ the operator H_m^ω is closed and one can compute $\mathrm{e}^{\mathrm{i}tH_m^\omega}$.

Let us now change the subject and recall some facts about the Lie algebra of real traceless 2×2 matrices $sl(2, \mathbb{R})$. It is spanned by

$$N = \frac{1}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A_+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

satisfying the commutation relations

$$[N, A_+] = A_+, \quad [N, A_-] = -A_-, \quad [A_+, A_-] = 2N.$$

Now

$$A_+^m := \frac{\mathrm{i}}{2} H_m, \quad N^m = -\frac{\mathrm{i}}{2} A, \quad A_-^m = -\frac{\mathrm{i}}{2} K,$$

satisfy the same commutation relations

$$[N^m, A_+^m] = A_+^m, \quad [N^m, A_-^m] = -A_-^m, \quad [A_+^m, A_-^m] = 2N^m.$$

They define a representation of the Lie algebra $sl(2, \mathbb{R})$ in (unbounded) operators on $L^2(\mathbb{R}_+)$.

$$sl(2, \mathbb{R}) \ni X \mapsto X^m.$$

The operators $\mathrm{i}H_m$, $\mathrm{i}K$, $\mathrm{i}A$ are unbounded, so the precise meaning of their commutators is problematic. However their exponentials are bounded, and the Lie-algebraic relations can be lifted to the level of exponentials:

$$\begin{aligned} a^{\mathrm{i}A}\mathrm{e}^{\frac{\mathrm{i}t}{2}H_m}a^{-\mathrm{i}A} &= \mathrm{e}^{\frac{\mathrm{i}t}{2}a^{-2}H_m}, \\ a^{\mathrm{i}A}\mathrm{e}^{\frac{\mathrm{i}s}{2}K}a^{-\mathrm{i}A} &= \mathrm{e}^{\frac{\mathrm{i}s}{2}a^{2}K}, \\ \mathrm{e}^{\frac{\mathrm{i}t}{2}H_m}\mathrm{e}^{-\frac{\mathrm{i}s}{2}K} &= \mathrm{e}^{-\frac{\mathrm{i}s}{2(1+ts)}K}\mathrm{e}^{\frac{\mathrm{i}t(1+ts)}{2}H_m}(1+ts)^{-\mathrm{i}A}, \quad a>0. \end{aligned}$$

The group $SL(2,\mathbb{R})$ consists of real 2×2 matrices with determinant 1:

$$SL(2,\mathbb{R}) := \left\{ h = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \mid h_{11}h_{22} - h_{12}h_{21} = 1 \right\}.$$

It is homotopic to a circle. Its universal covering will be denoted $\widetilde{SL}(2,\mathbb{R})$.

The representation of the Lie algebra

$$sl(2,\mathbb{R}) \ni X \mapsto X^m$$

can be integrated to a representation of the Lie group

$$\widetilde{SL}(2,\mathbb{R}) \ni h \mapsto h^m \in B(L^2(\mathbb{R}_+)).$$

For brevity let us write $G = SL(2,\mathbb{R})$. Let us partition G into 4 subsets

$$G = (-G_+) \sqcup G_0 \sqcup G_+ \sqcup (-G_0),$$

where

$$G_+ := \{h \in G \mid h_{12} > 0\}, \quad G_0 := \{h \in G \mid h_{12} = 0,\ h_{11} > 0\}.$$

Note that $G \backslash (-G_0)$ is an open neighborhood of the identity $\mathbb{1}$.

We have

$$h = \mathrm{e}^{\frac{h_{21}}{h_{11}} A_-} h_{11}^{2N}, \quad h \in G_0;$$

$$h = \mathrm{e}^{\frac{h_{22}-1}{h_{12}} A_-} \mathrm{e}^{h_{12} A_+} \mathrm{e}^{\frac{h_{11}-1}{h_{12}} A_-}, \quad h \in G_+ \cup (-G_+).$$

Knowing

$$\mathrm{e}^{tN^m}(x,y) = \delta(x\mathrm{e}^{-t} - y)\mathrm{e}^{-\frac{t}{2}}, \quad \mathrm{e}^{tA_-^m}(x,y) = \mathrm{e}^{-\frac{\mathrm{i}}{2}tx^2}\delta(x-y),$$

$$\mathrm{e}^{\frac{\mathrm{i}t}{2}A_+^m}(x,y) = \mathrm{e}^{\frac{\mathrm{i}\pi}{2}(m+1)}\sqrt{\frac{2}{\pi t}}\mathcal{J}_m\left(\frac{xy}{t}\right)\mathrm{e}^{\frac{-\mathrm{i}x^2-\mathrm{i}y^2}{2t}},$$

we can guess the form of a local representation of $SL(2,\mathbb{R})$

$$h^m(x,y) = \begin{cases} \mathrm{e}^{-\frac{\mathrm{i}h_{21}}{2h_{11}}x^2}\delta\Big(\frac{x}{h_{11}} - y\Big)\frac{1}{\sqrt{h_{11}}}; \\ \quad h \in G_0; \\ \\ \mathrm{e}^{\mathrm{i}\frac{\pi}{2}(m+1)\mathrm{sgn}(h_{12})}\sqrt{\frac{2}{\pi|h_{12}|}}\mathcal{J}_m\Big(\frac{xy}{|h_{12}|}\Big)\mathrm{e}^{-\frac{\mathrm{i}}{2h_{12}}(h_{11}x^2+h_{22}y^2)}, \\ \quad h \in G_+ \cup (-G_+). \end{cases}$$

This local representation can be extended to a true representation of $\widetilde{SL}(2,\mathbb{R})$. This representation is unitary for real $m > -1$. For all $\mathrm{Re}\,(m) > -1$ it acts boundedly in $L^2(\mathbb{R}_+)$. For all $m \in \mathbb{C}$ it acts boundedly on $\mathcal{H}_m$.

$$
\begin{aligned}
A_+^{m,\omega} &:= \frac{\mathrm{i}}{2} H_m^{\omega}, \\
N^{m,\omega} &:= -\frac{\mathrm{i}}{2} A \oplus (-\frac{\mathrm{i}}{2} A), \\
A_-^{m,\omega} &:= -\frac{\mathrm{i}}{2} K \oplus \frac{\mathrm{i}}{2} K
\end{aligned}
$$

also satisfy the commutation relations of $sl(2,\mathbb{R})$. For $|\mathrm{Re}\,(m)| < 1$ they can be integrated to a representation of $\widetilde{SL}(2,\mathbb{R})$ on $L^2(\mathbb{R}_+) \oplus L^2(\mathbb{R}_+)$. They can be integrated to a representation of $\widetilde{SL}(2,\mathbb{R})$ for all $m \in \mathbb{C}$ if we extend appropriately the Hilbert space.

Representations of $SL(2,\mathbb{R})$ and $\widetilde{SL}(2,\mathbb{R})$, at least unitary ones, are well-known. They were classified by Bergmann in the 40's, and then studied by Gelfand. Let us recall their basic theory on the level of $sl(2,\mathbb{R})$.

Let us first complexify $sl(2,\mathbb{R})$, obtaining $sl(2,\mathbb{C})$. Let us choose $N, A_+, A_- \in sl(2,\mathbb{C})$ such that $\mathrm{i}N \in sl(2,\mathbb{R})$ and

$$[N, A_+] = A_+, \quad [N, A_-] = -A_-, \quad [A_+, A_-] = 2N.$$

Let $m \in \mathbb{C}$. Then the following operators yield a (complex) representation of $sl(2, \mathbb{C})$:

$$\begin{aligned} A_-^m &= \partial_w - \frac{m+1}{2} w^{-1}, \\ N^m &= w \partial_w, \\ A_+^m &= -w^2 \partial_w - \frac{m+1}{2} w. \end{aligned}$$

The finite dimensional representation of spin l on

$$\left\{w^k \mid k = -l, -l+1, \dots, l\right\}, \qquad l = -\frac{m+1}{2},$$

is well known from the theory of angular momentum. Right now we are not interested in these representations, because they are not dirtectly related to Bessel operators.

We have also

the lowest weight representation, acting on

$$\left\{ w^k \mid k = \frac{m}{2} + \frac{1}{2} + n, \quad n = 0, 1, 2, \dots \right\},$$

the highest weight representation, acting on

$$\left\{ w^k \mid k = \frac{m}{2} - \frac{1}{2} - n, \quad n = 0, 1, 2, \dots \right\}.$$

They integrate to representations of $\widetilde{SL}(2, \mathbb{R})$. For $m \in \mathbb{N}$, also of $SL(2, \mathbb{R})$. In standard texts they are presented in terms of homographies on holomorphic functions on $\mathbb{C}_+$. They can be also presented using the operator H_m.

For each $(m, \eta) \in \mathbb{C}^2$ we have a representation of $sl(2,\mathbb{R})$ on

$$\{w^k \mid k \in \mathbb{Z} + \eta\}.$$

They are generically irreducible and integrate to representations of $\widetilde{SL}(2,\mathbb{R})$. If $\eta = 0$ or $\eta = \frac{1}{2}$, they integrate to representations of $SL(2,\mathbb{R})$. Their standard presentation involves homographies on $\mathbb{R}$.

They can be also presented using the operator H_m^ω with

$$\mathrm{e}^{\mathrm{i}\eta} = \mathrm{e}^{\mathrm{i}\pi(m+1)} \frac{(1 - \mathrm{e}^{-\mathrm{i}\pi m}\omega^2)}{(1 - \mathrm{e}^{\mathrm{i}\pi m}\omega^2)}$$

Messages.

1. If the Hilbert space is too small, enlarge it.
2. Organize your objects in holomorphic families.
3. Non-self-adjoint Schrödinger operators are almost as nice as self-adjoint ones.
4. Non-unitary representations of groups can be almost as good as unitary ones.

Thank you for your attention.