Introduction to hypergeometric type functions

J. Dereziński
Department of Mathematical Methods in Physics,
Faculty of Physics
University of Warsaw,
Pasteura 5, 02-093 Warszawa, Poland,
email jan.derezinski@fuw.edu.pl

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1 Generalized hypergeometric equations and functions

1.1 Generalized hypergeometric series

Recall that for $a \in \mathbb{C}$ and $n \in \mathbb{N}$ we introduce the *Pochhammer symbol* $(a)_n := a(a+1)\cdots(a+n-1)$. For $a_1, \dots, a_k \in \mathbb{C}$, $c_1, \dots, c_m \in \mathbb{C}\backslash\{0, -1, -2, \dots\}$, we define the *(generalized) hypergeometric series of type* ${}_kF_m$:

$$ {}_kF_m(a_1, \dots, a_k; c_1, \dots, c_m; z) := \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_k)_j z^j}{(c_1)_j \cdots (c_m)_j j!}. \tag{1.1} $$

Notice that

1. if $m+1 > k$, then (1.1) is convergent for $z \in \mathbb{C}$;
2. if $m+1 = k$, then (1.1) is convergent for $|z| < 1$;
3. if $m+1 < k$, then (1.1) is divergent (however sometimes we can give a meaning to the function ${}_kF_m$).

This follows by the d'Alembert criterion: if f_j is jth coefficient of (1.1), then

$$ \frac{f_{j+1}}{f_j} = \frac{(a_1+j)\cdots(a_k+j)}{(c_1+j)\cdots(c_m+j)}. $$

We can also use a different normalization:

$$ \begin{aligned} {}_k\mathbf{F}_m(a_1, \dots, a_k; c_1, \dots, c_m; z) &:= \frac{{}_kF_m(a_1, \dots, a_k; c_1, \dots, c_m; z)}{\Gamma(c_1)\cdots\Gamma(c_m)} \\ &= \sum_{j=0}^{\infty} \frac{(a_1)_j \cdots (a_k)_j z^j}{\Gamma(c_1+j)\cdots\Gamma(c_m+j) j!}. \end{aligned} \tag{1.2} $$

Then we do not have to restrict the values of $c_1, \ldots, c_m \in \mathbb{C}$. (If for some i $c_i \in \{0, -1, -2 \ldots\}$, then $\mathbf{F}$ is zero).

1.2 Generalized hypergeometric equations

Theorem 1.1 *The function (1.1) solves the equation*

$$(c_1 + z\partial_z) \cdots (c_m + z\partial_z)\partial_z F(a_1, \ldots, a_k; c_1, \ldots, c_m; z) \tag{1.3}$$
$$=(a_1 + z\partial_z) \cdots (a_k + z\partial_z) F(a_1, \ldots, a_k; c_1, \ldots, c_m; z). \tag{1.4}$$

Proof. We check that both (1.3) and (1.4) are equal to

$$a_1 \cdots a_k F(a_1 + 1, \ldots, a_k + 1; c_1, \ldots, c_m; z).$$

□

Note that the equation (1.4) is of the order $\max(k, m+1)$. Below we list all equations and hypergeometric functions with equations of the order at most 2.

1.3 Hypergeometric function or ${}_2F_1$

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!(c)_n} z^n.$$

The series is convergent for $|z| < 1$, it extends to a multivalued function on a covering of $\mathbb{C}\backslash\{0, 1\}$.

The function is a solution of the hypergeometric equation

$$\left(z(1-z)\partial_z^2 + (c - (a+b+1)z)\partial_z - ab\right) u(z) = 0$$

that is analytic around 0 and equals there 1.

1.4 Confluent function or ${}_1F_1$

$$F(a; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!(c)_n} z^n.$$

The series is convergent for all $z \in \mathbb{C}$. It defines a solution analytic around 0 and equal there 1 of the confluent equation

$$(z\partial_z^2 + (c - z)\partial_z - a)u(z) = 0,$$

1.5 Function ${}_0F_1$

$$F(-; c; z) = F(c; z) = \sum_{n=0}^{\infty} \frac{1}{n!(c)_n} z^n.$$

The series is convergent for all $z \in \mathbb{C}$. It defines a solution analytic around 0 and equal there 1 of the ${}_0F_1$ equation (related to the Bessel equation)

$$(z\partial_z^2 + c\partial_z - 1)u(z) = 0.$$

1.6 ${}_2F_0$ function

For $\arg z \neq 0$ we define

$$F(a,b;-;z) := \lim_{c\to\infty} F(a,b;c;cz).$$

It extends to an analytic function on the universal cover of $\mathbb{C}\backslash\{0\}$ with a branch point of an infinite order at 0. It has the following asymptotic expansion:

$$F(a,b;-;z) \sim \sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{n!} z^n, \ \ |\arg z - \pi| < \pi - \epsilon, \quad \epsilon > 0.$$

This function has a branch point at zero. Hence it cannot be defined with a series around zero. It solves the ${}_2F_0$ equation (related to the confluent equation)

$$\big(z^2\partial_z^2 + (-1 + (a+b+1)z)\partial_z + ab\big)\, u(z) = 0.$$

1.7 Power function ${}_1F_0$

$$F(a;-;z) = (1-z)^{-a} = \sum_{n=0}^{\infty} \tfrac{(a)_n}{n!} z^n$$

The series is convergent for $|z| < 1$, it extends to a multivalued function on a covering of $\mathbb{C}\backslash\{1\}$. It is a solution of

$$((1-z)\partial_z - a)u(z) = 0.$$

1.8 Exponential function ${}_0F_0$

$$F(-;-;z) = \mathrm{e}^z = \sum_{n=0}^{\infty} \tfrac{1}{n!} z^n.$$

It solves

$$(\partial_z - 1)u(z) = 0.$$

2 2nd order differential equations in complex domain

In this section we will discuss a general theory of equations of the form

$$\big(b(z)\partial_z^2 + c(z)\partial_z + d(z)\big)\, u(z) = 0. \tag{2.5}$$

z will be a complex variable. The functions b, c, d will be usually holomorphic or at least meromorphic in an open set $\Omega \subset \mathbb{C}$.

Discussing an equation such as (2.5), we will often introduce an operator

$$A(z, \partial_z) := b(z)\partial_z^2 + c(z)\partial_z + d(z). \tag{2.6}$$

We will say that (2.5) is given by the operator (2.6). Indeed, u solves (2.5) iff u is in the kernel of (2.6).

By dividing (2.5) by $b(z)$ we obtain

$$\left(\partial_z^2 + \frac{c(z)}{b(z)}\partial_z + \frac{d(z)}{b(z)}\right) u(z) = 0. \tag{2.7}$$

Thus we can usually assume that $b(z)$ is 1.

2.1 Wronskian

Let $u_1(z)$, $u_2(z)$ be a pair of functions. Their Wronskian is

$$W(u_1, u_2)(z) = W(z) := u_1(z)u_2'(z) - u_1'(z)u_2(z).$$

If they are solutions of (2.5), then the Wronskian satisfies

$$\left(\partial_z + \frac{c(z)}{b(z)}\right) W(z) = 0.$$

If

$$\tilde{u}_1(z) = a_{11}u_1(z) + a_{12}u_2(z), \qquad \tilde{u}_2(z) = a_{21}u_1(z) + a_{22}u_2(z)$$

is another pair of solutions, then

$$W(\tilde{u}_1, \tilde{u}_2) = (a_{11}a_{22} - a_{12}a_{21})W(u_1, u_2).$$

2.2 Regular points

Definition 2.1 *We say that $z_0 \in \Omega$ is a* regular point *of the equation* (2.6) *if $\frac{c(z)}{b(z)}$ and $\frac{d(z)}{b(z)}$ are analytic around z_0.*

Proposition 2.2 *Let $b(z)$, $c(z)$, $d(z)$ be holomorphic in a connected and simply connected open subset $\Omega \subset \mathbb{C}$ and z_0 is a regular point. Then the problem*

$$\begin{cases} \left(b(z)\partial_z^2 + c(z)\partial_z + d(z)\right) u(z) = 0 \\ u(z_0) = w_0, \ \partial_z u(z_0) = w_1, \end{cases} \tag{2.8}$$

has a unique solution in Ω.

Let us give the formula for the coefficients of the expansion

$$u(z) := \sum_{k=0}^{\infty} u_k z^k.$$

of (2.8):

$$\begin{cases} u_0 = w_0, \quad u_1 = w_1, \\ \sum\limits_{k=0}^{m} k(k-1)u_k b_{m-k} + \sum\limits_{k=0}^{m-1} k c_{m-k-1} u_k + \sum\limits_{k=0}^{m-2} d_{m-k-2} u_k = 0. \end{cases}$$

Definition 2.3 *Assume that $b(z)$, $c(z)$, $d(z)$ are holomorphic for $|z| > R$. We say that ∞ is a regular point of (2.6) if after the change of coordinates $w = z^{-1}$ we obtain a regular point at 0.*

Consider (2.6). The change $w = z^{-1}$ and division by w^4 leads to

$$\Big(b(w^{-1})\partial_w^2 + \big(2w^{-1}b(w^{-1}) - w^{-2}c(w^{-1})\big)\partial_w + w^{-4}d(w^{-1})\Big)u(w^{-1}) = 0.$$

Hence ∞ is a regular point if there exist (finite) limits

$$\lim_{z\to\infty}\Big(2z - z^2\frac{c(z)}{b(z)}\Big), \qquad \lim_{z\to\infty} z^4\frac{d(z)}{b(z)}.$$

Theorem 2.4 *Let ∞ be a regular point of (2.6). Then for any w_0, w_1 there exists a unique solution of the problem*

$$\begin{cases} \big(b(z)\partial_z^2 + c(z)\partial_z + d(z)\big)\, u(z) = 0 \\ \lim\limits_{z\to\infty} u(z) = w_0, \ \lim\limits_{z\to\infty}(u(z) - w_0)z = w_1. \end{cases} \tag{2.9}$$

2.3 Regular-singular points

Definition 2.5 *We say that an equation*

$$\big(b(z)\partial_z^2 + c(z)\partial_z + d(z)\big)\, u(z) = 0 \tag{2.10}$$

has a regular-singular point *at $z_0 \in \Omega$, if $\frac{c(z)}{b(z)}$ has at z_0 a pole of at most 1st order and $\frac{d(z)}{b(z)}$ has at z_0 a pole of at most 2nd order.*

For simplicity, assume that $z_0 = 0$. We can rewrite the above equation as

$$\big(p(z)z^2\partial_z^2 + q(z)z\partial_z + r(z)\big)\, u(z) = 0. \tag{2.11}$$

If $p(0) \neq 0$, then 0 is regular-singular iff p, q, r are analytic at 0.

Theorem 2.6 (Frobenius Method) *Assume that $p(0) \neq 0$ and p, q, r are holomorphic in an open connected simply connected set $\Omega \subset \mathbb{C}$ containing 0. Let $\lambda \in \mathbb{C}$ satisfy*

$$\lambda(\lambda - 1)p(0) + \lambda q(0) + r(0) = 0,$$
$$(\lambda + m)(\lambda + m - 1)p(0) + (\lambda + m)q(0) + r(0) \neq 0, \quad m = 1, 2, \dots.$$

Then there exists a unique function $\tilde{u}(z)$ holomorphic in Ω, such that $u(z) := z^\lambda \tilde{u}(z)$ is a solution of the problem

$$\begin{cases} \big(z^2p(z)\partial_z^2 + q(z)z\partial_z + r(z)\big)\, u(z) = 0, \\ \lim\limits_{z\to 0} z^{-\lambda}u(z) = 1, \end{cases} \tag{2.12}$$

We insert

$$u(z) := \sum_{k=0}^{\infty} u_k z^{\lambda+k}$$

into the differential equation. The coefficient at $z^{\lambda+m}$ for $m = 0, 1, 2, \ldots$ is

$$\sum_{k=0}^{m} \big((\lambda+k)(\lambda+k-1)p_{m-k} + (\lambda+k)q_{m-k} + r_{m-k}\big)u_k. \tag{2.13}$$

We want $u_0 = 1$. The cofficient at z^{λ} is

$$\big(\lambda(\lambda-1)p_0 + \lambda q_0 + r_0\big)u_0. \tag{2.14}$$

Thus, if we are looking for solutions (2.11), we should first find the roots λ_1, λ_2 of the so-called *indicial equation*

$$\lambda(\lambda-1)p(0) + \lambda q(0) + r(0) = 0.$$

These roots are called the *indices* of the regular-singular point z_0.

The next terms in the series we find from

$$\begin{aligned} &\big((\lambda+k)(\lambda+k-1)p_0 + \lambda q_0 + r_0\big)u_0 \\ = &\sum_{k=0}^{m-1} \big((\lambda+k)(\lambda+k-1)p_{m-k} + (\lambda+k)q_{m-k} + r_{m-k}\big)u_k. \end{aligned} \tag{2.15}$$

If $\lambda_1 - \lambda_2 \notin \mathbb{Z}$, then we can find two linearly independent solutions that behave at zero as z^{λ_1} and z^{λ_2}. If $\lambda_1 - \lambda_2 \in \mathbb{Z}$, then generally we can find only a solution behaving as z^{λ_1}, where $\lambda_1 - \lambda_2 \geq 0$.

Definition 2.7 *We say that a singular point z_0 is nonlogarithmic if all solutions are linear combinations of functions of the form* $\sum_{j=0}^{\infty}(z-z_0)^{\lambda+j}w_j$.

Clearly, if a singular point is nonlogarithmic, then its indices are integers.

Theorem 2.8 *Suppose that $m = 0, 1, 2, \ldots$ and the indices of 0 are $\{0, -m\}$.*

1. *If $m = 0$, then 0 is logarithmic and assuming with $v_0 = 0$, $w_0 = 1$ there exists a unique pair of solutions of the form*

$$\sum_{k=0}^{\infty} v_k z^k + \ln z \sum_{k=0}^{\infty} w_k z^k, \qquad \sum_{k=0}^{\infty} w_k z^k. \tag{2.16}$$

2. *If 0 is nonlogarithmic, then $m > 0$. Besides, assuming $v_{-m} = 1$ and $w_0 = 1$ there exists a unique pair of solutions of the form*

$$\sum_{k=-m}^{-1} v_k z^k, \qquad \sum_{k=0}^{\infty} w_k z^k. \tag{2.17}$$

3. If $m \geq 1$ and 0 is logarithmic, then assuming with $w_0 = 1$, $v_{-m} = 1$ and $v_0 = 0$, there exists a unique pair of solutions of the form

$$\sum_{k=-m}^{\infty} v_k z^k + \ln z \sum_{k=0}^{\infty} w_k z^k, \qquad \sum_{k=0}^{\infty} w_k z^k. \tag{2.18}$$

Proof. The existence of the solution $\sum_{k=0}^{\infty} w_k z^k$ is a special case of the generic Frobenius method. Let us show the existence of the other solution.

We insert the first expression of (2.18) in the differential equation. We obtain

$$\sum_{n=0}^{\infty} z^n \ln z \sum_{j=0}^{n} \Big((n-j)(n-j-1)p_j w_{n-j} + q_j(n-j)w_{n-j} + r_j w_{n-j}\Big)$$
$$+ \sum_{n=-m}^{\infty} z^n \left(\sum_{j=0}^{n+m} \Big((n-j)(n-j-1)p_j v_{n-j} + q_j(n-j)v_{n-j} + r_j v_{n-j}\Big) \right.$$
$$\left. + \sum_{j=0}^{n} \Big((2n-2j-1)p_j w_{n-j} + q_j w_{n-j}\Big) \right) = 0.$$

For $n = -m, \dots, -1$ we obtain the recurrence relations for v_n:

$$\sum_{j=0}^{n+m} \Big((n-j)(n-j-1)p_j v_{n-j} + q_j(n-j)v_{n-j} + r_j v_{n-j} = 0, \tag{2.19}$$

At the 0th level we have

$$\sum_{j=1}^{n+m} \Big((-j)(-j-1)p_j v_{n-j} + \sum_{j=1}^{m} q_j(-j)v_{-j} + \sum_{j=1}^{m} r_j v_{-j}\Big) \tag{2.20}$$
$$+(-p_0 + q_0)w_0 = 0. \tag{2.21}$$

The term (2.21) is mp_0w_0.

If $m = 0$, then (2.20) is zero and w_0 can be arbitrary, e.g. $w_0 = 1$. We first solve for w_j, $j = 1, 2, \dots$. v_0 is arbitrary, we set $v_0 = 0$. Then we solve for $v_1, \dots$.

Let $m \geq 1$. The equation at the level $n = -m$ is trivially zero, hence we can set $v_{-m} = 1$. Then we solve for $v_{-m+1}, \dots, v_{-1}$.

If $m \geq 1$ and (2.20) is zero, then there is no logarithmic term, we can set $v_0 = 1$ and we can solve for $v_1, \dots$, and then rename $v_0, v_1, \dots$ into $w_0, w_1, \dots$.

If $m \geq 1$ and (2.20) is nonzero, then first we solve for $w_0, w_1, \dots$ with $w_0 = 1$. Moreover, v_0 is arbitrary, and can be set 0. Then we determine $v_1, \dots$. □

Definition 2.9 *Assume that $p(z)$, $q(z)$, $r(z)$ are holomorphic for $|z| > R$. We say that ∞ is a regular-singular point of (2.6) if after the change of coordinates $w = z^{-1}$ we obtain a regular-singular point at 0.*

∞ is a regular singular point of (2.11) if

$$\lim_{z\to\infty}\frac{q(z)}{p(z)},\quad \lim_{z\to\infty}\frac{r(z)}{p(z)} \tag{2.22}$$

exist.

Proposition 2.10 *Let $p(z)$, $q(z)$, $r(z)$ be holomorphic in a connected simply connected open set $\Omega\subset\mathbb{C}$ containing $\{|z|>R\}$. Let $\lambda\in\mathbb{C}$ satisfy*

$$\begin{aligned}&\lambda(\lambda+1)p(\infty)-\lambda q(\infty)+r(\infty)=0,\\ &(\lambda+m)(\lambda+m+1)p(\infty)-(\lambda+m)q(\infty)+r(\infty)\neq 0,\quad m=1,2,\dots.\end{aligned}$$

Then there exists a unique function $\tilde{u}(z)$ holomorphic in Ω, such that $u(z):=z^{-\lambda}\tilde{u}(z)$ is a solution of

$$\begin{cases}\big(p(z)z^2\partial_z^2+q(z)z\partial_z+r(z)\big)\,u(z)=0,\\ \lim\limits_{z\to\infty} z^{\lambda}u(z)=1.\end{cases} \tag{2.23}$$

For simplicity, in what follows we assume $p(z)=1$.

Proposition 2.11 *Let*

$$\left(\partial_z^2+\frac{q(z)}{(z-z_0)}\partial_z+\frac{r(z)}{(z-z_0)^2}\right) \tag{2.24}$$

have indices $\rho_0,\tilde{\rho}_0$ at z_0 and $\rho_\infty,\tilde{\rho}_\infty$ at ∞. Then

$$(z-z_0)^{\mu}\left(\partial_z^2+\frac{q(z)}{(z-z_0)}\partial_z+\frac{r(z)}{(z-z_0)^2}\right)(z-z_0)^{-\mu} \tag{2.25}$$

has at z_0 indices $\rho_0+\mu,\tilde{\rho}_0+\mu$ and at ∞ indices $\rho_\infty-\mu,\tilde{\rho}_\infty-\mu$.

Proof. We can assume that $z_0=0$. We use $z^{\mu}\partial_z z^{-\mu}=\partial_z-\frac{\mu}{z}$. Then (2.25) is

$$\begin{aligned}&\left(\partial_z-\frac{\mu}{z}\right)^2+\frac{q(z)}{z}\left(\partial_z-\frac{\mu}{z}\right)+\frac{r(z)}{z^2}\\ =\;&\partial_z^2-2\frac{\mu}{z}\partial_z+\frac{\mu+\mu^2}{z^2}+\frac{q(z)}{z}\partial_z-\frac{q(z)\mu}{z^2}+\frac{r(z)}{z^2}\\ =\;&\partial_z^2+\frac{(-2\mu+q(z))}{z}\partial_z+\frac{\big(\mu+\mu^2-\mu q(z)+r(z)\big)}{z^2}.\end{aligned}$$

Therefore, the indicial equation at 0 is

$$\lambda(\lambda-1)+\lambda(q(0)-2\mu)+\mu+\mu^2-q(0)\mu+r(0) \tag{2.26}$$
$$=\;(\lambda-\mu)(\lambda-\mu-1)+q(0)(\lambda-\mu)+r(0), \tag{2.27}$$

and the indicial equation at ∞ is

$$\lambda(\lambda+1)-\lambda(q(\infty)-2\mu)+\mu+\mu^2-q(\infty)\mu+r(\infty) \tag{2.28}$$
$$=\;(\lambda+\mu)(\lambda+\mu+1)-q(\infty)(\lambda+\mu)+r(\infty). \tag{2.29}$$

□

Theorem 2.12 *Suppose that we change the variables in the equations, considering a (holomorphic) map $y \mapsto z(y)$. Assume that y_0 is mapped at z_0 and $\frac{\partial z}{\partial y}(y_0) \neq 0$. Then the indices of the transformed equation coincide with the indices of the original equation.*

Proof. We will assume that the equation has the form (2.24) and $z_0 = y_0 = 0$. Let us compute the change of the differentiation operators:

$$\partial_z = \frac{1}{\frac{\partial z}{\partial y}}\partial_y, \tag{2.30}$$

$$\partial_z^2 = -\frac{\frac{\partial^2 z}{\partial y^2}}{\left(\frac{\partial z}{\partial y}\right)^3}\partial_y + \frac{1}{\left(\frac{\partial z}{\partial y}\right)^2}\partial_y^2. \tag{2.31}$$

Therefore,

$$\left(\partial_z^2 + \frac{q(z)}{z}\partial_z + \frac{r(z)}{z^2}\right) \tag{2.32}$$

$$= \frac{1}{\left(\frac{\partial z}{\partial y}\right)^2}\left(\partial_y^2 + \Big(\frac{q(z(y))\frac{\partial z(y)}{\partial y}}{z(y)} - \frac{\frac{\partial^2 z(y)}{\partial y^2}}{\frac{\partial z(y)}{\partial y}}\Big)\partial_y + \frac{\left(\frac{\partial z}{\partial y}\right)^2}{z(y)^2}r(z(y))\right) \tag{2.33}$$

$$= \frac{1}{\left(\frac{\partial z}{\partial y}\right)^2}\left(\partial_y^2 + \frac{\tilde{q}(y)}{y}\partial_y + \frac{\tilde{r}(y)}{y^2}\right) \tag{2.34}$$

Now it is easy to see that $\tilde{q}(0) = q(0)$ and $\tilde{r}(0) = r(0)$. □

2.4 Equations with two regular-singular points on the Riemann sphere

Example 2.13 *Every 2nd order equation that in $\mathbb{C} \cup \{\infty\}$ has only regular points except for two regular-singular points at 0 and ∞ has the form*

$$(z^2\partial_z^2 + qz\partial_z + r)u(z) = 0. \tag{2.35}$$

It is sometimes called the **homogeneous Euler equation**. *Its indicial points are*

$$0: \qquad \lambda(\lambda - 1) + q\lambda + r = 0,$$
$$\infty: \qquad \lambda(\lambda + 1) - q\lambda + r = 0.$$

If $\rho, \tilde{\rho}$ are its indices at 0, then $-\rho, -\tilde{\rho}$ are its indices at ∞. Its solutions are z^ρ, $z^{\tilde{\rho}}$ if $\rho \neq \tilde{\rho}$ and z^ρ, $z^\rho \log z$ if $\rho = \tilde{\rho}$. The equation (2.35) can be rewritten as

$$(z^2\partial_z + (1 - \rho - \tilde{\rho})z\partial_z + \rho\tilde{\rho})u(z) = 0.$$

Example 2.14 *Every 2nd order equation that in $\mathbb{C} \cup \{\infty\}$ has only regular points except for two regular-singular points at z_1 and z_2 has the form*

$$\left(\partial_z^2 + \left(g_1(z - z_1)^{-1} + g_2(z - z_2)^{-1}\right)\partial_z + h(z - z_1)^{-2}(z - z_2)^{-2}\right)u(z) = 0, \tag{2.36}$$

where $g_1 + g_2 = 2$. Its indicial equations are

$$z_1: \quad \lambda(\lambda - 1) + g_1\lambda + h(z_1 - z_2)^{-2} = 0,$$

$$z_2: \quad \lambda(\lambda - 1) + g_2\lambda + h(z_1 - z_2)^{-2} = 0.$$

If ρ, $\tilde{\rho}$ are indices at z_1, then $-\rho$, $-\tilde{\rho}$ are indices at z_2. Solutions have the form $(z - z_1)^\rho(z - z_2)^{-\rho}$, $(z - z_1)^{\tilde{\rho}}(z - z_2)^{-\tilde{\rho}}$, if $\rho \neq \tilde{\rho}$ and $(z - z_1)^\rho(z - z_2)^{-\rho}$, $(z - z_1)^\rho(z - z_2)^{-\rho}\log(z - z_1)(z - z_2)^{-1}$, if $\rho = \tilde{\rho}$.

Equation (2.36) can be rewritten as

$$\Big(\partial_z^2 + \Big((1 - \rho - \tilde{\rho})(z - z_1)^{-1} + (1 + \rho + \tilde{\rho})(z - z_2)^{-1}\Big)\partial_z$$
$$+\rho\tilde{\rho}(z_1 - z_2)^2(z - z_1)^{-2}(z - z_2)^{-2}\Big)u(z) = 0.$$

3 Systems of 1st order equations

3.1 Regular points

This subsection can be skipped.

We will discuss differential equations

$$\partial_z v(z) = A(z)v(z). \tag{3.37}$$

where $A(z)$ is a matrix and $v(z) \in \mathbb{C}^n$.

Definition 3.1 *If $A(z)$ is analytic at z_0, then we say that z_0 is a regular point of (3.37).*

Theorem 3.2 *Let Ω be a connected simply connected open subset of $\mathbb{C}$. Let*

$$\Omega \ni z \mapsto A(z) = \begin{bmatrix} a_{11}(z) & \dots & a_{1n}(z) \\ & \dots & \\ a_{n1}(z) & \dots & a_{nn}(z) \end{bmatrix}$$

be a holomorphic function with values in $n \times n$ matrices and $w = \begin{bmatrix} w_1 \\ \dots \\ w_n \end{bmatrix} \in \mathbb{C}^n$. Then there exists a unique holomorphic function $\Omega \ni z \mapsto v(z) = \begin{bmatrix} v_1(z) \\ \dots \\ v_n(z) \end{bmatrix} \in \mathbb{C}^n$ that solves the problem

$$\begin{cases} \frac{\mathrm{d}v(z)}{\mathrm{d}z} = A(z)v(z), \\ v(z_0) = w. \end{cases} \tag{3.38}$$

Proof. Let us first restrict ourselves to a disk $K(z_0, r)$ such that $K(z_0, r)^{\mathrm{cl}} \subset \Omega$. We can also assume that $z_0 = 0$.

Let

$$A(z) = \sum_{k=0}^{\infty} A_k z^k$$

Then the series

$$v(z) := \sum_{k=0}^{\infty} v_k z^k,$$

where

$$\begin{cases} v_0 = w, \\ v_{m+1} := \frac{1}{m+1} \sum_{k=0}^{m} A_{m-k} v_k. \end{cases}$$

is the unique formal series solving (3.38).

Let us show that this series is convergent in $K(0, r)$. By the Cauchy inequality,

$$\|A_k\| \leq C r^{-k}.$$

If we set

$$\begin{cases} p_0 = \|w\| \\ p_{m+1} := \frac{1}{m+1} \sum_{k=0}^{m} C r^{-m+k} p_k, \end{cases}$$

then we can show inductively that

$$\|v_m\| \leq p_m. \tag{3.39}$$

Indeed, we have

$$\|v_0\| = p_0.$$

Assume that

$$\|v_k\| \leq p_k, \ k = 0, \ldots, m.$$

Then

$$\begin{aligned} \|v_{m+1}\| &\leq \frac{1}{m+1} \sum_{k=0}^{m} \|A_{m-k} v_k\| \\ &\leq \frac{1}{m+1} \sum_{k=0}^{m} \|A_{m-k}\| \|v_k\| \\ &\leq \frac{1}{m+1} \sum_{k=0}^{m} C r^{k-m} p_k = p_{m+1}. \end{aligned}$$

This proves (3.39).

If we subtract the formula

$$\begin{aligned} r(m+1) p_{m+1} &= \sum_{k=0}^{m} C r^{-m+k+1} p_k, \\ m p_m &= \sum_{k=0}^{m-1} C r^{-m+k+1} p_k, \end{aligned}$$

then we obtain

$$r(m+1)p_{m+1} = (Cr+m)p_m.$$

This immediately implies

$$\lim_{m\to\infty} \frac{p_{m+1}}{p_m} = r^{-1}.$$

Hence, by the d'Alembert criterion

$$\sum_{k=0}^{\infty} p_k z^k$$

is convergent in the disk $K(0,r)$. Therefore, so is

$$\sum_{k=0}^{\infty} v_k z^k$$

The above reasoning can be repeated for any disk contained in Ω. In this way, since Ω is connected, we can extend $v(z)$ to the whole Ω. Ω is simply connected, and therefore the resulting function will be univalued. □

Example 3.3

$$(\partial_z - 1)v(z) = 0, \quad v(0) = 1.$$

We set

$$v(z) = \sum_{n=0}^{\infty} v_n z^n.$$

We obtain a recurrence relation

$$nv_n = v_{n-1}.$$

Therefore,

$$v(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}, \ z \in \mathbb{C}.$$

Obviously, $v(z) = \mathrm{e}^z$.

Example 3.4 *Let* $\mu \in \mathbb{C}$, $z \neq -1$

$$\left(\partial_z - \mu(z+1)^{-1}\right) v(z) = 0, \quad v(0) = 1.$$

We set

$$v(z) = \sum_{n=0}^{\infty} v_n z^n.$$

We obtain a recurrence relation

$$nv_n = (\mu - n + 1)v_{n-1}.$$

Therefore,

$$v(z) = \sum_{n=0}^{\infty} \frac{\mu \dots (\mu - n + 1) z^n}{n!}, \ |z| < 1.$$

Obviously, $v(z) = (1+z)^\mu$.

Proof of Thm 2.4. Define

$$v(z) := \begin{bmatrix} u(z) \\ u'(z) \end{bmatrix}, \quad w := \begin{bmatrix} w_0 \\ w_1 \end{bmatrix}$$

and

$$A(z) := \begin{bmatrix} 0 & 1 \\ -d(z) & -c(z). \end{bmatrix}$$

Then (2.8) can be rewritten as

$$\begin{cases} \frac{\mathrm{d}v(z)}{\mathrm{d}z} = A(z)v(z), \\ v(z_0) = w. \end{cases}$$

We can apply Thm 3.2. □

Definition 3.5 *Assume that $A(z)$ is defined for $|z| > R$. We say that ∞ is a regular point of (3.37), if after the change of the variable $w = z^{-1}$ we obtain a reggular point at* 0.

Obviously, $\partial_z = -w^2\partial_w$. Hence, after the change of the variable (3.37) transforms into

$$\partial_w v(w^{-1}) = -w^{-2}A(w^{-1})v(w^{-1}).$$

Therefore, ∞ is a regular point iff there exists

$$\lim_{z\to\infty} z^2 A(z).$$

Theorem 3.6 *Let ∞ be a regular point of (3.40). Then for any $w \in \mathbb{C}^n$, there exists a unique holomorphic solution satisfying*

$$\begin{cases} \frac{\mathrm{d}v(z)}{\mathrm{d}z} = A(z)v(z), \\ \lim\limits_{z\to\infty} v(z) = w. \end{cases} \tag{3.40}$$

3.2 Regular-singular points

Definition 3.7 *We say that the equation*

$$\frac{\mathrm{d}v(z)}{\mathrm{d}z} = A(z)v(z) \tag{3.41}$$

has a regular-singular point *at z_0, if $A(z)$ has at z_0 a pole of at most 1st order.*

We can then rewrite (3.38) as

$$(z - z_0)\partial_z v(z) = B(z)v(z), \tag{3.42}$$

where $B(z)$ is holomorphic around z_0. The eigenvalues of the matrix $B(z_0)$ are called *indices of the singular point* z_0.

For simplicity, assume that $z_0 = 0$.

Theorem 3.8 (Frobenius method for systems of equations) *Let Ω be a connected simply connected open subset of $\mathbb{C}$ containing 0. Let*

$$\Omega \ni z \mapsto B(z) = \begin{bmatrix} b_{11}(z) & \dots & b_{1n}(z) \\ & \dots & \\ b_{n1}(z) & \dots & b_{nn}(z) \end{bmatrix}$$

be a holomorphic function with values in $n \times n$ matrices Let $w \in \mathbb{C}^n$ and $\lambda \in \mathbb{C}$ satisfy

$$\begin{aligned} &(B(0) - \lambda)w = 0, \\ &\lambda + m \text{ is not an eigenvalue of } B(0) \text{ for } m = 1, 2, \dots. \end{aligned} \tag{3.43}$$

Then there exists a unique function $\tilde{v}(z)$ holomorphic on Ω such that $v(z) := z^\lambda \tilde{v}(z)$ solves the problem

$$\begin{cases} z\frac{\mathrm{d}v(z)}{\mathrm{d}z} = B(z)v(z), \\ \lim\limits_{z\to 0} z^{-\lambda} v(z) = w. \end{cases} \tag{3.44}$$

Proof. Let us first consider a disc $K(0,r)$ such that $K(0,r)^{\mathrm{cl}} \subset \Omega$.

Let

$$B(z) = \sum_{k=0}^{\infty} B_k z^k$$

Then the series

$$v(z) := z^\lambda \sum_{k=0}^{\infty} v_k z^k,$$

where

$$\begin{cases} v_0 = w \\ v_m := (\lambda + m - B_0)^{-1} \sum\limits_{k=0}^{m-1} B_{m-k} v_k. \end{cases}$$

is the unique formal series solving (3.44).

Let us show that this series is convergent in the disk $K(0,r)$. By the Cauchy inequality,

$$\|B_k\| \leq Cr^{-k}.$$

If we set

$$\begin{cases} p_0 = \|w\| \\ p_m := \left\|(\lambda + m - B_0)^{-1}\right\| \sum\limits_{k=0}^{m-1} Cr^{-m+k} p_k, \end{cases}$$

then we can show by induction that

$$\|v_m\| \leq p_m.$$

If we subtract the formulas

$$\begin{aligned} r\left\|(\lambda+m+1-B_0)^{-1}\right\|^{-1} p_{m+1} &= \sum_{k=0}^{m} C r^{-m+k} p_k, \\ \left\|(\lambda+m-B_0)^{-1}\right\|^{-1} p_m &= \sum_{k=0}^{m-1} C r^{-m+k} p_k, \end{aligned}$$

then we obtain

$$r\left\|(\lambda+m+1-B_0)^{-1}\right\|^{-1} p_{m+1} = \left(C + \left\|(\lambda+m-B_0)^{-1}\right\|^{-1}\right) p_m.$$

It is easy to see that

$$\lim_{m\to\infty} m\left\|(\lambda+m-B_0)^{-1}\right\| = 1.$$

Hence,

$$\lim_{m\to\infty} \frac{p_{m+1}}{p_m} = r^{-1}.$$

Thus by the d'Alembert criterion, the series that defines $\tilde{v}(z)$ is convergent in the disk $K(0,r)$.

Using Them 3.2 we can extend $\tilde{v}(z)$ to the whole Ω. □

Example 3.9 *Let*

$$B = \begin{bmatrix} \lambda & & \dots & & \\ 1 & \lambda & \dots & & \\ & & \dots & & \\ & & \dots & \lambda & \\ & & \dots & 1 & \lambda \end{bmatrix}.$$

Consider the equation $z\partial_z v(z) = Bv(z)$. *We obtain*

$$\begin{aligned} z\partial_z v_1 &= \lambda v_1, \\ v_1 + z\partial_z v_2 &= \lambda v_2, \\ &\dots \\ v_{n-1} + z\partial_z v_n &= \lambda v_n. \end{aligned}$$

A basis of solution of this system is

$$\begin{bmatrix} 0 \\ \dots \\ 0 \\ z^\lambda \\ z^\lambda \log z \\ \dots \\ z^\lambda (\log z)^{m-1} \end{bmatrix}, \qquad m = 1, \dots, n.$$

Example 3.10 *The following equation has a regular-singular point at* 0*:*

$$\partial_z v(z) = (az^{-1} + b)v(z).$$

its solution is $v(z) = z^a \mathrm{e}^{bz}$

Proof of Thm 2.6 Define

$$v(z) := \begin{bmatrix} u(z) \\ zu'(z) \end{bmatrix}, \quad w := \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$$

and

$$B(z) := \begin{bmatrix} 0 & 1 \\ -c(z) & 1 - b(z) \end{bmatrix}.$$

We then have

$$B(z)v(z) = \begin{bmatrix} zu'(z) \\ -c(z)u(z) - b(z)zu'(z) + zu'(z) \end{bmatrix},$$

$$z\partial_z \begin{bmatrix} u(z) \\ zu'(z) \end{bmatrix} = \begin{bmatrix} zu'(z) \\ z^2u''(z) + zu'(z) \end{bmatrix},$$

$$z^{-\lambda}v(z) = \begin{bmatrix} \tilde{u}(z) \\ z\tilde{u}'(z) + \lambda\tilde{u}(z) \end{bmatrix}.$$

Hence (2.12) can be rewritten as

$$\begin{cases} z\frac{\mathrm{d}v(z)}{\mathrm{d}z} = B(z)v(z), \\ \lim\limits_{z\to 0} z^{-\lambda}v(z) = w. \end{cases}$$

We can apply Thm 3.8. □

Definition 3.11 *Assume that* $B(z)$ *is defined for* $|z| > R$*. We say that* ∞ *is a regular-singular point of (3.41), if after the change of the variable* $w = z^{-1}$ *we obtain a regular-singular point at* 0*.*

Thus (3.42) has a regular-singular point if $\lim\limits_{z\to\infty} B(z)$ exists.The eigenvalues of $-B(\infty)$ are called indices of ∞.

Theorem 3.12 *Let* Ω *be a connected simply connected subset of* $\mathbb{C}$ *containing* $\{|z| > R\}$*. Let*

$$\Omega \ni z \mapsto B(z) = \begin{bmatrix} a_{11}(z) & \dots & a_{1n}(z) \\ & \dots & \\ a_{n1}(z) & \dots & a_{nn}(z) \end{bmatrix}$$

be a holomorphic function with values in $n \times n$ *matrices. Let* $w \in \mathbb{C}^n$ *and* $\lambda \in \mathbb{C}$ *satisfy*

$$\begin{aligned} &(B(\infty) + \lambda)w = 0, \\ &\lambda + m \textit{ is not an eigenvalue of } -B(\infty) \textit{ for } m = 1, 2, \dots. \end{aligned} \tag{3.45}$$

Then there exists a unique function $\tilde{v}(z)$ holomorphic on Ω such that $v(z) := z^{-\lambda}\tilde{v}(z)$ solves

$$\begin{cases} z\frac{\mathrm{d}v(z)}{\mathrm{d}z} = B(z)v(z), \\ \lim\limits_{z\to\infty} z^{\lambda}v(z) = w. \end{cases} \tag{3.46}$$

Example 3.13 *Every 1st order equation on the Riemann sphere possessing only regular points except of regular-singular points at z_1, z_2 and ∞ has the form*

$$\partial_z v(z) = \Big(a_1(z-z_1)^{-1} + a_2(z-z_2)^{-1}\Big)v(z) \tag{3.47}$$

It has indices

$$z_1: \quad a_1, \qquad z_2: \quad a_2, \qquad \infty: \quad -a_1-a_2,$$

and a solution $(z-z_1)^{a_1}(z-z_2)^{a_2}$.

4 Hypergeometric equation

4.1 Riemann equations

Lemma 4.1 *Every 2nd order equation which on the Riemann sphere has only regular points except for 3 points at z_1, z_2 and ∞ is given by an operator of the form*

$$\begin{aligned} &\partial_z^2 + \Big(\frac{g_1}{z-z_1} + \frac{g_2}{z-z_2}\Big)\partial_z \\ &+\frac{h_1}{(z-z_1)^2} + \frac{h_2}{(z-z_2)^2} + \frac{k}{(z-z_1)(z-z_2)}. \end{aligned} \tag{4.48}$$

Proof. Consider

$$\partial_z^2 + c(z)\partial_z + d(z) \tag{4.49}$$

Clearly, if in $\mathbb{C}$ the only singular points are at z_1, z_2, and they are regular-singular, then

$$c(z) = c_{\mathrm{reg}}(z) + \frac{g_1}{z-z_1} + \frac{g_2}{z-z_2}, \tag{4.50}$$

$$d(z) = d_{\mathrm{reg}}(z) + \frac{h_1}{(z-z_1)^2} + \frac{h_2}{(z-z_2)^2} + \frac{k_1}{z-z_1} + \frac{k_2}{z-z_2}. \tag{4.51}$$

where c_{reg}, d_{reg} are entire functions. ∞ is a regular-singular point if the following limits also exist:

$$\lim_{z\to\infty} zc(z), \tag{4.52}$$

$$\lim_{z\to\infty} z^2 d(z). \tag{4.53}$$

(4.52) implies the existence of $\lim_{z\to\infty} zc_{\rm reg}(z)$. Thus, $zc_{\rm reg}(z)$ is a bounded entire function. By the Liouville Theorem, $zc_{\rm reg}$ is a constant. But $c_{\rm reg}$ is also an entire function. Hence $c_{\rm reg} = 0$

(4.53) implies the existence of a limit $\lim_{z\to\infty} zd(z)$, which in turn implies the existence of $\lim_{z\to\infty} zd_{\rm reg}(z)$. By the Liouville Theorem, $zd_{\rm reg}$ is a constant. But $d_{\rm reg}$ is also an entire function. Hence $d_{\rm reg} = 0$.

Using again (4.53), knowing that $d_{\rm reg} = 0$, we obtain $k_1 + k_2 = 0$. □

We can transform (4.48) further, obtaining

$$\partial_z^2 + \Big(\frac{g_1}{(z-z_1)} + \frac{g_2}{(z-z_2)}\Big)\partial_z + \frac{h_1(z_1-z_2)}{(z-z_1)^2(z-z_2)} + \frac{h_2(z_2-z_1)}{(z-z_2)^2(z-z_1)} + \frac{h}{(z-z_1)(z-z_2)}. \qquad (4.54)$$

with $h = k - h_1 - h_2$.

Suppose that the indices are

$$\begin{array}{ll} z_1: & \rho_1, \tilde\rho_1, \\ z_2: & \rho_2, \tilde\rho_2, \\ \infty: & \rho_3, \tilde\rho_3. \end{array}$$

Lemma 4.2 *The sum of indices of (4.48) is* 1*. The Riemann operator expressed in terms of the indeces is*

$$\begin{aligned} &\mathcal{P}\begin{pmatrix} z_1 & z_2 & \infty & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde\rho_1 & \tilde\rho_2 & \tilde\rho_3 & \end{pmatrix} \\ = \ & \partial_z^2 - \left(\frac{\rho_1+\tilde\rho_1-1}{z-z_1} + \frac{\rho_2+\tilde\rho_2-1}{z-z_2}\right)\partial_z \\ & + \frac{\rho_1\tilde\rho_1(z_1-z_2)}{(z-z_1)^2(z-z_2)} + \frac{\rho_2\tilde\rho_2(z_2-z_1)}{(z-z_2)^2(z-z_1)} + \frac{\rho_3\tilde\rho_3}{(z-z_1)(z-z_2)} \end{aligned} \qquad (4.55)$$

Proof. Its indicial equations are

$$\begin{array}{ll} z_1: & \lambda(\lambda-1) + g_1\lambda + h_1 = 0, \\ z_2: & \lambda(\lambda-1) + g_2\lambda + h_2 = 0, \\ \infty: & \lambda(\lambda+1) - (g_1+g_2)\lambda + h = 0. \end{array}$$

By the Vieta equations

$$\begin{aligned} -1 + g_1 &= -\rho_1 - \tilde\rho_1, \\ -1 + g_2 &= -\rho_2 - \tilde\rho_2, \\ 1 - g_1 - g_2 &= -\rho_\infty - \tilde\rho_\infty. \end{aligned}$$

We sum up these equations. □

It is easy to generalize (4.55) to an arbitrary triplet of points:

Theorem 4.3 *1. Suppose that we are given a 2nd order differential equation on the Riemann sphere having 3 singular points z_1, z_2, z_3, all of them regular singular points with the following indices*

$$\begin{array}{ll} z_1 : & \rho_1, \tilde\rho_1, \\ z_2 : & \rho_2, \tilde\rho_2, \\ z_3 : & \rho_3, \tilde\rho_3. \end{array}$$

Then the following condition is satisfied:

$$\rho_1 + \tilde\rho_1 + \rho_2 + \tilde\rho_2 + \rho_3 + \tilde\rho_3 = 1. \tag{4.56}$$

Such an equation, normalized to have coefficient 1 *at the 2nd derivative, is always equal to*

$$\mathcal{P}\left(\begin{array}{cccc} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde\rho_1 & \tilde\rho_2 & \tilde\rho_3 & \end{array}\right)\phi(z) = 0, \tag{4.57}$$

where

$$\mathcal{P}\left(\begin{array}{cccc} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde\rho_1 & \tilde\rho_2 & \tilde\rho_3 & \end{array}\right) := \partial_z^2 - \left(\tfrac{\rho_1+\tilde\rho_1-1}{z-z_1} + \tfrac{\rho_2+\tilde\rho_2-1}{z-z_2} + \tfrac{\rho_3+\tilde\rho_3-1}{z-z_3}\right)\partial_z$$
$$+\tfrac{\rho_1\tilde\rho_1(z_1-z_2)(z_1-z_3)}{(z-z_1)^2(z-z_2)(z-z_3)} + \tfrac{\rho_2\tilde\rho_2(z_2-z_3)(z_2-z_1)}{(z-z_2)^2(z-z_3)(z-z_1)} + \tfrac{\rho_3\tilde\rho_3(z_3-z_1)(z_3-z_2)}{(z-z_3)^2(z-z_1)(z-z_2)}.$$

2. Let $z \mapsto w(z) = \frac{az+b}{cz+d}$. (Transformations of this form are called homographies or Möbius transformations). We can always assume that $ad-bc = 1$. Then

$$\mathcal{P}\left(\begin{array}{cccc} w(z_1) & w(z_2) & w(z_3) & \\ \rho_1 & \rho_2 & \rho_3 & w, \partial_w \\ \tilde\rho_1 & \tilde\rho_2 & \tilde\rho_3 & \end{array}\right) = (cz{+}d)^4\mathcal{P}\left(\begin{array}{cccc} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde\rho_1 & \tilde\rho_2 & \tilde\rho_3 & \end{array}\right),$$

3.

$$(z-z_1)^{-\lambda}(z-z_2)^{\lambda}\mathcal{P}\left(\begin{array}{cccc} z_1 & z_2 & z_3 & \\ \rho_1 & \rho_2 & \rho_3 & z, \partial_z \\ \tilde\rho_1 & \tilde\rho_2 & \tilde\rho_3 & \end{array}\right)(z-z_1)^{\lambda}(z-z_2)^{-\lambda}$$
$$= \mathcal{P}\left(\begin{array}{cccc} z_1 & z_2 & z_3 & \\ \rho_1-\lambda & \rho_2+\lambda & \rho_3 & z, \partial_z \\ \tilde\rho_1-\lambda & \tilde\rho_2+\lambda & \tilde\rho_3 & \end{array}\right).$$

Clearly, in all above formulas one of z_i can equal ∞, with an obvious meaning of various expressions.

4.2 Hypergeometric equation

By Thm 4.3 (2), we can assume that the points z_1, z_2, z_3 are any triplet of distinct points on the Riemann sphere. We choose them to be $0, 1, \infty$.

By Thm 4.3 (3), we can assume that ρ_1, ρ_2 are arbitrary numbers. We choose them to be both 0. The sum of remaining indices must be 1. Hence, we have 3 parameters left. We set
0, indices: 0, $1-c$;
1, indices: 0, $c-a-b$;
∞, indices: a, b. Thus

$$\begin{aligned}&\mathcal{P}\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z, \partial_z \\ 1-c & c-a-b & b & \end{pmatrix}\\ =\ & \partial_z^2 - \Big(\frac{1-c-1}{z} + \frac{c-a-b-1}{(z-1)}\Big)\partial_z + \frac{ab}{z(z-1)}. \end{aligned} \tag{4.58}$$

Define

$$\begin{aligned}\mathcal{F}(a,b;c;z,\partial_z) &:= z(1-z)\mathcal{P}\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z, \partial_z \\ 1-c & c-a-b & b & \end{pmatrix} && (4.59)\\ &= z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab. && (4.60)\end{aligned}$$

Rewrite the equation

$$\mathcal{F}(a,b;c;z,\partial_z)F(z) = 0$$

in the form

$$\big(z^2\partial_z^2 + (a+b+1)z\partial_z + ab\big)F(z) = (z\partial_z^2 + c\partial_z)F(z). \tag{4.61}$$

Substituting $F = \sum_{n=0}^{\infty} F_n z^n$ into (4.61) we obtain

$$\sum_{n=0}^{\infty}(n+a)(n+b)F_n z^n = \sum_{n=0}^{\infty} n(n+c-1)F_n z^{n-1}. \tag{4.62}$$

This leads to the recurrence relation

$$(n+a)(n+b)F_n = F_{n+1}(n+1)(n+c). \tag{4.63}$$

For $a \in \mathbb{C}$ we define

$$(a)_n := a(a+1)\cdots(a+n-1).$$

The solution analytic at 0 and equal there 1 is the *hypergeometric function*

$$F(a,b;c;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!},$$

$F(a,b;c;z)$ is defined for $c \neq 0, -1, -2, \ldots$. Sometimes, it is more convenient to consider

$$\mathbf{F}(a,b;c;z) := \frac{F(a,b,c,z)}{\Gamma(c)} = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{\Gamma(c+j)} \frac{z^j}{j!}$$

defined for all a, b, c.

4.3 Solution $\sim z^{1-c}$ at 0

We have the identity

$$\begin{aligned} & z^{c-1} \mathcal{F}(a,b;c) z^{1-c} \\ = \; & \mathcal{F}(b+1-c, a+1-c; 2-c) \end{aligned}$$

Therefore, the solution of (4.60) behaving as z^{1-c} at zero is

$$z^{1-c} F(b+1-c, a+1-c; 2-c; z) \tag{4.64}$$

4.4 Solutions having definite behaviors at 1

$w = 1 - z$ is a substitution that exchanges 0 and 1:

$$\begin{aligned} & \mathcal{F}(a,b;c;z,\partial_z) := \\ = \; & \mathcal{F}(a,b;a+b+1-c;w,\partial_w). \end{aligned}$$

Therefore, the solution analytic at 1 and having there the value 1 is

$$F(a,b;a+b+1-c;1-z).$$

There is also a solution behaving as $(1-z)^{c-a-b}$ at 1:

$$(1-z)^{c-a-b} F(-b+c, -a+c; 1+c-a-b; 1-z).$$

4.5 Solutions having definite behaviors at ∞

∞ is a regular-singular point with indices a, b. $w = z^{-1}$ is the substitution that exchanges 0 and ∞

$$(-z)^{1+a} \mathcal{F}(a,b;c;z,\partial_z)(-z)^{-a} \tag{4.65}$$
$$= \mathcal{F}(a, a-c+1; a-b+1; w, \partial_w). \tag{4.66}$$

Hence, the solution that behaves at ∞ as z^{-a} is

$$z^{-a} F(a, a-c+1; a-b+1; z^{-1}).$$

The second solution is obtained by exchanging a and b:

$$z^{-b} F(b-c+1, b; b-a+1; z^{-1}).$$

4.6 Identities

The following substitution does not move 0, and exchanges 1 and ∞: $z \mapsto w = \frac{z}{z-1}$. It leads to

$$\begin{aligned} & -(1-z)^{1+a}\mathcal{F}(a,b;c;z,\partial_z)(1-z)^{-a} \\ = \quad & \mathcal{F}\left(a, c-b; c; w, \partial_w\right) \end{aligned} \tag{4.67}$$

An analogous identity is obtained if we exchange a and b. This yields

$$\begin{aligned} & F(a,b;c;z) \\ & = (1-z)^{c-a-b}F\left(c-a, c-b; c; z\right) \\ & = (1-z)^{-a}F\left(a, c-b; c; \tfrac{z}{z-1}\right) \\ & = (1-z)^{-b}F\left(c-a, b; c; \tfrac{z}{z-1}\right). \end{aligned}$$

4.7 Integral representations

Theorem 4.4 *Let the curve* $[0,1] \ni \tau \overset{\gamma}{\mapsto} t(\tau)$ *satisfy*

$$t^{a-c+1}(1-t)^{c-b}(t-z)^{-a-1}\Big|_{t(0)}^{t(1)} = 0.$$

Then

$$\int_\gamma t^{a-c}(1-t)^{c-b-1}(t-z)^{-a}\mathrm{d}t \tag{4.68}$$

solves the hypergeometric equation.

Proof. We check that

$$\begin{aligned} & \left(z(1-z)\partial_z^2 + (c-(a+b+1)z)\partial_z - ab\right) t^{a-c}(1-t)^{c-b-1}(t-z)^{-a} \\ & = -a\partial_t t^{a-c+1}(1-t)^{c-b}(t-z)^{-a-1}. \end{aligned}$$

□

This implies the following representation of the hypergeometric function:

$$\int_1^\infty t^{a-c}(t-1)^{c-b-1}(t-z)^{-a} \tag{4.69}$$

$$= \quad \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}F(a,b;c;z), \qquad \mathrm{Re}(c-b) > 0,\ \mathrm{Re} b > 0. \tag{4.70}$$

Indeed, notice that (4.69) satisfies the assumptions of Thm 4.4, it is analytic around zero and at zero equals

$$\int_1^\infty t^{-c}(t-1)^{c-b-1}\mathrm{d}t = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}.$$

Setting $z = 1$ in (4.69) we obtain

$$\int_1^\infty t^{a-c}(t-1)^{c-a-b-1}\mathrm{d}t = \frac{\Gamma(c-a-b)\Gamma(b)}{\Gamma(c-a)}. \tag{4.71}$$

Therefore,

$$F(a,b;c;1) = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}c > \mathrm{Re}(a+b). \tag{4.72}$$

5 Confluent equation

5.1 ${}_1F_1$ equation as a limit of the hypergeometric equation

Let $a, c \in \mathbb{C}$. The *confluent* or the ${}_1F_1$ *equation* is given by the operator

$$\mathcal{F}(a;c;z,\partial_z) := z\partial_z^2 + (c-z)\partial_z - a. \tag{5.73}$$

The confluent equation is a limiting case of the hypergeometric equation:

$$\begin{aligned}
&\lim_{b\to\infty} \frac{1}{b}\mathcal{F}(a,b;c;z/b,\partial_{z/b}) \\
=&\lim_{b\to\infty} \frac{1}{b}\Big(\frac{z}{b}\Big(1-\frac{z}{b}\Big)b^2\partial_z^2 + \Big(c-(a+b+1)\,\frac{z}{b}\Big)\,b\partial_z - ab\Big) \\
=&\mathcal{F}(a;c;z,\partial_z).
\end{aligned}$$

Thus we move the singularity from 1 to b and let it coalesce with the singularity at ∞. Not surprisingly, the singularity at ∞ becomes irregular.

5.2 Confluent function

0 stays a regular-singular point with indices 0, $1-c$. Thus the general Frobenius theory guarantees that if $c \neq 0, -1, -2, \dots$, the equation possesses a solution analytic around 0 and equal 1 at 0. This solution is called *Kummer's confluent function* and is given by the following series, convergent for all $z \in \mathbb{C}$:

$$F(a;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{n!(c)_n} z^n.$$

Note that under the above restriction on c this is the unique solution of the confluent equation analytic at zero and equal there 1.

Sometimes it is better to use the function

$$\mathbf{F}(a;c;z) = \frac{F(a;c;z)}{\Gamma(c)} = \sum_{n=0}^{\infty} \frac{(a)_n}{n!\Gamma(c+n)} z^n,$$

which is always defined, where used the identity

$$\Gamma(c+n) = \Gamma(c)(c)_n.$$

5.3 Solution $\sim z^{1-c}$ at 0

By the Frobenius theory the confluent equation possesses another solution that behave at zero as z^{1-c} times an analytic function equal 1 at 0 (at least, when $c \neq 1, 2, \dots$). Fortunately, this other solution can also be expressed in terms of the confluent function. Indeed, we have the identity

$$z^{c-1}\mathcal{F}(a;c)z^{1-c} \quad = \quad \mathcal{F}(1+a-c;2-c).$$

Hence

$$z^{1-c}F(a-c+1;2-c;z)=\sum_{n=0}^{\infty}\frac{(a-c+1)_n}{n!(2-c)_n}z^{1-c+n}.$$

is a solution of the confluent equation.

5.4 First Kummer's identity

Using $\mathrm{e}^{-z}\partial_z\mathrm{e}^z=\partial_z+1$ we obtain the identity

$$\begin{aligned}&\mathrm{e}^{-z}(z\partial_z^2+(c-z)\partial_z-a)\mathrm{e}^z\\ =\;&z\partial_z^2+(c+z)\partial_z+c-a.\end{aligned}\tag{5.74}$$

Substitute $z=-w$ and multiply by -1, obtaining

$$w\partial_w^2+(c-w)\partial_w-c+a.$$

Thus

$$\mathrm{e}^{-z}\mathcal{F}(a;c;z,\partial_z)\mathrm{e}^z=\mathcal{F}(c-a;c;w,\partial_w).\tag{5.75}$$

Hence $\mathrm{e}^zF(c-a;c;-z)$ is a solution of the confluent equation analytic around 0 and equal 1 at 0. But we know that, at least for $c\neq 0,-1,-2,\dots$, such a solution is $F(a;c;z)$. Therefore we obtain the identity

$$F(a;c;z)=\mathrm{e}^zF(c-a;c;-z),\tag{5.76}$$

5.5 Integral representations

If $[0,1]\ni\tau\mapsto \mathrm{s}(\tau)\in\Omega$ is a curve and f is a function on Ω, we introduce the notation

$$f\Big|_{\gamma(0)}^{\gamma(1)}:=f(\gamma(1))-f(\gamma(0)).$$

Theorem 5.1 *Let the curve γ satisfy*

$$\mathrm{e}^{zs}s^a(1-s)^{c-a}\Big|_{\gamma(0)}^{\gamma(1)}=0.\tag{5.77}$$

Then

$$\int_\gamma \mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1}\mathrm{d}s\tag{5.78}$$

is a solution of the confluent equation

Proof.

$$\begin{aligned}&\big(z\partial_z^2+(c-z)\partial_z-a\big)\mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1}\\ =\;&z\mathrm{e}^{zs}s^{a+1}(1-s)^{c-a-1}+(c-z)\mathrm{e}^{zs}s^a(1-s)^{c-a-1}-a\mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1}\\ =\;&-z\mathrm{e}^{zs}s^a(1-s)^{c-a}-a\mathrm{e}^{zs}s^{a-1}(1-s)^{c-a}+(c-a)\mathrm{e}^{zs}s^a(1-s)^{c-a-1}\\ =\;&-\partial_s\mathrm{e}^{zs}s^a(1-s)^{c-a}.\end{aligned}$$

We apply $\int_\gamma \mathrm{d}s$ to both sides of the above equation. We obtain

$$\begin{aligned}
&\big(z\partial_z^2 + (c-z)\partial_z - a\big)\int_\gamma \mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1}\mathrm{d}s \\
=&-\int_\gamma \partial_s \mathrm{e}^{zs}s^{a}(1-s)^{c-a}\mathrm{d}s \\
=&\mathrm{e}^{zs}s^{a}(1-s)^{c-a}\Big|_{\gamma(0)}^{\gamma(1)},
\end{aligned}$$

which by assumption vanishes. □

We would like to find a curve that satisfies the assumptions of the above theorem. One way to find such a curve is to "pin" it at points where the function contained in (5.77) vanishes.

Now for $\mathrm{Re}a > 0$ this function vanishes at $t = 0$, and for $\mathrm{Re}(c-a) > 0$ it vanishes at $t = 1$. Therefore, for $\mathrm{Re}a > 0, \quad \mathrm{Re}(c-a) > 0$,

$$\int_0^1 \mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1}\mathrm{d}s = \Gamma(a)\Gamma(c-a)\mathbf{F}(a;c;z). \tag{5.79}$$

Indeed, as we explained above, the assumptions of Theorem 5.1 are satisfied, hence (5.79) is a solution of the confluent equation. It is clearly analytic around 0. We check that at zero it equals

$$\int_0^1 s^{a-1}(1-s)^{c-a-1}\mathrm{d}s = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}.$$

Hence (5.79) is true, at least for $c \neq 0, -1, -2, \dots$. But by continuity of both sides, the identity is true also in these exceptional points. □

5.6 Laguerre polynomials

For $n = -a \in \{0, 1, 2, \dots\}$, we have $(a)_{n+1} = (a)_{n+2} = \cdots 0$. Therefore, $F(-n;c;z)$ is an nth degree polynomial. These are the so-called *Laguerre polynomials*, defined according to the standard convention as

$$\begin{aligned}
L_n^\alpha(z) := \frac{(1+\alpha)_n}{n!}F(-n;1+\alpha;z) &= \frac{(1+\alpha)_n}{n!}\sum_{j=0}^{n}\frac{(-n)_j z^j}{(1+\alpha)_j j!} \\
&= \sum_{j=0}^{n}\frac{(\alpha+j+1)_{n-j}(-z)^j}{j!(n-j)!}.
\end{aligned}$$

They can be represented as an integral with γ encircling 0:

$$L_n^\alpha(z) = \frac{(-1)^n}{2\pi\mathrm{i}}\int_{[0^+]} \mathrm{e}^{tz}t^{-n-1}(1-t)^{\alpha+n}\mathrm{d}t. \tag{5.80}$$

Indeed, clearly the contour satisfies the assumptions of Theorem 5.1 (because it is closed). Hence it is a solution of the confluent equation. It is clear that it is analytic at zero. We check that at zero it yields $\frac{(1+\alpha)_n}{n!}$.

5.7 The ${}_2F_0$ equation

Parallel to the ${}_1F_1$ equation we will consider the ${}_2F_0$ *equation*, given by the operator

$$\mathcal{F}(a,b;-;z,\partial_z) := z^2\partial_z^2 + (-1+(1+a+b)z)\partial_z + ab, \tag{5.81}$$

where $a, b \in \mathbb{C}$. This equation is another limiting case of the hypergeometric equation:

$$\begin{aligned}\lim_{c\to\infty} \mathcal{F}(a,b;c;cz,\partial_{(cz)}) &= cz(1-cz)\frac{1}{c^2}\partial_z^2 + (c-(a+b+1)cz)\frac{1}{c}\partial_z - ab \\ &= -\mathcal{F}(a,b;-;z,\partial_z). \end{aligned} \tag{5.82}$$

Thus the singularity at 1 is moved to $\frac{1}{c} \to 0$, so that it coalesces with 0 and forms an irregular singularity.

5.8 Point ∞ for the confluent equation

We have

$$\begin{aligned} & z^{a+1}(z\partial_z^2 + (c-z)\partial_z - a)z^{-a} \\ = \;& z^2\partial_z^2 + z(-2a+c-z)\partial_z + a(1+a-c). \qquad (5.83) \\ = \;& z^2\partial_z^2 + z(1-a-b-z)\partial_z + ab, \qquad (5.84) \end{aligned}$$

where we set $b := 1+a-c$. Substituting $w = -z^{-1}$ (with the inverse $z = -w^{-1}$), using $\partial_z = w^2\partial_w$, we obtain that (5.84) is

$$w^2\partial_w^2 + (-1+(1+a+b)w)\partial_w + ab.$$

We thus obtained the ${}_2F_0$ equation.

If

$$(w^2\partial_w^2 + (-1+(1+a+b)w)\partial_w + ab)g(w) = 0,$$

then

$$(z\partial_z^2 + (c-z)\partial_z - a)z^{-a}g(-z^{-1}) = 0. \tag{5.85}$$

Conversely, if

$$(z\partial_z^2 + (c-z)\partial_z - a)f(z) = 0,$$

then

$$(w^2\partial_w^2 + (-1+(1+a+b)w)\partial_w + ab)w^{-a}f(-w^{-1}) = 0.$$

5.9 Asymptotic series

Let function f be defined on $K(z_0,r) \cap \{\alpha_1 < \arg(z-z_0) < \alpha_2\}$. We write

$$f(z) \sim \sum_{j=0}^{\infty} a_j(z-z_0)^j,$$

if for any n there exists C_n such that

$$\Big|f(z) - \sum_{j=0}^{n} a_j (z-z_0)^j\Big| \leq C_n |z-z_0|^{n+1}.$$

Clearly, if $f(z) = \sum\limits_{j=0}^{\infty} a_j (z-z_0)^j$ for $z \in K(z_0, r)$, then $f(z) \sim \sum\limits_{j=0}^{\infty} a_j (z-z_0)^j$.

Example. For $-\frac{\pi}{2} + \epsilon < \arg z < \frac{\pi}{2} - \epsilon$

$$\mathrm{e}^{-\frac{1}{z}} \sim \sum_{j=0}^{\infty} 0 z^j.$$

Example. For $-\frac{\pi}{4} + \epsilon < \arg(z) < \frac{\pi}{4} - \epsilon$ and $-\frac{\pi}{4} + \epsilon < \arg(-z) < \frac{\pi}{4} - \epsilon$

$$\mathrm{e}^{-\frac{1}{z^2}} \sim \sum_{j=0}^{\infty} 0 z^j.$$

Indeed,

$$\partial_z^n \mathrm{e}^{-\frac{1}{z^2}} = \mathrm{e}^{-\frac{1}{z^2}} P_n\left(\frac{1}{z}\right), \tag{5.86}$$

where P_n is a polynomial. Moreover, $\left|\mathrm{e}^{-\frac{1}{z^2}}\right| = \mathrm{e}^{-\mathrm{Re}(\frac{1}{z^2})}$. We have $\mathrm{Re}(\frac{1}{z^2}) > 0$ for $-\frac{\pi}{4} + \epsilon < \arg(z) < \frac{\pi}{4}$ and $-\frac{\pi}{4} + \epsilon < \arg(-z) < \frac{\pi}{4}$, therefore in these sectors we have a rapid convergence to zero of (5.86). (In the other two sectors it explodes). Therefore $\mathrm{e}^{-\frac{1}{z^2}}$ has zero directional derivatives in theses sectors. Hence its asymptotic series is zero by the Taylor formula.

Example: Error Function.

$$\mathrm{Erf}(z) := \int_0^z \mathrm{e}^{-t^2} \mathrm{d}t.$$

Clearly, $\lim\limits_{\mathrm{Re} z \to \infty} \mathrm{Erf}(z) = \frac{1}{2}\sqrt{\pi}$. We integrate by parts:

$$\frac{1}{2}\sqrt{\pi} - \mathrm{Erf}(z) = \int_z^{\infty} \mathrm{e}^{-t^2} \mathrm{d}t = \frac{\mathrm{e}^{-z^2}}{2z} + \int_z^{\infty} \mathrm{e}^{-t^2} \frac{\mathrm{d}t}{2t^2} = \frac{\mathrm{e}^{-z^2}}{2z} + \mathrm{e}^{-z^2} O\left(\frac{1}{z^2}\right).$$

Repeating integration by parts we obtain for $-\frac{\pi}{2} + \epsilon < \arg z < \frac{\pi}{2} - \epsilon$

$$\mathrm{e}^{z^2}\Big(\frac{1}{2}\sqrt{\pi} - \mathrm{Erf}(z)\Big) \sim \frac{1}{2z} \sum_{k=1}^{\infty} (-1)^k \frac{1 \cdot 3 \cdots (2k-1)}{(2z^2)^k}\Big).$$

5.10 $_2F_0$ function

We try to solve (5.85) with a power series

$$g(w) = \sum_{n=0}^{\infty} g_n w^n.$$

We obtain

$$\sum_{n=0}^{\infty}\big(n(n-1)g_nw^n - ng_nw^{n-1} + (1+a+b)ng_nw^n + abg_nw^n\big) = 0$$

Hence

$$(n-1+a)(n-1+b)g_{n-1} = ng_n.$$

This gives the coefficients

$$g_n = \frac{(a)_n(b)_n}{n!}g_0$$

and leads to a divergent series.

Theorem 5.2 *Let a contour γ satisfy*

$$\mathrm{e}^{-t}t^a(1-wt)^{1-b}\Big|_{\gamma(0)}^{\gamma(1)} = 0 \tag{5.87}$$

Then

$$\int_\gamma \mathrm{e}^{-t}t^{a-1}(1-wt)^{-b}\mathrm{d}t$$

is a solution of (5.85).

Proof. By Thm 5.1

$$\int_\gamma \mathrm{e}^{zs}s^{a-1}(1-s)^{c-a-1}\mathrm{d}s$$

is a solution of the confluent equation. Therefore,

$$w^{-a}\int_\gamma \mathrm{e}^{-sw^{-1}}s^{a-1}(1-s)^{c-a-1}\mathrm{d}s,$$

for $b = 1+a-c$ is a solution of (5.85). Next we substitute $t = \frac{s}{w}$. □

For $w \in \mathbb{C}\backslash[0,\infty[$, $\mathrm{Re}a > 0$ we define

$$F(a,b;-;w) := \frac{1}{\Gamma(a)}\int_0^\infty \mathrm{e}^{-t}t^{a-1}(1-wt)^{-b}\mathrm{d}t. \tag{5.88}$$

For other values of a we extend (5.88) by analytic continuation. Note that the integrand does not have a singularity for $t \in]0,\infty[$ (because we assumed that $w \notin [0,\infty[$. We obtain an analytic function on $\mathbb{C}\backslash[0,\infty[$. It is easy to see that 0 is a branch point of this function–thus the natural domain of this function is the same as for the logarithm.

Proposition 5.3 *We have the following asymptotic expansion:*

$$F(a,b;-;w) \sim \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!}w^n.$$

More precisely, for any n, for $|\arg w| \geq \epsilon > 0$, $|w| < 1$

$$\left| w^{-n-1}\Big(F(a,b;-;w) - \sum_{j=0}^{n} \frac{(a)_j(b)_j}{j!} w^j \Big) \right| = 0.$$

Proof. It is easy to see that, at least for $\mathrm{Re}(a) > 0$ the function $F(a,b;-;w)$ is continuous at $w = 0$ within the sector $|\arg w| \geq \epsilon > 0$.

To prove this, we use Taylor's formula with a remainder

$$f(z) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)z^j}{j!} + z^n \int_0^1 \frac{f^{(n)}(sz)n(1-s)^{n-1}}{n!} \mathrm{d}s,$$

which implies

$$(1-z)^{-b} = \sum_{j=0}^{n-1} \frac{(b)_j z^j}{j!} + \frac{(b)_n z^n}{n!} \int_0^1 n(1-s)^{n-1}(1-zs)^{-b-n} \mathrm{d}s.$$

Hence

$$\begin{aligned}
& F(a,b;-;w) \\
= & \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-t} t^{a-1} (1-wt)^{-b} \mathrm{d}t \\
= & \sum_{j=0}^{n-1} \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-t} t^{a-1} \frac{(b)_j w^j t^j}{j!} \mathrm{d}t \\
& + \frac{1}{\Gamma(a)} \int_0^\infty \mathrm{e}^{-t} t^{a-1} \frac{(b)_n w^n t^n}{n!} \int_0^1 (1-wts)^{-b-n} n(1-s)^{n-1} \mathrm{d}s \\
= & \sum_{j=0}^{n-1} \frac{(b)_j \Gamma(a+j) w^j}{\Gamma(a) j!} \\
& + \frac{w^n (b)_n}{\Gamma(a) n!} \int_0^1 n(1-s)^{n-1} \mathrm{d}s \int_0^\infty \mathrm{e}^{-t} t^{a-1+n} (1-wts)^{-b-n} \mathrm{d}t \\
= & \sum_{j=0}^{n-1} \frac{(b)_j (a)_j w^j}{j!} \\
& + \frac{w^n (b)_n (a)_n}{n!} \int_0^1 n(1-s)^{n-1} \mathrm{d}s F(a+n, b+n; -; ws).
\end{aligned}$$

Now, for large n, $\mathrm{Re}(a+n) > 0$. We know that $F(a+n, b+n; -; ws)$ is then bounded. Hence the remainder is of the order $O(w^n)$. □

5.11 Solutions of the confluent equation with definite behavior at ∞

Consider the analytic function on the upper halfplane given by

$$s \mapsto \mathrm{e}^{zs} s^{a-1} (1-s)^{c-a-1},$$

where we use the principal branch of power functions. Assume that $\mathrm{Re}a > 0$, $\mathrm{Re}(c-a) > 0$. Remember that

$$G(z) = \int_0^1 \mathrm{e}^{zs} s^{a-1}(1-s)^{c-a-1}\mathrm{d}s = \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F(a;c;z)$$

is one of solutions of the confluent equation. If $\mathrm{Im}z > 0$, then we can write the following solutions

$$\begin{aligned} G_0(z) &= \int_0^{-\mathrm{e}^{-\mathrm{i}\phi}\infty} \mathrm{e}^{zs} s^{a-1}(1-s)^{c-a-1}\mathrm{d}s, \\ G_1(z) &= \int_1^{-\mathrm{e}^{-\mathrm{i}\phi}\infty} \mathrm{e}^{zs} s^{a-1}(1-s)^{c-a-1}\mathrm{d}s, \end{aligned}$$

where $|\phi - \arg z| < \frac{\pi}{2}$ guarantees that e^{zs} along the halfline where we integrate converges fast to zero (thus the appropriate condition is fulfilled). Notice that

$$G(z) + G_1(z) - G_0(z) = 0. \tag{5.89}$$

Substituting $s = -z^{-1}t$, where $t \in [0, \infty[$, for $\mathrm{Re}a > 0$ we obtain

$$\begin{aligned} G_0(z) &= \int_0^\infty \mathrm{e}^{-t}(-tz^{-1})^{a-1}(1+z^{-1}t)^{c-a-1}(-z^{-1})\mathrm{d}t \\ &= (-z)^{-a}\Gamma(a)F(a, a+1-c; -, -z^{-1}). \end{aligned}$$

Substituting $s = 1 - z^{-1}t$, where $t \in [0, \infty[$, for $\mathrm{Re}(c-a) > 0$ we can write

$$\begin{aligned} G_1(z) &= -\mathrm{e}^z \int_0^\infty \mathrm{e}^{-t}(1-z^{-1}t)^{a-1} z^{-c+a} t^{c-a-1}\mathrm{d}t \\ &= -\mathrm{e}^z z^{-c+a}\Gamma(c-a)F(c-a, 1-a; -, z^{-1}). \end{aligned}$$

By (5.89), we obtain

$$\frac{F(a;c;z)}{\Gamma(c)} = (-z)^{-a}\frac{F(a, a+1-c; -; -z^{-1})}{\Gamma(c-a)} + z^{-c+a}\frac{\mathrm{e}^z F(c-a, 1-a; -; z^{-1})}{\Gamma(a)}$$

5.12 Hydrogen atom

We transform the confluent operator

$$\begin{aligned} & \mathrm{e}^{-z/2}\left(z\partial_z^2 + (c-z)\partial_z - a\right)\mathrm{e}^{z/2} && (5.90) \\ =\ & z\partial_z^2 + c\partial_z + \frac{c}{2} - a - \frac{z}{4}; && (5.91) \end{aligned}$$

Next,

$$\begin{aligned} & z^{-(1-c)/2}\left(z\partial_z^2 + c\partial_z + \frac{c}{2} - a - \frac{z}{4}\right) z^{(1-c)/2} && (5.92) \\ =\ & z\partial_z^2 + \partial_z - \frac{z}{4} + \frac{c}{2} - a - \frac{(1-c)^2}{4z}. && (5.93) \end{aligned}$$

We divide (5.91) by z and substitute $z = 2w$. We obtain

$$\partial_w^2 + \frac{1}{w}\partial_w - 1 + (c - 2a)\frac{1}{w} - \left(\frac{1-c}{2}\right)^2 \frac{1}{w^2}, \tag{5.94}$$

We will see that this is the equation for the radial wave function for the Coulomb potential in dimension 2. (It is easy to transform into an analogous equation in higher dimensions).

The stationary Schrödinger equation with the Coulomb potential is

$$\left(-\Delta - \frac{\beta}{r} - E\right)\Phi = 0. \tag{5.95}$$

In the polar coordinates it reads

$$\left(-\partial_r^2 - \frac{1}{r}\partial_r - \frac{1}{r^2}\partial_\phi^2 - \frac{\beta}{r} - E\right)\Phi = 0. \tag{5.96}$$

We make an ansatz

$$\Phi(r,\phi) = f(r)g(\phi). \tag{5.97}$$

We obtain

$$\frac{r^2\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{\beta}{r} + E\right)f(r)}{f(r)} = \frac{\partial_\phi^2 g(\phi)}{g(\phi)} = A, \tag{5.98}$$

where A does not depend on r or ϕ.

Now

$$\frac{\partial_\phi^2 g(\phi)}{g(\phi)} = A \tag{5.99}$$

is solved by

$$g(\phi) = B_+ \mathrm{e}^{\mathrm{i}\sqrt{A}\phi} + B_- \mathrm{e}^{-\mathrm{i}\sqrt{A}\phi}. \tag{5.100}$$

To guarantee the continuity of g we need to assume that $m := \sqrt{A} \in \mathbb{Z}$ We are left with the radial equation

$$\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{\beta}{r} - \frac{m^2}{r^2} + E\right)f(r) = 0. \tag{5.101}$$

Let us assume that $E = -k^2 < 0$. We change the variable $r = \frac{w}{k}$. (5.101) transforms into

$$\left(\partial_w^2 + \frac{1}{w}\partial_w + \frac{\beta}{kr} - \frac{m^2}{r^2} - 1\right)f(r) = 0. \tag{5.102}$$

Compared with (5.94) we obtain

$$c - 2a = \frac{\beta}{k}, \quad \frac{c-1}{2} = m,$$

or

$$c = 1 + 2m, \quad a = \frac{1}{2} + m - \frac{\beta}{2k}. \tag{5.103}$$

Therefore, if f satisfies the confluent equation with these parameters, then $\mathrm{e}^{-w} w^{(-1+c)/2} f(2w) = \mathrm{e}^{-w} w^m f(2w)$ satisfies (5.94). If we want the function to be square integrable at zero, we need $m > -1$, hence $m = 0, 1, 2, \dots$. If we want the function to be square integrable at infinity, we need to assume that f is a Laguerre polynomial. This means $-a = n$ is a nonnegative integer. This yields the spectrum of the (2-dimensional) "Hydrogen":

$$E = -k^2 = -\frac{\beta^2}{4(\frac{1}{2} + m + n)^2}, \quad m = 0, 1, 2, \dots, \quad , n = 0, 1, 2, \dots. \tag{5.104}$$

6 Poisson summation formula and Jacobi's theta function

6.1 Alternative convention for Fourier transformation

In the literature one can find (at least) two conventions for Fourier transformations. In the following table we compare two conventions. The first is more common. In this section we adopt the second, which has many advantages.

	Standard convention	Convention with 2π in exponent
direct transform	$\hat{f}(\xi) := \int f(x)\mathrm{e}^{-\mathrm{i}x\xi}\mathrm{d}x,$	$\hat{f}(\xi) := \int f(x)\mathrm{e}^{-\mathrm{i}2\pi x\xi}\mathrm{d}x;$
inverse transform	$f(x) := \frac{1}{2\pi}\int \hat{f}(x)\mathrm{e}^{\mathrm{i}x\xi}\mathrm{d}\xi,$	$f(x) := \int \hat{f}(\xi)\mathrm{e}^{\mathrm{i}2\pi x\xi}\mathrm{d}\xi;$
periodic functions	period 2π	period 1;
	$\hat{f}_k = \int_{-\pi}^{\pi} f(x)\mathrm{e}^{-\mathrm{i}xk}\mathrm{d}x,$	$\hat{f}_k = \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x)\mathrm{e}^{-\mathrm{i}2\pi xk}\mathrm{d}x;$
	$f(x) = \frac{1}{2\pi}\sum_k \hat{f}_k \mathrm{e}^{\mathrm{i}kx}$	$f(x) = \sum_k \hat{f}_k \mathrm{e}^{\mathrm{i}2\pi kx};$
Gaussian	$f(x) = \mathrm{e}^{-x^2}$	$f(x) = \mathrm{e}^{-\pi x^2};$
	$\hat{f}(\xi) = \mathrm{e}^{-\frac{1}{4}\xi^2}$	$\hat{f}(\xi) = \mathrm{e}^{-\pi\xi^2}.$

6.2 Poisson summation formula

In the following theorem we adopt the convention with 2π in the exponent.

Theorem 6.1 *Let $f, \hat{f} \in L^1$, so that they are continuous functions. Assume also that $\sum_j |\hat{f}(j)| < \infty$ and $\sum_j |f(j)| < \infty$. Then*

$$\sum_{k=-\infty}^{\infty} f(k) = \sum_{j=-\infty}^{\infty} \hat{f}(j). \tag{6.105}$$

Proof. Let $j \in \mathbb{Z}$.

$$\int_{-n-\frac{1}{2}}^{n+\frac{1}{2}} f(x)\mathrm{e}^{-\mathrm{i}2\pi xj}\mathrm{d}x = \sum_{k=-n}^{n} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(x+k)\mathrm{e}^{-2\pi \mathrm{i} jx}\mathrm{d}x. \tag{6.106}$$

Define

$$g(x) := \lim_{n\to\infty} \sum_{k=-n}^{n} f(x+k). \tag{6.107}$$

g is a periodic function with period 1. Using that $f \in L^1(\mathbb{R})$ and the dominated convergence theorem we show that $g \in L^1[-\frac{1}{2}, \frac{1}{2}]$. Letting $n \to \infty$ in (6.106) we obtain

$$\hat{f}(j) = \int_{-\frac{1}{2}}^{\frac{1}{2}} g(x)\mathrm{e}^{-2\pi \mathrm{i} jx}\mathrm{d}x. \tag{6.108}$$

Since $\sum\limits_{j=-\infty}^{\infty} |\hat{f}(j)| < \infty$, we can apply the inversion of the Fourier transformation for periodic functions, which yields

$$g(x) = \sum_{j=-\infty}^{\infty} \hat{f}(j)\mathrm{e}^{2\pi \mathrm{i} jx}. \tag{6.109}$$

Setting $x = 0$ and using $\sum\limits_{k=-\infty}^{\infty} |f(k)| < \infty$, we see that $g(0) = \sum\limits_{k=-\infty}^{\infty} f(k)$. (A priori, the limit in (6.107) was valid only for almost all x.) Therefore,

$$\sum_{k=-\infty}^{\infty} f(k) = \sum_{j=-\infty}^{\infty} \hat{f}(j). \tag{6.110}$$

□

Note that under slightly weaker assumptions we can prove a slightly weaker asymmetric statement, by essentially the same proof:

Theorem 6.2 *If $f \in L^1$ and $\sum\limits_{j=-\infty}^{\infty} |\hat{f}(j)| < \infty$, then for almost all x,*

$$\sum_{k=-\infty}^{\infty} f(x+k) = \sum_{j=-\infty}^{\infty} \hat{f}(j)\mathrm{e}^{2\pi \mathrm{i} jx}. \tag{6.111}$$

Let us rewrite the Poisson summation formula with the standard Fourier transformation:

$$\sum_{n=-\infty}^{\infty} f(x - Ln) = \sum_{j=-\infty}^{\infty} \hat{f}\Big(\frac{2\pi}{L}j\Big). \tag{6.112}$$

One can interpret it as follows: if we want to approximate a function which decays sufficiently fast at ∞ by a function periodic of period L, then we have two ways to do it: either we do it in the position space, or we first take the Fourier transform, restrict it to the *reciprocal lattice* which has the period $\frac{2\pi}{L}$, and then take the inverse Fourier sum. Both methods yield the same answer.

6.3 Heat equation on the line

Consider the heat equation on the line with the initial condition given by the deltafunction at the origin:

$$\partial_t g(x,t) = \nu \partial_x^2 g(x,t), \tag{6.113}$$

$$\lim_{t\searrow 0} g(x,t) = \delta(x), \tag{6.114}$$

To solve it we use the Fourier transformation in x:

$$\partial_t \hat{g}(x,t) := \frac{1}{2\pi}\int \mathrm{e}^{\mathrm{i}xk}\hat{g}(x,t),$$

$$\partial_t \hat{g}(k,t) = -\nu k^2 \hat{g}(k,t),$$

$$\lim_{t\searrow 0} \hat{g}(k,t) = 1.$$

This is solved by $\hat{g}(k,t) = \mathrm{e}^{-\nu t k^2}$. Hence

$$g(x,t) = \frac{1}{2\pi}\int \mathrm{e}^{-\nu t k^2 + \mathrm{i}kx}\mathrm{d}x$$

$$= \frac{1}{2\pi}\int \mathrm{e}^{-\nu t (k + \mathrm{i}\frac{\mathrm{i}x}{2\nu t})^2 - \frac{x^2}{4\nu t}}\mathrm{d}x \;=\; \frac{1}{2\sqrt{\pi\nu t}}\mathrm{e}^{-\frac{x^2}{4\nu t}}. \tag{6.115}$$

Secifying $\nu = \frac{1}{4\pi}$ we obtain

$$g(x,t) = t^{-\frac{1}{2}} \exp\Big(-\frac{\pi x^2}{t}\Big).$$

6.4 Heat equation on the circle

Consider the heat equation for $t > 0$ on the circle of perimeter 1, identified with $\mathbb{R}/\mathbb{Z}$. We choose the initial condition to be the deltafunction at 0. We can write it on the line as the solution with the initial condition given by the Dirac comb:

$$\partial_t f(x,t) = \frac{1}{4\pi}\partial_x^2 f(x,t), \tag{6.116}$$

$$\lim_{t\searrow 0} f(x,t) = \sum_{m=-\infty}^{\infty} \delta(x-m), \tag{6.117}$$

One can solve it in the position representation by periodizing the solution for the line

$$f(x,t) = t^{-\frac{1}{2}} \sum_{m=-\infty}^{\infty} \exp\Big(-\frac{\pi(x-m)^2}{t}\Big). \tag{6.118}$$

By the Poisson summation formula we have

$$f(x,t) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 t + 2\pi \mathrm{i} n x), \tag{6.119}$$

This is precisely the expression that we would obtain if we applied the Fourier approach to the heat equation on the circle.

6.5 Jacobi's theta function

One of the most famous special functions is Jacobi's theta function. There are several conventions for its arguments. Here are two of them:

$$q = \exp(\pi \mathrm{i}\tau), \quad \eta = \exp(2\pi \mathrm{i} z).$$

$$\theta(z, \tau) = \sum_{n=-\infty}^{\infty} q^{n^2} \eta^n, \quad |q| < 1; \tag{6.120}$$

$$= \sum_{n=-\infty}^{\infty} \exp(\pi \mathrm{i} n^2 \tau + 2\pi \mathrm{i} n z), \quad \mathrm{Im}\tau > 0; \tag{6.121}$$

Here are the basic identities in the z, τ notation:

$$\theta(z + 1, \tau) = \theta(z, \tau), \tag{6.122}$$

$$\theta(z + \tau, \tau) = \exp(-\pi \mathrm{i}\tau - 2\mathrm{i}z)\theta(z, \tau), \tag{6.123}$$

$$\theta(z + a + b\tau) = \exp(-\pi \mathrm{i} b^2 z - 2\pi \mathrm{i} b z)\theta(z, \tau), \tag{6.124}$$

$$\theta\Big(\frac{z}{\tau}, \frac{-1}{\tau}\Big) = (-\mathrm{i}\tau)^{\frac{1}{2}} \exp\Big(\frac{\pi}{\tau} \mathrm{i} z^2\Big)\theta(z, \tau). \tag{6.125}$$

Let us repeat the formulas (6.122)–(6.125) in the convention

$$z = x, \quad \tau = \mathrm{i}t :$$

$$\theta(x, \mathrm{i}t) = \sum_{n=-\infty}^{\infty} \exp(-\pi n^2 t + 2\pi \mathrm{i} n x), \quad \mathrm{Re}t > 0; \tag{6.126}$$

$$\theta(x + 1, \mathrm{i}t) = \theta(x, \mathrm{i}t), \tag{6.127}$$

$$\theta(x + \mathrm{i}t, \mathrm{i}t) = \exp(\pi t - 2\mathrm{i}x)\theta(x, \mathrm{i}t), \tag{6.128}$$

$$\theta\Big(\frac{-\mathrm{i}x}{t}, \frac{\mathrm{i}}{t}\Big) = t^{\frac{1}{2}} \exp\Big(\frac{\pi}{t} x^2\Big)\theta(x, \mathrm{i}t). \tag{6.129}$$

By the Poisson summation formula (6.126) can be rewritten as

$$\theta(x, \mathrm{i}t) = \sum_{n=-\infty}^{\infty} t^{-\frac{1}{2}} \exp\Big(-\frac{\pi(x - n)^2}{t}\Big). \tag{6.130}$$

(6.129) follows from the comparison of (6.126) and (6.130).

Note that

$$f(x, t) = \theta(x, \mathrm{i}t)$$

is precisely the solution of the heat equation on the circle we considered above.

7 Dzeta function

7.1 Riemann's dzeta function and prime numbers

For $\mathrm{Re} s > 1$ Riemann's dzeta function is defined as

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}. \tag{7.131}$$

Every positive integer has a unique representation $n = 2^{\alpha} m$, where m is odd. Therefore,

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \sum_{m \text{ is odd}} \frac{1}{m^s} \sum_{\alpha=0}^{\infty} \frac{1}{2^{s\alpha}} = \frac{1}{(1 - \frac{1}{2^s})} \sum_{m \text{ is odd}} \frac{1}{m^s}. \tag{7.132}$$

Hence

$$\zeta(s)(1 - 2^{-s}) = \sum_{m \text{ is odd}} \frac{1}{m^s}. \tag{7.133}$$

Likewise, if $p_1 = 2$, $p_2 = 3$, $p_3 = 5, \dots$ are the prime numbers in the increasing order, then

$$\zeta(s)(1 - p_1^{-s}) \cdots (1 - p_k^{-s}) = \sum_{\substack{m \text{ not divisible} \\ \text{by } p_1, \dots, p_k}} \frac{1}{m^s}, \tag{7.134}$$

Proposition 7.1 *We have*

$$\prod_{j=1}^{\infty} (1 - p_j^{-s}) = \frac{1}{\zeta(s)}. \tag{7.135}$$

Moreover, $\zeta(s) \neq 0$ *for* $\mathrm{Re} s > 1$.

Proof. First note that the lhs of (7.135) is an absolutely convergent product, because

$$p_j^{-s} \leq j^{-s},$$

and $\sum j^{-s} < \infty$. By continuing (7.1) we obtain

$$\lim_{n \to \infty} \zeta(s) \prod_{j=1}^{n} (1 - p_j^{-s}) = 1, \tag{7.136}$$

which implies (7.135).

All the factors of (7.135) are nonzero. Hence (7.135) is nonzero. □

7.2 Holomorphic extension of the dzeta function and Riemann's reflection formula–1st proof

Define

$$\eta(s) := \pi^{-\frac{s}{2}}\Gamma\big(\tfrac{s}{2}\big)\zeta(s), \tag{7.137}$$

$$\xi(s) := \frac{1}{2}s(s-1)\eta(s) = \frac{1}{2}s(s-1)\pi^{-\frac{s}{2}}\Gamma\big(\tfrac{s}{2}\big)\zeta(s). \tag{7.138}$$

The main aim of this subsection is a proof of the following theorem:

Theorem 7.2 *1. The function ζ extends holomorphically to $\mathbb{C}\backslash\{1\}$ and satisfies*

$$\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\sin(\tfrac{\pi}{2}s)\zeta(1-s). \tag{7.139}$$

2. The function η extends to holomorphically to $\mathbb{C}\backslash\{0,1\}$ and satisfies

$$\eta(s) = \eta(1-s). \tag{7.140}$$

3. ξ extends to an entire function on $\mathbb{C}$ satisfying

$$\xi(s) = \xi(1-s), \quad \xi(0) = \xi(1) = \frac{1}{2}. \tag{7.141}$$

We will present two proofs of this theorem.

In this subsection we describe the first one. It proves directly (2), from which (1) and (3) immediately follow. It uses the following auxiliary function:

$$\phi(x) := \sum_{n=1}^{\infty} \mathrm{e}^{-n^2\pi x}. \tag{7.142}$$

Note that

$$1 + 2\phi(x) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{-n^2\pi x} = \theta(0, \mathrm{i}t). \tag{7.143}$$

By the Poisson summation formula

$$1 + 2\phi(x) = \theta(0, \mathrm{i}x) = \frac{1}{\sqrt{x}}\theta\Big(0, \frac{\mathrm{i}}{x}\Big) = \frac{1}{\sqrt{x}}\Big(1 + 2\phi\Big(\frac{1}{x}\Big)\Big) \tag{7.144}$$

Theorem 7.3 *We have the identities*

$$\eta(s) = \int_0^\infty \frac{\mathrm{d}x}{x}\phi(x)x^{\frac{s}{2}} \tag{7.145}$$

$$= \int_1^\infty \frac{\mathrm{d}x}{x}\phi(x)\big(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\big) - \frac{1}{s(1-s)} \tag{7.146}$$

The first is valid for $\mathrm{Re}s > 1$*, the second for all s. In particular, by (7.146), η extends analytically to $\mathbb{C}\backslash\{0,1\}$ and (7.140) holds.*

Proof. For any $\mathrm{Re} s > 0$ we have

$$\frac{1}{n^s}\pi^{-\frac{s}{2}}\Gamma\Big(\frac{s}{2}\Big) = \int_0^\infty \frac{\mathrm{d}x}{x}\mathrm{e}^{-n^2\pi x}x^{\frac{s}{2}}. \tag{7.147}$$

For $\mathrm{Re} s > 1$ we can sum up (7.147) obtaining (7.145). Now,

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}x}{x}\phi(x)x^{\frac{s}{2}} &= \int_1^\infty \frac{\mathrm{d}x}{x}\phi(x)x^{\frac{s}{2}} + \int_1^\infty \frac{\mathrm{d}x}{x}\phi\Big(\frac{1}{x}\Big)x^{-\frac{s}{2}} \\ &= \int_1^\infty \frac{\mathrm{d}x}{x}\phi(x)\Big(x^{\frac{s}{2}} + x^{\frac{1-s}{2}}\Big) + \frac{1}{2}\int_1^\infty \frac{\mathrm{d}x}{x}\Big(x^{\frac{1-s}{2}} - x^{-\frac{s}{2}}\Big)\end{aligned}$$

But

$$\frac{1}{2}\int_1^\infty \frac{\mathrm{d}x}{x}\Big(x^{\frac{1-s}{2}} - x^{-\frac{s}{2}}\Big) = \frac{1}{s-1} - \frac{1}{s} = -\frac{1}{s(1-s)}. \tag{7.148}$$

This implies (7.146) for $\mathrm{Re} s > 1$. But (7.146) is analytic except for $0, 1$. Hence the formula can be extended.

(7.140) follows from (7.146). □

Proof of Theorem 7.2. (2), and hence (3) follow directly from Theorem 7.3.

Applying $\Gamma(s) = \pi^{-\frac{1}{2}}2^{s-1}\Gamma(\frac{s}{2})\Gamma(\frac{s}{2}+\frac{1}{2})$ and $\Gamma(\frac{s}{2}+\frac{1}{2})\Gamma(-\frac{s}{2}+\frac{1}{2}) = \frac{\pi}{\cos(\frac{\pi}{2}s)}$ we see that (7.164) is equivalent to

$$\Gamma\Big(\frac{s}{2}\Big)\zeta(s) = \pi^{s-\frac{1}{2}}\Gamma\Big(-\frac{s}{2}+\frac{1}{2}\Big)\zeta(1-s). \tag{7.149}$$

This proves (1). □

Theorem 7.4 *For* $0 < \mathrm{Re} s < 1$.

$$\begin{aligned}\eta(s) &= \int_1^\infty \frac{\mathrm{d}x}{x}\Big(\phi(x)x^{\frac{1}{4}} - \tfrac{1}{2}x^{-\frac{1}{4}}\Big)\Big(x^{\frac{1}{2}(s-\frac{1}{2})} + x^{-\frac{1}{2}(s-\frac{1}{2})}\Big) &\quad (7.150)\\ &= \int_0^\infty \frac{1}{2}\frac{\mathrm{d}x}{x}\Big(\phi(x)x^{\frac{1}{4}} - \tfrac{1}{2}x^{-\frac{1}{4}}\Big)\Big(x^{\frac{1}{2}(s-\frac{1}{2})} + x^{-\frac{1}{2}(s-\frac{1}{2})}\Big) &\quad (7.151)\end{aligned}$$

Proof. For $0 < \mathrm{Re} s < 1$,

$$-\frac{1}{2}\int_1^\infty \frac{\mathrm{d}x}{x}\Big(x^{\frac{s-1}{2}} + x^{-\frac{s}{2}}\Big) = -\frac{1}{1-s} - \frac{1}{s} = -\frac{1}{s(1-s)}. \tag{7.152}$$

Hence (7.146) implies (7.150)

(7.150) implies (7.151) by (7.144). □

7.3 Holomorphic extension of the dzeta function and Riemann's reflection formula–2nd proof

In this subsection we prove Theorem 7.2 by directly showing (1).

Theorem 7.5 *For any s with $\mathrm{Re}s > 1$ we have*

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{x^{s-1}\mathrm{e}^{-x}}{1-\mathrm{e}^{-x}} \mathrm{d}x. \tag{7.153}$$

ζ extends to a holomorphic function on $\mathbb{C}\backslash\{0\}$. It has the following integral representation valid for all s except for $s = 1, 2, \dots$ (because of singularities of the Gamma function):

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi\mathrm{i}} \int_{]-\infty,0^+,-\infty[} \frac{z^{s-1}\mathrm{e}^{z}}{1-\mathrm{e}^{z}} \mathrm{d}z. \tag{7.154}$$

Proof. (7.153) follows by summing up

$$\int_0^\infty x^{s-1}\mathrm{e}^{-nx}\mathrm{d}x = \Gamma(s)\frac{1}{n^s}. \tag{7.155}$$

(7.154) follows by summing up

$$\frac{1}{2\pi\mathrm{i}} \int_{]-\infty,0^+,-\infty[} z^{s-1}\mathrm{e}^{nz}\mathrm{d}z = \frac{1}{\Gamma(1-s)n^s}. \tag{7.156}$$

(7.154) is holomorphic on the whole $\mathbb{C}$ except maybe at the singularities of $\Gamma(1-s)$, which are $1, 2, \dots$. But we already know that ζ is holomorphic for $\mathrm{Re}s > 1$. Hence the only singularity can be at 1. □

We can analyze the behavior of ζ close to the singularity:

Theorem 7.6

$$\zeta(s) = \frac{1}{s-1} + \gamma + O(s). \tag{7.157}$$

Proof.

$$\begin{aligned}
\zeta(s) &= \sum_{n=1}^\infty \int_n^\infty \frac{s}{x^{s+1}}\mathrm{d}x & (7.158)\\
&= \sum_{n=1}^\infty \int_n^{n+1} \frac{sn}{x^{s+1}}\mathrm{d}x & (7.159)\\
&= \int_1^\infty \frac{s}{x^s}\mathrm{d}x + \sum_{n=1}^\infty \int_n^{n+1} \frac{s(n-x)}{x^{s+1}}\mathrm{d}x. & (7.160)
\end{aligned}$$

Now,

$$\int_1^\infty \frac{s}{x^s}\mathrm{d}x = \frac{s}{s-1} = 1 + \frac{1}{s-1},$$

$$\lim_{s\searrow 1} \int_n^{n+1} \frac{s(n-x)}{x^{s+1}}\mathrm{d}x = \int_n^{n+1} \frac{n-x}{x^2}\mathrm{d}x = \frac{1}{n+1} - \log(n+1) + \log(n).$$

Therefore,

$$\begin{aligned}
& \lim_{s\searrow 0}\Big(\zeta(s)-\frac{1}{s-1}\Big) && (7.161)\\
= \;& 1+\sum_{n=1}^{\infty}\Big(\frac{1}{n+1}-\log(n+1)+\log(n)\Big) && (7.162)\\
= \;& \sum_{n=1}^{\infty}\Big(\frac{1}{n}-\log(n+1)+\log(n)\Big) \;=\; \gamma. && (7.163)
\end{aligned}$$

□

Theorem 7.7

$$\zeta(s)=2^s\pi^{s-1}\Gamma(1-s)\sin(\tfrac{\pi}{2}s)\zeta(1-s). \tag{7.164}$$

or equivalently

$$\zeta(s)\Gamma(s)\cos(\tfrac{\pi}{2}s)=2^{s-1}\pi^s\zeta(1-s). \tag{7.165}$$

Proof. Assume that $\mathrm{Re}s<0$. The function $\frac{z^{s-1}\mathrm{e}^z}{1-\mathrm{e}^z}$ has simple poles at $z\in\mathrm{i}2\pi\mathbb{Z}\backslash\{0\}$. We compute the residues:

$$\mathrm{Res}\frac{z^{s-1}\mathrm{e}^z}{1-\mathrm{e}^z}\Big|_{z=\mathrm{i}2\pi n}=-(\mathrm{i}2\pi n)^{s-1}. \tag{7.166}$$

Now for $n>0$,

$$\begin{aligned}
& (\mathrm{i}2\pi n)^{s-1}+(\mathrm{i}2\pi(-n))^{s-1} && (7.167)\\
= \;& (2\pi)^{s-1}(\mathrm{e}^{\mathrm{i}\frac{\pi}{2}(s-1)}+\mathrm{e}^{-\mathrm{i}\frac{\pi}{2}(s-1)})n^{s-1} && (7.168)\\
= \;& 2^s\pi^{s-1}\sin\Big(\frac{\pi}{2}s\Big)n^{s-1}. && (7.169)
\end{aligned}$$

On $\mathbb{C}\backslash]-\infty,0]$, treated as the domain of z^s, we consider the circle of radius $(2N+1)\pi$ and centered at 0. We treat it as a curve γ_N starting at $-(2N+1)\pi-\mathrm{i}0$ and ending at $-(2N+1)\pi+\mathrm{i}0$. Let $\delta_N:=[-(2N+1)\pi-\mathrm{i}0,0^+,-(2N+1)\pi+\mathrm{i}0]$. Then

$$\begin{aligned}
& \frac{1}{2\pi\mathrm{i}}\int_{\gamma_N}\frac{z^{s-1}\mathrm{e}^z}{1-\mathrm{e}^z}\mathrm{d}z-\frac{1}{2\pi\mathrm{i}}\int_{\delta_N}\frac{z^{s-1}\mathrm{e}^z}{1-\mathrm{e}^z}\mathrm{d}z && (7.170)\\
= \;& -\sum_{n=1}^{N}2^s\pi^{s-1}\sin(\frac{\pi}{2}s)n^{s-1} && (7.171)\\
\to \;& -2^s\pi^{s-1}\sin(\frac{\pi}{2}s)\zeta(1-s). && (7.172)
\end{aligned}$$

But on γ_N

$$\Big|\frac{\mathrm{e}^z}{1-\mathrm{e}^z}\Big|<K \tag{7.173}$$

$$|z^{s-1}| < |(2N+1)\pi|^{\mathrm{Re}s-1} \tag{7.174}$$

Hence the first term of (7.170) converges to 0. Clearly, the second term of (7.170) converges to $-\zeta(s)$.

We pass from (7.164) to (7.165) by $\Gamma(1-s)\Gamma(s) = \frac{\pi}{\sin(\pi s)}$ and $\sin \pi s = 2\sin(\frac{\pi}{2}s)\cos(\frac{\pi}{2}s)$. □

7.4 Bernoulli numbers

The Bernoulli numbers are defined by

$$\frac{x}{\mathrm{e}^x - 1} = \sum_{n=0}^{\infty} B_n \frac{x^n}{n!}. \tag{7.175}$$

The function

$$\frac{x}{\mathrm{e}^x - 1} + \frac{x}{2} = \frac{x}{2}\coth\left(\frac{x}{2}\right) \tag{7.176}$$

is even. Hence for odd n we have $B_n = 0$ except for $B_1 = -\frac{1}{2}$. Otherwise, $B_0 = 1$, $B_2 = \frac{1}{6}$, etc. We also have

$$x\coth(x) = \sum_{k=0}^{\infty} B_{2k}\frac{(2x)^{2k}}{(2k)!}, \tag{7.177}$$

$$x\cot(x) = \sum_{k=0}^{\infty}(-1)^k B_{2k}\frac{(2x)^{2k}}{(2k)!}. \tag{7.178}$$

Theorem 7.8 *For positive even integers the dzeta function can be expressed in terms of Bernoulli numbers:*

$$\zeta(2k) = \frac{(-1)^{k+1}2^{2k-1}\pi^{2k}}{(2k)!}B_{2k}. \tag{7.179}$$

For all negative integers the dzeta function can be expressed in terms of Bernoulli numbers:

$$\zeta(-n) = -\frac{B_{n+1}}{n+1}. \tag{7.180}$$

(In particular, for even negative integers the dzeta function is zero).

Proof. To prove (7.179) we use the well known identity

$$\pi\cot\pi x = \frac{1}{x} + \sum_{n=1}^{\infty}\frac{2x}{x^2-n^2}. \tag{7.181}$$

Setting $z=\pi x$ into (7.181) we obtain

$$\begin{aligned} z\cot z &= 1+2\sum_{n=1}^{\infty}\frac{z^2}{z^2-n^2\pi^2} && (7.182)\\ &= 1-2\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\frac{z^{2k}}{n^{2k}\pi^{2k}} && (7.183)\\ &= 1-2\sum_{k=1}^{\infty}\frac{z^{2k}}{\pi^{2k}}\zeta(2k). && (7.184)\end{aligned}$$

To prove (7.180) we can use the reflection formula

$$\zeta(s)=2^s\pi^{s-1}\Gamma(1-s)\sin(\tfrac{\pi}{2}s)\zeta(1-s). \tag{7.185}$$

Setting $s=-2k$, $k=0,1,\dots$ we see that $\zeta(-2k)=0$ because $\sin(-\pi k)=0$. Setting $s=-2k+1$, $m=1,2,\dots$ we obtain

$$\zeta(-2k+1)=2^{-2k+1}\pi^{-2k}(2k-1)!(-1)^{k+1}\zeta(2k)=-\frac{B_{2k}}{2k}, \tag{7.186}$$

Alternatively, (7.180) follows from (7.154), which for $s=-n$ can be rewritten as

$$\zeta(-n)=\frac{n!}{2\pi\mathrm{i}}\int_{[0+]}\frac{z^{-1-n}}{\mathrm{e}^{-z}-1}\mathrm{d}z. \tag{7.187}$$

7.5 Zeros of the dzeta function

Theorem 7.9 *η and ξ have no zeros except in $0\le \mathrm{Re}s\le 1$. The only zeros of ζ away from $0\le \mathrm{Re}s\le 1$ are at $-2,-4,\dots$.*

Proof. ζ has no zeros for $\mathrm{Re}s>1$ by Prop. 7.1. Hence so does not η for $\mathrm{Re}s>1$. By reflection, η has no zeros for $\mathrm{Re}s<0$. The only singularities of Γ are at $0,-1,-2,\dots$. Hence the only zeros of ζ in $\mathrm{Re}s<0$ can be at $-2,-4,\dots$. □

Let $\frac12+\mathrm{i}Z$ be the set of all zeros of the dzeta function away from $-2,-4,\dots$. Z coincides with the set of zeros of η and of ξ. Note that $Z=-Z$ by the reflection identity

$$\eta(z)=\eta(1-z),\quad \xi(z)=\xi(1-z).$$

Moreover, $\overline{Z}=-Z$ by

$$\overline{\zeta(z)}=\zeta(\overline{z}),\quad \overline{\eta(z)}=\eta(\overline{z}),\quad \overline{\xi(z)}=\xi(\overline{z}).$$

Theorem 7.9 says that $Z\subset\{|\mathrm{Im}(z)|<\frac12\}$.

The Riemann conjecture says that $Z\subset\mathbb{R}$.

Let $\frac12+\mathrm{i}Z_+$ be the set of zeros of the dzeta function with positive imaginary part. Clearly, $Z=Z_+\cup(-Z_+)$.

The function ξ is entire and $\xi(0) = \frac{1}{2}$. Therefore, one can try to recover it from the position of its zeros:

$$\begin{aligned} 2\xi(s) &= \prod_{\lambda\in Z}\Big(1 - \frac{s}{\frac{1}{2}+\mathrm{i}\lambda}\Big) \\ &= \prod_{\lambda\in Z_+}\Big(\frac{s}{\frac{1}{2}+\mathrm{i}\lambda} - 1\Big)\Big(\frac{s}{\frac{1}{2}-\mathrm{i}\lambda} - 1\Big). \qquad (7.188) \end{aligned}$$

This is indeed true, and follows from a more general theorem proven much later by Hadamard. Riemann guessed (7.188), which implies

$$\log\big(2\xi(s)\big) = \sum_{\lambda\in Z}\log\Big(\frac{s}{\frac{1}{2}+\mathrm{i}\lambda} - 1\Big). \qquad (7.189)$$

Using

$$\zeta(s) = \frac{\pi^{\frac{s}{2}}}{s(s-1)\Gamma(\frac{s}{2})}\xi(s) \qquad (7.190)$$

we obtain

$$\log(\zeta(s)) = \log\Big(\frac{\pi^{\frac{s}{2}}}{s(s-1)\Gamma(\frac{s}{2})}\Big) + \log(\xi(s)). \qquad (7.191)$$

Hence

$$\log(\zeta(s)) = \log\Big(\frac{\pi^{\frac{s}{2}}}{s(s-1)\Gamma(\frac{s}{2})}\Big) + \sum_{\lambda\in Z}\log\Big(\frac{s}{\frac{1}{2}+\mathrm{i}\lambda} - 1\Big). \qquad (7.192)$$

7.6 Riemann's formula

Introduce

$$\begin{aligned} \pi(x) &:= \#\{\text{primes } \le x\}, \qquad (7.193) \\ \Pi(x) &= \sum_{m=1}^{\infty}\frac{1}{m}\pi(x^{\frac{1}{m}}). \qquad (7.194) \end{aligned}$$

Then for $\mathrm{Re}s > 1$,

$$\begin{aligned} \log\zeta(s) &= -\sum_p \log(1-p^{-s}) \qquad (7.195) \\ &= -\sum_p\sum_{m=1}^{\infty}\frac{p^{-sm}}{m}, \qquad (7.196) \\ &= -\int_0^{\infty} x^{-s}\Pi'(x)\mathrm{d}x. \qquad (7.197) \end{aligned}$$

For any $a > 1$, we can write Π' as the Fourier transform

$$\begin{aligned} \log\zeta(a+\mathrm{i}t) &= -\int_0^{\infty} x^{-a-\mathrm{i}t}\Pi'(x)\mathrm{d}x \qquad (7.198) \\ &= -\int \mathrm{e}^{-au-\mathrm{i}tu}\Pi'(\mathrm{e}^u)\mathrm{e}^u\mathrm{d}u. \qquad (7.199) \end{aligned}$$

Inverting the Fourier transform we obtain

$$\begin{aligned}\Pi'(x) &= -\frac{x^{a-1}}{2\pi}\int x^{\mathrm{i}t}\log\zeta(a+\mathrm{i}t)\mathrm{d}t. \\ &= -\frac{1}{2\pi\mathrm{i}}\int_{a+\mathrm{i}\mathbb{R}} x^{s-1}\zeta(s)\mathrm{d}s.\end{aligned} \tag{7.200}$$

We use now (7.192) and (7.200). This yields

$$\Pi'(x) = \frac{1}{\log(x)} - \sum_{\lambda\in Z}\frac{x^{-\frac{1}{2}+\mathrm{i}\lambda}}{\log(x)} - \frac{1}{x(x^2-1)\log(x)}. \tag{7.201}$$

Then we use $\Pi(1-0)=0$ to obtain

$$\Pi(x) = \mathrm{Li}(x) - \sum_{\lambda\in Z}\mathrm{Li}(x^{\frac{1}{2}+\mathrm{i}\lambda}) - \log(2) + \int_x^\infty \frac{\mathrm{d}t}{t(t^2-1)\log(t)}. \tag{7.202}$$

7.7 The Li function

We define

$$\mathrm{Li}(x) := \int_0^x \frac{\mathrm{d}y}{\log(y)}, \tag{7.203}$$

where the singularity at zero is integrated in the sense of the principal value. Note that for $\mathrm{Re}(\mu)>0$

$$\mathrm{Li}(x^{\frac{1}{\mu}}) := \int_0^x \frac{t^{\mu-1}\mathrm{d}t}{\log(t)}. \tag{7.204}$$

In fact, we can change the variables $y=t^\mu$ in (7.203).

Let us recall the Euler constant and the Fourier transformation of the logarithm

$$-\gamma = \int_0^\infty \mathrm{e}^{-k}\log(k)\mathrm{d}k, \tag{7.205}$$

$$\int \log(\pm\mathrm{i}s+0)\mathrm{e}^{-\mathrm{i}sk}\mathrm{d}s = -2\pi\frac{\theta(\mp k)}{|k|} - 2\pi\gamma\delta(k), \tag{7.206}$$

Here $\frac{\theta(\mp k)}{|k|}$ is understood as the distribution

$$\int \frac{\theta(\pm k)}{|k|}\phi(k)\mathrm{d}k = \lim_{\epsilon\searrow 0}\Big(\int_\epsilon^\infty \frac{1}{|k|}\phi(\pm k)\mathrm{d}k + \log(\epsilon)\phi(0)\Big). \tag{7.207}$$

In (7.206) we choose the + case, set $x=\mathrm{e}^{-k}$, and we rewrite it as follows for $\mathrm{Re}(a)\geq\mathrm{Re}(\mu)$,

$$\frac{1}{2\pi\mathrm{i}}\int_{a+\mathrm{i}\mathbb{R}} x^s\log(s-\mu)\mathrm{d}s = -\frac{x^\mu\theta(x-1)}{\log x} - \gamma\delta(x-1). \tag{7.208}$$

Here again $\frac{\theta(x-1)}{\log x}$ is a distribution regularized at zero:

$$\int \frac{\theta(x-1)}{\log x}\phi(x)\mathrm{d}x = \lim_{\epsilon\searrow 0}\Big(\int_{1+\epsilon}^{\infty}\frac{1}{\log(x)}\phi(x)\mathrm{d}x + \log(\epsilon)\phi(0)\Big). \tag{7.209}$$

We will see that the primitive of (7.208) can be expressed in terms of the Li function.

Proposition 7.10 *For $x > 1$*

$$\mathrm{Li}(x) = \int_0^x \Big(\frac{\theta(x-1)}{\log x} + \gamma\delta(x-1)\Big)\mathrm{d}x. \tag{7.210}$$

Proof. First we compute

$$\begin{aligned}\int_0^{\mathrm{e}^{-\epsilon}} \frac{\mathrm{d}t}{\log(t)} &= -\int_\epsilon^\infty \frac{\mathrm{e}^{-t}}{t}\mathrm{d}t \\ -\int_\epsilon^\infty \mathrm{e}^{-t}\log(t)\mathrm{d}t + \mathrm{e}^{-\epsilon}\log(\epsilon) &= \gamma + \log(\epsilon) + o(\epsilon).\end{aligned}$$

Now

$$\begin{aligned}\mathrm{Li}(x) &= \lim_{\epsilon\searrow 0}\Big(\int_0^{1-\epsilon}\frac{1}{\log(y)} + \int_{1+\epsilon}^x \frac{1}{\log(y)}\Big) \\ &= \lim_{\epsilon\searrow 0}\Big(\gamma + \log(\epsilon) + \int_{1+\epsilon}^x \frac{1}{\log(y)}\Big).\end{aligned}$$

7.8 The Hurwitz dzeta function

For $\mathrm{Re}s > 0$ and $a \not\in \{\dots,-2,-1,0\}$, we define

$$\zeta(s,a) := \sum_{n=1}^{\infty}\frac{1}{(a+n)^s}. \tag{7.211}$$

Theorem 7.11 *For any s, a with $\mathrm{Re}s > 0$ and $\mathrm{Re}a > 0$, we have*

$$\zeta(s) = \frac{1}{\Gamma(s)}\int_0^\infty \frac{x^{s-1}\mathrm{e}^{-ax}}{1-\mathrm{e}^{-x}}\mathrm{d}x. \tag{7.212}$$

$s \mapsto \zeta(s,a)$ extends to a holomorphic function on $\mathbb{C}\backslash\{0\}$. It has the following integral representation valid for all s except for $s = 0,1,2,\dots$:

$$\zeta(s) = \frac{\Gamma(1-s)}{2\pi\mathrm{i}}\int_{]-\infty,0^+,-\infty[}\frac{z^{s-1}\mathrm{e}^{az}}{1-\mathrm{e}^{z}}\mathrm{d}z. \tag{7.213}$$

For $s \approx 0$ we have

$$\zeta(s) = \frac{1}{s-1} + O(s^0). \tag{7.214}$$

7.9 The Hurwitz identity

Theorem 7.12 *Let* $0 < \mathrm{Re} a \leq 1$. *Then*

$$\zeta(s,a) = 2^s \pi^{s-1} \Gamma(1-s) \sum_{n=1}^{\infty} \frac{\sin(2\pi n a + \frac{\pi}{2} s)}{n^{1-s}}. \tag{7.215}$$