Exactly solvable Schrödinger operators related to the hypergeometric equation

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Abstract

We study one-dimensional Schrödinger operators defined as closed operators that are exactly solvable in terms of the Gauss hypergeometric function. We allow the potentials to be complex. These operators fall into three groups. The first group can be reduced to the Gegenbauer equation, up to an affine transformation, a special case of the hypergeometric equation. The two other groups, which we call *hypergeometric of the first*, resp. *second kind*, can be reduced to the general Gauss hypergeometric equation. Each of the group is subdivided in three families, acting to on the Hilbert space $L^2]-1,1[$, $L^2(\mathbb{R}_+)$ resp. $L^2(\mathbb{R})$. Motivated by geometric applications of these families, we call them *spherical*, *hyperbolic*, resp. *deSitterian*. All these families are known from applications in Quantum Mechanics: e.g. spherical hypergeometric Schrödinger operators of the first kind are often called *trigonometric Pöschl-Teller Hamiltonians*. For operators belonging to each family we compute their spectrum and determine their Green function (the integral kernel of their resolvent). We also describe transmutation identities that relate these Green functions. These identities interchange spectral parameters with coupling constants across different operator families. Finally, we describe how these operators arise from separation of variables of (pseudo-)Laplacians on symmetric manifolds. Our paper can be viewed as a sequel to [DL], where closed realizations of one-dimensional Schrödinger operators solvable in terms Kummer's confluent equation were studied.

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Contents

1 Introduction

One-dimensional Schrödinger operators are operators of the form

$$L := -\partial_x^2 + V(x), \tag{1.1}$$

where $V(x)$ is the *potential*, which in this paper is allowed to be complex-valued. Our paper is devoted to several families of operators of the form (1.1), interpreted as *closed operators* on $L^2(]a,b[)$ for appropriate $-\infty \leq a < b \leq +\infty$, which can be reduced to the *(Gauss) hypergeometric equation*

$$\big(z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab\big)f(z) = 0, \tag{1.2}$$

and whose Green functions can be expressed in terms of the *(Gauss) hypergeometric function.*

We will also consider operators of the form (1.1) that can be reduced to the *Gegenbauer equation.*

$$\Big((1-w^2)\partial_w^2 - 2(1+\alpha)w\partial_w + \lambda^2 - \big(\alpha + \tfrac{1}{2}\big)^2\Big)g(w) = 0. \tag{1.3}$$

The Gegenbauer equation is up to an affine transformation a special case of the hypergeometric equation. Its special property is the mirror symmetry.

Our paper can be viewed as the sequel to [DL], where one of the authors (JD) together with Jinyeop Lee studied a similar problem for the *confluent equation.* We will mostly use the same terminology and methods. We try to make the present paper reasonably self-contained, however the reader is encouraged to consult [DL], especially concerning the general theory of closed realizations of operators of the form (1.1).

In the remaining part of the introduction we give a summary of the results of our paper. In the later section these results will be discussed in detail.

1.1 3×3 families of hypergeometric Hamiltonians

Abusing the terminology, for the sake of brevity, we will use the term *Hamiltonian* for one-dimensional Schrödinger operators. We study three categories of Hamiltonians:

(1) Those reducible to the Gegenbauer equation; they will depend on a single complex parameter, and can be viewed as a subclass of hypergeometric Hamiltonians, both of the first and second kind.

(2) Those reducible to the hypergeometric equation by the substitution $z = \sin^2\frac{r}{2}$ (or similar); they will be called *hypergeometric of the first kind*; they will depend on two complex parameters.

(3) Those reducible to the hypergergeometric equation by the substitution $z = \frac{1}{1+\mathrm{e}^{2r}}$ (or similar); they will be called *hypergeometric of the second kind*; they will depend on two complex parameters.

Within each category we will consider 3 families, which differ by the choice of the interval $]a,b[$. This interval can be viewed as a subset of the complex plane. We will always assume that the endpoints are singular points of the equation. For each of $3\times 3 = 9$ cases, for a set of parameters with a nonempty interior the operator L, defined originally on $C_\mathrm{c}^\infty]a,b[$, possesses a unique closed realization in the sense of $L^2]a,b[$. This realization depends holomorphically on parameters, and extends to a holomorphic family of closed operators on a larger domain. We will call it the *basic family* of closed realizations of L. For some special ranges of parameters there exist other closed realizations of L with *mixed boundary conditions*—we will not consider them in this paper.

For each operator $L_\bullet$ from those families we will find its spectrum, denoted $\sigma(L_\bullet)$. In all cases with real potentials, these operators will be self-adjoint (so that $\sigma(L_\bullet) \subset \mathbb{R}$). More generally, the resolvent set (the complement of the spectrum) of these operators will be nonempty. For z in the resolvent set we will find the resolvent, that is $(L_\bullet - z)^{-1}$, which we will usually denote $\frac{1}{L_\bullet - z}$. The *Green function of* $L_\bullet - z$, that is integral kernel of the resolvent $\frac{1}{L_\bullet - z}$, will be denoted $\frac{1}{(L_\bullet - z)}(x, y)$, with $x, y \in]a, b[$. We will find expressions of Green functions of Hamiltonians from all 3×3 families in terms of the Gamma function and the Gauss hypergeometric function.

These 9 families were discovered in the early days of Quantum Mechanics by physicists trying to find exactly solvable models for various quantum systems. In the literature they are usually named after the researchers who discovered them. Instead of the traditional names, we will prefer to use different names: spherical, hyperbolic and deSitterian. In the spherical case $]a, b[$ is $]-1, 1[$, in the hyperbolic case it is $]0, +\infty[=: \mathbb{R}_+$, and in the deSitterian case it is $\mathbb{R}$.

Our names indicate a major geometric application of these families: spherical, hyperbolic, resp. deSitterian Gegenbauer Hamiltonians appear when we separate variables for the (pseudo-)Laplacian on the sphere, on the hyperbolic space, resp. on the deSitter space.

1.2 Review of 9 families

Let us briefly review the 9 families described in the paper, referring the reader to the main text for precise statements. We will write $\mathcal{A}_{\mathrm{un}}$ for the domain of uniqueness, that is, the set of parameters (a subset of $\mathbb{C}$ or $\mathbb{C} \times \mathbb{C}$) for which there exists a unique closed realization of a given differential expression. This realization in all cases depends holomorphically on its parameters, and extends to a larger domain, denoted $\mathcal{A}_{\mathrm{hol}}$. $\mathcal{A}_{\mathrm{sa}}$ will indicate the set of parameters for which the operator is self-adjoint. Note that the operators are essentially self-adjoint on $C_{\mathrm{c}}^\infty]a, b[$ if and only if the parameter belongs to $\mathcal{A}_{\mathrm{sa}} \cap \mathcal{A}_{\mathrm{un}}$.

1. Gegenbauer Hamiltonians

   (a) Spherical Gegenbauer Hamiltonian, $L^2]0, \pi[$:

   $$L_\alpha^{\mathrm{s}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right) \frac{1}{\sin^2 r}, \tag{1.4}$$
   $$\mathcal{A}_{\mathrm{un}} = \{\operatorname{Re}\alpha \geq 1\}, \quad \mathcal{A}_{\mathrm{hol}} = \{\operatorname{Re}\alpha > -1\}, \quad \mathcal{A}_{\mathrm{sa}} =]-1, +\infty[.$$

   (b) Hyperbolic Gegenbauer Hamiltonian, $L^2(\mathbb{R}_+)$:

   $$L_\alpha^{\mathrm{h}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right) \frac{1}{\sinh^2 r}, \tag{1.5}$$
   $$\mathcal{A}_{\mathrm{un}} = \{\operatorname{Re}\alpha \geq 1\}, \quad \mathcal{A}_{\mathrm{hol}} = \{\operatorname{Re}\alpha > -1\}, \quad \mathcal{A}_{\mathrm{sa}} =]-1, +\infty[.$$

   (c) DeSitterian Gegenbauer Hamiltonian, $L^2(\mathbb{R})$:

   $$L_\alpha^{\mathrm{dS}} := -\partial_r^2 - \left(\alpha^2 - \frac{1}{4}\right) \frac{1}{\cosh^2 r}, \tag{1.6}$$
   $$\mathcal{A}_{\mathrm{un}} = \mathcal{A}_{\mathrm{hol}} = \mathbb{C}, \quad \mathcal{A}_{\mathrm{sa}} = \mathbb{R}.$$

2. Hypergeometric Hamiltonians of the first kind:

   (a) Spherical hypergeometric Hamiltonian of the first kind, $L^2]0, \pi[$:

   $$L_{\alpha,\beta}^{\mathrm{s}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right) \frac{1}{4\sin^2 \frac{r}{2}} + \left(\beta^2 - \frac{1}{4}\right) \frac{1}{4\cos^2 \frac{r}{2}}, \tag{1.7}$$
   $$\mathcal{A}_{\mathrm{un}} = \{\operatorname{Re}\alpha, \operatorname{Re}\beta \geq 1\}, \quad \mathcal{A}_{\mathrm{hol}} = \{\operatorname{Re}\alpha, \operatorname{Re}\beta > -1\}, \quad \mathcal{A}_{\mathrm{sa}} =]-1, +\infty[\times]-1, +\infty[.$$

(b) Hyperbolic hypergeometric Hamiltonian of the first kind, $L^2(]0,\infty[)$:

$$L^{\mathrm{h}}_{\alpha,\beta} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{4\sinh^2\frac{r}{2}} - \left(\beta^2 - \frac{1}{4}\right)\frac{1}{4\cosh^2\frac{r}{2}}, \tag{1.8}$$
$$\mathcal{A}_{\mathrm{un}} = \{\operatorname{Re}\alpha \geq 1\}\times\mathbb{C}, \quad \mathcal{A}_{\mathrm{hol}} = \{\operatorname{Re}\alpha > -1\}\times\mathbb{C}, \quad \mathcal{A}_{\mathrm{sa}} =]-1,+\infty[\times\mathbb{R}.$$

(c) DeSitterian hypergeometric Hamiltonian of the first kind, $L^2(\mathbb{R})$,

$$L^{\mathrm{dS}}_{\alpha,\beta} := -\partial_r^2 - \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}\left(\frac{1}{2} + \frac{\mathrm{i}\sinh r}{2}\right) - \left(\beta^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}\left(\frac{1}{2} - \frac{\mathrm{i}\sinh r}{2}\right), \tag{1.9}$$
$$\mathcal{A}_{\mathrm{un}} = \mathcal{A}_{\mathrm{hol}} = \mathbb{C}\times\mathbb{C}, \quad \mathcal{A}_{\mathrm{sa}} = \{\alpha = \overline{\beta}\}.$$

3. Hypergeometric Hamiltonians of the second kind:

(a) Spherical hypergeometric Hamiltonian of the second kind, $L^2]0,\pi[$:

$$K^{\mathrm{s}}_{\tau,\mu} := -\partial_u^2 + \tau\frac{\cos u}{\sin u} + \left(\frac{\mu^2}{4} - \frac{1}{4}\right)\frac{1}{\sin^2 u}, \tag{1.10}$$
$$\mathcal{A}_{\mathrm{un}} = \mathbb{C}\times\{\operatorname{Re}\mu \geq 2\}, \quad \mathcal{A}_{\mathrm{hol}} = \mathbb{C}\times\{\operatorname{Re}\mu > -2\}\setminus\{(0,-1)\}, \quad \mathcal{A}_{\mathrm{sa}} = \mathbb{R}\times]-2,+\infty[.$$

(b) Hyperbolic hypergeometric Hamiltonian of the second kind, $L^2(]0,\infty[)$:

$$K^{\mathrm{h}}_{\kappa,\mu} := -\partial_u^2 + \kappa\frac{\cosh u}{\sinh u} + \left(\frac{\mu^2}{4} - \frac{1}{4}\right)\frac{1}{\sinh^2 u}, \tag{1.11}$$
$$\mathcal{A}_{\mathrm{un}} = \mathbb{C}\times\{\operatorname{Re}\mu \geq 2\}, \quad \mathcal{A}_{\mathrm{hol}} = \mathbb{C}\times\{\operatorname{Re}\mu > -2\}\setminus\{(0,-1)\}, \quad \mathcal{A}_{\mathrm{sa}} = \mathbb{R}\times]-2,+\infty[.$$

(c) DeSitterian hypergeometric Hamiltonian of the second kind, $L^2(\mathbb{R})$,

$$K^{\mathrm{dS}}_{\kappa,\mu} := -\partial_u^2 - \kappa\frac{\sinh w}{\cosh w} - \left(\frac{\mu^2}{4} - \frac{1}{4}\right)\frac{1}{\cosh^2 w}, \tag{1.12}$$
$$\mathcal{A}_{\mathrm{un}} = \mathcal{A}_{\mathrm{hol}} = \mathbb{C}\times\mathbb{C}, \quad \mathcal{A}_{\mathrm{sa}} = \mathbb{R}\times\mathbb{R}.$$

Gegenbauer Hamiltonians are special cases of both hypergeometric Hamiltonians of the first and second type. In fact, we have the following coincidencies:

$$L^{\mathrm{s}}_{\alpha} = L^{\mathrm{s}}_{\alpha,\alpha} = K^{\mathrm{s}}_{0,2\alpha}; \tag{1.13}$$
$$L^{\mathrm{h}}_{\alpha} = L^{\mathrm{h}}_{\alpha,\alpha} = K^{\mathrm{h}}_{0,2\alpha}; \tag{1.14}$$
$$L^{\mathrm{dS}}_{\alpha} = L^{\mathrm{dS}}_{\alpha,\alpha} = K^{\mathrm{dS}}_{0,2\alpha}. \tag{1.15}$$

Let us also list the following identities that we prove:

$$K^{\mathrm{s}}_{\tau,-1} = K^{\mathrm{s}}_{\tau,1}, \quad \tau \neq 0, \tag{1.16}$$
$$K^{\mathrm{h}}_{\kappa,-1} = K^{\mathrm{h}}_{\kappa,1}. \quad \kappa \neq 0. \tag{1.17}$$

These identities imply that $(0,-1)$ are singularities of the functions $(\tau,\mu)\mapsto K^{\mathrm{s}}_{\tau,\mu}$ and $(\kappa,\mu)\mapsto K^{\mathrm{h}}_{\kappa,\mu}$.

Going back to (1.13) and (1.14), note that

$$L^{\mathrm{s}}_{\alpha} = K^{\mathrm{s}}_{0,2\alpha}, \tag{1.18}$$
$$L^{\mathrm{h}}_{\alpha} = K^{\mathrm{h}}_{0,2\alpha}, \tag{1.19}$$

are the identities for holomorphic functions only for $\alpha \neq -\frac{1}{2}$, because of the above mentioned singularity. Thus the identities

$$L^{\mathrm{s}}_{-\frac{1}{2}} = K^{\mathrm{s}}_{0,-1}, \tag{1.20}$$

$$L^{\mathrm{h}}_{-\frac{1}{2}} = K^{\mathrm{h}}_{0,-1}. \tag{1.21}$$

should be used as *definitions* of $K^{\mathrm{s}}_{0,-1}$ and $K^{\mathrm{h}}_{0,-1}$.

1.3 Transmutations of Green functions

Green functions of distinct Hamiltonians from the above list are linked by identities, which we find quite curious. We call them *transmutation identities*, since the spectral parameter undergoes a change into a coupling constant and the other way around. They follow from various identities satisfied by the hypergeometric function and are similar to the transmutation identities for Hamiltonians related to the confluent equation described in [DL].

Here is the list of transmutation identities considered in our paper. In each case, first we indicate the change of variables involved in a given transmutation. Then we describe two versions of the identity for Green functions.

Proposition 1.1. Gegenbauer spherical — Gegenbauer deSitterian:

$$]0,\pi[\ni r \mapsto q \in \mathbb{R}, \quad \cot r = \sinh q; \tag{1.22}$$

$$\sin^{\frac{1}{2}} r \frac{1}{(L^{\mathrm{s}}_{\lambda} - \alpha^2)}(r,r') \sin^{\frac{1}{2}} r' = \frac{1}{(L^{\mathrm{dS}}_{\alpha} - \lambda^2)}(q,q'), \tag{1.23}$$

$$\frac{1}{(L^{\mathrm{s}}_{\lambda} - \alpha^2)}(r,r') = \cosh^{\frac{1}{2}} q \frac{1}{(L^{\mathrm{dS}}_{\alpha} - \lambda^2)}(q,q') \cosh^{\frac{1}{2}} q'. \tag{1.24}$$

Proposition 1.2. 1st kind spherical — 1st kind hyperbolic:

$$]0,\pi[\ni r \mapsto q \in \mathbb{R}_+, \quad \tan\frac{r}{2} = \sinh\frac{q}{2}; \tag{1.25}$$

$$\left(\cos\frac{r}{2}\right)^{-\frac{1}{2}} \frac{1}{\left(L^{\mathrm{s}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)}(r,r') \left(\cos\frac{r'}{2}\right)^{-\frac{1}{2}} = \frac{1}{\left(L^{\mathrm{h}}_{\alpha,\mu} + \frac{\beta^2}{4}\right)}(q,q'), \tag{1.26}$$

$$\frac{1}{\left(L^{\mathrm{s}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)}(r,r') = \left(\cosh\frac{q}{2}\right)^{-\frac{1}{2}} \frac{1}{\left(L^{\mathrm{h}}_{\alpha,\mu} + \frac{\beta^2}{4}\right)}(q,q') \left(\cosh\frac{q'}{2}\right)^{-\frac{1}{2}}. \tag{1.27}$$

Proposition 1.3. 2nd kind hyperbolic — 2nd kind deSitterian

$$\mathbb{R}_+ \ni u \mapsto w \in \mathbb{R}, \quad \mathrm{e}^{2u} = 1 + \mathrm{e}^{2w}; \tag{1.28}$$

$$(1-\mathrm{e}^{-2u})^{-\frac{1}{2}} \frac{1}{\left(K^{\mathrm{h}}_{\nu-\frac{\beta^2}{2},\mu} + \nu + \frac{\beta^2}{2}\right)}(u,u')(1-\mathrm{e}^{-2u'})^{-\frac{1}{2}} = \frac{1}{\left(K^{\mathrm{dS}}_{\nu+\frac{\mu^2}{2},\beta} + \nu - \frac{\mu^2}{2}\right)}(w,w'), \tag{1.29}$$

$$\frac{1}{\left(K^{\mathrm{h}}_{\nu-\frac{\beta^2}{2},\mu} + \nu + \frac{\beta^2}{2}\right)}(u,u') = (1+\mathrm{e}^{-2w})^{-\frac{1}{2}} \frac{1}{\left(K^{\mathrm{dS}}_{\nu+\frac{\mu^2}{2},\beta} + \nu - \frac{\mu^2}{2}\right)}(w,w')\ (1+\mathrm{e}^{-2w'})^{-\frac{1}{2}}. \tag{1.30}$$

Proposition 1.4. 1st kind spherical — 2nd kind deSitterian

$$]0,\pi[\ni r\mapsto u\in\mathbb{R},\quad \cos r=\tanh u; \tag{1.31}$$

$$\sin^{\frac12} r\ \frac{1}{\left(L_{\alpha,\beta}^{\mathrm{s}}+\frac{\mu^2}{4}\right)}(r,r')\ \ \sin^{\frac12} r'=\frac{1}{\left(K_{\kappa,\mu}^{\mathrm{dS}}+\delta\right)}(u,u'), \tag{1.32}$$

$$\frac{1}{\left(L_{\alpha,\beta}^{\mathrm{s}}+\frac{\mu^2}{4}\right)}(r,r')=\cosh^{\frac12} u\frac{1}{\left(K_{\kappa,\mu}^{\mathrm{dS}}+\delta\right)}(u,u')\cosh^{\frac12} u', \tag{1.33}$$

$$\delta=\frac{\alpha^2+\beta^2}{2},\quad \kappa=\frac{\alpha^2-\beta^2}{2}. \tag{1.34}$$

Proposition 1.5. 1st kind hyperbolic — 2nd kind hyperbolic

$$\mathbb{R}_+\ni r\mapsto u\in\mathbb{R}_+,\quad \cosh r=\coth u; \tag{1.35}$$

$$\sinh^{\frac12} r\ \frac{1}{\left(L_{\alpha,\beta}^{\mathrm{h}}+\frac{\mu^2}{4}\right)}(r,r')\ \ \sinh^{\frac12} r'=\frac{1}{\left(K_{\kappa,\mu}^{\mathrm{h}}+\delta\right)}(u,u'), \tag{1.36}$$

$$\frac{1}{\left(L_{\alpha,\beta}^{\mathrm{h}}+\frac{\mu^2}{4}\right)}(r,r')=\sinh^{\frac12} u\frac{1}{\left(K_{\kappa,\mu}^{\mathrm{h}}+\delta\right)}(u,u')\sinh^{\frac12} u', \tag{1.37}$$

$$\delta=\frac{\alpha^2+\beta^2}{2},\quad \kappa=\frac{\alpha^2-\beta^2}{2}. \tag{1.38}$$

Proposition 1.6. 1st kind deSitterian — 2nd kind spherical

$$\mathbb{R}\ni r\mapsto u\in]0,\pi[,\quad \sinh r=-\cot u; \tag{1.39}$$

$$\cosh^{\frac12} r\ \frac{1}{\left(L_{\alpha,\beta}^{\mathrm{dS}}+\frac{\mu^2}{4}\right)}(r,r')\ \ \cosh^{\frac12} r'=\frac{1}{\left(K_{\tau,\mu}^{\mathrm{s}}+\delta\right)}(u,u'), \tag{1.40}$$

$$\frac{1}{\left(L_{\alpha,\beta}^{\mathrm{dS}}+\frac{\mu^2}{4}\right)}(r,r')=\sin^{\frac12} u\frac{1}{\left(K_{\tau,\mu}^{\mathrm{s}}+\delta\right)}(u,u')\sin^{\frac12} u', \tag{1.41}$$

$$\delta=\frac{\alpha^2+\beta^2}{2},\quad \tau=\frac{\mathrm{i}(\alpha^2-\beta^2)}{2}. \tag{1.42}$$

1.4 Geometric applications

The main original application of hypergeometric Hamiltonians was Quantum Mechanics, as we describe in Subsection 1.6. However, probably the most important context where hypergeometric Hamiltonians appear is geometry, more precisely, the theory of symmetric spaces and Lie groups. In fact, when we separate variables for invariant differential operators, e.g. (pseudo-)Laplacians, on symmetric (pseudo-)Riemannian spaces we often obtain some forms of hypergeometric Hamiltonians.

This fact plays an important role in Quantum Field Theory on curved spacetimes, where Green functions of the d'Alembertian on deSitter and anti-deSitter spaces appear naturally; see e.g. [DeGa].

This geometric interpretation is especially striking for Gegenbauer Hamiltonians. In the following list we show how they arise after separation of variables of various d-dimensional (pseudo-)Laplacians and restriction to $d-1$-dimensional spherical harmonics of degree l. In all cases $\alpha=\frac{d}{2}-1+l$:

1. $\Delta_d^{\rm s}$, Laplacian on unit sphere $\mathbb{S}^d$, reduces to spherical Gegenbauer Hamiltonian $L_\alpha^{\rm s}$:

$$(\sin r)^{\frac{d-1}{2}}\big(-\Delta_d^{\rm s}\big)(\sin r)^{-\frac{d-1}{2}}+\big(\tfrac{d-1}{2}\big)^2=-\partial_r^2+\frac{\big(\frac{d-2}{2}\big)^2-\frac14-\Delta_{d-1}^{\rm s}}{\sin^2 r}. \tag{1.43}$$

2. $\Delta_d^{\rm h}$, Laplacian on hyperbolic space $\mathbb{H}^d$, reduces to hyperbolic Gegenbauer Hamiltonian $L_\alpha^{\rm h}$:

$$(\sinh r)^{\frac{d-1}{2}}\big(-\Delta_d^{\rm h}\big)(\sinh r)^{-\frac{d-1}{2}}-\big(\tfrac{d-1}{2}\big)^2=-\partial_r^2+\frac{\big(\frac{d-2}{2}\big)^2-\frac14-\Delta_{d-1}^{\rm s}}{\sinh^2 r}. \tag{1.44}$$

3. $\Box_d^{\rm dS}$, d'Alembertian on de Sitter space ${\rm dS}^d$, reduces to deSitterian Gegenbauer Hamiltonian $L_\alpha^{\rm dS}$:

$$(\cosh r)^{\frac{d-1}{2}}\Box_d^{\rm dS}(\cosh r)^{-\frac{d-1}{2}}-\big(\tfrac{d-1}{2}\big)^2=-\partial_r^2-\frac{\big(\frac{d-2}{2}\big)^2-\frac14-\Delta_{d-1}^{\rm s}}{\cosh^2 r}. \tag{1.45}$$

All three types of hypergeometric Hamiltonians of the 1st kind have natural geometric interpretations as well. In the following list we show how they arise from separtion of variables in a (pseudo-)Laplacian on a (pseudo-)sphere in "double spherical coordinates". In all three examples the coordinates in the ambient pseudo-Euclidean space are partitioned into two groups.

In the first two cases, these groups are of dimension p and q, and then spherical coordinates are considered within each group. After restriction to products of spherical harmonics of degree l and j a hypergeometric Hamiltonian arises, expressed in the relative variable. We have $\alpha=\frac{p}{2}-1+l$, $\beta=\frac{q}{2}-1+j$.

The third case is somewhat different: $p=q$ is the dimension of holomorphic and antiholomorphic (complex) spherical coordinates. The spherical harmonics are not the usual ones: they are holomorphic and antiholomorphic harmonics on the complex $p-1$-dimensional sphere.

1. $\Delta_{p+q-1}^{\rm s}$, Laplacian on unit sphere $\mathbb{S}^{p+q-1}$, reduces to spherical hypergeometric Hamiltonian of 1st kind $L_{\alpha,\beta}^{\rm s}$:

$$\Big(\sin\frac{r}{2}\Big)^{\frac{p-1}{2}}\Big(\cos\frac{r}{2}\Big)^{\frac{q-1}{2}}\big(-\Delta_{p+q-1}^{\rm s}\big)\Big(\sin\frac{r}{2}\Big)^{-\frac{p-1}{2}}\Big(\cos\frac{r}{2}\Big)^{-\frac{q-1}{2}}+\big(\tfrac{p+q-2}{2}\big)^2 \tag{1.46}$$

$$=4\bigg(-\partial_r^2+\frac{\big(\frac{p-2}{2}\big)^2-\frac14-\Delta_{p-1}^{\rm s}}{4\sin^2\frac{r}{2}}+\frac{\big(\frac{q-2}{2}\big)^2-\frac14-\Delta_{q-1}^{\rm s}}{4\cos^2\frac{r}{2}}\bigg). \tag{1.47}$$

2. $\Delta_{p-1,q}$, pseudo-Laplacian on the hyperboloid $\mathbb{H}^{p-1,q}$, reduces to hyperbolic hypergeometric Hamiltonian of 1st kind $L_{\alpha,\beta}^{\rm h}$:

$$\Big(\cosh\frac{r}{2}\Big)^{\frac{p-1}{2}}\Big(\sinh\frac{r}{2}\Big)^{\frac{q-1}{2}}\big(-\Delta_{p-1,q}\big)\Big(\cosh\frac{r}{2}\Big)^{-\frac{p-1}{2}}\Big(\sinh\frac{r}{2}\Big)^{-\frac{q-1}{2}}-\big(\tfrac{p+q-2}{2}\big)^2 \tag{1.48}$$

$$=4\bigg(-\partial_r^2-\frac{\big(\frac{p-2}{2}\big)^2-\frac14-\Delta_{p-1}^{\rm s}}{4\cosh^2\frac{r}{2}}+\frac{\big(\frac{q-2}{2}\big)^2-\frac14-\Delta_{q-1}^{\rm s}}{4\sinh^2\frac{r}{2}}\bigg). \tag{1.49}$$

3. $\Delta_{p-1,p}$, pseudo-Laplacian on the hyperboloid $\mathbb{H}^{p-1,p}$, reduces to deSitterian hypergeometric Hamiltonian of 1st kind $L_{\alpha,\beta}^{\rm dS}$:

$$\big(\cosh r\big)^{\frac{p-1}{2}}\big(\Delta_{p-1,p}\big)(\cosh r)^{-\frac{p-1}{2}}-(p-1)^2 \tag{1.50}$$

$$=4\bigg(-\partial_r^2-\frac{\big(-\Delta_{p-1}^{\rm s,\mathbb{C}}+(\frac{p-1}{2})^2-\frac14\big)}{2(1+{\rm i}\sinh r)}-\frac{\big(-\overline{\Delta_{p-1}^{\rm s,\mathbb{C}}}+(\frac{p-1}{2})^2-\frac14\big)}{2(1-{\rm i}\sinh r)}\bigg). \tag{1.51}$$

Unfortunately, we have not found a direct geometric interpretation of hypergeometric Hamiltonians of the second kind.

Remark 1.7. *We use the geometric interpretation of Gegenbauer Hamiltonians as the main justification for our names of types: spherical, hyperbolic and deSitterian. For coherence, we extend these names to hypergeometric Hamiltonians of both kinds. The names "spherical" and "hyperbolic" seem quite non-controversial, and are used in the literature in similar contexts. The name "deSitterian" is our invention. Its justification is somewhat weaker and based only on the Gegenbauer Hamiltonian: the connection of deSitterian hypergeometric Hamiltonians of the first kind and de Sitter spaces is less obvious.*

1.5 Boundary conditions

For some potentials the operator L initially defined on $C_{\mathrm{c}}^{\infty}]a,b[$ possesses many closed realizations. These realizations $L_\bullet$ differ only by the behavior of elements of their domain near the endpoints—in other words, they differ by *boundary conditions*. The need for boundary conditions depends on the behavior of the potential V near these endpoints.

Let us consider e.g. the right endpoint b. There are two possibilities:

1. One does not need to impose boundary conditon at b. This will be denoted $\nu_b(L) = 0$.
2. There is a 1-parameter family of boundary conditions at b. This is denoted $\nu_b(L) = 2$.

Analogous definitions are valid for the other endpoint a.

The 9 families considered in this paper illustrate Hamiltonians with various kinds of behaviors of the potential near endpoints. Let us list the behaviors encountered among these 9 families. We restrict ourselves to the right endpoint b, an analogous list applies to the left endpoint a. Our description is somewhat informal; for rigorous statements we refer to [DL].

1. **Short range potential.** $b = +\infty$ and $V(x)$ is integrable near $+\infty$. Then $\nu_b(L) = 0$. Moreover, eigenfunctions in $\mathcal{D}(L_\bullet)$ with eigenvalue $-k^2$, $\mathrm{Re}(k) > 0$ behave as e^{-kx} with $\mathrm{Re}(k) > 0$.
2. **Shifted short range potential.** $b = +\infty$ and $V(x)$ is a constant plus integrable near $+\infty$. Same as before, except that the eigenfunctions $\sim \mathrm{e}^{-kx}$ have energy $-k^2 + V(+\infty)$.
3. **Bessel type.** b is finite and
$$V(x) \sim \Big(m^2 - \frac{1}{4}\Big)\frac{1}{(x-b)^2}. \tag{1.52}$$
Then $\nu_b(L) = 2$ iff $|\mathrm{Re}(m)| < 1$, otherwise $\nu_b(L) = 0$. The behavior of elements of $\mathcal{D}(L_\bullet)$ near b is $|x-b|^{\frac{1}{2}+m}$ for $\mathrm{Re}(m) \geq 1$, a linear combination of $|x-b|^{\frac{1}{2}+m}$ and $|x-b|^{\frac{1}{2}-m}$ for $0 \leq \mathrm{Re}\, m < 1$, $m \neq 0$ and a linear combination of $|x-b|^{\frac{1}{2}}$ and $|x-b|^{\frac{1}{2}} \ln|x-b|$ for $m = 0$. Note that $m = \frac{1}{2}$ corresponds to the Dirichlet b.c. and $m = -\frac{1}{2}$ to the Neumann b.c.

   The families of operators considered in this paper will always have only "homogeneous" or "basic" boundary conditions given by $|x-b|^{\frac{1}{2}+m}$, with $\mathrm{Re} m > -1$. Thus in particular for $|\mathrm{Re}\, m| < 1$, $m \neq 0$, each differential expression will have two closed realizations. We will not consider mixed boundary conditions, which are discussed e.g. in. [GTV, DeGe, DeRi].
4. **Whittaker type.** b is finite and
$$V(x) \sim \left(m^2 - \frac{1}{4}\right)\frac{1}{(x-b)^2} - \frac{\beta}{|x-b|}. \tag{1.53}$$

The conditions are essentially the same as in the Bessel type, with one difference. For $\frac{1}{2} \leq \operatorname{Re} m < 1$, $(\beta, m) \neq \left(0, \frac{1}{2}\right)$ the behavior of functions in the domain of closed realizations of L are linear combinations of $|x-b|^{\frac{1}{2}+m}$ and $|x-b|^{\frac{1}{2}-m}\left(1 - \frac{\beta|x-b|}{1-2m}\right)$.

In the families of operators considered in this paper we will only consider "basic" boundary conditions given by $|x-b|^{\frac{1}{2}+m}\left(1 - \frac{\beta|x-b|}{1-2m}\right)$, with $\operatorname{Re} m > -1$. See e.g. [DL, DeRi]

Potential	Boundary type at a	Boundary type at b
Spherical of the first kind	Bessel	Bessel
Hyperbolic of the first kind	Bessel	Short range potential
Desitterian of the first kind	Short range potential	Short range potential
Spherical of the second type	Whittaker	Whittaker
Hyperbolic of the second type	Whittaker	Short range potential
Desitterian of the second type	Shifted short range potential	Shifted short range potential

1.6 Comparison with literature and historic remarks

One can roughly divide mathematical literature related to the topic of this paper into two parts: algebraic and functional-analytic.

In algebraic papers one considers differential expressions without a Hilbert space setting and without asking for self-adjointness or closedness. Functional analytic papers treat differential operators as unbounded operators on a certain Hilbert space, usually self-adjoint, sometimes only closed.

Needless to say, the algebraic literature is vast. In fact, the hypergeometric equation is one of the most classic subjects of mathematics, with history going back about three centuries. From this category one should mention [CKS, DW] which contain an algebraic analysis of all 9 families of Schrödinger operators solvable in terms of the hypergeometric function.

1-dimensional Schrödinger operators are naturally a special case of Sturm-Liouville operators, whose history goes back to [Lio]. There exists large contemporary literature about self-adjoint or closed realizations of Sturm-Liouville operators, see e.g. [GeZin, GTV, DuSch, EE]. In our paper we use mostly [DeGe], which is summarized in Section 2 of [DL].

Each Sturm-Liouville operator can appear in many equivalent forms, often with distinct names. By a unitary transformation, called the Liouville transformation, essentially each of them can be transformed into a Schrödinger operator, often called its *Liouville form* [Lio].

For instance, in the literature one often considers the Sturm-Liouville operator called the *Jacobi operator*,

$$-(1-x)^{-\alpha}(1+x)^{-\beta}\partial_x(1-x)^{\alpha+1}(1+x)^{\beta+1}\partial_x, \tag{1.54}$$

which acts on the Hilbert space $L^2\big(]-1,1[, (1-x)^{\alpha}(1+x)^{\beta}\big)$ see e.g. [GPLS] and [Koo]. Its eigenfunctions are the famous Jacobi polynomials. By a simple transformation the Jacobi operator is unitarily equivalent to trigonometric Pöschl-Teller Hamiltonian (which we call spherical hypergeometric Hamiltonian of the first kind).

Another common differential operator is the Legendre operator

$$(1+x^2)\partial_x^2 - 2x\partial_x + \mu(\mu+1) - \frac{\alpha^2}{1-x^2}. \tag{1.55}$$

Acting on the Hilbert space $L^2\big(]-1,1[, \sqrt{1-x^2}\big)$, used in the study of spherical harmonics. It is also equivalent to the trigonometric Pöschl-Teller Hamiltonian with $\alpha = \beta$. Needless to say, spherical harmonics possess a very large literature.

The trigonometric Pöschl-Teller Hamiltonian itself is also often studied, see e.g. [FS].

The Scarf Hamiltonian with $\alpha = \beta$ (which we call the deSitterian Gegenbauer Hamiltonian) has also a large literature because of its special properties: for certain values of parameters it is reflectionless.

An interesting review of various exactly solvable Schrödinger operators is contained in [Everitt]. It does not, however, contain all 9 families that we consider.

To our knowledge, a complete analysis of all 9 families interpreted as closed operators, including the formulas for the integral kernels of their resolvents and the computation of their spectra, seems to appear for the first time in the literature. The transmutation formulas described in our paper are probably new. They are analogous to the transmutation formulas for Hamiltonians related to the confluent equation [DL]. Another novelty of our paper are the identities (1.16) and (1.17). They are analogous to an identity for the Whittaker operators described in [DeRi].

Let us briefly outline the history of these Hamiltonians in physics here. In the physics community, the study of Schrödinger operators exactly solvable in terms of hypergeometric functions started in the 1930's when physicists studied biatomic and polyatomic molecules dynamics and the exact solution to the Schrödinger equation. The main reason these physicists considered such Hamiltonians was often due to the limitations of the perturbation method. Therefore, various kinds of exactly solvable Hamiltonians were suggested which fit the experimental data. That was the primary motivation behind the Hamiltonians proposed by Morse [Mor], Rosen-Morse [MoR], Eckart [Eck], and Manning-Rosen [MaR].

The second motivation for this line of research seems to be the search for exact solutions to the Schrödinger equation. In the work of Eckart [Eck] and Rosen-Morse [MoR], it is clearly noted that these potentials are new exactly solvable potentials, although their main motivation was still the study of polyatomic molecules. It seems that for Pöschl and Teller [PT], the motivation leaned more towards the fact that their proposed potential was exactly solvable—they even referred to it as exakt integrierbar (exactly integrable).

The case of the Scarf Hamiltonian (1.9) is a bit different. In the original paper, Scarf did not consider the Hamiltonian in (4.46). We traced this naming to [CKS], where authors refered to it as the hyperbolic Scarf, or Scarf II. However, the origin of this naming is unclear to us.

The uniform study of the Schrödinger equation of hypergeometric type was started by Bose [Bos] and continued by Natanzon [Nat], Ginocchio [Gin], and Milson [Mil]. For a more systematic study and a detailed history of the topic, we refer to [DW]. In a separate line of research, the study of these potentials appeared in the factorization method of Infeld and Hull [HI], and later in the context of supersymmetric quantum mechanics and the so-called shape invariance [CKS, Cot].

Table C presents a comparison of the various names used in the literature and our suggested terminology.

The geometric interpretation of hypergeometric Hamiltonians is closely related to the analysis of hypergeometric equation based on Lie groups and Lie algebras, which possess large literature [M1, V, Wa, D2]. The interpretation of the deSitterian hypergeometric Hamiltonian of the 1st kind in terms of complex spheres in 1.50 seems to be new.

1.7 Plan of the paper

In Sect. 2 we recall the definitions of hypergeometric and Gegenbauer function (the latter, following the conventions of [DGR]), and we sketch the Liouville method that allows us to transform a Sturm-Liouville operator into a 1-dimensional Schrödinger operator.

The core of the paper are the sections 3, 4 and 5, where the 3×3 families of Hamiltonians are introduced and studied. For each member of each family its spectrum and Green functions are computed. We also prove various identities that we already listed in the introduction, including the transmutation identities.

Section 6 is devoted to 1-dimensional Laplacians with various boundary conditions. We explain why they are special cases of hypergeometric Hamiltonians.

In Sect. 7 we describe how Gegenbauer Hamiltonians and hypergeometric Hamiltonians of the first kind arise when we separate variables in (pseudo-)Laplacians on some symmetric spaces.

In Appendix A we collect identities about hypergeometric and Gegenbauer functions that we use in our paper.

In Appendix B we give a very concise account of the theory of closed realizations of 1d Schrödinger operators. This account is incomplete and not fully rigorous—the reader should consult Sect. 2 of [DL] for a more detailed and rigorous exposition, or [DeGe], where a complete theory with proofs is given. Of course, the topic is classic and contained in other texts such as [DuSch, EE, DeGe].

In our paper we try to use notation and conventions that make our formulas, especially for Green functions, as simple, elegant and symmetric as possible. This often motivates us to introduce our conventions, different from the standard ones. In particular, we do not use the standard conventions for associated Legendre functions, which could be used for Gegenbauer Hamiltonians. For readers used to associated Legendre functions, in Appendix C we recall their definitions and compare them with the special functions that we use.

2 Preliminaries

2.1 Hypergeometric equation

The *hypergeometric equation* is given by the *hypergeometric operator*

$$\mathcal{F}(a,b;c;z,\partial_z) := z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab, \tag{2.1}$$

where a,b,c are arbitrary complex parameters. One of solutions of the hypergeometric equation is the famous hypergeometric function $F(a,b;c;z)$. It is usually convenient to apply to $F(a,b;c;z)$ the so-called *Olver's normalization*, which yields the function

$$\mathbf{F}(a,b;c;z) := \frac{F(a,b;c;z)}{\Gamma(c)} = \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{\Gamma(c+j)j!} z^j. \tag{2.2}$$

The hypergeometric equation is closely related to the so-called *Riemann equation*, that is the class of equations on the Riemann sphere $\mathbb{C}\cup\{\infty\}$ having 3 regular singular points. Suppose these points are z_1, z_2, z_3, and the corresponding indices are $\rho_1, \tilde\rho_1$, resp. ρ_2, $\tilde\rho_2$, resp. ρ_3, $\tilde\rho_3$. Then we have the constraint

$$\rho_1 + \tilde\rho_1 + \rho_2 + \tilde\rho_2 + \rho_3 + \tilde\rho_3 = 1. \tag{2.3}$$

Without limiting the generality we can put z_3 at ∞. The corresponding Riemann equation is given by the *Riemann operator*

$$\begin{aligned}
&\mathcal{P}\begin{pmatrix} z_1 & z_2 & \infty & \\ \rho_1 & \rho_2 & \rho_3 & z,\partial_z \\ \tilde\rho_1 & \tilde\rho_2 & \tilde\rho_3 & \end{pmatrix} \\
:= \;& \partial_z^2 - \left(\frac{\rho_1+\tilde\rho_1-1}{z-z_1} + \frac{\rho_2+\tilde\rho_2-1}{z-z_2}\right)\partial_z \\
&+\frac{\rho_1\tilde\rho_1(z_1-z_2)}{(z-z_1)^2(z-z_2)} + \frac{\rho_2\tilde\rho_2(z_2-z_1)}{(z-z_2)^2(z-z_1)} + \frac{\rho_3\tilde\rho_3}{(z-z_1)(z-z_2)}.
\end{aligned} \tag{2.4}$$

Here is the relation between the hypergeometric operator and the Riemann operator:

$$\begin{aligned}
\mathcal{F}(a,b;c;z,\partial_z) &= z(1-z)\mathcal{P}\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z,\partial_z \\ 1-c & c-a-b & b & \end{pmatrix} \\
&= z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab.
\end{aligned} \tag{2.5}$$

Note that the symmetries of the hypergeometric equation are better visible if we replace a, b, c with α, β, μ:

$$\begin{array}{rll} \alpha = c-1 & \beta = a+b-c, & \mu = a-b; \\ a = \frac{1+\alpha+\beta-\mu}{2}, & b = \frac{1+\alpha+\beta+\mu}{2}, & c = 1+\alpha; \end{array} \tag{2.6}$$

so that the hypergeometric equation has the form

$$\mathcal{F}\left(\tfrac{\alpha+\beta+\mu+1}{2}, \tfrac{\alpha+\beta-\mu+1}{2}; 1+\alpha; z, \partial_z\right) F(z) = 0. \tag{2.7}$$

2.2 Gegenbauer equation

The *Gegenbauer equation* is essentially the special case of the hypergeometric equation with the symmetry $w \to -w$ and the singular points put at $-1, 1, \infty$. It is given by the *Gegenbauer operator*

$$\mathcal{G}_{\alpha,\lambda}(w, \partial_w) := (1-w^2)\partial_w^2 - 2(1+\alpha)w\partial_w + \lambda^2 - \big(\alpha + \tfrac{1}{2}\big)^2. \tag{2.8}$$

Here is its relationship to the Riemann operator:

$$\mathcal{G}_{\alpha,\lambda}(w, \partial_w) = (1-w^2)\mathcal{P}\left(\begin{array}{cccc} -1 & 1 & \infty & \\ 0 & 0 & \alpha+\lambda+\frac{1}{2} & w, \partial_w \\ -\alpha & -\alpha & \alpha-\lambda+\frac{1}{2} & \end{array}\right). \tag{2.9}$$

Following [DGR], we introduce two special solutions of the Gegenbauer equation

$$\mathbf{S}_{\alpha,\lambda}(w) = \frac{1}{\Gamma(\alpha+1)} F\left(\frac{1}{2}+\alpha+\lambda, \frac{1}{2}+\alpha-\lambda; \alpha+1; \frac{1-w}{2}\right), \tag{2.10}$$

$$\mathbf{Z}_{\alpha,\lambda}(w) = \frac{(w\pm1)^{-\frac{1}{2}-\alpha-\lambda}}{\Gamma(\lambda+1)} F\Big(\frac{1}{2}+\alpha+\lambda, \frac{1}{2}+\lambda; 2\lambda+1; \frac{2}{1\pm w}\Big). \tag{2.11}$$

2.3 Liouville transformation

An operator of the form

$$-\partial_r^2 + V(r) \tag{2.12}$$

will be called a *(1-dimensional) Schrödinger operator*.

Let us briefly describe how to transform a 2nd order equation

$$\big(p(z)\partial_z^2 + q(z)\partial_z + r(z)\big)\, u(z), \tag{2.13}$$

into an eigenvalue equation of a certain Schrödinger operator. Consider the operator

$$p(z)\partial_z^2 + q(z)\partial_z + r(z), \tag{2.14}$$

that defines the equation (2.13). We first multiply (2.14) from the left by a function f, from the right by a function g, obtaining

$$f(z)\big(p(z)\partial_z^2 + q(z)\partial_z + r(z)\big)g(z). \tag{2.15}$$

We choose f, g in such a way, that (2.15) has the form

$$-t(z)\partial_z^2 - \frac{1}{2}t'(z)\partial_z + v(z), \tag{2.16}$$

for some function $z \mapsto t(z)$. Then we change the variable z into r, such that

$$\left(\frac{\mathrm{d}z}{\mathrm{d}r}\right)^2 = t(z). \tag{2.17}$$

We obtain

$$-\partial_r^2 + v\big(z(r)\big). \tag{2.18}$$

More details can be found in [DW].

Operators of the form (2.14) are often called *Sturm-Liouville operators* and the transformation that leads from (2.14) to (2.18)—a *Liouville transformation.*

For brevity, we will usually use the term *Hamiltonian* instead of *(one-dimesional) Schrödinger operator*. Thus e.g. the *hypergeometric operator* means the operator (2.1), whereas *hypergeometric Hamiltonians* will be various Schrödinger operators obtained by transforming the hypergeometric equation.

3 Gegenbauer Hamiltonians

Let us transform the Gegenbauer operator (2.8) as follows:

$$\begin{aligned} & -(1-w^2)^{\frac{\alpha}{2}+\frac14}\mathcal{G}_{\alpha,\lambda}(w,\partial_w)(1-w^2)^{-\frac{\alpha}{2}-\frac14} \\ = \;& -(1-w^2)\mathcal{P}\begin{pmatrix} -1 & 1 & \infty & \\ \frac{\alpha}{2}+\frac14 & \frac{\alpha}{2}+\frac14 & \lambda & w,\partial_w \\ -\frac{\alpha}{2}+\frac14 & -\frac{\alpha}{2}+\frac14 & -\lambda & \end{pmatrix} \\ = \;& -(1-w^2)\partial_w^2 + w\partial_w + \left(\alpha^2-\frac14\right)\frac{1}{1-w^2} - \lambda^2. \end{aligned} \tag{3.1}$$

Thus if we set

$$L_\alpha := -(1-w^2)\partial_w^2 + w\partial_w + \left(\alpha^2-\frac14\right)\frac{1}{1-w^2}, \tag{3.2}$$

and $G(w)$ solves the Gegenbauer equation, then

$$\big(L_\alpha - \lambda^2\big)(1-w^2)^{\frac{\alpha}{2}+\frac14}G(w) = 0. \tag{3.3}$$

We have reinterpreted the Gegenbauer equation as the eigenequation of a certain operator L_α with the eigenvalue λ^2. It is natural to interpret this operator as acting on functions on an interval contained in $\mathbb{C}$, whose endpoints are singularities of the Gegenbauer equation. In each of these cases we perform the Liouville transformation, which yields a 1-dimensional Hamiltonian. We will consider three cases:

1. $w \in]-1,1[$. This leads to an operator on $L^2]0,\pi[$, which we call the *spherical Gegenbauer Hamiltonian.*
2. $w \in]1,\infty[$. This leads to an operator on $L^2(\mathbb{R}_+)$, which we call the *hyperbolic Gegenbauer Hamiltonian.*
3. $w \in \mathrm{i}\mathbb{R}$. This leads to an operator on $L^2(\mathbb{R})$, which we call the *deSitterian Gegenbauer Hamiltonian.*

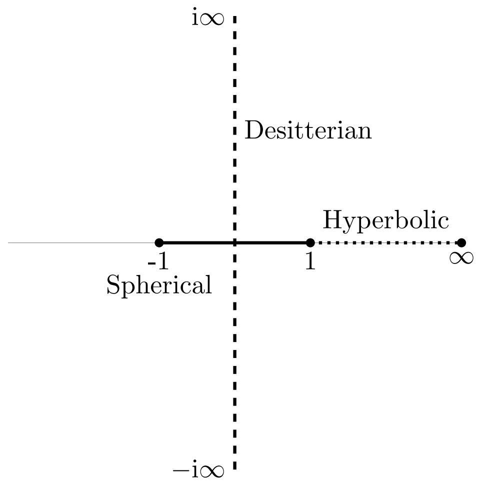

Figure 1: Gegenbauer equation on the w plane. The hyperbolic Hamiltonian acts on the interval marked with a dotted line, the spherical Hamiltonian—with a thick line and the deSitterian Hamiltonian—with a dashed line.

3.1 Spherical case

For $r \in]0, \pi[$, in (3.1) set

$$w = \cos r, \quad \text{which solves} \ \ w' = -(1-w^2)^{\frac{1}{2}}. \tag{3.4}$$

This leads to the Schrödinger equation

$$\left(L_\alpha^{\mathrm{s}} - \lambda^2\right)\phi(r) = 0, \tag{3.5}$$

where

$$L_\alpha^{\mathrm{s}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\sin^2 r}. \tag{3.6}$$

It has the mirror symmetry $r \to \pi - r$. It is obtained when we separate variables of the Laplacian on the sphere in any dimensions, see e.g. Subsect. 7.1. Hence our name “spherical”.

Let us define the function on $]0, \pi[$

$$\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(r) \ := \ \left(\frac{\sin r}{2}\right)^{\alpha+\frac{1}{2}} \mathbf{S}_{\alpha,\lambda}(\cos r). \tag{3.7}$$

It has the following asymptotic behaviour near 0:

$$\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(r) \sim \frac{1}{\Gamma(1+\alpha)}\left(\frac{r}{2}\right)^{\frac{1}{2}+\alpha}. \tag{3.8}$$

The following four functions solve the eigenequation (3.5):

$$\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(r), \quad \mathcal{P}_{-\alpha,\lambda}^{\mathrm{s}}(r), \quad \mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(\pi - r), \quad \mathcal{P}_{-\alpha,\lambda}^{\mathrm{s}}(\pi - r). \tag{3.9}$$

The following symmetries hold

$$\mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(r) = \mathcal{P}_{\alpha,-\lambda}^{\mathrm{s}}(r), \qquad \mathcal{P}_{\alpha,\lambda}^{\mathrm{s}}(\pi - r) = \mathcal{P}_{\alpha,-\lambda}^{\mathrm{s}}(\pi - r). \tag{3.10}$$

The following theorem describes the basic family of closed realizations of L_α^{s} on $L^2]0, \pi[$.

Theorem 3.1. *For* $\operatorname{Re}\alpha \geq 1$ *there exists a unique closed operator* L_α^{s} *in the sense of* $L^2]0,\pi[$*, which on* $C_{\mathrm{c}}^\infty]0,\pi[$ *is given by (3.6). The family* $\alpha \mapsto L_\alpha^{\mathrm{s}}$ *is holomorphic and possesses a unique holomorphic extension to* $\operatorname{Re}\alpha > -1$*. It has only discrete spectrum:*

$$\sigma(L_\alpha^{\mathrm{s}}) = \sigma_{\mathrm{d}}(L_\alpha^{\mathrm{s}}) = \left\{ \left(k+\alpha\right)^2 \ : \ k \in \mathbb{N}_0 + \frac{1}{2} \right\}. \tag{3.11}$$

Outside of the spectrum its resolvent is

$$\begin{aligned} \frac{1}{(L_\alpha^{\mathrm{s}} - \lambda^2)}(x,y) =& \Gamma\left(\alpha - \lambda + \frac{1}{2}\right)\Gamma\left(\alpha + \lambda + \frac{1}{2}\right) \\ & \times \begin{cases} \mathcal{P}^{\mathrm{s}}_{\alpha,\lambda}(x)\mathcal{P}^{\mathrm{s}}_{\alpha,\lambda}(\pi - y), & \text{if} \quad 0 < x < y < \pi; \\ \mathcal{P}^{\mathrm{s}}_{\alpha,\lambda}(y)\mathcal{P}^{\mathrm{s}}_{\alpha,\lambda}(\pi - x), & \text{if} \quad 0 < y < x < \pi. \end{cases} \end{aligned} \tag{3.12}$$

Proof. $\mathcal{P}^{\mathrm{s}}_{\alpha,\lambda}(r)$ and $\mathcal{P}^{\mathrm{s}}_{-\alpha,\lambda}(r)$ can be used as a basis of solutions of (3.5). The connection formula found using (A.23) is

$$\mathcal{P}^{\mathrm{s}}_{\alpha,\lambda}(\pi - r) = -\frac{\cos\pi\lambda}{\sin\pi\alpha}\mathcal{P}^{\mathrm{s}}_{\alpha,\lambda}(r) + \frac{\pi}{\sin\pi\alpha}\frac{1}{\Gamma(\frac{1}{2}+\alpha+\lambda)\Gamma(\frac{1}{2}+\alpha-\lambda)}\mathcal{P}^{\mathrm{s}}_{-\alpha,\lambda}(r). \tag{3.13}$$

From

$$\mathcal{P}^{\mathrm{s}}_{-\alpha,\lambda}(r) \sim \frac{1}{\Gamma(1-\alpha)}\left(\frac{r}{2}\right)^{\frac{1}{2}-\alpha}, \tag{3.14}$$

we obtain

$$\mathcal{W}(\mathcal{P}^{\mathrm{s}}_{-\alpha,\lambda}, \mathcal{P}^{\mathrm{s}}_{\alpha,\lambda}) = \frac{\sin\pi\alpha}{\pi}. \tag{3.15}$$

This yields the Wronskians

$$\mathcal{W}\left(\mathcal{P}^{\mathrm{s}}_{\alpha,\lambda}(\pi - r), \mathcal{P}^{\mathrm{s}}_{\alpha,\lambda}(r)\right) = \frac{1}{\Gamma\left(\alpha - \lambda + \frac{1}{2}\right)\Gamma\left(\alpha + \lambda + \frac{1}{2}\right)} \tag{3.16}$$

$$\mathcal{W}\left(\mathcal{P}^{\mathrm{s}}_{\alpha,\lambda}(\pi - r), \mathcal{P}^{\mathrm{s}}_{-\alpha,\lambda}(r)\right) = \frac{\cos(\pi\lambda)}{\pi}. \tag{3.17}$$

For Re $\alpha > -1$ the integral kernel (3.12) is square integrable, and hence it defines a Hilbert-Schmidt operator. It depends analytically on α. Hence it defines a holomorphic family of operators. For Re $\alpha \geq 1$, $r^{\frac{1}{2}-\alpha}$ is not L^2-integrable. Hence for such α (3.6) possesses a unique closed realization.

The singularities of the Gamma function yield the discrete spectrum. □

3.2 Hyperbolic case

In (3.1), for $r \in \mathbb{R}_+$ we set

$$w = \cosh r, \quad \text{which solves} \quad w' = (w^2 - 1)^{\frac{1}{2}}. \tag{3.18}$$

This leads to the Schrödinger equation

$$\left(L_\alpha^{\mathrm{h}} + \lambda^2\right)\phi(r) = 0, \tag{3.19}$$

where

$$L_\alpha^{\mathrm{h}} := -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\sinh^2 r}. \tag{3.20}$$

It is obtained when we separate variables of the Laplacian on the hyperbolic space of any dimensions, see e.g. Subsect. 7.2. Hence our name "hyperbolic".

Let us define functions on $\mathbb{R}_+$

$$\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r) := \Big(\frac{\sinh r}{2}\Big)^{\alpha+\frac12}\ \mathbf{S}_{\alpha,\lambda}(\cosh r), \tag{3.21}$$

$$\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r) := \frac{(\sinh r)^{\alpha+\frac12}}{2^\lambda}\ \mathbf{Z}_{\alpha,\lambda}(\cosh r). \tag{3.22}$$

They have the following asymptotic behavior:

$$\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r) \sim \frac{1}{\Gamma(1+\alpha)}\Big(\frac{r}{2}\Big)^{\frac12+\alpha}, \quad r\sim 0; \tag{3.23}$$

$$\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r) \sim \frac{1}{\Gamma(1+\lambda)}\mathrm{e}^{-\lambda r}, \quad r\to+\infty. \tag{3.24}$$

The following four functions solve the eigenequation (3.19):

$$\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r), \quad \mathcal{P}^{\mathrm{h}}_{-\alpha,\lambda}(r), \quad \mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r), \quad \mathcal{Q}^{\mathrm{h}}_{\alpha,-\lambda}(r). \tag{3.25}$$

The following symmetries are obvious:

$$\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r) = \mathcal{P}^{\mathrm{h}}_{\alpha,-\lambda}(r), \qquad \mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r) = \mathcal{Q}^{\mathrm{h}}_{-\alpha,\lambda}(r). \tag{3.26}$$

The following theorem describes the basic family of closed realizations of L^{h}_α on $L^2[0,\infty[$.

Theorem 3.2. *For* $\operatorname{Re}\alpha \geq 1$ *there exists a unique closed operator* L^{h}_α *in the sense of* $L^2(\mathbb{R}_+)$*, which on* $C^\infty_{\mathrm{c}}(\mathbb{R}_+)$ *is given by* (3.20)*. The family* $\alpha\mapsto L^{\mathrm{h}}_\alpha$ *is holomorphic and possesses a unique holomorphic extension to* $\operatorname{Re}\alpha > -1$*. Here is its discrete spectrum and spectrum:*

$$\sigma_{\mathrm{d}}(L^{\mathrm{h}}_\alpha) = \Big\{-\Big(\frac12+\alpha\Big)^2\Big\}, \quad -1<\operatorname{Re}\alpha<-\frac12; \tag{3.27}$$

$$\sigma_{\mathrm{d}}(L^{\mathrm{h}}_\alpha) = \emptyset, \quad -\frac12<\operatorname{Re}\alpha; \tag{3.28}$$

$$\sigma(L^{\mathrm{h}}_\alpha) = [0,+\infty[\cup\sigma_{\mathrm{d}}(L^{\mathrm{h}}_\alpha). \tag{3.29}$$

Outside of the spectrum, for $\operatorname{Re}\lambda>0$*, its resolvent is*

$$\begin{aligned}\frac{1}{(L^{\mathrm{h}}_\alpha+\lambda^2)}(x,y) =& \frac{\Gamma\left(\frac12+\alpha+\lambda\right)}{\sqrt{\pi}}\\ &\times\begin{cases}\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(x)\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(y) & \textit{if}\quad 0<x<y<\infty;\\ \mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(x)\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(y) & \textit{if}\quad 0<y<x<\infty.\end{cases}\end{aligned} \tag{3.30}$$

Proof. Consider $\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r)$ and $\mathcal{P}^{\mathrm{h}}_{-\alpha,\lambda}(r)$ as a basis of solutions of (3.19). The connection formula is

$$\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r) = -\frac{\sqrt{\pi}}{\sin\pi\alpha\ \Gamma\left(\frac12-\alpha+\lambda\right)}\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r) + \frac{\sqrt{\pi}}{\sin\pi\alpha\ \Gamma\left(\frac12+\alpha+\lambda\right)}\mathcal{P}^{\mathrm{h}}_{-\alpha,\lambda}(r). \tag{3.31}$$

Similarly as in the spherical case, we obtain

$$\mathcal{W}(\mathcal{P}^{\mathrm{h}}_{-\alpha,\lambda},\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}) = \frac{\sin\alpha}{\pi}. \tag{3.32}$$

This yields the Wronskians

$$\mathcal{W}\left(\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r), \mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r)\right) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{1}{2}+\alpha+\lambda\right)}.$$

For $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\lambda > 0$ the functions $\mathcal{P}^{\mathrm{h}}_{\alpha,\lambda}(r)$ resp. $\mathcal{Q}^{\mathrm{h}}_{\alpha,\lambda}(r)$, are square integrable at the endpoints. We check that under these conditions the integral kernel 3.30 defines a bounded operator, depending analytically on α.

For $\operatorname{Re}\alpha \geq 1$, $r^{\frac{1}{2}-\alpha}$ is not square integrable near 0. Therefore, for such α (3.20) possesses a unique closed realization.

Looking for singularities of the Gamma function we find the discrete spectrum:

$$\sigma_{\mathrm{d}}(L^{\mathrm{h}}_{\alpha}) = \left\{ -\left(n+\frac{1}{2}+\alpha\right)^2 \,\middle|\, n \in \mathbb{N}_0, \quad \operatorname{Re}\left(n+\frac{1}{2}+\alpha\right) < 0 \right\}. \tag{3.33}$$

It is easy to see that this coincides with (3.28) □

3.3 DeSitterian case

For $r \in \mathbb{R}$, in (3.1) we set

$$w = -\mathrm{i}\sinh r, \quad \text{which solves} \;\; w' = (w^2-1)^{\frac{1}{2}}. \tag{3.34}$$

This leads to the Schrödinger equation

$$\left(L^{\mathrm{dS}}_{\alpha} + \lambda^2\right)\phi(r) = 0, \tag{3.35}$$

where

$$L^{\mathrm{dS}}_{\alpha} := -\partial_r^2 - \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{\cosh^2 r}. \tag{3.36}$$

It has the mirror symmetry $r \to -r$. It is obtained when we separate variables in the d'Alembertian on the deSitter space of any dimension, see Subsect. 7.3. Hence our name "deSitterian".

The following theorem describes all closed realizations of L^{dS}_{α} on $L^2(\mathbb{R})$.

For $r \geq 0$ we introduce the following function which solves the eigenequation

$$\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(r) := \mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(\frac{1}{2}+\alpha+\lambda)}\frac{(\cosh r)^{\alpha+\frac{1}{2}}}{2^{\lambda}}\;\mathcal{Z}_{\alpha,\lambda}(-\mathrm{i}\sinh r). \tag{3.37}$$

We extend it to $r \leq 0$ by analytic continuation. Here is its asymptotics:

$$\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(r) \sim \frac{1}{\Gamma(\lambda+1)}\mathrm{e}^{-\lambda r}, \quad r \to +\infty. \tag{3.38}$$

Thus the following functions solve the eigenequation (3.35):

$$\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(r), \qquad \mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(-r), \qquad \mathcal{Q}^{\mathrm{dS}}_{\alpha,-\lambda}(r), \qquad \mathcal{Q}^{\mathrm{dS}}_{\alpha,-\lambda}(-r). \tag{3.39}$$

Let us describe closed realizations of L^{dS}_{α} on $L^2(\mathbb{R})$:

Theorem 3.3. *For any $\alpha\in\mathbb{C}$ there exists a unique closed operator L_α^{dS} in the sense of $L^2(\mathbb{R})$ that on $C_{\mathrm{c}}^\infty(\mathbb{R})$ is given by (3.36). The function $\mathbb{C}\ni\alpha\mapsto L_\alpha^{\mathrm{dS}}$ is holomorphic. It satisfies $L_\alpha^{\mathrm{dS}}=L_{-\alpha}^{\mathrm{dS}}$. Outside of the spectrum, for $\operatorname{Re}\lambda>0$, its resolvent is*

$$\begin{aligned}&\frac{1}{(L_\alpha^{\mathrm{dS}}+\lambda^2)}(x,y)\\ =&\frac{\Gamma\left(\frac12-\alpha+\lambda\right)\Gamma\left(\frac12+\alpha+\lambda\right)}{2}\begin{cases}\mathcal{Q}_{\alpha,\lambda}^{\mathrm{dS}}(x)\mathcal{Q}_{\alpha,\lambda}^{\mathrm{dS}}(-y) & -\infty<x<y<\infty;\\ \mathcal{Q}_{\alpha,\lambda}^{\mathrm{dS}}(y)\mathcal{Q}_{\alpha,\lambda}^{\mathrm{dS}}(-x) & -\infty<y<x<\infty.\end{cases}\end{aligned} \tag{3.40}$$

To describe the discrete spectrum and spectrum of L_α^{dS}, without loss of generality we can assume that $\operatorname{Re}\alpha\geq 0$. Then

$$\sigma_{\mathrm{d}}(L_\alpha^{\mathrm{dS}})=\left\{-(\alpha-k)^2\ \middle|\ k\in\mathbb{N}_0+\frac12,\quad k<\operatorname{Re}\alpha\right\} \tag{3.41}$$

$$\sigma(L_\alpha^{\mathrm{dS}})=[0,\infty[\cup\sigma_{\mathrm{d}}(L_\alpha^{\mathrm{dS}}). \tag{3.42}$$

Proof. We can use the proof of the theorem 4.3. Note that

$$\mathcal{Q}_{\alpha,\lambda}^{\mathrm{dS}}(r)=\frac{\Gamma\big(\frac12+\lambda\big)}{\sqrt{\pi}}\mathcal{Q}_{\alpha,\alpha,2\lambda}^{\mathrm{dS}}(r)=\frac{\Gamma(1+2\lambda)}{2^{2\lambda}\Gamma(1+\lambda)}\mathcal{Q}_{\alpha,\alpha,2\lambda}^{\mathrm{dS}}(r). \tag{3.43}$$

Hence, the Wronskians are

$$\mathcal{W}\left(\mathcal{Q}_{\alpha,\lambda}^{\mathrm{dS}}(-r),\mathcal{Q}_{\alpha,\lambda}^{\mathrm{dS}}(r)\right)=\frac{2}{\Gamma\left(\frac12+\alpha+\lambda\right)\Gamma\left(\frac12-\alpha+\lambda\right)}, \tag{3.44}$$

$$\mathcal{W}\left(\mathcal{Q}_{\alpha,-\lambda}^{\mathrm{dS}}(-r),\mathcal{Q}_{\alpha,\lambda}^{\mathrm{dS}}(r)\right)=\frac{2\cos(\pi\alpha)}{\pi}. \tag{3.45}$$

We check that the integral kernel 3.40 defines a bounded operator.

The singularities of the Gamma function are at $\lambda=-\frac12+\alpha-n$ and $\lambda=-\frac12-\alpha-n$, $n\in\mathbb{N}_0$. This gives the following discrete spectrum:

$$\sigma_{\mathrm{d}}(L_\alpha^{\mathrm{dS}})=\left\{-\left(\frac12+\alpha+n\right)^2\ \middle|\ n\in\mathbb{N}_0,\quad \operatorname{Re}\left(\frac12+\alpha+n\right)<0\right\} \tag{3.46}$$

$$\cup\left\{-\left(\frac12-\alpha+n\right)^2\ \middle|\ n\in\mathbb{N}_0,\quad \operatorname{Re}\left(\frac12-\alpha+n\right)<0\right\}. \tag{3.47}$$

For Re$\alpha\geq 0$ the right hand side of (3.46) is empty. This yields (3.41). □

Proof of Prop. 1.1. The transformation $\cot r=\sinh q$ implies

$$\cos r=\tanh q,\quad \tan r=\sinh q, \tag{3.48}$$

$$\frac{\mathrm{d}q}{\mathrm{d}r}=-\frac{1}{\sin r}=\cosh q. \tag{3.49}$$

The Whipple transformation (A.26), (A.27) on the interval $w\in]-1,1[$ has two versions: above this interval and below, that is

$$\mathbf{Z}_{\alpha,\lambda}(w\pm\mathrm{i}0)=(\pm\mathrm{i}\sqrt{1-w^2})^{-\frac12-\alpha-\lambda}\mathbf{S}_{\lambda,\alpha}\left(\frac{w}{\pm\mathrm{i}\sqrt{1-w^2}}\right). \tag{3.50}$$

This yields

$$\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(q) = \Big(\frac{2}{\sin r}\Big)^{\frac12} \mathcal{P}^{\mathrm{s}}_{\lambda,\alpha}(r), \tag{3.51}$$

$$\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(-q) = \Big(\frac{2}{\sin r}\Big)^{\frac12} \mathcal{P}^{\mathrm{s}}_{\lambda,\alpha}(\pi - r). \tag{3.52}$$

This yields

$$(\sin r)^{\frac12} \frac{1}{(L^{\mathrm{s}}_{\lambda} - \alpha^2)} (r, r') (\sin r')^{\frac12} = \frac{1}{(L^{\mathrm{dS}}_{\alpha} - \lambda^2)} (q, q'), \tag{3.53}$$

which proves Prop. 1.1. □

Remark 3.4. *Using* (3.49)*,* (3.51) *and* (3.52) *we obtain*

$$\mathcal{W}\big(\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(-q), \mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(q)\big) = 2\mathcal{W}\big(\mathcal{P}^{\mathrm{s}}_{\lambda,\alpha}(\pi - r), \mathcal{P}^{\mathrm{s}}_{\lambda,\alpha}(r)\big), \tag{3.54}$$

$$\mathcal{W}\big(\mathcal{Q}^{\mathrm{dS}}_{\alpha,\lambda}(-q), \mathcal{Q}^{\mathrm{dS}}_{\alpha,-\lambda}(q)\big) = 2\mathcal{W}\big(\mathcal{P}^{\mathrm{s}}_{\lambda,\alpha}(\pi - r), \mathcal{P}^{\mathrm{s}}_{-\lambda,\alpha}(r)\big). \tag{3.55}$$

Therefore, (3.16) *and* (3.17) *imply* (3.44) *and* (3.45)*. This can be used in an alternative proof of Thm 3.3.*

4 Hypergeometric Hamiltonians of the first kind

Let us transform the hypergeometric operator as follows:

$$\begin{aligned}
& -z^{\frac{\alpha}{2}+\frac14}(1-z)^{\frac{\beta}{2}+\frac14}\mathcal{F}\left(\tfrac{\alpha+\beta+\mu+1}{2}, \tfrac{\alpha+\beta-\mu+1}{2}; 1+\alpha; z, \partial_z\right) z^{-\frac{\alpha}{2}-\frac14}(1-z)^{-\frac{\beta}{2}-\frac14} \\
= \; & -z(1-z)\mathcal{P}\left(\begin{matrix} 0 & 1 & \infty & \\ \frac{\alpha}{2}+\frac14 & \frac{\beta}{2}+\frac14 & \frac{\mu}{2} & z, \partial_z \\ -\frac{\alpha}{2}+\frac14 & -\frac{\beta}{2}+\frac14 & -\frac{\mu}{2} & \end{matrix}\right) \\
& -z(1-z)\left(\partial_z^2 + \left(\frac{1}{2z} - \frac{1}{2(1-z)}\right)\partial_z\right) \\
& + \left(\alpha^2 - \frac14\right)\frac{1}{4z} + \left(\beta^2 - \frac14\right)\frac{1}{4(1-z)} - \frac{\mu^2}{4}.
\end{aligned} \tag{4.1}$$

Thus if we set

$$L_{\alpha,\beta} := -z(1-z)\left(\partial_z^2 + \left(\frac{1}{2z} - \frac{1}{2(1-z)}\right)\partial_z\right) + \left(\alpha^2 - \frac14\right)\frac{1}{4z} + \left(\beta^2 - \frac14\right)\frac{1}{4(1-z)} \tag{4.2}$$

and $F(z)$ solves the hypergeometric equation (2.7), then

$$\Big(L_{\alpha,\beta} - \frac{\mu^2}{4}\Big) z^{\frac{\alpha}{2}+\frac14}(1-z)^{\frac{\beta}{2}+\frac14}F(z) = 0. \tag{4.3}$$

The hypergeometric equation has been reinterpreted as the eigenequation of the operator $L_{\alpha,\beta}$ with the eigenvalue $\frac{\mu^2}{4}$. It is natural to interpret this operator as acting on functions on an interval whose endpoints are singularities of the hypergeometric equations. In each of these cases we perform the Liouville transformation, which yields a 1-dimensional Hamiltonian. We will consider three cases:

1. $z \in]0,1[$, which leads to an operator on $L^2]0,\pi[$, which we call the *spherical hypergeometric Hamiltonian of the 1st kind*;

2. $z \in]-\infty, 0[$, which leads to an operator on $L^2(\mathbb{R}_+)$, which we call the *hyperbolic hypergeometric Hamiltonian of the 1st kind*;

3. $z \in \frac{1}{2} + \mathrm{i}\mathbb{R}$, which leads to an operator on $L^2(\mathbb{R})$, which we call the *deSitterian hypergeometric Hamiltonian of the 1st kind*.

It will be natural to introduce the parameters

$$\delta := \frac{1}{2}(\alpha^2 + \beta^2), \tag{4.4}$$

$$\kappa := \frac{1}{2}(\alpha^2 - \beta^2), \qquad \tau := \frac{\mathrm{i}}{2}(\alpha^2 - \beta^2) = \mathrm{i}\kappa. \tag{4.5}$$

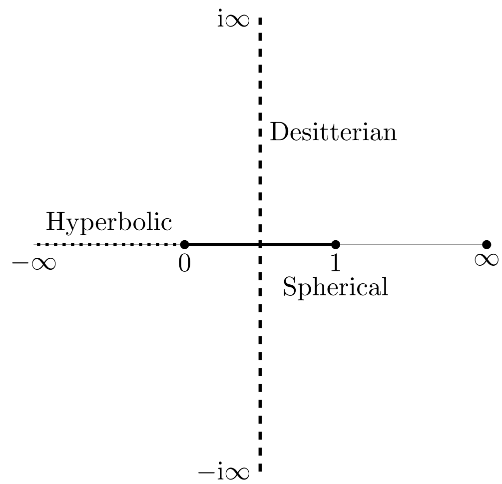

Figure 2: Hypergeometirc Hamiltonians of the first kind on the z-plane. The hyperbolic Hamiltonian act on the interval with the dotted line, spherical Hamiltonian—the thick line, and deSitterian Hamiltonian—the dashed line.

4.1 Spherical case

For $r \in]0,\pi[$, set in (4.2)

$$z = \sin^2\frac{r}{2} = \frac{1-\cos r}{2}, \quad \text{which solves} \;\; z' = z^{\frac{1}{2}}(1-z)^{\frac{1}{2}}. \tag{4.6}$$

This leads to the Schrödinger equation

$$\left(L^{\mathrm{s}}_{\alpha,\beta} - \frac{\mu^2}{4}\right)\phi(r) = 0, \tag{4.7}$$

where

$$\begin{aligned} L^{\mathrm{s}}_{\alpha,\beta} &:= -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{4\sin^2\frac{r}{2}} + \left(\beta^2 - \frac{1}{4}\right)\frac{1}{4\cos^2\frac{r}{2}} \\ &= -\partial_r^2 + \left(\delta - \frac{1}{4}\right)\frac{1}{\sin^2 r} + \kappa\frac{\cos r}{\sin^2 r}. \end{aligned} \tag{4.8}$$

In [DW] $L^{\rm s}_{\alpha,\beta}$ is called the *trigonometric Pöschl-Teller Hamiltonian*

The case $\alpha = \beta$ is especially important and coincides with the spherical Gegenbauer Hamiltonian:

$$L^{\rm s}_{\alpha} := L^{\rm s}_{\alpha,\alpha}. \tag{4.9}$$

For $r \in]0, \pi[$, define the function

$$P^{\rm s}_{\alpha,\beta,\mu}(r) := \Big(\sin\frac{r}{2}\Big)^{\alpha+\frac12}\Big(\cos\frac{r}{2}\Big)^{\beta+\frac12}\mathbf{F}\Big(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha+\beta-\mu+1}{2}; 1+\alpha; \sin^2\Big(\frac{r}{2}\Big)\Big) \tag{4.10}$$

$$= \Big(\sin\frac{r}{2}\Big)^{\alpha+\frac12}\Big(\cos\frac{r}{2}\Big)^{-\alpha-\mu-\frac12}\mathbf{F}\Big(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha-\beta+\mu+1}{2}; 1+\alpha; -\tan^2\Big(\frac{r}{2}\Big)\Big). \tag{4.11}$$

Note that

$$P^{\rm s}_{\alpha,\beta,\mu}(r) = P^{\rm s}_{\alpha,-\beta,\mu}(r) = P^{\rm s}_{\alpha,\beta,-\mu}(r). \tag{4.12}$$

Asymptotically our function behaves like

$$P^{\rm s}_{\alpha,\beta,\mu}(r) \sim \frac{1}{\Gamma(1+\alpha)}\Big(\frac{r}{2}\Big)^{\alpha+\frac12}, \quad r \sim 0. \tag{4.13}$$

Now the following functions solve the eigenvalue problem (4.7):

$$P^{\rm s}_{\alpha,\beta,\mu}(r), \quad P^{\rm s}_{-\alpha,\beta,\mu}(r), \quad P^{\rm s}_{\beta,\alpha,\mu}(\pi - r), \quad P^{\rm s}_{-\beta,\alpha,\mu}(\pi - r). \tag{4.14}$$

The following theorem describes the basic holomorphic family of closed realizations of $L^{\rm s}_{\alpha,\beta}$ on $L^2]0, \pi[$.

Theorem 4.1. *For* $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta \geq 1$ *there exists a unique closed operator* $L^{\rm s}_{\alpha,\beta}$ *in the sense of* $L^2]0, \pi[$, *which on* $C^\infty_{\rm c}]0, \pi[$ *is given by* (4.8). *The family* $\alpha, \beta \mapsto L^{\rm s}_{\alpha,\beta}$ *is holomorphic and possesses a unique holomorphic extension to* $\mathrm{Re}\,\alpha, \mathrm{Re}\,\beta > -1$. *It has only discrete spectrum:*

$$\sigma(L^{\rm s}_{\alpha,\beta}) = \sigma_{\rm d}(L^{\rm s}_{\alpha,\beta}) = \Big\{\Big(k + \frac{\alpha+\beta}{2}\Big)^2 \ : \ k \in \mathbb{N}_0 + \frac12\Big\}. \tag{4.15}$$

Outside of the spectrum its resolvent is

$$\begin{aligned}\frac{1}{\Big(L^{\rm s}_{\alpha,\beta} - \frac{\mu^2}{4}\Big)}(x, y) =& \Gamma\Big(\frac{1+\alpha+\beta+\mu}{2}\Big)\Gamma\Big(\frac{1+\alpha+\beta-\mu}{2}\Big)\\ &\times \begin{cases} P^{\rm s}_{\alpha,\beta,\mu}(x)\ P^{\rm s}_{\beta,\alpha,\mu}(\pi - y) & \text{if} \quad 0 < x < y < \pi; \\ P^{\rm s}_{\alpha,\beta,\mu}(y)\ P^{\rm s}_{\beta,\alpha,\mu}(\pi - x) & \text{if} \quad 0 < y < x < \pi. \end{cases}\end{aligned} \tag{4.16}$$

Proof. Considering $P^{\rm s}_{\alpha,\beta,\mu}(r)$, and $P^{\rm s}_{-\alpha,\beta,\mu}(r)$ as a basis of solutions of (4.7), we can rewrite the connection formula (A.9) as

$$P^{\rm s}_{\beta,\alpha,\mu}(\pi - r) = \frac{\pi\ P^{\rm s}_{\alpha,\beta,\mu}(r)}{\sin(-\pi\alpha)\Gamma\big(\frac{1-\alpha+\beta+\mu}{2}\big)\Gamma\big(\frac{1-\alpha+\beta-\mu}{2}\big)} + \frac{\pi\ P^{\rm s}_{-\alpha,\beta,\mu}(r)}{\sin(\pi\alpha)\Gamma\big(\frac{1+\alpha+\beta+\mu}{2}\big)\Gamma\big(\frac{1+\alpha+\beta-\mu}{2}\big)}. \tag{4.17}$$

Using (4.13) and arguing as in the proof of (3.15), we obtain

$$\mathcal{W}\left(P^{\rm s}_{-\alpha,\beta,\mu}(r), P^{\rm s}_{\alpha,\beta,\mu}(r)\right) = \frac{\sin\pi\alpha}{\pi}. \tag{4.18}$$

From the connection formula and (4.18) we obtain the Wronskian

$$\mathcal{W}\left(P^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-r), P^{\mathrm{s}}_{\alpha,\beta,\mu}(r)\right) = \frac{1}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)}. \tag{4.19}$$

The L^2 integrability conditions at the endpoints are $\operatorname{Re}\alpha > -1, \operatorname{Re}\beta > -1$, for $P^{\mathrm{s}}_{\alpha,\beta,\mu}(r)$ and $P^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-r)$ respectively. With these conditions, we can write the candidate for the resolvent (4.16). The L^2 norm of this integral kernel is finite. Hence it defines a bounded (even Hilbert-Schmidt) operator. For $\operatorname{Re}\alpha \geq 1, \operatorname{Re}\beta \geq 1$ it is a unique candidate for the resolvent. □

4.2 Hyperbolic case

For $r \in \mathbb{R}_+$, in (4.2) we set

$$z = -\sinh^2\frac{r}{2} = \frac{1-\cosh r}{2}, \quad \text{which solves } \ z' = -(-z)^{\frac{1}{2}}(1-z)^{\frac{1}{2}}. \tag{4.20}$$

This leads to the Schrödinger equation

$$\left(L^{\mathrm{h}}_{\alpha,\beta} + \frac{\mu^2}{4}\right)\phi(r) = 0, \tag{4.21}$$

where

$$\begin{aligned} L^{\mathrm{h}}_{\alpha,\beta} \quad &:= \quad -\partial_r^2 + \left(\alpha^2 - \frac{1}{4}\right)\frac{1}{4\sinh^2\frac{r}{2}} - \left(\beta^2 - \frac{1}{4}\right)\frac{1}{4\cosh^2\frac{r}{2}} \\ &= \quad -\partial_r^2 + \left(\delta - \frac{1}{4}\right)\frac{1}{\sinh^2 r} + \kappa\frac{\cosh r}{\sinh^2 r}. \end{aligned} \tag{4.22}$$

In [DW] $L^{\mathrm{h}}_{\alpha,\beta}$ is called the *hyperbolic Pöschl-Teller Hamiltonian.*

The case $\alpha = \beta$ is especially important and coincides with the hyperbolic Gegenbauer operator:

$$L^{\mathrm{h}}_{\alpha} := L^{\mathrm{h}}_{\alpha,\alpha}. \tag{4.23}$$

For $r \in \mathbb{R}_+$, let us define

$$P^{\mathrm{h}}_{\alpha,\beta,\mu}(r) := \left(\sinh\frac{r}{2}\right)^{\alpha+\frac{1}{2}}\left(\cosh\frac{r}{2}\right)^{\beta+\frac{1}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha+\beta-\mu+1}{2}; 1+\alpha; -\sinh^2\frac{r}{2}\right) \tag{4.24}$$

$$= \left(\sinh\frac{r}{2}\right)^{\alpha+\frac{1}{2}}\left(\cosh\frac{r}{2}\right)^{-\alpha-\mu-\frac{1}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha-\beta+\mu+1}{2}; 1+\alpha; \tanh^2\frac{r}{2}\right), \tag{4.25}$$

$$Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r) := \left(\sinh\frac{r}{2}\right)^{-\mu-\beta-\frac{1}{2}}\left(\cosh\frac{r}{2}\right)^{\beta+\frac{1}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{-\alpha+\beta+\mu+1}{2}; 1+\mu; -\sinh^{-2}\frac{r}{2}\right) \tag{4.26}$$

$$= \left(\sinh\frac{r}{2}\right)^{\alpha+\frac{1}{2}}\left(\cosh\frac{r}{2}\right)^{-\alpha-\mu-\frac{1}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha-\beta+\mu+1}{2}; 1+\mu; \cosh^{-2}\frac{r}{2}\right). \tag{4.27}$$

Note that

$$P^{\mathrm{h}}_{\alpha,\beta,\mu}(r) = P^{\mathrm{h}}_{\alpha,-\beta,\mu}(r) = P^{\mathrm{h}}_{\alpha,\beta,-\mu}(r), \tag{4.28}$$

$$Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r) = Q^{\mathrm{h}}_{\alpha,-\beta,\mu}(r) = Q^{\mathrm{h}}_{-\alpha,\beta,\mu}(r), \tag{4.29}$$

Asymptotically,

$$P^{\mathrm{h}}_{\alpha,\beta,\mu}(r) \sim \frac{1}{\Gamma(1+\alpha)}\Big(\frac{r}{2}\Big)^{\alpha+\frac12}, \quad r\to 0; \tag{4.30}$$

$$Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r) \sim \frac{2^{\mu}}{\Gamma(1+\mu)}\mathrm{e}^{-\frac{\mu}{2}r}, \quad r\to +\infty. \tag{4.31}$$

Now the following functions solve the eigenvalue problem (4.45):

$$P^{\mathrm{h}}_{\alpha,\beta,\mu}(r), \quad P^{\mathrm{h}}_{-\alpha,\beta,\mu}(r), \quad Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r), \quad Q^{\mathrm{h}}_{\alpha,\beta,-\mu}(r). \tag{4.32}$$

The following theorem describes the basic holomorphic family of closed realizations of $L^{\mathrm{h}}_{\alpha,\beta}$ on the Hilbert space $L^2(\mathbb{R}_+)$.

Theorem 4.2. *For* $\operatorname{Re}\alpha \geq 1$ *there exists a unique closed operator* $L^{\mathrm{h}}_{\alpha,\beta}$ *in the sense of* $L^2(\mathbb{R}_+)$*, which on* $C_{\mathrm{c}}^{\infty}(\mathbb{R}_+)$ *is given by (4.22). The family* $\alpha,\beta \mapsto L^{\mathrm{h}}_{\alpha,\beta}$ *is holomorphic and possesses a unique holomorphic extension to* $\operatorname{Re}\alpha > -1$*. Its discrete spectrum and spectrum are*

$$\sigma_{\mathrm{d}}(L^{\mathrm{h}}_{\alpha,\beta}) = \Big\{-\Big(\frac{\alpha+\beta}{2}+k\Big)^2 \quad | \quad k\in\mathbb{N}_0+\frac12, \quad k< -\operatorname{Re}\frac{\alpha+\beta}{2}\Big\}, \tag{4.33}$$

$$\cup\Big\{-\Big(\frac{\alpha-\beta}{2}+k\Big)^2 \quad | \quad k\in\mathbb{N}_0+\frac12, \quad k< -\operatorname{Re}\frac{\alpha-\beta}{2}\Big\}, \tag{4.34}$$

$$\sigma(L^{\mathrm{h}}_{\alpha,\beta}) = [0,\infty[\cup \sigma_{\mathrm{d}}(L^{\mathrm{h}}_{\alpha,\beta}). \tag{4.35}$$

Outside of the spectrum, for $\operatorname{Re}\mu > 0$*, its resolvent is*

$$\frac{1}{(L^{\mathrm{h}}_{\alpha,\beta}+\frac{\mu^2}{4})}(x,y) = \Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right) \times \begin{cases} P^{\mathrm{h}}_{\alpha,\beta,\mu}(x)\ Q^{\mathrm{h}}_{\alpha,\beta,\mu}(y) & \text{if} \quad 0<x<y<\infty; \\ P^{\mathrm{h}}_{\alpha,\beta,\mu}(y)\ Q^{\mathrm{h}}_{\alpha,\beta,\mu}(x) & \text{if} \quad 0<y<x<\infty. \end{cases} \tag{4.36}$$

Proof. Considering $P^{\mathrm{h}}_{-\alpha,\beta,\mu}(r)$, and $P^{\mathrm{h}}_{-\alpha,\beta,\mu}(r)$ as a basis of solutions of (4.45), we can rewrite connection formula (A.13) as

$$Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r) = -\frac{\pi\ P^{\mathrm{h}}_{\alpha,\beta,\mu}(r)}{\sin\pi\alpha\ \Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} + \frac{\pi\ P^{h}_{-\alpha,\beta,\mu}(r)}{\sin(\alpha\pi)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}. \tag{4.37}$$

Using (4.30) and again arguing as in the proof of (3.15), we obtain we obtain

$$\mathcal{W}\left(P^{\mathrm{h}}_{-\alpha,\beta,\mu}(r), P^{\mathrm{h}}_{\alpha,\beta,\mu}(r)\right) = \frac{\sin\pi\alpha}{\pi}. \tag{4.38}$$

This yields the Wronskian

$$\mathcal{W}\left(Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r), P^{\mathrm{h}}_{\alpha,\beta,\mu}(r)\right) = \frac{1}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}.$$

The L^2 integrability conditions at the endpoints are $\operatorname{Re}\alpha > -1$ and $\operatorname{Re}\mu > 0$ for $P^{\mathrm{h}}_{\alpha,\beta,\mu}(r)$ and $\mathcal{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(r)$, respectively. Using the Schur Test we check that integral kernel (4.36) defines a bounded operator. We also see that for $\operatorname{Re}\alpha > -1, \operatorname{Re}\mu > 0$ it is a unique candidate for the resolvent. □

Proof of Prop. 1.2. The transformation

$$\tan\frac{r}{2}=\sinh\frac{q}{2}. \tag{4.39}$$

implies

$$\sin\frac{r}{2}=\tanh\frac{q}{2},\quad \frac{\mathrm{d}q}{\mathrm{d}r}=\frac{1}{\cos\frac{r}{2}}=\cosh\frac{q}{2}; \tag{4.40}$$

$$P^{\mathrm{s}}_{\alpha,\beta,\mu}(r)=\frac{1}{\left(\cosh\frac{q}{2}\right)^{\frac12}}P^{\mathrm{h}}_{\alpha,\mu,\beta}(q), \tag{4.41}$$

$$P^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-r)=\frac{1}{\left(\cosh\frac{q}{2}\right)^{\frac12}}Q^{\mathrm{h}}_{\alpha,\mu,\beta}(q). \tag{4.42}$$

We obtain the transmutation relation

$$\frac{1}{\left(L^{\mathrm{s}}_{\alpha,\beta}+\frac{\mu^2}{4}\right)}(r,r')=\left(\cosh\frac{q}{2}\right)^{-\frac12}\frac{1}{\left(L^{\mathrm{h}}_{\alpha,\mu}+\frac{\beta^2}{4}\right)}(q,q')\ \left(\cosh\frac{q'}{2}\right)^{-\frac12}. \tag{4.43}$$

□

4.3 DeSitterian case

For $r\in\mathbb{R}$, in (4.2) we set

$$z=\frac12-\mathrm{i}\cosh\frac{r}{2}\sinh\frac{r}{2}=\frac{1-\mathrm{i}\sinh r}{2},\quad \text{which solves}\ \ z'=(-z)^{\frac12}(1-z)^{\frac12}. \tag{4.44}$$

This leads to the Schrödinger equation

$$\left(L^{\mathrm{dS}}_{\alpha,\beta}+\frac{\mu^2}{4}\right)\phi(r)=0, \tag{4.45}$$

where

$$L^{\mathrm{dS}}_{\alpha,\beta}\ :=\ -\partial_r^2-\left(\alpha^2-\frac14\right)\frac{1}{\cosh^2 r}\left(\frac12+\frac{\mathrm{i}\sinh r}{2}\right)-\left(\beta^2-\frac14\right)\frac{1}{\cosh^2 r}\left(\frac12-\frac{\mathrm{i}\sinh r}{2}\right) \tag{4.46}$$

$$=\ -\partial_r^2-\left(\delta-\frac14\right)\frac{1}{\cosh^2 r}-\tau\frac{\sinh r}{\cosh^2 r}. \tag{4.47}$$

This Hamiltonian was proposed and solved by F. Scarf [Sca] and in [DW] it is called the *Scarf Hamiltonian*.

The case $\alpha=\beta$ is especially important and coincides with the deSitterian Gegenbauer Hamiltonian:

$$L^{\mathrm{dS}}_{\alpha}:=L^{\mathrm{dS}}_{\alpha,\alpha}. \tag{4.48}$$

Define for $r\geq 0$

$$Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r):=\Big(\frac{\mathrm{i}+\sinh r}{2}\Big)^{-\frac{\beta}{2}-\frac{\mu}{2}-\frac14}\Big(\frac{-\mathrm{i}+\sinh r}{2}\Big)^{\frac{\beta}{2}+\frac14}\times\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{-\alpha+\beta+\mu+1}{2};1+\mu;\frac{2}{1-\mathrm{i}\sinh r}\right) \tag{4.49}$$

$$=\Big(\frac{\mathrm{i}+\sinh r}{2}\Big)^{\frac{\alpha}{2}+\frac14}\Big(\frac{-\mathrm{i}+\sinh r}{2}\Big)^{-\frac{\alpha}{2}-\frac{\mu}{2}-\frac14}\times\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha-\beta+\mu+1}{2};1+\mu;\frac{2}{1+\mathrm{i}\sinh r}\right). \tag{4.50}$$

We extend $r \to Q^{\rm dS}_{\alpha,\beta,\mu}(r)$ to $r<0$ by analyticity. It satisfies

$$Q^{\rm dS}_{\alpha,\beta,\mu}(r) \sim \frac{2^{\mu}{\rm e}^{-\frac{\mu}{2}r}}{\Gamma(1+\mu)}, \quad r\to+\infty. \tag{4.51}$$

Note that

$$Q^{\rm dS}_{\alpha,\beta,\mu}(r) = Q^{\rm dS}_{\alpha,-\beta,\mu}(r) = Q^{\rm dS}_{-\alpha,\beta,\mu}(r). \tag{4.52}$$

Now the following functions solve the eigenvalue problem (4.45):

$$Q^{\rm dS}_{\alpha,\beta,\mu}(r), \quad Q^{\rm dS}_{\alpha,\beta,-\mu}(r), \quad Q^{\rm dS}_{\beta,\alpha,\mu}(-r), \quad Q^{\rm dS}_{\beta,\alpha,-\mu}(-r). \tag{4.53}$$

The following theorem describes all closed realizations of $L^{\rm dS}_{\alpha,\beta}$ on $L^2(\mathbb{R})$.

Theorem 4.3. *For any $\alpha,\beta\in\mathbb{C}$ there exists a unique closed operator $L^{\rm dS}_{\alpha}$ in the sense of $L^2(\mathbb{R})$ that on $C_{\rm c}^\infty(\mathbb{R})$ is given by (4.46). The function $\mathbb{C}\ni(\alpha,\beta)\mapsto L^{\rm dS}_{\alpha}$ is holomorphic. We have $L^{\rm dS}_{\alpha,\beta} = L^{\rm dS}_{-\alpha,\beta} = L^{\rm dS}_{\alpha,-\beta}$.*

Outside of the spectrum, for $\operatorname{Re}\mu>0$, its resolvent is

$$\begin{aligned}\frac{1}{(L^{\rm dS}_{\alpha,\beta}+\frac{\mu^2}{4})}(x,y) =& \frac{\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)}{2\pi}\\ &\times\begin{cases} Q^{\rm dS}_{\alpha,\beta,\mu}(x)Q^{\rm dS}_{\beta,\alpha,\mu}(-y) & \text{if} \quad -\infty<x<y<\infty;\\ Q^{\rm dS}_{\beta,\alpha,\mu}(-y)Q^{\rm dS}_{\alpha,\beta,\mu}(x) & \text{if} \quad -\infty<y<x<\infty.\end{cases}\end{aligned} \tag{4.54}$$

To describe the discrete spectrum of $L^{\rm dS}_{\alpha,\beta}$ assume without loss of generality that $\operatorname{Re}(\alpha+\beta)\geq 0$. We also assume that $\operatorname{Re}(\alpha-\beta)\geq0$ (the case $\operatorname{Re}(\alpha-\beta)\leq 0$ is analogous). Then

$$\begin{aligned}\sigma_{\rm d}(L^{\rm dS}_{\alpha.\beta}) =& \left\{-\left(\frac{\alpha+\beta}{2}-k\right)^2 \;\middle|\; k\in\mathbb{N}_0+\frac{1}{2},\quad k<\operatorname{Re}\frac{\alpha+\beta}{2}\right\}\\ &\cup\left\{-\left(\frac{\alpha-\beta}{2}-k\right)^2 \;\middle|\; k\in\mathbb{N}_0+\frac{1}{2},\quad k<\operatorname{Re}\frac{\alpha-\beta}{2}\right\},\end{aligned} \tag{4.55}$$

$$\sigma(L^{\rm dS}_{\alpha,\beta}) = [0,\infty[\cup\sigma_{\rm d}(L^{\rm dS}_{\alpha,\beta}). \tag{4.56}$$

Proof. In the connection formula (A.15) we insert $z=\frac{1-{\rm i}s}{2}$ and multiply it by $\left(\frac{1-{\rm i}s}{2}\right)^{\frac{\alpha}{2}+\frac14}\left(\frac{1+{\rm i}s}{2}\right)^{\frac{\beta}{2}+\frac14}$, obtaining

$$\begin{aligned}\Big(\frac{1-{\rm i}s}{2}\Big)^{\frac{\alpha}{2}+\frac14}\Big(\frac{1+{\rm i}s}{2}\Big)^{\frac{\beta}{2}+\frac14}\mathbf{F}_{\alpha,\beta,\mu}\Big(\frac{1-{\rm i}s}{2}\Big) =& \frac{\pi\left(\frac{1-{\rm i}s}{2}\right)^{\frac{\alpha}{2}+\frac14}\left(\frac{1+{\rm i}s}{2}\right)^{\frac{\beta}{2}+\frac14}\left(\frac{{\rm i}s-1}{2}\right)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\mu,\beta,\alpha}(\frac{2}{1-{\rm i}s})}{\sin(-\pi\mu)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)}\\ &+\frac{\pi\left(\frac{1-{\rm i}s}{2}\right)^{\frac{\alpha}{2}+\frac14}\left(\frac{1+{\rm i}s}{2}\right)^{\frac{\beta}{2}+\frac14}\left(\frac{{\rm i}s-1}{2}\right)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{-\mu,\beta,\alpha}(\frac{2}{1-{\rm i}s})}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}.\end{aligned} \tag{4.57}$$

We transform separately (4.57) above and below the real line: obtaining resp.

$$\frac{\pi e^{i\frac{\pi}{2}(-\alpha-\frac12-\frac{\mu}{2})}\left(\frac{s+i}{2}\right)^{-\frac{\beta}{2}-\frac14-\frac{\mu}{2}}\left(\frac{s-i}{2}\right)^{\frac{\beta}{2}+\frac14}\mathbf{F}_{\mu,\beta,\alpha}(\frac{2}{1-is})}{\sin(-\pi\mu)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)}$$
$$+\frac{\pi e^{i\frac{\pi}{2}(-\alpha-\frac12+\frac{\mu}{2})}\left(\frac{s+i}{2}\right)^{-\frac{\beta}{2}-\frac14+\frac{\mu}{2}}\left(\frac{s-i}{2}\right)^{\frac{\beta}{2}+\frac14}\mathbf{F}_{-\mu,\beta,\alpha}(\frac{2}{1-is})}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)},\quad s>0 \tag{4.58}$$
$$\frac{\pi e^{i\frac{\pi}{2}(\alpha+\frac12+\frac{\mu}{2})}\left(\frac{-s-i}{2}\right)^{-\frac{\beta}{2}-\frac14-\frac{\mu}{2}}\left(\frac{-s+i}{2}\right)^{\frac{\beta}{2}+\frac14}\mathbf{F}_{\mu,\beta,\alpha}(\frac{2}{1-is})}{\sin(-\pi\mu)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)}$$
$$+\frac{\pi e^{i\frac{\pi}{2}(\alpha+\frac12-\frac{\mu}{2})}\left(\frac{-s-i}{2}\right)^{-\frac{\beta}{2}-\frac14+\frac{\mu}{2}}\left(\frac{-s+i}{2}\right)^{\frac{\beta}{2}+\frac14}\mathbf{F}_{-\mu,\beta,\alpha}(\frac{2}{1-is})}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}.\quad s<0 \tag{4.59}$$

Inserting $s=\sinh(-r)=-\sinh r$ into (4.59), and using uniqueness of analytic continuation we get

$$\begin{aligned}p^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) :=&\left(\frac{1-i\sinh r}{2}\right)^{\frac{\alpha}{2}+\frac14}\left(\frac{1+i\sinh r}{2}\right)^{\frac{\beta}{2}+\frac14}\mathbf{F}_{\alpha,\beta,\mu}\left(\frac{1-i\sinh r}{2}\right)\\ =&\frac{\pi e^{\frac{i\pi}{2}\left(\alpha+\frac{\mu}{2}+\frac12\right)}}{\sin(-\pi\mu)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)}Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r)\\ &+\frac{\pi e^{\frac{i\pi}{2}\left(\alpha-\frac{\mu}{2}+\frac12\right)}}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r).\end{aligned} \tag{4.60}$$

By replacing α to $-\alpha$ we obtain the second identity

$$\begin{aligned}p^{\mathrm{dS}}_{-\alpha,\beta,\mu}(r) =&\frac{\pi e^{\frac{i\pi}{2}\left(-\alpha+\frac{\mu}{2}+\frac12\right)}}{\sin(-\pi\mu)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)}Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r)\\ &+\frac{\pi e^{\frac{i\pi}{2}\left(-\alpha-\frac{\mu}{2}+\frac12\right)}}{\sin(\pi\mu)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)}Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r).\end{aligned} \tag{4.61}$$

We can rewrite them via

$$\begin{aligned}\begin{bmatrix}p^{\mathrm{dS}}_{\alpha,\beta,\mu}(r)\\ p^{\mathrm{dS}}_{-\alpha,\beta,\mu}(r)\end{bmatrix} =&\frac{\pi}{\sin(\pi\mu)}\begin{bmatrix}-\frac{e^{\frac{i\pi}{2}\left(\alpha+\frac{\mu}{2}+\frac12\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)} & \frac{e^{\frac{i\pi}{2}\left(\alpha-\frac{\mu}{2}+\frac12\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}\\ -\frac{e^{\frac{i\pi}{2}\left(-\alpha+\frac{\mu}{2}+\frac12\right)}}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & \frac{e^{\frac{i\pi}{2}\left(-\alpha-\frac{\mu}{2}+\frac12\right)}}{\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)}\end{bmatrix}\\ &\times\begin{bmatrix}Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r)\\ Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r)\end{bmatrix}.\end{aligned} \tag{4.62}$$

We evaluate

$$\det\begin{bmatrix}-\frac{e^{\frac{i\pi}{2}\left(\alpha+\frac{\mu}{2}+\frac12\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)} & \frac{e^{\frac{i\pi}{2}\left(\alpha-\frac{\mu}{2}+\frac12\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)}\\ -\frac{e^{\frac{i\pi}{2}\left(-\alpha+\frac{\mu}{2}+\frac12\right)}}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & \frac{e^{\frac{i\pi}{2}\left(-\alpha-\frac{\mu}{2}+\frac12\right)}}{\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)}\end{bmatrix} = -i\frac{\sin(\pi\mu)\sin(\pi\alpha)}{\pi^2} \tag{4.63}$$

Therefore we have

$$\begin{bmatrix} Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r) \\ Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r) \end{bmatrix} = \frac{-\mathrm{i}\pi}{\sin(\pi\alpha)} \begin{bmatrix} \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} & -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)} \\ \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)}. \end{bmatrix} \times \begin{bmatrix} p^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) \\ p^{\mathrm{dS}}_{-\alpha,\beta,\mu}(r) \end{bmatrix} \tag{4.64}$$

In (4.62), we exchange α and β, and replace r with $-r$, and we obtain

$$\begin{bmatrix} p^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r) \\ p^{\mathrm{dS}}_{-\beta,\alpha,\mu}(-r) \end{bmatrix} = \frac{\pi}{\sin(\pi\mu)} \begin{bmatrix} -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\beta+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)} & \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\beta-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} \\ -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\beta+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\beta-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} \end{bmatrix} \times \begin{bmatrix} Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) \\ Q^{\mathrm{dS}}_{\alpha,\beta,-\mu}(r) \end{bmatrix}. \tag{4.65}$$

Using (A.12), we know that

$$\begin{bmatrix} p^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) \\ p^{\mathrm{dS}}_{-\alpha,\beta,\mu}(r) \end{bmatrix} = \frac{\pi}{\sin(\pi\beta)} \begin{bmatrix} -\frac{1}{\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)} & \frac{1}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)} \\ -\frac{1}{\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} & \frac{1}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} \end{bmatrix} \times \begin{bmatrix} p^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r) \\ p^{\mathrm{dS}}_{-\beta,\alpha,\mu}(-r) \end{bmatrix}. \tag{4.66}$$

Thus we obtained the connection formula

$$\begin{aligned} \begin{bmatrix} Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r) \\ Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r). \end{bmatrix} &= \frac{-\mathrm{i}\pi^3}{\sin(\pi\alpha)\sin(\pi\beta)\sin(\pi\mu)} \begin{bmatrix} \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} & -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\alpha-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)} \\ \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\alpha+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)} \end{bmatrix} \\ &\times \begin{bmatrix} -\frac{1}{\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)} & \frac{1}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)} \\ -\frac{1}{\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} & \frac{1}{\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} \end{bmatrix} \\ &\times \begin{bmatrix} -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\beta+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)} & \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(\beta-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} \\ -\frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\beta+\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} & \frac{\mathrm{e}^{\frac{\mathrm{i}\pi}{2}\left(-\beta-\frac{\mu}{2}+\frac{1}{2}\right)}}{\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} \end{bmatrix} \begin{bmatrix} Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) \\ Q^{\mathrm{dS}}_{\alpha,\beta,-\mu}(r) \end{bmatrix}. \end{aligned} \tag{4.67}$$

The result of multiplication after simplification with the reflection formula for the gamma functions is

$$\begin{aligned} &\begin{bmatrix} -\frac{\mathrm{e}^{-\frac{\mathrm{i}\pi\mu}{2}}\cos(\pi\alpha)+\mathrm{e}^{\frac{\mathrm{i}\pi\mu}{2}}\cos(\pi\beta)}{\pi} & \frac{2\pi}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} \\ \frac{-2\pi}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} & \frac{\mathrm{e}^{-\frac{\mathrm{i}\pi\mu}{2}}\cos(\pi\beta)+\mathrm{e}^{\frac{\mathrm{i}\pi\mu}{2}}\cos(\pi\alpha)}{\pi} \end{bmatrix} \\ &\times \begin{bmatrix} Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) \\ Q^{\mathrm{dS}}_{\alpha,\beta,-\mu}(r) \end{bmatrix} \frac{-\pi}{\sin(\pi\mu)} = \begin{bmatrix} Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r) \\ Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r) \end{bmatrix}. \end{aligned} \tag{4.68}$$

Using

$$\mathcal{W}\big(Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r), Q^{\mathrm{dS}}_{\alpha,\beta,-\mu}(r)\big) = \frac{\sin\pi\mu}{\pi} \tag{4.69}$$

we obtain

$$\mathcal{W}\big(Q^{\rm dS}_{\beta,\alpha,\mu}(-r),Q^{\rm dS}_{\alpha,\beta,\mu}(r)\big)=\frac{2\pi}{\Gamma\Big(\frac{1-\alpha-\beta+\mu}{2}\Big)\Gamma\Big(\frac{1+\alpha-\beta+\mu}{2}\Big)\Gamma\Big(\frac{1-\alpha+\beta+\mu}{2}\Big)\Gamma\Big(\frac{1+\alpha+\beta+\mu}{2}\Big)} \tag{4.70}$$

$$\mathcal{W}\big(Q^{\rm dS}_{\beta,\alpha,-\mu}(-r),Q^{\rm dS}_{\alpha,\beta,\mu}(r)\big)=\frac{\mathrm{e}^{\mathrm{i}\pi\frac{\mu}{2}}\cos\pi\beta+\mathrm{e}^{-\mathrm{i}\pi\frac{\mu}{2}}\cos\pi\alpha}{\pi}. \tag{4.71}$$

The L^2 integrable condition at both endpoints is $\mathrm{Re}(\mu)>0$ for both $Q^{\rm dS}_{\alpha,\beta,\mu}(r)$, and $Q^{\rm dS}_{\beta,\alpha,\mu}(-r)$. Therefore, for $\mathrm{Re}\,\mu>0$, it is a unique candidate for the resolvent. Using the Schur Test we see that the integral kernel (4.54) defines a bounded operator.

From the singularities of the Gamma function we obtain

$$\sigma_{\rm d}(L^{\rm dS}_\alpha)=\Big\{-\Big(n+\frac{1+\alpha+\beta}{2}\Big)^2\ \Big|\ n\in\mathbb{N}_0,\quad \mathrm{Re}\Big(n+\frac{1+\alpha+\beta}{2}\Big)<0\Big\} \tag{4.72}$$

$$\cup\Big\{-\Big(n+\frac{1+\alpha-\beta}{2}\Big)^2\ \Big|\ n\in\mathbb{N}_0,\quad \mathrm{Re}\Big(n+\frac{1+\alpha-\beta}{2}\Big)<0\Big\} \tag{4.73}$$

$$\cup\Big\{-\Big(n+\frac{1-\alpha+\beta}{2}\Big)^2\ \Big|\ n\in\mathbb{N}_0,\quad \mathrm{Re}\Big(n+\frac{1-\alpha+\beta}{2}\Big)<0\Big\} \tag{4.74}$$

$$\cup\Big\{-\Big(n+\frac{1-\alpha-\beta}{2}\Big)^2\ \Big|\ n\in\mathbb{N}_0,\quad \mathrm{Re}\Big(n+\frac{1-\alpha-\beta}{2}\Big)<0\Big\}. \tag{4.75}$$

If $\mathrm{Re}(\alpha+\beta)\geq 0$ and $\mathrm{Re}(\alpha-\beta)\geq 0$, then (4.72) and (4.73) are empty and we obtain (4.55). □

5 Hypergeometric Hamiltonians of the second kind

We transform the hypergeometric equation in a different way:

$$\begin{aligned}&-4z^{1+\frac{\alpha}{2}}(1-z)^{1+\frac{\beta}{2}}\mathcal{F}\big(\tfrac{\alpha+\beta+\mu+1}{2},\tfrac{\alpha+\beta-\mu+1}{2};1+\alpha;z,\partial_z\big)z^{-\frac{\alpha}{2}}(1-z)^{-\frac{\beta}{2}}\\ =&\ -4z^2(z-1)^2\mathcal{P}\begin{pmatrix}0&1&\infty&\\ \frac{\alpha}{2}&\frac{\beta}{2}&\frac{\mu}{2}+\frac12&z,\partial_z\\ -\frac{\alpha}{2}&-\frac{\beta}{2}&-\frac{\mu}{2}+\frac12&\end{pmatrix}\\ =&\ -4z^2(1-z)^2\Big(\partial_z^2+\Big(\frac1z-\frac{1}{1-z}\Big)\partial_z\Big)+\alpha^2(1-z)+\beta^2z-(\mu^2-1)z(1-z). \end{aligned}\tag{5.1}$$

We rearrange the terms in (5.1) containing α and β as follows:

$$\alpha^2(1-z)+\beta^2z=\delta+\kappa(1-2z) \tag{5.2}$$

$$=\delta+\tau\mathrm{i}(2z-1), \tag{5.3}$$

where δ,κ,τ are defined in (4.5). Thus if we set

$$K_{\kappa,\mu}:=-4z^2(1-z)^2\Big(\partial_z^2+\Big(\frac1z-\frac{1}{1-z}\Big)\partial_z\Big)+\kappa(1-2z)-(\mu^2-1)z(1-z) \tag{5.4}$$

and if $F(z)$ solves the hypergeometric equation (2.7), then

$$\Big(K_{\kappa,\mu} + \delta\Big) z^{\frac{\alpha}{2}} (1-z)^{\frac{\beta}{2}} F(z) = 0. \tag{5.5}$$

We have reinterpreted (5.1) as the eigenequation of the operator $K_{\kappa,\mu}$ with eigenvalue $-\delta$.

Again, it is natural to interpret this operator as acting on functions on an interval whose endpoints are singularities of the hypergeometric equations. In each of these cases we perform the Liouville transformation, which yields a 1-dimensional Hamiltonian. We will consider three cases:

1. $z \in \frac{1}{2} + \mathrm{i}\mathbb{R}$, which leads to an operator on $L^2]0,\pi[$, which we call the *spherical hypergeometric Hamiltonian of the 2nd kind*;

2. $z \in]-\infty, 0[$, which leads to an operator on $L^2]0,\infty[$ which we call the *hyperbolic hypergeometric Hamiltonian of the 2nd kind*;

3. $z \in]0,1[$, which leads to an operator on $L^2(\mathbb{R})$, which we call *the deSitterian hypergeometric Hamiltonian of the 2nd kind.*

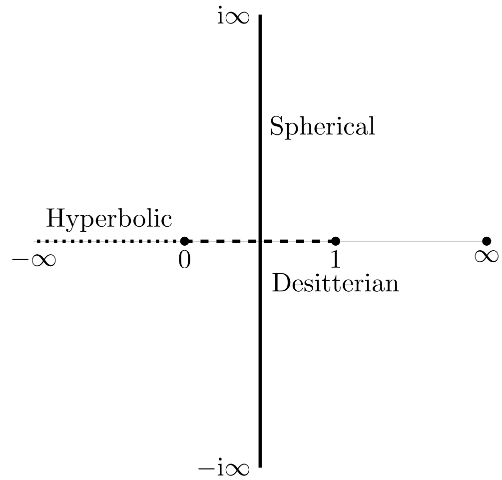

Figure 3: Hypergeometric Hamiltonian of the second kind on the z-plane. The hyperbolic Hamiltonian act on the interval with dotted line, the spherical—with the thick line, and the Desitterian—with the dashed line.

5.1 Spherical case

For $u \in]0,\pi[$, in (5.1) and (5.3) we set

$$z = \frac{1}{1 - \mathrm{e}^{2\mathrm{i}u}}, \quad \text{which solves} \quad z' = 2\mathrm{i}z(1-z). \tag{5.6}$$

This leads to the Schrödinger equation

$$\big(K^{\mathrm{s}}_{\tau,\mu} - \delta\big)\, \phi(u) = 0, \tag{5.7}$$

where

$$K^{\mathrm{s}}_{\tau,\mu}(u) := -\partial_u^2 + \tau \frac{\cos u}{\sin u} + \left(\frac{\mu^2}{4} - \frac{1}{4}\right) \frac{1}{\sin^2 u}. \tag{5.8}$$

This Hamiltonian is known as the *Rosen-Morse Hamiltonian* (see [DW]).

In the case $\tau = 0$ we have the coincidence:

$$L^{\mathrm{s}}_{\alpha} = K^{\mathrm{s}}_{0,2\alpha}. \tag{5.9}$$

We define for $u \in]0, \frac{\pi}{2}]$

$$\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u) \tag{5.10}$$

$$:= \Big(\frac{\mathrm{i}}{1-\mathrm{e}^{2\mathrm{i}u}}\Big)^{\frac{-1-\beta-\mu}{2}} \Big(\frac{-\mathrm{i}}{1-\mathrm{e}^{-2\mathrm{i}u}}\Big)^{\frac{\beta}{2}} \mathbf{F}\Big(\frac{\alpha+\beta+\mu+1}{2}, \frac{-\alpha+\beta+\mu+1}{2}; \mu+1; 1-\mathrm{e}^{2\mathrm{i}u}\Big) \tag{5.11}$$

$$= \Big(\frac{\mathrm{i}}{1-\mathrm{e}^{2\mathrm{i}u}}\Big)^{\frac{\alpha}{2}} \Big(\frac{-\mathrm{i}}{1-\mathrm{e}^{-2\mathrm{i}u}}\Big)^{\frac{-1-\alpha-\mu}{2}} \mathbf{F}\Big(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha-\beta+\mu+1}{2}; \mu+1; 1-\mathrm{e}^{-2\mathrm{i}u}\Big). \tag{5.12}$$

For $u \in [\frac{\pi}{2}, \pi[$ it is continued analytically. It has the asymptotics

$$\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u) \sim \frac{1}{\Gamma(\mu+1)}(2u)^{\frac{\mu}{2}+\frac{1}{2}}, \quad u \sim 0. \tag{5.13}$$

Note that

$$\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u) = \mathbb{Q}^{\mathrm{s}}_{-\alpha,\beta,\mu}(u) = \mathbb{Q}^{\mathrm{s}}_{\alpha,-\beta,\mu}(u), \tag{5.14}$$

$$\mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-u) = \mathbb{Q}^{\mathrm{s}}_{\beta,-\alpha,\mu}(\pi-u) = \mathbb{Q}^{\mathrm{s}}_{-\beta,\alpha,\mu}(\pi-u), \tag{5.15}$$

$$\Big(\frac{2}{\cosh r}\Big)^{\frac{1}{2}} Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r) = \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u), \tag{5.16}$$

$$\Big(\frac{2}{\cosh r}\Big)^{\frac{1}{2}} Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r) = \mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-u), \quad \sinh r = -\cot u. \tag{5.17}$$

Now the following functions solve the eigenvalue problem (5.7):

$$\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u), \quad \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,-\mu}(u), \quad \mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-u), \quad \mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,-\mu}(\pi-u). \tag{5.18}$$

The following theorem describes the basic closed realization of $K^{\mathrm{s}}_{\tau,\mu}$:

Theorem 5.1. *For $\operatorname{Re}\mu \geq 2$, $\tau \in \mathbb{C}$, there exists a unique closed operator $K^{\mathrm{s}}_{\tau,\mu}$ in the sense of $L^2]0,\pi[$, which on $C^\infty_{\mathrm{c}}]0,\pi[$ is given by (5.8). The family $\tau, \mu \mapsto K^{\mathrm{s}}_{\tau,\mu}$ is holomorphic and possesses a unique holomorphic extension to $\operatorname{Re}\mu > -2$, except for a singularity at $(\tau,\mu) = (0,-1)$. It has only discrete spectrum:*

$$\sigma(K^{\mathrm{s}}_{\tau,\mu}) = \Big\{ -\frac{\tau^2}{(2k+\mu)^2} + \Big(k+\frac{\mu}{2}\Big)^2 \quad | \quad k \in \mathbb{N}_0 + \frac{1}{2} \Big\}. \tag{5.19}$$

Set

$$\alpha := \sqrt{\delta - \mathrm{i}\tau}, \quad \beta := \sqrt{\delta + \mathrm{i}\tau}. \tag{5.20}$$

(It does not matter which sign of the square root is taken). Outside of the spectrum the resolvent of $K^{\mathrm{s}}_{\tau,\mu}$ is

$$\frac{1}{(K^{\mathrm{s}}_{\tau,\mu}-\delta)}(x,y) = \frac{\Gamma\Big(\frac{1-\alpha-\beta+\mu}{2}\Big)\Gamma\Big(\frac{1+\alpha-\beta+\mu}{2}\Big)\Gamma\Big(\frac{1-\alpha+\beta+\mu}{2}\Big)\Gamma\Big(\frac{1+\alpha+\beta+\mu}{2}\Big)}{4\pi} \times \begin{cases} \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(x)\mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-y), & \text{if} \quad 0<x<y<\pi; \\ \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(y)\mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-x), & \text{if} \quad 0<y<x<\pi. \end{cases} \tag{5.21}$$

Proof. The relation $\sinh r = -\cot u$ implies

$$\frac{2}{1-\mathrm{i}\sinh r} = 1-\mathrm{e}^{2\mathrm{i}u}, \quad \frac{2}{1+\mathrm{i}\sinh r} = 1-\mathrm{e}^{-2\mathrm{i}u}, \tag{5.22}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \sin u = \frac{1}{\cosh r}. \tag{5.23}$$

Therefore, using (4.70) and (4.71), we obtain

$$\mathcal{W}\left(\mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi-u), \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u)\right) = 2\mathcal{W}\left(Q^{\mathrm{dS}}_{\beta,\alpha,\mu}(-r), Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r)\right) \tag{5.24}$$

$$= \frac{4\pi}{\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)}, \tag{5.25}$$

$$\mathcal{W}\big(\mathbb{Q}^{\mathrm{s}}_{\beta,\alpha,-\mu}(\pi-u), \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u)\big) = 2\mathcal{W}\left(Q^{\mathrm{dS}}_{\beta,\alpha,-\mu}(-r), Q^{\mathrm{dS}}_{\alpha,\beta,\mu}(r)\right) \tag{5.26}$$

$$= \frac{2(\mathrm{e}^{\mathrm{i}\pi\frac{\mu}{2}}\cos\pi\beta + \mathrm{e}^{-\mathrm{i}\pi\frac{\mu}{2}}\cos\pi\alpha)}{\pi}. \tag{5.27}$$

The L^2 integrability condition for $\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u)$ at 0 and $\mathbb{Q}^{s}_{\beta,\alpha,\mu}(\pi-u)$ at π is $\mathrm{Re}\,\mu > -2$. The L^2 norm of this kernel 5.21 is finite. For $\mathrm{Re}(\mu) \geq 2$ it is a unique candidate for the resolvent.

As a byproduct we obtain a proof of Prop. 1.4 about the transmutation identity $L^{\mathrm{dS}} \to K^{\mathrm{s}}$.

The singularities of the Gamma functions in (5.21) are at

$$1 + \epsilon_1\alpha + \epsilon_2\beta + \mu = -2n, \quad n \in \mathbb{N}_0, \tag{5.28}$$

where $\epsilon_1, \epsilon_2 \in \{1, -1\}$. This implies

$$\alpha^2 = (2n+1+\mu)^2 + 2\epsilon_2\beta(2n+1+\mu) + \beta^2. \tag{5.29}$$

Hence,

$$\beta = \frac{\epsilon_2\kappa}{2n+1+\mu} - \epsilon_2\Big(n + \frac{\mu}{2} + \frac{1}{2}\Big), \tag{5.30}$$

$$\alpha = -\frac{\epsilon_2\kappa}{2n+1+\mu} - \epsilon_2\Big(n + \frac{\mu}{2} + \frac{1}{2}\Big). \tag{5.31}$$

This shows

$$\delta = \frac{\kappa^2}{(2n+1+\mu)^2} + \Big(n + \frac{1}{2} + \frac{\mu}{2}\Big)^2. \tag{5.32}$$

Replacing κ^2 with $-\tau^2$ we obtain (5.19). □

For $\mu \in \mathbb{Z}$ we have an additional identity for the $\mathbb{Q}^{\mathrm{s}}$ function, which follows directly from (A.17):

$$\Big(\frac{\alpha+\beta-\mu+1}{2}\Big)_\mu \Big(\frac{\alpha-\beta-\mu+1}{2}\Big)_\mu \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,\mu}(u) = \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,-\mu}(u). \tag{5.33}$$

Using this with $\mu = 1$ we obtain the following unexpected identity:

Theorem 5.2. *For any $\tau \neq 0$, we have $K^{\mathrm{s}}_{\tau,1} = K^{\mathrm{s}}_{\tau,-1}$.*

Proof. Setting $\mu = 1$ in (5.62) we obtain

$$\frac{\alpha^2-\beta^2}{4}\mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,1}(u) = \mathbb{Q}^{\mathrm{s}}_{\alpha,\beta,-1}(u). \tag{5.34}$$

Setting $\mu = 1$ and $\mu = -1$ in the prefactor of the right hand side of (5.21) we obtain

$$\frac{\Gamma\left(1-\frac{\alpha+\beta}{2}\right)\Gamma\left(1+\frac{\alpha-\beta}{2}\right)\Gamma\left(1-\frac{\alpha-\beta}{2}\right)\Gamma\left(1+\frac{\alpha+\beta}{2}\right)}{4\pi} = \frac{(\alpha^2-\beta^2)\pi}{16\sin\frac{\pi}{2}(\alpha+\beta)\sin\frac{\pi}{2}(\alpha-\beta)}, \tag{5.35}$$

$$\frac{\Gamma\left(-\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\alpha-\beta}{2}\right)\Gamma\left(-\frac{\alpha-\beta}{2}\right)\Gamma\left(\frac{\alpha+\beta}{2}\right)}{4\pi} = \frac{\pi}{(\alpha^2-\beta^2)\sin\frac{\pi}{2}(\alpha+\beta)\sin\frac{\pi}{2}(\alpha-\beta)}. \tag{5.36}$$

We thus obtain

$$\frac{1}{(K^{\mathrm{s}}_{\tau,1}+\delta)} = \frac{1}{(K^{\mathrm{s}}_{\tau,-1}+\delta)}. \tag{5.37}$$

□

Note that (5.21) is ill defined for $(\tau,\mu) = (0,-1)$. Moreover, we know that $\{2 < \operatorname{Re}\mu\} \ni \mu \mapsto L^{\mathrm{s}}_{\frac{\mu}{2}}$ is analytic, and for $\mu \neq -1$.

We have $K^{\mathrm{s}}_{0,\mu} = L^{\mathrm{s}}_{\frac{\mu}{2}}$. Therefore, it is natural to set

$$K^{\mathrm{s}}_{0,-1} := L^{\mathrm{s}}_{-\frac{1}{2}}. \tag{5.38}$$

We know that $L^{\mathrm{s}}_{-\frac{1}{2}} \neq L^{\mathrm{s}}_{\frac{1}{2}}$. Therefore, Thm 5.2 implies that the point $(\tau,\mu) = (0,-1)$ is a singularity of the function $(\tau,\mu) \mapsto K^{\mathrm{s}}_{\tau,\mu}$. See [DeRi], where a similar phenomenon is described for the Whittaker operator.

5.2 Hyperbolic case

For $u \in \mathbb{R}_+$, in (5.1) and (5.2) we set

$$z = \frac{1}{1-\mathrm{e}^{2u}}, \quad \text{which solves} \quad z' = 2z(z-1). \tag{5.39}$$

This leads to the Schrödinger equation

$$\left(K^{\mathrm{h}}_{\kappa,\mu}+\delta\right)\phi(u) = 0, \tag{5.40}$$

where

$$K^{\mathrm{h}}_{\kappa,\mu} := -\partial_u^2 + \kappa\frac{\cosh u}{\sinh u} + \left(\frac{\mu^2}{4}-\frac{1}{4}\right)\frac{1}{\sinh^2 u}. \tag{5.41}$$

This Hamiltonian in [DW] is called the *Eckart Hamiltonian.*

In the case $\kappa = 0$ we have the coincidence

$$L^{\mathrm{h}}_{\alpha} = K^{\mathrm{h}}_{0,2\alpha}. \tag{5.42}$$

We define

$$\mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(u) := (\mathrm{e}^{2u}-1)^{-\frac{\alpha}{2}}(1-\mathrm{e}^{-2u})^{-\frac{\beta}{2}}\,\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha+\beta-\mu+1}{2};\alpha+1;\frac{1}{1-\mathrm{e}^{2u}}\right) \tag{5.43}$$

$$= (\mathrm{e}^{2u}-1)^{-\frac{\alpha}{2}}(1-\mathrm{e}^{-2u})^{\frac{1+\alpha+\mu}{2}}\,\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha-\beta+\mu+1}{2};\alpha+1;\mathrm{e}^{-2u}\right), \tag{5.44}$$

$$\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(u) := (\mathrm{e}^{2u}-1)^{\frac{1+\beta+\mu}{2}}(1-\mathrm{e}^{-2u})^{-\frac{\beta}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{-\alpha+\beta+\mu+1}{2};\mu+1;1-\mathrm{e}^{2u}\right) \tag{5.45}$$

$$= (\mathrm{e}^{2u}-1)^{-\frac{\alpha}{2}}(1-\mathrm{e}^{-2u})^{\frac{1+\alpha+\mu}{2}}\mathbf{F}\left(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha-\beta+\mu+1}{2};\mu+1;1-\mathrm{e}^{-2u}\right). \tag{5.46}$$

Note that

$$\mathbb{P}^{\mathrm{h}}_{\alpha,\beta\mu}(u) \sim \frac{1}{\Gamma(1+\alpha)}\mathrm{e}^{-\alpha u}, \quad u \sim +\infty; \tag{5.47}$$

$$\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta\mu}(u) \sim \frac{1}{\Gamma(1+\mu)}(2u)^{\frac{\mu}{2}+\frac{1}{2}}, \quad u \sim 0. \tag{5.48}$$

We have

$$\mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(u) = \mathbb{P}^{\mathrm{h}}_{\alpha,-\beta,\mu}(u) = \mathbb{P}^{\mathrm{h}}_{\alpha,\beta,-\mu}(u), \tag{5.49}$$

$$\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(u) = \mathbb{Q}^{\mathrm{h}}_{-\alpha,\beta,\mu}(u) = \mathbb{Q}^{\mathrm{h}}_{\alpha,-\beta,\mu}(u); \tag{5.50}$$

$$\Big(\frac{2}{\sinh r}\Big)^{\frac{1}{2}} P^{\mathrm{h}}_{\alpha,\beta,\mu}(r) = \mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(u), \tag{5.51}$$

$$\Big(\frac{2}{\sinh r}\Big)^{\frac{1}{2}} Q^{\mathrm{h}}_{\alpha,\beta,\mu}(r) = \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(u), \qquad \cosh r = \coth u. \tag{5.52}$$

Now the following functions solve the eigenvalue problem (5.40):

$$\mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(u), \quad \mathbb{P}^{\mathrm{h}}_{-\alpha,\beta,\mu}(u), \quad \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(u), \quad \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,-\mu}(u). \tag{5.53}$$

Let us describe the basic closed realization of $K^{\mathrm{h}}_{\kappa,\mu}$ on $L^2(\mathbb{R}_+)$.

Theorem 5.3. *For $\kappa \in \mathbb{C}, \operatorname{Re}\mu \geq 2$ there exists a unique closed operator $K^{\mathrm{h}}_{\kappa,\mu}$ in the sense of $L^2(\mathbb{R}_+)$, which on $C_{\mathrm{c}}^{\infty}(\mathbb{R}_+)$ is given by (5.41). The family $\kappa, \mu \mapsto K^{\mathrm{h}}_{\kappa,\mu}$ is holomorphic and possesses a unique holomorphic extension to $\operatorname{Re}\mu > -2$, $(\kappa,\mu) \neq (0,1)$. Its discrete spectrum and spectrum are*

$$\begin{aligned}\sigma_{\mathrm{d}}(K^{\mathrm{h}}_{\kappa,\mu}) =& \Big\{ -\frac{\kappa^2}{(2k+\mu)^2} - \Big(k+\frac{\mu}{2}\Big)^2 \ \Big|\ k \in \mathbb{N}_0 + \frac{1}{2}, \quad \operatorname{Re}\Big(\frac{\kappa}{2k+\mu} + k + \frac{\mu}{2}\Big) < 0 \Big\},\\ \sigma(K^{\mathrm{h}}_{\kappa,\mu}) =& [0,+\infty[\cup \sigma_{\mathrm{d}}(K^{\mathrm{h}}_{\kappa,\mu}).\end{aligned} \tag{5.54}$$

Set

$$\alpha := \sqrt{\delta+\kappa}, \quad \operatorname{Re}\alpha > 0, \quad \beta := \sqrt{\delta-\kappa} \tag{5.55}$$

(The choice of the square root for β does not matter). Outside of its spectrum the resolvent of $K^{\mathrm{h}}_{\kappa,\mu}$ is

$$\begin{aligned}\frac{1}{(K^{\mathrm{h}}_{\kappa,\mu}+\delta)}(x,y) =& \frac{\Gamma\Big(\frac{1+\alpha+\beta+\mu}{2}\Big)\Gamma\Big(\frac{1+\alpha-\beta+\mu}{2}\Big)}{2}\\ &\times \begin{cases} \mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(x)\ \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(y) & \text{if} \quad 0<y<x<\infty,\\ \mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(y)\ \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(x) & \text{if} \quad 0<x<y<\infty.\end{cases}\end{aligned} \tag{5.56}$$

Proof. The relation $\cosh r = \coth u$ implies

$$\sinh^2\frac{r}{2} = \frac{1}{\mathrm{e}^{2u}-1}, \quad \cosh^2\frac{r}{2} = \frac{1}{\mathrm{e}^{-2u}-1}; \tag{5.57}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{1}{\sinh r} = \sinh u, \tag{5.58}$$

Therefore,

$$\mathcal{W}\left(\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}, \mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}\right) = 2\mathcal{W}\left(Q^{\mathrm{h}}_{\alpha,\beta,\mu}, P^{\mathrm{h}}_{\alpha,\beta,\mu}\right) \tag{5.59}$$

$$= \frac{2}{\Gamma\Big(\frac{1+\alpha+\beta+\mu}{2}\Big)\Gamma\Big(\frac{1+\alpha-\beta+\mu}{2}\Big)}. \tag{5.60}$$

The L^2 integrability conditions at endpoints are $\operatorname{Re}\alpha > 0$, and $\operatorname{Re}\mu > -2$ for $\mathbb{P}^{\mathrm{h}}_{\alpha,\beta,\mu}(r)$ and $\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(r)$, respectively. For such parameters the integral kernel (5.56) defines a bounded operator. For $\operatorname{Re}\alpha > 0$, and $\operatorname{Re}\mu \geq 2$, it is the unique candidate for the resolvent.

As a byproduct we obtain a proof of Prop. 1.5 about the transmutation identity $L^{\mathrm{h}} \to K^{\mathrm{h}}$.

The determination of the discrete spectrum follows by similar arguments as for Thm 5.2. □

For $\mu \in \mathbb{Z}$ we have an additional identity for the $\mathbb{Q}^{\mathrm{h}}$ function, which follows directly from (A.17):

$$\Big(\frac{\alpha+\beta-\mu+1}{2}\Big)_{\mu}\Big(\frac{\alpha-\beta-\mu+1}{2}\Big)_{\mu}\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,\mu}(u) = \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,-\mu}(u). \tag{5.61}$$

Using this with $\mu = 1$ we obtain

Theorem 5.4. *For any $\kappa \neq 0$, we have $K^{\mathrm{h}}_{\kappa,1} = K^{\mathrm{h}}_{\kappa,-1}$.*

Proof. Setting $\mu = 1$ in (5.62) we obtain

$$\frac{\alpha^2-\beta^2}{4}\mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,1}(u) = \mathbb{Q}^{\mathrm{h}}_{\alpha,\beta,-1}(u). \tag{5.62}$$

Moreover,

$$\Gamma\left(1-\frac{\alpha+\beta}{2}\right)\Gamma\left(1+\frac{\alpha-\beta}{2}\right) = \frac{(\alpha^2-\beta^2)}{4}\Gamma\left(\frac{\alpha+\beta}{2}\right)\Gamma\left(\frac{\alpha-\beta}{2}\right), \tag{5.63}$$

We thus obtain

$$\frac{1}{(K^{\mathrm{h}}_{\tau,1}+\delta)} = \frac{1}{(K^{\mathrm{h}}_{\tau,-1}+\delta)}. \tag{5.64}$$

□

Similarly as in the spherical case, (5.56) is ill defined for $(\kappa,\mu) = (0,-1)$. Moreover, we know that $\{2 < \mathrm{Re}\mu\} \ni \mu \mapsto L^{\mathrm{h}}_{\frac{\mu}{2}}$ is analytic, and for $\mu \neq -1$ we have $K^{\mathrm{h}}_{0,\mu} = L^{\mathrm{h}}_{\frac{\mu}{2}}$. Therefore, it is natural to set

$$K^{\mathrm{h}}_{0,-1} := L^{\mathrm{h}}_{-\frac{1}{2}}. \tag{5.65}$$

We know that $L^{\mathrm{h}}_{-\frac{1}{2}} \neq L^{\mathrm{h}}_{\frac{1}{2}}$. Therefore, Thm 5.4 implies that the point $(\kappa,\mu) = (0,-1)$ is a singularity of the function $(\kappa,\mu) \mapsto K^{\mathrm{h}}_{\kappa,\mu}$.

5.3 DeSitterian case

For $u \in \mathbb{R}$, in (5.1) and (5.2) we set

$$z = \frac{1}{1+\mathrm{e}^{2u}}, \quad \text{which solves} \quad z' = 2z(z-1). \tag{5.66}$$

This leads to the Schrödinger equation

$$\left(K^{\mathrm{dS}}_{\kappa,\mu}+\delta\right)\phi(u) = 0, \tag{5.67}$$

where

$$K^{\mathrm{dS}}_{\kappa,\mu} := -\partial_u^2 - \kappa\frac{\sinh u}{\cosh u} - \left(\frac{\mu^2}{4}-\frac{1}{4}\right)\frac{1}{\cosh^2 u}. \tag{5.68}$$

In [DW] it is called the *Manning-Rosen Hamiltonian.*

In the case $\kappa = 0$ we have the coincidence:

$$L_\alpha^{\mathrm{dS}} = K_{0,2\alpha}^{\mathrm{dS}}. \tag{5.69}$$

We define

$$\mathbb{P}_{\alpha,\beta,\mu}^{\mathrm{dS}}(u) := (1+\mathrm{e}^{2u})^{-\frac{\alpha}{2}}(1+\mathrm{e}^{-2u})^{-\frac{\beta}{2}}\mathbf{F}\Big(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha+\beta-\mu+1}{2}; \alpha+1; \frac{1}{1+\mathrm{e}^{2u}}\Big) \tag{5.70}$$

$$:= (1+\mathrm{e}^{2u})^{-\frac{\alpha}{2}}(1+\mathrm{e}^{-2u})^{\frac{1+\alpha+\mu}{2}}\mathbf{F}\Big(\frac{\alpha+\beta+\mu+1}{2}, \frac{\alpha-\beta+\mu+1}{2}; \alpha+1; -\mathrm{e}^{-2u}\Big). \tag{5.71}$$

We have the asymptotics

$$\mathbb{P}_{\alpha,\beta,\mu}^{\mathrm{dS}}(u) \sim \frac{1}{\Gamma(1+\alpha)}\mathrm{e}^{-\alpha u}, \quad u \sim +\infty. \tag{5.72}$$

Note that

$$\mathbb{P}_{\alpha,\beta,\mu}^{\mathrm{dS}}(u) = \mathbb{P}_{\alpha,-\beta,\mu}^{\mathrm{dS}}(u) = \mathbb{P}_{\alpha,\beta,-\mu}^{\mathrm{dS}}(u), \tag{5.73}$$

$$\Big(\frac{2}{\sin r}\Big)^{\frac{1}{2}} P_{\alpha,\beta,\mu}^{\mathrm{s}}(r) = \mathbb{P}_{\alpha,\beta,\mu}^{\mathrm{dS}}(u), \quad \cos r = \tanh u. \tag{5.74}$$

Now the following functions solve the eigenvalue problem (5.67):

$$\mathbb{P}_{\alpha,\beta,\mu}^{\mathrm{dS}}(u), \quad \mathbb{P}_{-\alpha,\beta,\mu}^{\mathrm{dS}}(u), \quad \mathbb{P}_{\beta,\alpha,\mu}^{\mathrm{dS}}(-u), \quad \mathbb{P}_{-\beta,\alpha,\mu}^{\mathrm{dS}}(-u). \tag{5.75}$$

We will find all closed realization of $K_{\kappa,\mu}^{\mathrm{dS}}$ in the sense of $L^2(\mathbb{R})$.

Theorem 5.5. *For any $\kappa, \mu \in \mathbb{C}$ there exists a unique closed operator $K_{\kappa,\mu}^{\mathrm{dS}}$ in the sense of $L^2(\mathbb{R})$ that on $C_{\mathrm{c}}^\infty(\mathbb{R})$ is given by (5.68). The function $\mathbb{C}^2 \ni (\kappa,\mu) \mapsto K_{\kappa,\mu}^{\mathrm{dS}}$ is holomorphic. The discrete spectrum and spectrum of $K_{\kappa,\mu}^{\mathrm{dS}}$ are*

$$\begin{aligned}\sigma_{\mathrm{d}}(K_{\kappa,\mu}^{\mathrm{dS}}) = &\Big\{-\frac{\kappa^2}{(2k+\mu)^2} - \Big(k+\frac{\mu}{2}\Big)^2 \ \Big| \ k \in \mathbb{N}_0 + \frac{1}{2}, \quad k < -\Big|\mathrm{Re}\,\frac{\kappa}{2k+\mu}\Big| - \frac{\mu}{2}\Big\} \\ &\cup \Big\{-\frac{\kappa^2}{(2k-\mu)^2} - \Big(k-\frac{\mu}{2}\Big)^2 \ \Big| \ k \in \mathbb{N}_0 + \frac{1}{2}, \quad k < -\Big|\mathrm{Re}\,\frac{\kappa}{2k-\mu}\Big| + \frac{\mu}{2}\Big\},\end{aligned} \tag{5.76}$$

$$\sigma(K_{\kappa,\mu}^{\mathrm{dS}}) = [\kappa, +\infty[\cup [-\kappa, +\infty[\cup \sigma_{\mathrm{d}}\left(K_{\kappa,\mu}^{\mathrm{dS}}\right). \tag{5.77}$$

Here, for $z \in \mathbb{C}$ we use the notation $[z, +\infty[:= \{z+t \quad | \quad t \in [0,+\infty[\}$.

Set

$$\alpha := \sqrt{\delta+\kappa}, \quad \mathrm{Re}\,\alpha > 0; \quad \beta := \sqrt{\delta-\kappa}, \quad \mathrm{Re}\,\beta > 0. \tag{5.78}$$

Outside of its spectrum, the resolvent of $K_{\kappa,\mu}^{\mathrm{dS}}$ is

$$\frac{1}{(K_{\kappa,\mu}^{\mathrm{dS}}+\delta)}(x,y) = \frac{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)}{2} \times \begin{cases}\mathbb{P}_{\beta,\alpha,\mu}^{\mathrm{dS}}(x)\ \mathbb{P}_{\alpha,\beta,\mu}^{\mathrm{dS}}(y), & \text{if} \quad -\infty < x < y < \infty; \\ \mathbb{P}_{\beta,\alpha,\mu}^{\mathrm{dS}}(y)\ \mathbb{P}_{\alpha,\beta,\mu}^{\mathrm{dS}}(x), & \text{if} \quad -\infty < y < x < \infty.\end{cases} \tag{5.79}$$

We have $K_{\kappa,\mu}^{\mathrm{dS}} = K_{\kappa,-\mu}^{\mathrm{dS}}$

Proof. The relation $\cos r = \tanh u$ implies

$$\sin^2\frac{r}{2} = \frac{1}{1+\mathrm{e}^{2u}}, \quad \cos^2\frac{r}{2} = \frac{1}{1+\mathrm{e}^{-2u}}; \tag{5.80}$$

$$\frac{\mathrm{d}u}{\mathrm{d}r} = \frac{1}{\sin r} = \cosh u. \tag{5.81}$$

This yields (5.74). Using (5.81) and (5.74), and then (4.19), we obtain

$$\mathcal{W}\left(\mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,\mu}(u), \mathbb{P}^{\mathrm{dS}}_{\beta,\alpha,\mu}(-u)\right) = 2\mathcal{W}\left(P^{\mathrm{s}}_{\alpha,\beta,\mu}(r), P^{\mathrm{s}}_{\beta,\alpha,\mu}(\pi - r)\right) \tag{5.82}$$

$$= \frac{2}{\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)}. \tag{5.83}$$

The L^2 integrability condition at $+\infty$ of $\mathbb{P}_{\alpha,\beta,\mu}(u)$ is $\mathrm{Re}(\alpha) > 0$, and at $-\infty$ of $\mathbb{P}_{\beta,\alpha,\mu}(-u)$ is $\mathrm{Re}(\beta) > 0$. For such parameters, the integral kernel (5.79) defines a bounded operator and is a unique candidate for the resolvent.

As a byproduct we obtain a proof of Prop. 1.4 about the transmutation identity $L^{\mathrm{s}} \to L^{\mathrm{dS}}$.

The determination of the discrete spectrum is similar as in the hyperbolic case. □

Proof of Prop. 1.3. The change of variables

$$1 + \mathrm{e}^{2w} = \mathrm{e}^{2u} \tag{5.84}$$

implies

$$-\mathrm{e}^{2w} = 1 - \mathrm{e}^{2u}, \quad -\mathrm{e}^{-2w} = \frac{1}{1-\mathrm{e}^{2u}}; \tag{5.85}$$

$$\frac{\mathrm{d}w}{\mathrm{d}u} = \frac{1}{1-\mathrm{e}^{-2u}} = 1 + \mathrm{e}^{-2w}, \tag{5.86}$$

$$\mathbb{P}^{\mathrm{dS}}_{\alpha,\beta,\mu}(w) = \left(1 - \mathrm{e}^{-2u}\right)^{-\frac{1}{2}} \mathbb{P}^{\mathrm{h}}_{\alpha,\mu,\beta}(u), \tag{5.87}$$

$$\mathbb{P}^{\mathrm{dS}}_{\beta,\alpha,\mu}(-w) = \left(1 - \mathrm{e}^{-2u}\right)^{-\frac{1}{2}} \mathbb{Q}^{\mathrm{h}}_{\alpha,\mu,\beta}(u). \tag{5.88}$$

We redefine parameters

$$\delta' := \frac{1}{2}(\alpha^2 - \mu^2), \qquad \kappa' := \frac{1}{2}(\alpha^2 + \mu^2). \tag{5.89}$$

We obtain the transmutation identity

$$\left(1 - \mathrm{e}^{-2u}\right)^{-\frac{1}{2}} \frac{1}{\left(K^{\mathrm{h}}_{\kappa,\mu} + \delta\right)}(u,u')\left(1 - \mathrm{e}^{-2u'}\right)^{-\frac{1}{2}} = \frac{1}{\left(K^{\mathrm{dS}}_{\kappa',\beta} + \delta'\right)}(w,w'). \tag{5.90}$$

□

6 The Laplacian on an interval, halfline and line

The Laplacians on $]0,\pi[$, $\mathbb{R}_+$ and $\mathbb{R}$ with the Dirichlet or Neumann boundary conditions at endpoints belong to the most widely used operators. Their Green functions can be easily computed in terms of elementary functions, without using hypergeometric functions. In this section we will check that they are special cases of hypergeometric Hamiltonians. We will see that this coincidence is related to various identities for hypergeometric functions from Appendix A.6.

6.1 Laplacian on an interval

Consider the Laplacian $-\partial_x^2$ on the interval $]0,\pi[$. The Dirichlet and Neumann boundary condition will be denoted D and N resp. Putting them at both 0 and π leads to 4 operators on $L^2]0,\pi[$. They are special cases of the spherical hypergeometric Hamiltonian of the first kind:

$$L_{\rm DD} := L^{\rm s}_{\frac12,\frac12} = L^{\rm s}_{\frac12}, \tag{6.1}$$

$$L_{\rm ND} := L^{\rm s}_{-\frac12,\frac12}, \tag{6.2}$$

$$L_{\rm DN} := L^{\rm s}_{\frac12,-\frac12}, \tag{6.3}$$

$$L_{\rm NN} := L^{\rm s}_{-\frac12,-\frac12} = L^{\rm s}_{-\frac12}. \tag{6.4}$$

Resolvents of these operators can be computed in terms of elementary functions. Indeed, the following functions solve

$$(-\partial_x^2 + k^2)\phi(x) = 0 \tag{6.5}$$

and satisfy the Dirichlet/Neumann boundary conditions at 0, resp. at π:

$$\text{Dirichlet:} \quad \sinh kx, \quad \sinh k(\pi - x); \tag{6.6}$$

$$\text{Neumann:} \quad \cosh kx, \quad \cosh k(\pi - x). \tag{6.7}$$

They have the Wronskians:

$$\mathcal{W}\big(\sinh k(\pi-x), \sinh kx\big) = k\sinh \pi k, \tag{6.8}$$

$$\mathcal{W}\big(\cosh k(\pi-x), \sinh kx\big) = k\cosh \pi k, \tag{6.9}$$

$$\mathcal{W}\big(\sinh k(\pi-x), \cosh kx\big) = k\cosh \pi k, \tag{6.10}$$

$$\mathcal{W}\big(\cosh k(\pi-x), \cosh kx\big) = k\sinh \pi k. \tag{6.11}$$

By the usual methods, we compute the spectra of operators (6.1)–(6.4), and for k^2 outside of the spectra their Green functions :

$$\begin{aligned}\sigma(L_{\rm DD}) &= \{n^2 \mid n \in \mathbb{N}_0\},\\ \frac{1}{L_{\rm DD}+k^2}(x,y) &= \frac{1}{k\sinh \pi k}\begin{cases}\sinh kx \, \sinh k(\pi-y), & \text{if} \quad 0<x<y<\pi,\\ \sinh ky \, \sinh k(\pi-x), & \text{if} \quad 0<y<x<\pi;\end{cases}\end{aligned} \tag{6.12}$$

$$\begin{aligned}\sigma(L_{\rm ND}) &= \{(n+\tfrac12)^2 \mid n \in \mathbb{N}_0\},\\ \frac{1}{L_{\rm ND}+k^2}(x,y) &= \frac{1}{k\cosh \pi k}\begin{cases}\cosh kx \, \sinh k(\pi-y), & \text{if} \quad 0<x<y<\pi,\\ \cosh ky \, \sinh k(\pi-x), & \text{if} \quad 0<y<x<\pi;\end{cases}\end{aligned} \tag{6.13}$$

$$\begin{aligned}\sigma(L_{\rm DN}) &= \{(n+\tfrac12)^2 \mid n \in \mathbb{N}_0\},\\ \frac{1}{L_{\rm DN}+k^2}(x,y) &= \frac{1}{k\cosh \pi k}\begin{cases}\sinh kx \, \cosh k(\pi-y), & \text{if} \quad 0<x<y<\pi,\\ \sinh ky \, \cosh k(\pi-x), & \text{if} \quad 0<y<x<\pi;\end{cases}\end{aligned} \tag{6.14}$$

$$\begin{aligned}\sigma(L_{\rm NN}) &= \{n^2 \mid n \in \mathbb{N}\},\\ \frac{1}{L_{\rm NN}+k^2}(x,y) &= \frac{1}{k\sinh \pi k}\begin{cases}\cosh kx \, \cosh k(\pi-y), & \text{if} \quad 0<x<y<\pi,\\ \cosh ky \, \cosh k(\pi-x), & \text{if} \quad 0<y<x<\pi.\end{cases}\end{aligned} \tag{6.15}$$

Let us check that (6.12)–(6.15) agree with the more general formula (4.16) involving the hypergeometric function and the Gamma function. We identify $k = \frac{\mathrm{i}\mu}{2}$. By (A.32) and (A.33) we obtain

$$P^{\mathrm{s}}_{\frac12,\pm\frac12,\mu}(x) = \frac{\sinh kx}{k\sqrt{\pi}}, \tag{6.16}$$

$$P^{\mathrm{s}}_{-\frac12,\pm\frac12,\mu}(x) = \frac{\cosh kx}{\sqrt{\pi}}, \tag{6.17}$$

Finally, we use

$$\Gamma\left(1+\mathrm{i}k\right)\Gamma\left(1-\mathrm{i}k\right) = \frac{k\pi}{\sinh k\pi}, \tag{6.18}$$

$$\Gamma\left(\frac12+\mathrm{i}k\right)\Gamma\left(\frac12-\mathrm{i}k\right) = \frac{\pi}{\cosh \pi k}, \tag{6.19}$$

$$\Gamma(\mathrm{i}k)\,\Gamma\left(-\mathrm{i}k\right) = \frac{\pi}{k\sinh k\pi}. \tag{6.20}$$

6.2 Laplacian on the halfline

Consider the Laplacian $-\partial_x^2$ on the half-line $\mathbb{R}_+$. Setting the Dirichlet and Neumann boundary conditions at 0 we obtain 2 operators on $L^2(\mathbb{R}_+)$, which are special cases of the hyperbolic Gegenbauer Hamiltonian:

$$L_{\mathrm{D}} := L^{\mathrm{h}}_{\frac12,\frac12} = L^{\mathrm{h}}_{\frac12,-\frac12} = L^{\mathrm{h}}_{\frac12}, \tag{6.21}$$

$$L_{\mathrm{N}} := L^{\mathrm{h}}_{-\frac12,\frac12} = L^{\mathrm{h}}_{-\frac12,-\frac12} = L^{\mathrm{h}}_{-\frac12}. \tag{6.22}$$

Let us compute their resolvents. The following functions solve

$$(-\partial_x^2 + k^2)\phi(x) = 0 \tag{6.23}$$

and satisfy the Dirichlet/Neumann boundary conditions at 0 and decay at $+\infty$:

$$\text{Dirichlet:}\quad \sinh kx; \tag{6.24}$$

$$\text{Neumann:}\quad \cosh kx; \tag{6.25}$$

$$\text{decaying at } +\infty:\quad \mathrm{e}^{-kx},\quad \operatorname{Re} k > 0. \tag{6.26}$$

They have the Wronskians:

$$\mathcal{W}\big(\mathrm{e}^{-kx}, \sinh kx\big) = k, \tag{6.27}$$

$$\mathcal{W}\big(\mathrm{e}^{-kx}, \cosh kx\big) = k. \tag{6.28}$$

Now for $\operatorname{Re} k > 0$,

$$(L_{\mathrm{D}} + k^2)^{-1}(x,y) = \frac1k \begin{cases} \sinh kx\ \mathrm{e}^{-ky}, & \text{if}\quad 0<x<y, \\ \sinh ky\ \mathrm{e}^{-kx}, & \text{if}\quad 0<y<x; \end{cases} \tag{6.29}$$

$$(L_{\mathrm{N}} + k^2)^{-1}(x,y) = \frac1k \begin{cases} \cosh kx\ \mathrm{e}^{-ky}, & \text{if}\quad 0<x<y, \\ \cosh ky\ \mathrm{e}^{-kx}, & \text{if}\quad 0<y<x. \end{cases} \tag{6.30}$$

To check that (6.29) and (6.30) agree with (4.36), identify $k = \frac{\mu}{2}$. By (A.34), (A.35) and (A.36), we have

$$P^{\mathrm{h}}_{\frac{1}{2},\pm\frac{1}{2},\pm\mu}(x) = \frac{\sinh kx}{k\sqrt{\pi}}, \tag{6.31}$$

$$P^{\mathrm{h}}_{-\frac{1}{2},\pm\frac{1}{2},\pm\mu}(x) = \frac{\cosh kx}{\sqrt{\pi}}, \tag{6.32}$$

$$Q^{\mathrm{h}}_{\pm\frac{1}{2},\pm\frac{1}{2},\mu}(x) = \frac{2^{2k}}{\Gamma(1+2k)}\mathrm{e}^{-kx}. \tag{6.33}$$

Finally, use

$$\Gamma(k)\Gamma\left(k+\frac{1}{2}\right) = \frac{2^{-2k}\sqrt{\pi}}{k}\Gamma(2k+1), \tag{6.34}$$

$$\Gamma\left(k+\frac{1}{2}\right)\Gamma\left(k+1\right) = 2^{-2k}\sqrt{\pi}\Gamma(2k+1). \tag{6.35}$$

6.3 Laplacian on the line

Consider the Laplacian $-\partial_x^2$ on the line $\mathbb{R}$, denoted L. It is a special case of the deSitterian hypergeometric Hamiltonian of the first kind:

$$L := L^{\mathrm{dS}}_{\pm\frac{1}{2},\pm\frac{1}{2}} = L^{\mathrm{dS}}_{\pm\frac{1}{2},\mp\frac{1}{2}} = L^{\mathrm{dS}}_{\pm\frac{1}{2}}. \tag{6.36}$$

It is well known how to compute its resolvent: The following functions solve

$$(-\partial_x^2 + k^2)\phi(x) = 0 \tag{6.37}$$

$$\text{decaying at } +\infty\text{:} \quad \mathrm{e}^{-kx}, \quad \mathrm{Re}(k) > 0, \tag{6.38}$$

$$\text{decaying at } -\infty\text{:} \quad \mathrm{e}^{kx}, \quad \mathrm{Re}(k) > 0. \tag{6.39}$$

They have the Wronskian:

$$\mathcal{W}\big(\mathrm{e}^{-kx}, \mathrm{e}^{kx}\big) = 2k. \tag{6.40}$$

Now for $\mathrm{Re}\, k > 0$,

$$(L+k^2)^{-1}(x,y) = \frac{1}{2k}\begin{cases}\mathrm{e}^{kx}\mathrm{e}^{-ky}, & \text{if} \quad x < y,\\ \mathrm{e}^{ky}\mathrm{e}^{-kx}, & \text{if} \quad y < x.\end{cases} \tag{6.41}$$

To see that (6.41) agrees with (4.54), we identify $k = \frac{\mu}{2}$, use

$$Q^{\mathrm{dS}}_{\pm\frac{1}{2},\pm\frac{1}{2},\mu}(x) = Q^{\mathrm{dS}}_{\pm\frac{1}{2},\mp\frac{1}{2},\mu}(x) = \frac{2^{2k}}{\Gamma(2k+1)}\mathrm{e}^{-kx}, \tag{6.42}$$

which follows from (A.36), and

$$\Gamma(k)\,\Gamma\left(k+\frac{1}{2}\right)^2\Gamma\left(k+1\right) = \frac{2^{-4k}\pi}{k}\Gamma(2k+1)^2. \tag{6.43}$$

7 Geometric applications

In this section we show major applications of hypergeometric Hamiltonians in geometry. We will obtain these Hamiltonians as the results of separation of variables of (pseudo-)Laplacians on various (pseudo-)spheres.

Recall that every (pseudo-)Riemannian manifold is equipped with a certain natural differential operator called the *(pseudo-)Laplacian.* Suppose we fix coordinates $x = x_1, \ldots, x_d$, the (pseudo-)metric is given by the field of symmetric invertible matrices $[g_{ij}]$, so that the the "line element" is

$$\mathrm{d}s^2 = \sum_{1\leq i,j\leq d} g_{ij}\mathrm{d}x_i\mathrm{d}x_j. \tag{7.1}$$

Then the pseudo-Laplacian is given by

$$\Delta = \frac{1}{\sqrt{|\det g|}}\partial_{x_i} g^{ij}\sqrt{|\det g|}\partial_{x_j}, \tag{7.2}$$

where $[g^{ij}]$ is the inverse of $[g_{ij}]$. In the case of the Riemannian signature, Δ is called the *Laplacian* (or the *Laplace-Beltrami operator*). For the Lorentzian signature the usual name is the *d'Alembertian.*

(Pseudo-)spheres in a *(pseudo-)Euclidean space* $\mathbb{R}^{p,q}$ inherit a (pseudo-)Riemannian structure from the ambient space. If the ambient space is Euclidean, they are called *spheres.* Otherwise, they are various kinds of *hyperboloids.*

Below we describe a few examples of separation of variables for a (pseudo-)Laplacian on a (pseudo-)sphere in appropriate coordinate systems. We will see that after an appropriate gauging, subtraction of a constant and restriction to an invariant subspace one obtains various hypergeometric Hamiltonians. These computations motivate the names "spherical", "hyperbolic" and "deSitterian" that we use in our paper for various types of hypergeometric Hamiltonians.

The best known among these Laplacians is Δ_d^{s}, the Laplacian on the d-dimensional sphere $\mathbb{S}^d$. As it is well-known, it has the spectrum

$$\sigma(\Delta_d^{\mathrm{s}}) = \{-l(l+d-1) \mid l \in \mathbb{N}_0\}. \tag{7.3}$$

(This is, incidentally, a consequence of computations in Subsection 7.1 and the properties of the spherical Gegenbauer Hamiltonian). Eigenfunctions of $\Delta_{\mathrm{d}}^{\mathrm{s}}$ with eigenvalue $-l(l+d-1)$ will be called *d-dimensional spherical harmonics of order l.*

7.1 Sphere

The unit d-dimensional sphere is defined as

$$\mathbb{S}^d := \{X \in \mathbb{R}^{d+1} \mid X_0^2 + X_1^2 + \cdots + X_d^2 = 1\}. \tag{7.4}$$

We will denote elements of $\mathbb{S}^{d-1}$ by $\hat{X}$ and the corresponding element of length by $\mathrm{d}\hat{X}^2$. We will use the coordinates $(r, \hat{X})$ on $\mathbb{S}^d$

$$X_0 = \cos r, \quad X_i = \sin r\hat{X}_i, \quad i = 1, \ldots, d, \quad \hat{X} \in \mathbb{S}^{d-1}. \tag{7.5}$$

One can also use slightly different coordinates $(w, \hat{X})$ with

$$\cos r = w, \quad \sin r = \sqrt{1-w^2}. \tag{7.6}$$

In these coordinates we first write the line element, then the Laplacian:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \sin^2 r \mathrm{d}\hat{X}^2 \tag{7.7}$$

$$= \frac{\mathrm{d}w^2}{1-w^2} + (1-w^2)\mathrm{d}\hat{X}^2; \tag{7.8}$$

$$\Delta_d^{\mathrm{s}} = \partial_r^2 + (d-1)\cot r \partial_r + \frac{\Delta_{d-1}^{\mathrm{s}}}{\sin^2 r} \tag{7.9}$$

$$= (1-w^2)\partial_w^2 - dw\partial_w + \frac{\Delta_{d-1}^{\mathrm{s}}}{1-w^2}. \tag{7.10}$$

Finally, we perform an appropriate gauging:

$$(\sin r)^{\frac{d-1}{2}}\big(-\Delta_d^{\mathrm{s}}\big)(\sin r)^{-\frac{d-1}{2}} + \big(\tfrac{d-1}{2}\big)^2 = -\partial_r^2 + \frac{\big(\frac{d-2}{2}\big)^2 - \frac{1}{4} - \Delta_{d-1}^{\mathrm{s}}}{\sin^2 r}. \tag{7.11}$$

Thus the Laplacian on $\mathbb{S}^d$ on $(d-1)$-dimensional spherical harmonics of order l (7.11) reduces to the spherical Gegenbauer Hamiltonian L_α^{s} with $\alpha = (\frac{d}{2} - 1 + l)$.

7.2 Hyperbolic space

The d-dimensional hyperbolic space is defined as

$$\mathbb{H}^d := \{X \in \mathbb{R}^{d+1} \mid -X_0^2 + X_1^2 + \cdots + X_d^2 = -1, \quad X_0 > 0\}. \tag{7.12}$$

We will use the following cordinates on $\mathbb{H}^d$: $(r, \hat{X})$, where $\hat{X} \in \mathbb{S}^{d-1}$:

$$X_0 = \cosh r, \quad i = 1, \dots, d, \quad X_i = \sinh r\, \hat{X}_i. \tag{7.13}$$

One can also use slightly different coordinates $(w, \hat{X})$ with

$$\cosh r = w, \quad \sinh r = \sqrt{w^2 - 1}. \tag{7.14}$$

In these coordinates we first write the line element, then the Laplacian:

$$\mathrm{d}s^2 = \mathrm{d}r^2 + \sinh^2 r \mathrm{d}\hat{X}^2 \tag{7.15}$$

$$= \frac{\mathrm{d}w^2}{w^2-1} + (w^2-1)\mathrm{d}\hat{X}^2; \tag{7.16}$$

$$\Delta_d^{\mathrm{h}} = \partial_r^2 + (d-1)\coth r \partial_r + \frac{\Delta_{d-1}^{\mathrm{s}}}{\sinh^2 r} \tag{7.17}$$

$$= (w^2-1)\partial_w^2 + dw\partial_w + \frac{\Delta_{d-1}^{\mathrm{s}}}{w^2-1}. \tag{7.18}$$

Finally, we perform an appropriate gauging:

$$(\sinh r)^{\frac{d-1}{2}}\big(-\Delta_d^{\mathrm{h}}\big)(\sinh r)^{-\frac{d-1}{2}} - \big(\tfrac{d-1}{2}\big)^2 = -\partial_r^2 + \frac{\big(\frac{d-2}{2}\big)^2 - \frac{1}{4} - \Delta_{d-1}^{\mathrm{s}}}{\sinh^2 r}. \tag{7.19}$$

Thus the Laplacian on $\mathbb{H}^d$ on $d-1$-dimensional spherical harmonics of order l reduces to the hyperbolic Gegenbauer Hamiltonian L_α^{h} with $\alpha = \frac{d}{2} - 1 + l$.

7.3 DeSitter space

The deSitter space is defined as

$$\mathrm{dS}^d := \{X \in \mathbb{R}^{d+1} \mid -X_0^2 + X_1^2 + \cdots + X_d^2 = 1\}. \tag{7.20}$$

We will use the cordinates $(t, \hat{X})$ on dS^d given by

$$X_0 = \sinh t, \quad X_i = \cosh t\, \hat{X}_i,, \quad i = 1, \dots, d, \quad \hat{X} \in \mathbb{S}^{d-1}. \tag{7.21}$$

Alternative coordinates are

$$\sinh t = w, \quad \cosh t = \sqrt{1 + w^2}. \tag{7.22}$$

In these coordinates we first write the line element, then the d'Alembertian:

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + \cosh^2 t \mathrm{d}\hat{X}^2 \tag{7.23}$$

$$= -\frac{\mathrm{d}w^2}{w^2+1} + (w^2+1)\mathrm{d}\hat{X}^2 \tag{7.24}$$

$$\Box_d^{\mathrm{dS}} = -\partial_t^2 + (d-1)\tanh t \partial_t + \frac{\Delta_{d-1}^{\mathrm{s}}}{\cosh^2 t} \tag{7.25}$$

$$= -(w^2+1)\partial_w^2 - dw\partial_w + \frac{\Delta_{d-1}^{\mathrm{s}}}{w^2+1}. \tag{7.26}$$

Finally, we perform an appropriate gauging:

$$(\cosh t)^{\frac{d-1}{2}} \Box_d^{\mathrm{dS}} (\cosh t)^{-\frac{d-1}{2}} - \left(\tfrac{d-1}{2}\right)^2 = -\partial_t^2 - \frac{\left(\frac{d-2}{2}\right)^2 - \frac{1}{4} - \Delta_{d-1}^{\mathrm{s}}}{\cosh^2 t}. \tag{7.27}$$

Thus the d'Alembertian on dS^d on $d-1$-dimensional spherical harmonics of order l reduces to the deSitterian Gegenbauer Hamiltonian L_α^{dS} with $\alpha = \frac{d}{2} - 1 + l$.

7.4 Sphere in double spherical coordinates

Consider the unit sphere of dimension $p + q - 1$ with coordinates partitioned in two groups:

$$\mathbb{S}^{p+q-1} := \{(X, Y) \in \mathbb{R}^{p+q} \mid X_1^2 + \cdots + X_p^2 + Y_1^2 + \cdots + Y_q^2 = 1\}. \tag{7.28}$$

We consider also two spheres of dimension $p - 1$ and $q - 1$:

$$\mathbb{S}^{p-1} = \{X \in \mathbb{R}^p \ : \ X_1^2 + \cdots + X_p^2 = 1\}, \quad \mathbb{S}^{q-1} = \{Y \in \mathbb{R}^q \ : \ Y_1^2 + \cdots + Y_q^2 = 1\} \tag{7.29}$$

$\mathbb{S}^{p+q-1}$ is parametrized by $(\tau, \hat{X}, \hat{Y})$, with $0 \le \tau \le \frac{\pi}{2}$, $\hat{X} \in \mathbb{S}^{p-1}$, $\hat{Y} \in \mathbb{S}^{q-1}$:

$$X_i = \sin\tau\, \hat{X}_i, \quad i = 1, \dots, p; \qquad Y_j = \cos\tau\, \hat{Y}_j, \quad i = 1, \dots, q. \tag{7.30}$$

Alternatively, one can use coordinates $(w, \hat{X})$ where

$$\sin^2\tau = w, \quad \cos^2\tau = 1 - w. \tag{7.31}$$

We compute the line element and the Laplacian:

$$\mathrm{d}s^2 = \frac{\mathrm{d}w^2}{4w(1-w)} + w\mathrm{d}\hat{X}^2 + (1-w)\mathrm{d}\hat{Y}^2 \tag{7.32}$$

$$= \mathrm{d}\tau^2 + \sin^2\tau\mathrm{d}\hat{X}^2 + \cos^2\tau\mathrm{d}\hat{Y}^2 \tag{7.33}$$

$$\Delta^{\mathrm{s}}_{p+q-1} = 4w(1-w)\partial_w^2 + 2\big(p(1-w) - qw\big)\partial_w + \frac{\Delta^{\mathrm{s}}_{p-1}}{w} + \frac{\Delta^{\mathrm{s}}_{q-1}}{1-w} \tag{7.34}$$

$$=\partial_\tau^2 + \big((p-1)\cot\tau - (q-1)\tan\tau\big)\partial_\tau + \frac{\Delta^{\mathrm{s}}_{p-1}}{\sin^2\tau} + \frac{\Delta^{\mathrm{s}}_{q-1}}{\cos^2\tau}. \tag{7.35}$$

We perform an appropriate gauging:

$$(\sin\tau)^{\frac{p-1}{2}}(\cos\tau)^{\frac{q-1}{2}}\big(-\Delta^{\mathrm{s}}_{p+q-1}\big)(\sin\tau)^{-\frac{p-1}{2}}(\cos\tau)^{-\frac{q-1}{2}} + \big(\tfrac{p+q-2}{2}\big)^2 \tag{7.36}$$

$$= -\partial_\tau^2 + \frac{\big(\frac{p-2}{2}\big)^2 - \frac14 - \Delta^{\mathrm{s}}_{p-1}}{\sin^2\tau} + \frac{\big(\frac{q-2}{2}\big)^2 - \frac14 - \Delta^{\mathrm{s}}_{q-1}}{\cos^2\tau}. \tag{7.37}$$

Finally, we make a substitution $\tau = \frac{r}{2}$:

$$(7.37) = 4\left(-\partial_r^2 + \frac{\big(\frac{p-2}{2}\big)^2 - \frac14 - \Delta^{\mathrm{s}}_{p-1}}{4\sin^2\frac{r}{2}} + \frac{\big(\frac{q-2}{2}\big)^2 - \frac14 - \Delta^{\mathrm{s}}_{q-1}}{4\cos^2\frac{r}{2}}\right). \tag{7.38}$$

Thus on products of a spherical harmonic of order j and l we obtain the spherical hypergeometric Hamiltonian of the first kind $L^{\mathrm{s}}_{\alpha,\beta}$ with

$$\alpha = \frac{p}{2} - 1 + j, \quad \beta = \frac{q}{2} - 1 + l. \tag{7.39}$$

7.5 Hyperboloid in double spherical coordinates

Consider the hyperboloid of signature $p-1, q$ embedded in the pseudoEuclidean space of signature (p,q):

$$\mathbb{H}^{p-1,q} := \{(X,Y) \in \mathbb{R}^{p+q} \mid -X_1^2 - \cdots - X_p^2 + Y_1^2 + \cdots + Y_q^2 = -1\}. \tag{7.40}$$

Let $\mathbb{S}^{p-1}$ and $\mathbb{S}^{q-1}$ be as in (7.29). $\mathbb{H}^{p-1,q}$ is parametrized by $(\tau, \hat{X}, \hat{Y})$, with $0 \le \tau \le \infty$, $\hat{X} \in \mathbb{S}^{p-1}$, $\hat{Y} \in \mathbb{S}^{q-1}$:

$$X_i = \cosh\tau\,\hat{X}_i, \quad i = 1, \dots, p; \qquad Y_j = \sinh\tau\,\hat{Y}_j, \quad i = 1, \dots, q. \tag{7.41}$$

Alternatively, one can use coordinates $(w, \hat{X}, \hat{Y})$ where

$$\cosh^2 r = w, \quad \sinh^2 r = w - 1. \tag{7.42}$$

The line element and the pseudo-Laplacian in these coordinates:

$$\mathrm{d}s^2 =\frac{\mathrm{d}w^2}{4w(w-1)} - w\mathrm{d}\hat{X}^2 + (w-1)\mathrm{d}\hat{Y}^2 \tag{7.43}$$

$$=\mathrm{d}\tau^2 - \cosh^2\tau\mathrm{d}\hat{X}^2 + \sinh^2\tau\mathrm{d}\hat{Y}^2 \tag{7.44}$$

$$\Delta_{p-1,q} = 4w(w-1)\partial_w^2 + 2\big(p(w-1) + qw\big)\partial_w - \frac{\Delta^{\mathrm{s}}_{p-1}}{w} + \frac{\Delta^{\mathrm{s}}_{q-1}}{w-1} \tag{7.45}$$

$$=\partial_\tau^2 + \big((p-1)\tanh\tau + (q-1)\coth\tau\big)\partial_\tau - \frac{\Delta^{\mathrm{s}}_{p-1}}{\cosh^2\tau} + \frac{\Delta^{\mathrm{s}}_{q-1}}{\sinh^2\tau}. \tag{7.46}$$

We perform an appropriate gauging:

$$(\cosh\tau)^{\frac{p-1}{2}}(\sinh\tau)^{\frac{q-1}{2}}\big(-\Delta_{p-1,q}\big)(\cosh\tau)^{-\frac{p-1}{2}}(\sinh\tau)^{-\frac{q-1}{2}}-\big(\tfrac{p+q-2}{2}\big)^2 \tag{7.47}$$

$$=-\partial_\tau^2-\frac{\big(\frac{p-2}{2}\big)^2-\frac14-\Delta_{p-1}^{\mathrm{s}}}{\cosh^2\tau}+\frac{\big(\frac{q-2}{2}\big)^2-\frac14-\Delta_{q-1}^{\mathrm{s}}}{\sinh^2\tau}. \tag{7.48}$$

Finally, we substitute $\tau=\frac{r}{2}$:

$$(7.48)=4\Bigg(-\partial_r^2-\frac{\big(\frac{p-2}{2}\big)^2-\frac14-\Delta_{p-1}^{\mathrm{s}}}{4\cosh^2\frac{r}{2}}+\frac{\big(\frac{q-2}{2}\big)^2-\frac14-\Delta_{q-1}^{\mathrm{s}}}{4\sinh^2\frac{r}{2}}\Bigg). \tag{7.49}$$

Thus on the product of a spherical harmonic of order j and l we obtain the hyperbolic hypergeometric Hamiltonian of the first kind $L_{\alpha,\beta}^{\mathrm{h}}$ with

$$\alpha=\frac{q}{2}-1+l,\quad \beta=\frac{p}{2}-1+j. \tag{7.50}$$

7.6 Complex manifolds

All the manifolds that we used so far were real. In the next subsection we will need a complex (analytic) manifold. They have essentially the same formalism as real manifolds. Let us briefly sketch its elements. For more details, see [LeB].

Suppose that a complex manifold is equipped with local complex coordinates $z=(z_1,\dots,z_d)$ and the holomorphic line element

$$\sum_{1\le i,j\le d} g_{ij}\mathrm{d}z_i\mathrm{d}z_j, \tag{7.51}$$

where g_{ij} is a complex symmetric invertible matrix. The corresponding complex Laplacian is defined by essentially the same formula as in the real case:

$$\Delta=\frac{1}{\sqrt{\det g}}\partial_{z_i}g^{ij}\sqrt{\det g}\partial_{z_j}. \tag{7.52}$$

(Note that there is no absolute value).

Suppose that the manifold is equipped with a conjugation, in the coordinates given by $z_i\mapsto\overline{z}_i$. We then also have the anti-holomorphic line element

$$\sum_{1\le i,j\le d}\overline{g}_{ij}\mathrm{d}\overline{z}_i\mathrm{d}\overline{z}_j. \tag{7.53}$$

and the corresponding conjugate Laplacian

$$\overline{\Delta}=\frac{1}{\sqrt{\det\overline{g}}}\partial_{\overline{z}_i}\overline{g^{ij}}\sqrt{\det\overline{g}}\partial_{\overline{z}_j}. \tag{7.54}$$

As our first example consider the space $\mathbb{C}^{d+1}$ equipped with the line element

$$\mathrm{d}Z^2=\mathrm{d}Z_0^2+\mathrm{d}Z_1^2+\cdots+\mathrm{d}Z_p^2. \tag{7.55}$$

The corresponding Laplacian is obviously

$$\Delta=\partial_{Z_0}^2+\cdots+\partial_{Z_p}^2. \tag{7.56}$$

Note that our standard identification of $\mathbb{C}^{d+1}$ with $\mathbb{R}^{2(d+1)}$ will be

$$Z_i = \frac{1}{\sqrt{2}}(X_i + \mathbf{i}Y_i) \tag{7.57}$$

(and not $Z_i = X_i + \mathbf{i}Y_i$. Therefore,

$$\partial_{Z_i} = \frac{1}{\sqrt{2}}\big(\partial_{X_i} - \mathrm{i}\partial_{Y_i}\big), \tag{7.58}$$

(and not, as usual, $\partial_{Z_i} = \frac{1}{2}\big(\partial_{X_i} - \mathrm{i}\partial_{Y_i}\big)$.) Clearly, with this definition $\langle \mathrm{d}Z_i|\partial_{Z_j}\rangle = \delta_{ij}$.

Another example of a complex manifold is the unit sphere [HIU]

$$\mathbb{S}^d_{\mathbb{C}} := \{Z \in \mathbb{C}^{d+1} \big| Z_0^2 + Z_1^2 + \cdots + Z_d^2 = 1\}. \tag{7.59}$$

Introducing the complex polar coordinanes

$$R := \sqrt{Z_0^2 + \cdots + Z_d^2}, \quad \hat{Z}_i := \frac{Z_i}{R}, \tag{7.60}$$

we have the direct analog of the formula from the real case:

$$\mathrm{d}Z^2 = \mathrm{d}R^2 + R^2\mathrm{d}\hat{Z}^2, \tag{7.61}$$

where $\mathrm{d}\hat{Z}^2$ is the complex line element on $\mathbb{S}^d_{\mathbb{C}}$. The corresponding complex Laplacian is given by the same expressions as in the real case:

$$\Delta^{\mathrm{s}}_{d,\mathbb{C}} = \sum_{0\leq i<j\leq d} (Z_i\partial_{Z_j} - Z_j\partial_{Z_i})^2 \tag{7.62}$$

$$= R^2\Delta_{d+1,\mathbb{C}} - R^2\partial_R^2 - d\,R\,\partial_R. \tag{7.63}$$

7.7 Hyperboloid $\mathbb{H}^{p-1,p}$ in complex coordinates

Consider the hyperboloid $\mathbb{H}^{p-1,p}$ embedded in $\mathbb{R}^{2p}$, defined as in Subsection 7.5, with the coordinates $X_i, Y_i \in \mathbb{R}$, $i = 1, \dots, p$. We identify $\mathbb{R}^{2p}$ with $\mathbb{C}^p$ as in (7.57), so that we obtain two representations of $\mathbb{H}^{p-1,p}$, a real and a complex one:

$$\mathbb{H}^{p-1,p} = \{(X,Y) \in \mathbb{R}^{2p} \mid -Y_1^2 - \cdots - Y_p^2 + X_1^2 + \cdots + X_p^2 = -1\} \tag{7.64}$$

$$= \{Z \in \mathbb{C}^p \mid Z_1^2 + \cdots + Z_p^2 + \overline{Z}_1^2 + \cdots + \overline{Z}_p^2 = 1\} \tag{7.65}$$

The (real) line element on $\mathbb{R}^{2p} = \mathbb{C}^p$ can be written as

$$-\mathrm{d}Y_1^2 - \cdots - \mathrm{d}Y_p^2 + \mathrm{d}X_1^2 + \cdots + \mathrm{d}X_p^2 \tag{7.66}$$

$$= \mathrm{d}Z_1^2 + \cdots + \mathrm{d}Z_p^2 + \mathrm{d}\overline{Z}_1^2 + \cdots + \mathrm{d}\overline{Z}_1^2 \tag{7.67}$$

$$= \mathrm{d}R^2 + R^2\mathrm{d}\hat{Z}^2 + \mathrm{d}\overline{R}^2 + \overline{R}^2\mathrm{d}\overline{\hat{Z}}^2 \tag{7.68}$$

Now on $\mathbb{H}^{p-1,p}$ we have $R^2 + \overline{R}^2 = 1$. Therefore $R^2 = \frac{1+\mathrm{i}\sinh r}{2}$ for a unique $r \in \mathbb{R}$. Thus we can parametrize $\mathbb{H}^{p-1,p}$ with $r \in \mathbb{R}, \hat{Z} \in \mathbb{S}^{p-1}_{\mathbb{C}}$

$$Z_i = \sqrt{\frac{1+\mathrm{i}\sinh r}{2}}\hat{Z}_i, \tag{7.69}$$

where we take the principal branch of square root. The (real) line element and the pseudo-Laplacian are

$$\mathrm{d}s^2 = -\frac{1}{4}\mathrm{d}r^2 + \frac{1+\mathrm{i}\sinh r}{2}\mathrm{d}\hat{Z}^2 + \frac{1-\mathrm{i}\sinh r}{2}\mathrm{d}\overline{\hat{Z}}^2, \tag{7.70}$$

$$\Delta_{p-1,p} = -4\partial_r^2 - 4(p-1)\tanh r\partial_r + \frac{2}{1+\mathrm{i}\sinh r}\Delta_{p-1}^{\mathrm{s},\mathbb{C}} + \frac{2}{1-\mathrm{i}\sinh r}\overline{\Delta_{p-1}^{\mathrm{s},\mathbb{C}}}. \tag{7.71}$$

We perform an appropriate gauging:

$$\big(\cosh r\big)^{\frac{p-1}{2}}\big(\Delta_{p-1,p}\big)(\cosh r)^{-\frac{p-1}{2}} - (p-1)^2 \tag{7.72}$$

$$=4\Bigg(- \partial_r^2 - \frac{\big(-\Delta_{p-1}^{\mathrm{s},\mathbb{C}} + (\frac{p-1}{2})^2 - \frac{1}{4}\big)}{2(1+\mathrm{i}\sinh r)} - \frac{\big(-\overline{\Delta_{p-1}^{\mathrm{s},\mathbb{C}}} + (\frac{p-1}{2})^2 - \frac{1}{4}\big)}{2(1-\mathrm{i}\sinh r)}\Bigg). \tag{7.73}$$

Thus on joint eigenvectors of $\Delta_{p-1}^{\mathrm{s},\mathbb{C}}$ and $\overline{\Delta_{p-1}^{\mathrm{s},\mathbb{C}}}$ of degree l, resp. j we obtain the deSitterian hypergeometric Hamiltonian of the first kind $L_{\alpha,\beta}^{\mathrm{dS}}$ with

$$\alpha = \frac{p}{2} - 1 + l, \quad \beta = \frac{p}{2} - 1 + j. \tag{7.74}$$

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A Identities for the hypergeometric function

In this section, we review identities for the hypergeometric and Gegenbauer functions with the emphasis on the connection formulas and Kummer's table. There are many reviews available including [WW] and [NIST1]. Our presentation is close to [D1], [D2].

A.1 Kummer's table

Recall that the hypergeometric equation is given by the operator $\mathcal{F}(a,b;c;z,\partial_z)$ defined in (2.1), and the hypergeometric function with Olver's normalization $\mathbf{F}(a,b;c;z)$ is defined in (2.2). In this section, for brevity and transparency, we change the notation following [D1] and [D2], writing

$$\mathbf{F}_{\alpha,\beta,\mu}(z) := \mathbf{F}\left(\frac{1+\alpha+\beta+\mu}{2}, \frac{1+\alpha+\beta-\mu}{2}; 1+\alpha; z\right). \tag{A.1}$$

It is obvious from the definition that we have the following identity

$$\mathbf{F}_{\alpha,\beta,\mu}(z) = \mathbf{F}_{\alpha,\beta,-\mu}(z). \tag{A.2}$$

The following 6 functions form a set of standard solutions of the hypergeometric equation. Each of the solutions can be expressed in 4 ways (actually, 4×2 ways if we include the trivial identity (A.2)). This yields $6\times 4 = 24$ expressions usually called *Kummer's table*:

$$\begin{aligned}\mathbf{F}_{\alpha,\beta,\mu}(z) &= (1-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{\alpha,-\mu,-\beta}(\frac{z}{z-1}) \\ &= (1-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\alpha,\mu,\beta}(\frac{z}{z-1}) \\ &= (1-z)^{-\beta}\mathbf{F}_{\alpha,-\beta,-\mu}(z).\end{aligned} \tag{A.3}$$

$$
\begin{aligned}
(-z)^{\alpha}\mathbf{F}_{-\alpha,\beta,-\mu}(z) &= (-z)^{\alpha}(1-z)^{\frac{-1+\alpha-\beta-\mu}{2}}\mathbf{F}_{-\alpha,\mu,-\beta}(\frac{z}{z-1}) \\
&= (-z)^{\alpha}(1-z)^{\frac{-1+\alpha-\beta+\mu}{2}}\mathbf{F}_{-\alpha,-\mu,\beta}(\frac{z}{z-1}) \\
&= (-z)^{\alpha}(1-z)^{-\beta}\mathbf{F}_{-\alpha,-\beta,\mu}(z).
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
(-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{-\mu,\beta,-\alpha}(z^{-1}) &= (1-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{-\mu,\alpha,-\beta}\left(\frac{1}{1-z}\right) \\
&= (-z)^{-\alpha}(1-z)^{\frac{-1+\alpha-\beta-\mu}{2}}\mathbf{F}_{-\mu,-\alpha,\beta}\left(\frac{1}{1-z}\right) \\
&= (-z)^{\frac{-1-\alpha-\beta+\mu}{2}}(1-z)^{-\beta}\mathbf{F}_{\alpha,-\beta,-\mu}\left(z^{-1}\right).
\end{aligned} \tag{A.5}
$$

$$
\begin{aligned}
(-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\mu,\beta,\alpha}(z^{-1}) &= (1-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\mu,-\alpha,-\beta}\left(\frac{1}{1-z}\right) \\
&= (-z)^{-\alpha}(1-z)^{\frac{-1+\alpha-\beta+\mu}{2}}\mathbf{F}_{\mu,\alpha,\beta}\left(\frac{1}{1-z}\right) \\
&= (-z)^{\frac{-1-\alpha-\beta-\mu}{2}}(1-z)^{-\beta}\mathbf{F}_{-\alpha,-\beta,\mu}\left(z^{-1}\right).
\end{aligned} \tag{A.6}
$$

$$
\begin{aligned}
\mathbf{F}_{\beta,\alpha,\mu}(1-z) &= (-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{\beta,-\mu,-\alpha}(1-\frac{1}{z}) \\
&= (-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\beta,\mu,\alpha}(1-\frac{1}{z}) \\
&= (-z)^{-\alpha}\mathbf{F}_{\beta,-\alpha,-\mu}(1-z).
\end{aligned} \tag{A.7}
$$

$$
\begin{aligned}
(1-z)^{-\beta}\mathbf{F}_{-\beta,-\mu,-\alpha}(1-z) &= (-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{\beta,-\mu,-\alpha}(1-\frac{1}{z}) \\
&= (-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\beta,\mu,\alpha}(1-\frac{1}{z}) \\
&= (-z)^{-\alpha}\mathbf{F}_{\beta,-\alpha,-\mu}(1-z).
\end{aligned} \tag{A.8}
$$

A.2 Connection formulas

Here are connection formulas. For $z \notin]-\infty,0] \cup [1,\infty[$:

$$
\begin{aligned}
&\mathbf{F}_{\beta,\alpha,\mu}(1-z) \qquad\qquad (A.9)\\
&= \frac{\pi\ \mathbf{F}_{\alpha,\beta,\mu}(z)}{\sin(-\pi\alpha)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} + \frac{\pi\ z^{-\alpha}\ \mathbf{F}_{-\alpha,\beta,-\mu}(z)}{\sin(\pi\alpha)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)},\\
&(1-z)^{-\beta}\mathbf{F}_{-\beta,\alpha,-\mu}(1-z) \qquad\qquad (A.10)\\
&= \frac{\pi\ \mathbf{F}_{\alpha,\beta,\mu}(z)}{\sin(-\pi\alpha)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} + \frac{\pi\ z^{-\alpha}\ \mathbf{F}_{-\alpha,\beta,-\mu}(z)}{\sin(\pi\alpha)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)},
\end{aligned}
$$

$$\begin{aligned}
&\mathbf{F}_{\alpha,\beta,\mu}(z) && \text{(A.11)}\\
&= \frac{\pi\ \mathbf{F}_{\beta,\alpha,\mu}(1-z)}{\sin(-\pi\beta)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)} + \frac{\pi\ (1-z)^{-\beta}\ \mathbf{F}_{-\beta,\alpha,-\mu}(1-z)}{\sin(\pi\beta)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)},\\
&z^{-\alpha}\mathbf{F}_{-\alpha,\beta,\mu}(z) && \text{(A.12)}\\
&= \frac{\pi\ \mathbf{F}_{\beta,\alpha,\mu}(z)}{\sin(-\pi\beta)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)} + \frac{\pi\ (1-z)^{-\beta}\ \mathbf{F}_{-\beta,\alpha,-\mu}(1-z)}{\sin(\pi\beta)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)}.
\end{aligned}$$

For $z \notin [0, \infty[$:

$$\begin{aligned}
&(-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\mu,\beta,\alpha}\left(z^{-1}\right) && \text{(A.13)}\\
&= \frac{\pi\ \mathbf{F}_{\alpha,\beta,\mu}(z)}{\sin(-\pi\alpha)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)} + \frac{\pi\ (-z)^{-\alpha}\mathbf{F}_{-\alpha,\beta,\mu}(z)}{\sin(\pi\alpha)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)};\\
&(-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{-\mu,\beta,\alpha}\left(z^{-1}\right) && \text{(A.14)}\\
&= \frac{\pi\ \mathbf{F}_{\alpha,\beta,\mu}(z)}{\sin(-\pi\alpha)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)} + \frac{\pi\ (-z)^{-\alpha}\ \mathbf{F}_{-\alpha,\beta,\mu}(z)}{\sin(\pi\alpha)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)}.
\end{aligned}$$

$$\begin{aligned}
&\mathbf{F}_{\alpha,\beta,\mu}(z) && \text{(A.15)}\\
&= \frac{\pi(-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\mu,\beta,\alpha}(z^{-1})}{\sin(-\pi\mu)\Gamma\left(\frac{1+\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta-\mu}{2}\right)} + \frac{\pi(-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{-\mu,\beta,\alpha}(z^{-1})}{\sin(\pi\mu)\Gamma\left(\frac{1+\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1+\alpha-\beta+\mu}{2}\right)},\\
&(-z)^{-\alpha}\mathbf{F}_{-\alpha,\beta,\mu}(z) && \text{(A.16)}\\
&= \frac{\pi(-z)^{\frac{-1-\alpha-\beta-\mu}{2}}\mathbf{F}_{\mu,\beta,\alpha}(z^{-1})}{\sin(-\pi\mu)\Gamma\left(\frac{1-\alpha+\beta-\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta-\mu}{2}\right)} + \frac{\pi(-z)^{\frac{-1-\alpha-\beta+\mu}{2}}\mathbf{F}_{-\mu,\beta,\alpha}(z^{-1})}{\sin(\pi\mu)\Gamma\left(\frac{1-\alpha+\beta+\mu}{2}\right)\Gamma\left(\frac{1-\alpha-\beta+\mu}{2}\right)}.
\end{aligned}$$

A.3 Degenerate case

Let $\mu \in \mathbb{Z}$. Then we have the identity

$$\Big(\frac{\alpha+\beta-\mu+1}{2}\Big)_\mu\Big(\frac{\alpha-\beta-\mu+1}{2}\Big)_\mu F\Big(\frac{\alpha+\beta+\mu+1}{2},\frac{\alpha-\beta+\mu+1}{2};1+\mu;z\Big) \tag{A.17}$$

$$=z^{-\mu}F\Big(\frac{\alpha+\beta-\mu+1}{2},\frac{\alpha-\beta-\mu+1}{2};1-\mu;z\Big). \tag{A.18}$$

A.4 Gegenbauer functions

In the remaining part of this section we review some of the relations for Gegenbauer function, following mostly [DGR].

Recall that the Gegenbauer equation is defined by the operator $\mathcal{G}_{\alpha,\lambda}(w,\partial_w)$ defined in (2.8), and the two Gegenbauer functions that we use were defined in (2.10) and (2.11). Let us rewrite their definitions using the notation introduced in (A.1).

The following function satisfies the Gegenbauer equation and has value 1 at 1:

$$S_{\alpha,\pm\lambda}(w) = F_{\alpha,\alpha,\lambda}\left(\frac{1-w}{2}\right). \tag{A.19}$$

It can be easily seen from (A.3) that the solution behaving as $(\frac{w+1}{2})^{-\alpha}$ at 1 is

$$S_{-\alpha,\pm\lambda}(w) = \left(\frac{1+w}{2}\right)^{-\alpha} F_{-\alpha,\alpha,2\lambda}\left(\frac{1-w}{2}\right) \tag{A.20}$$

Solution behaving as $w^{-\frac{1}{2}-\lambda-\alpha}$ at $+\infty$ is

$$Z_{\alpha,\lambda}(w) = (w\pm 1)^{-\frac{1}{2}-\alpha-\lambda} F_{2\lambda,\alpha,\alpha}\left(\frac{2}{1\pm w}\right) \tag{A.21}$$

It is useful to introduce Olver's normalization

$$\mathbf{S}_{\alpha,\lambda}(w) = \frac{S_{\alpha,\lambda}(w)}{\Gamma(1+\alpha)}, \qquad \mathbf{Z}_{\alpha,\lambda}(w) = \frac{Z_{\alpha,\lambda}(w)}{\Gamma(1+\lambda)}. \tag{A.22}$$

We can read their connection formula from the connection formulas of hypergeometric functions (A.13) (A.9), for $\mathrm{Im}(w) < 0$:

$$\mathbf{Z}_{\alpha,\lambda}(w) = \frac{\sqrt{\pi}2^{-\alpha+\lambda-\frac{1}{2}}}{\sin(-\pi\alpha)\,\Gamma\left(-\alpha+\lambda+\frac{1}{2}\right)}\mathbf{S}_{\alpha,\lambda}(w) + \frac{\sqrt{\pi}2^{\alpha+\lambda-\frac{1}{2}}(w^2-1)_\bullet^{-\alpha}}{\sin\pi\alpha\,\Gamma\left(\alpha+\lambda+\frac{1}{2}\right)}\mathbf{S}_{-\alpha,\lambda}(w), \tag{A.23}$$

$$\mathbf{S}_{\alpha,\lambda}(-w) = \frac{\cos\pi\lambda}{\sin(-\pi\alpha)}\mathbf{S}_{\alpha,\lambda}(w) + \frac{\pi\,(1-w^2)^{-\alpha}}{\sin\pi\alpha\,\Gamma\left(\frac{1}{2}+\alpha+\lambda\right)\Gamma\left(\frac{1}{2}+\alpha-\lambda\right)}\mathbf{S}_{-\alpha,\lambda}(w). \tag{A.24}$$

Note that the first connection formula is valid when $w \notin]-\infty,1]$, and the second connection formula is valid when $w \notin]-\infty,-1] \cup [1,\infty[$. Here we borrow a notation from [DGR] where

$$(w^2-1)_\bullet^\alpha := (w-1)^\alpha(w+1)^\alpha. \tag{A.25}$$

The functions $(w^2-1)^\alpha$ and $(w^2-1)_\bullet^\alpha$ coincide only if $\mathrm{Re}(w) > 0$. In general the function $(w^2-1)_\bullet^\alpha$ is holomorphic on $\mathbb{C}\backslash]-\infty,1]$ while $(w^2-1)^\alpha$ is holomorphic on $\mathbb{C}\setminus\{[-1,1]\cup\mathrm{i}\mathbb{R}\}$.

A.5 Whipple transformation

Gegenbauer equation has an extra symmetry compared to hypergeoemetric symmetry called Whipple transformation. On the level of its standard solutions it has the following form:

$$\mathbf{Z}_{\alpha,\lambda}(w) = (w^2-1)_\bullet^{-\frac{1}{4}-\frac{\alpha}{2}-\frac{\lambda}{2}}\mathbf{S}_{\lambda,\alpha}\left(\frac{w}{(w^2-1)_\bullet^{\frac{1}{2}}}\right), \tag{A.26}$$

$$\mathbf{S}_{\alpha,\lambda}(w) = (w^2-1)_\bullet^{-\frac{1}{4}-\frac{\alpha}{2}-\frac{\lambda}{2}}\mathbf{Z}_{\lambda,\alpha}\left(\frac{w}{(w^2-1)_\bullet^{\frac{1}{2}}}\right), \quad \mathrm{Re}(w) > 0. \tag{A.27}$$

(A.27) is obtain by inverting (A.26) and using the fact that $w \mapsto \frac{w}{(w^2-1)_\bullet^{\frac{1}{2}}}$ is an involution if and only if $\mathrm{Re}(w) > 0$.

A.6 Half integer case

Gegenbauer functions with $\alpha = \pm\frac{1}{2}$ have simple expressions in terms of elementary functions. To see this we change variables in Gegenbauer operators. For $w \in]-1,1[$, we substitute $w = \cos\phi$:

$$\mathcal{G}_{-\frac{1}{2},\lambda}(w, \partial_w) = \partial_\phi^2 + \lambda^2; \tag{A.28}$$

$$\mathcal{G}_{\frac{1}{2},\lambda}(w, \partial_w) = \frac{1}{\sin\phi}\big(\partial_\phi^2 + \lambda^2\big)\sin\phi. \tag{A.29}$$

For $w \in]1, +\infty[$, we substitute $w = \cosh\theta$:

$$\mathcal{G}_{-\frac{1}{2},\lambda}(w, \partial_w) = -\partial_\theta^2 + \lambda^2; \tag{A.30}$$

$$\mathcal{G}_{\frac{1}{2},\lambda}(w, \partial_w) = \frac{1}{\sinh\theta}\big(-\partial_\theta^2 + \lambda^2\big)\sinh\theta. \tag{A.31}$$

(A.28)—(A.31) easily imply the first column of the following identities for Gegenbauer functions. (The second column simply follows from the definitions of Gegenbauer functions).

$$\cos\lambda\phi = S_{-\frac{1}{2},\lambda}(\cos\phi) = F\Big(\lambda, -\lambda; \frac{1}{2}; \sin^2\frac{\phi}{2}\Big), \tag{A.32}$$

$$\frac{\sin\lambda\phi}{\lambda\sin\phi} = S_{\frac{1}{2},\lambda}(\cos\phi) = F\Big(1+\lambda, 1-\lambda; \frac{3}{2}; \sin^2\frac{\phi}{2}\Big), \tag{A.33}$$

$$\cosh\lambda\theta = S_{-\frac{1}{2},\lambda}(\cosh\theta) = F\Big(\lambda, -\lambda; \frac{1}{2}; \sinh^2\frac{\theta}{2}\Big), \tag{A.34}$$

$$\frac{\sinh\lambda\theta}{\lambda\sinh\theta} = S_{\frac{1}{2},\lambda}(\cosh\theta) = F\Big(1+\lambda, 1-\lambda; \frac{3}{2}; \sinh^2\frac{\theta}{2}\Big), \tag{A.35}$$

$$2^\lambda e^{-\lambda\theta} = Z_{-\frac{1}{2},\lambda}(\cosh\theta) = \Big(2\sinh^2\frac{\theta}{2}\Big)^{-\lambda} F\Big(\lambda, \lambda+\frac{1}{2}; 1+2\lambda; -\frac{1}{\sinh^2\frac{\theta}{2}}\Big), \tag{A.36}$$

$$\frac{2^\lambda e^{-\lambda\theta}}{\sinh\theta} = Z_{\frac{1}{2},\lambda}(\cosh\theta) = \Big(2\sinh^2\frac{\theta}{2}\Big)^{-\lambda-1} F\Big(\lambda+\frac{1}{2}, \lambda+1; 1+2\lambda; -\frac{1}{\sinh^2\frac{\theta}{2}}\Big). \tag{A.37}$$

The above formulas can be found e.g. in equations (4.25)–(4.28) of [DGR] (in a slightly different form). They are straightforward generalizations of well-known formulas for Chebyshev polynomials. In fact, for $n \in \mathbb{N}_0$ the usual Chebyshev polynomials are special cases of Gegenbauer functions with $\alpha = \pm\frac{1}{2}$:

$$T_n(w) = S_{-\frac{1}{2},n}(w), \quad U_n(w) = (n+1)S_{\frac{1}{2},n+1}(w). \tag{A.38}$$

B Closed realizations of 1d Schrödinger operators

B.1 Minimal and maximal realization

The theory of self-adjoint realizations of 1d Schrödinger operators with real potentials is well-known and discussed in various sources [GeZin, GTV]. Somewhat less known is the theory of their closed realizations, which allows for complex potentials—however it is also a classic subject covered in various texts [DuSch, EE, DeGe]. We will treat [DeGe] as the basic source for this topic. It is concisely repeated in Sect. 2 of [DL].

For the convenience of the reader let us summarize some points from [DeGe, DL].

We consider an interval $]a,b[$ and an operator L acting on $f \in C_{\mathrm{c}}^{\infty}]a,b[$ given by (1.1), that is

$$Lf := \big(-\partial_x^2 + V(x)\big)f. \tag{B.1}$$

A *closed realization* of L is a closed operator $L_\bullet$ in the sense of $L^2]a,b[$ that restricted to $C_{\mathrm{c}}^{\infty}]a,b[$ coincides with L.

There always exists the maximal closed realization, denoted $L^{\max}$ and the minimal closed realization, denoted $L^{\min}$. $L^{\max}$ restricted to $\mathcal{D}(L^{\min})$ coincides with $L^{\min}$.

In many cases $\mathcal{D}(L^{\min}) = \mathcal{D}(L^{\max})$, and then there exists a unique closed realization of L.

Sometimes $\mathcal{D}(L^{\max})$ is larger than $\mathcal{D}(L^{\min})$, and then there exist also closed realizations of L, denote them $L_\bullet$, which satisfy

$$\mathcal{D}(L^{\min}) \subset \mathcal{D}(L_\bullet) \subset \mathcal{D}(L^{\max}) \tag{B.2}$$

and $L_\bullet = L^{\max}\Big|_{\mathcal{D}(L_\bullet)}$.

B.2 Resolvent

Let $L_\bullet$ be a realization of L with separated boundary conditions. Following [DL, DeGe] we will now sketch how to find the spectrum of $L_\bullet$, denoted $\sigma(L_\bullet)$, and how to compute the integral kernel of $\frac{1}{L_\bullet - z}$ for $z \in \mathbb{C}$ outside of $\sigma(L_\bullet)$.

First let us recall the definition of the Wronskian of two complex functions Φ_1, Φ_2 on $]a,b[$:

$$\mathcal{W}(\Phi_1,\Phi_2)(x) := \Phi_1(x)\Phi_2'(x) - \Phi_1'(x)\Phi_2(x). \tag{B.3}$$

It is easily checked that if both Φ_1 and Φ_2 are eigenfunctions of $-\partial_x^2 + V(x)$ with the same eigenvalue, then $\mathcal{W}(\Phi_1,\Phi_2)(x)$ does not depend on x, so that we can write $\mathcal{W}(\Phi_1,\Phi_2)$.

Consider a closed realization of L, denoted $L_\bullet$. Let $\mathcal{D}(L_\bullet) \subset L^2]a,b[$ denote the domain of $L_\bullet$. Suppose $z \in \mathbb{C}$ and let $\Psi_a(z,\cdot), \Psi_b(z,\cdot)$ be functions in $AC^1]a,b[$ solving the eigenvalue equation

$$\big(-\partial_x^2 + V(x) - z\big)\Psi_a(z,x) = 0, \tag{B.4}$$
$$\big(-\partial_x^2 + V(x) - z\big)\Psi_b(z,x) = 0, \tag{B.5}$$

$\Psi_a(z,\cdot)$ is in $\mathcal{D}(L_\bullet)$ near a and $\Psi_b(z,\cdot)$ is in $\mathcal{D}(L_\bullet)$ near b. Set

$$\mathcal{W}(x) := \mathcal{W}\big(\Psi_b(z,\cdot), \Psi_a(z,\cdot)\big). \tag{B.6}$$

Define the integral kernel

$$R_\bullet(z;x,y) := \frac{1}{\mathcal{W}(z)} \begin{cases} \Psi_a(z,x)\,\Psi_b(z,y) & \text{if } a < x < y < b, \\ \Psi_a(z,y)\,\Psi_b(z,x) & \text{if } a < y < x < b. \end{cases} \tag{B.7}$$

Note that (B.7) does not depend on the choice of $\Psi_a(z,\cdot)$ and $\Psi_b(z,\cdot)$.

Suppose that (B.7) defines a bounded operator $R_\bullet(z)$, Then $z \notin \sigma(L_\bullet)$ and

$$\frac{1}{L_\bullet - z} = R_\bullet(z). \tag{B.8}$$

Conversely, if $z \notin \sigma(L_\bullet)$, then the functions Ψ_a, Ψ_b with the above properties exist and the operator $R_\bullet(z)$ is bounded.

C Associated Legendre functions vs. Gegenbauer functions

In the literature, many authors use a Legendre function instead of Gegenbauer functions. Here, we briefly discuss the relations between these functions and Gegenbauer functions. For Legendre function we use [NIST2] as our reference and [D1] as our reference for Gegenbauer function.

The Legendre differential operator is

$$\mathcal{L}_\mu^\alpha := (1+z^2)\partial_z^2 - 2z\partial_z + \mu(\mu+1) - \frac{\alpha^2}{1-z^2}. \tag{C.1}$$

It is equivalent to the Gegenbauer operator, in fact

$$(1-w^2)^{\mp\frac{\alpha}{2}}\mathcal{L}_\mu^\alpha(1-w^2)^{\pm\frac{\alpha}{2}} \tag{C.2}$$
$$=(1-w^2)\partial_w^2 - 2(\pm\alpha+1)w\partial_w + (\mu\mp\alpha)(\mu\pm\alpha+1) = \mathcal{G}_{\pm\alpha,\mu+\frac{1}{2}}. \tag{C.3}$$

Certain distinguished funnctions annihilated by this operator are called associated Legendre functions. There various choices for these functions, which we quote following [NIST2]:
The associated Legendre function of the first kind is

$$\mathbf{P}_\mu^\alpha(z) = \left(\frac{z+1}{z-1}\right)^{\frac{\alpha}{2}} \mathbf{F}\left(\mu+1, -\mu; 1-\alpha; \frac{1-z}{2}\right) \tag{C.4}$$
$$= \frac{2^\alpha}{(z^2-1)_\bullet^\alpha}\mathbf{S}_{-\alpha,\mu+\frac{1}{2}}(z) \tag{C.5}$$

The Ferrers function of the first kind is

$$\mathcal{P}_\mu^\alpha(z) = \left(\frac{z+1}{1-z}\right)^{\frac{\alpha}{2}} \mathbf{F}\left(\mu+1, -\mu; 1-\alpha; \frac{1-z}{2}\right) \tag{C.6}$$
$$= \frac{2^\alpha}{(1-z^2)_\bullet^\alpha}\mathbf{S}_{-\alpha,\mu+\frac{1}{2}}(z) \tag{C.7}$$

And **the associated Legendre function of the second kind** is

$$\mathbf{Q}_\mu^\alpha(z) = \mathrm{e}^{\mathrm{i}\pi\alpha}\frac{\pi^{\frac{1}{2}}\Gamma(\alpha+\mu+1)\left(z^2-1\right)^{\frac{\alpha}{2}}}{2^{\mu+1}z^{\alpha+\mu+1}}\mathbf{F}\left(\frac{\alpha+\mu}{2}+1, \frac{\alpha+\mu}{2}+\frac{1}{2}; \frac{3}{2}+\mu; z^{-2}\right) \tag{C.8}$$
$$= \mathrm{e}^{\mathrm{i}\pi\alpha}\frac{\pi^{\frac{1}{2}}\Gamma(\alpha+\mu+1)\left(z^2-1\right)^{\frac{\alpha}{2}}}{2^{\mu+1}}\mathbf{Z}_{\alpha,\mu+\frac{1}{2}}(z) \tag{C.9}$$

Note that Legendre functions are represented with upper and lower indices. One should not confuse them with other, quite analogous functions defined in the text, whose parameters are lower indices.

The Legendre functions are closely related to the functions $\mathcal{P}^{\mathrm{s}}$, $\mathcal{P}^{\mathrm{h}}$, and $\mathcal{Q}^{\mathrm{h}}$ that we introduced in the section on Gegenbauer Hamiltonians, which you can see on the right of the following comparison:

$$\mathcal{P}_\mu^\alpha(\cos r) = \left(\frac{2}{\sin r}\right)^{\frac{1}{2}}\mathcal{P}^{\mathrm{s}}_{-\alpha,\mu+\frac{1}{2}}(r), \tag{C.10}$$
$$\mathbf{P}_\mu^\alpha(\cosh r) = \left(\frac{2}{\sinh r}\right)^{\frac{1}{2}}\mathcal{P}^{\mathrm{h}}_{-\alpha,\mu+\frac{1}{2}}(r), \tag{C.11}$$
$$\mathbf{Q}_\mu^\alpha(\cosh r) = \frac{1}{(2\sinh r)^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}\pi\alpha}\sqrt{\pi}\Gamma(\alpha+\mu+1)\mathcal{Q}^{\mathrm{h}}_{-\alpha,\mu+\frac{1}{2}}(r). \tag{C.12}$$

Conflict of interest. This manuscript has no conflict of interest.

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Table 1: Names of hypergeometric Hamiltonians

Our suggestion	Name in [DW]	Name in [CKS]	Alternative names
Spherical of 1st kind	Trigonometric Pöschl-Teller	Scarf I	Pöschl-Teller of 1st kind, Trigonometric Scarf
Hyperbolic of 1st kind	Hyperbolic Pöschl-Teller	Scarf II	Pöschl-Teller of 2nd kind,
DeSitterian of 1st kind	Scarf	Generalized Pöschl-Teller	Hyperbolic Scarf
Spherical of 2nd kind	Rosen-Morse	Rosen-Morse I	Trigonometric Rosen-Morse
Hyperbolic of 2nd kind	Eckart	Eckart	Generalized Morse, Hulthén
DeSitterian of 2nd kind	Manning-Rosen	Rosen-Morse II	Hyperbolic Rosen-Morse, Woods-Saxon