A unified approach
to hypergeometric class functions

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Abstract

Hypergeometric class equations are given by second order differential operators in one variable whose coefficient at the second derivative is a polynomial of degree ≤ 2, at the first derivative of degree ≤ 1 and the free term is a number. Their solutions, called hypergeometric class functions, include the Gauss hypergeometric function and its various limiting cases. The paper presents a unified approach to these functions. The main structure behind this approach is a family of complex 4-dimensional Lie algebras, originally due to Willard Miller. Hypergeometric class functions can be interpreted as eigenfunctions of the quadratic Casimir operator in a representation of Miller's Lie algebra given by differential operators in three complex variables. One obtains a unified treatment of various properties of hypergeometric class functions such as recurrence relations, discrete symmetries, power series expansions, integral representations, generating functions and orthogonality of polynomial solutions.

Contents

1 Introduction

This paper is devoted to the family of equations of the form

$$\left(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta\right) f(z) = 0, \tag{1.1}$$

where $\sigma(z)$, $\tau(z)$, η are polynomials with

$$\deg\sigma \leq 2, \ \deg\tau \leq 1, \ \ \deg\eta = 0. \tag{1.2}$$

Their solutions include some of the most useful special functions with many applications in physics and mathematics.

In the literature one can find several names for this family. In this paper, we will use the name *hypergeometric class equations* for equations of the form (1.1) satisfying (1.2).[1]

Let us start with a short review of basic nontrivial types of hypergeometric class equations. We will always assume that $\sigma \neq 0$. Every class will be simplified by dividing by a constant and an affine change of the complex variable z.

(1) **The $_2F_1$ or Gauss hypergeometric equation**

$$\left(z(1-z)\partial_z^2 + (c - (a+b+1)z)\partial_z - ab\right) f(z) = 0.$$

(2) **The $_2F_0$ equation**

$$\left(z^2\partial_z^2 + (-1 + (1+a+b)z)\partial_z + ab\right) f(z) = 0.$$

(3) **The $_1F_1$ or Kummer's confluent equation**

$$(z\partial_z^2 + (c-z)\partial_z - a)f(z) = 0.$$

(4) **The $_0F_1$ equation**, closely related to the Bessel equation,

$$(z\partial_z^2 + c\partial_z - 1)f(z) = 0.$$

(5) **The Hermite equation**

$$(\partial_z^2 - 2z\partial_z - 2a)f(z) = 0.$$

In our work we collect various results about hypergeometric class equations that can be stated and proven in a unified way, with as few restrictions on σ, τ, η as possible. We believe that such an approach has a considerable pedagogical and theoretical value. The pedagogical advantage of the unified approach is obvious: it reduces the need for repetitive arguments. From the theoretical point of view, in this way we easily see the *coalescence* of various types. The properties of hypergeometric class equations that we describe depend *analytically* on the the coefficients σ, τ, η. These properties include a pair of recurrence relations, a discrete symmetry, integral representations, power expansions around singular points, generating functions, the Rodriguez formulas for polynomials and their orthogonality.

The unified approach has its limitations. There are some properties that are not easy to formulate in a unified way. For instance, only one pair of recurrence relations depends analytically on the coefficients. If we restrict ourselves to specific types, then we often find a bigger number of recurrence relations (eg. at least 12 in the case of the Gauss hypergeometric equation). Another family of important properties not included in our presentation are quadratic identities, which link various types of hypergeometric class functions. Thus this work should be compared to other studies of hypergeometric class equations, such as [De1], where specific types are described one by one.

The central structure behind the properties described in this paper is a certain family of Lie algebras $m_{\alpha,\beta}$, described in Section 2, which we propose to call *Miller's Lie algebra*, since it was probably

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[1]In the book by Nikiforov-Uvarov [NU], and also in [De1, De2], this family is called *equations of the hypergeometric type*. However, Slavianov-Lay's book [SL] suggests to use the term "type" for smaller families, such as the families (1)–(5) listed below. [DIL] calls this family the *grounded Riemann class*. The name *Riemann class*, following the suggestions of Slavianov-Lay, is reserved in [DIL] for a somewhat wider family, where $\eta(z)$ is allowed to be a rational function such that $\eta\sigma$ is a polynomial of degree ≤ 2. See Appendix B for more comments.

first introduced by Willard Miller in [M1]. Miller's Lie algebra is defined as the span of four elements $N, A_+, A_-, \mathbb{1}$ satisfying the commutation relations

$$
\begin{aligned}
[N, A_+] &= A_+, \\
[N, A_-] &= -A_-, \\
[A_+, A_-] &= 2\alpha N + \beta \mathbb{1},
\end{aligned}
\tag{1.3}
$$

where α, β are complex parameters.

Let σ be a polynomial of degree ≤ 2 (as in (1.1)) and κ a polynomial of degree ≤ 1. Miller's Lie algebra with $\alpha = \frac{\sigma''}{2}$ and $\beta = \kappa'$ can be represented by the following 1st order differential operators acting on $\mathbb{C}^3$:

$$
\begin{aligned}
N &:= t\partial_t - s\partial_s, \\
A_+ &:= t\partial_z + \sigma'(z)\partial_s, \\
A_- &:= s\partial_z + \sigma'(z)\partial_t + \frac{\kappa(z)}{t}.
\end{aligned}
\tag{1.4}
$$

The operators (1.4) can be restricted to functions on the quadric

$$
\sigma(z) - ts = 0.
\tag{1.5}
$$

Miller's Lie algebra $m_{\alpha,\beta}$ commutes with

$$
\mathcal{C}_{\alpha,\beta} := \frac{1}{2}(A_- A_+ + A_+ A_-) + \alpha N^2 + \beta N,
\tag{1.6}
$$

which will be called the *Casimir operator* of Miller's Lie algebra. In the representation (1.4), the operator (1.6) restricted to the quadric 1.5 and to the eigenspace $N = 0$ is a 2nd order differential operator in z, which we denote $\mathcal{H}(\sigma, \kappa)$. Its eigenvalue equation

$$
\bigl(\mathcal{H}(\sigma, \kappa) + \omega\bigr)f = 0
\tag{1.7}
$$

is precisely the hypergeometric class equation (1.1), where

$$
\tau(z) = \kappa(z) + \sigma'(z), \quad \eta = \frac{\kappa'}{2} + \omega.
\tag{1.8}
$$

The operators A_+ and A_- can be treated as "root operators" wrt the "Cartan algebra" spanned by N. They lead to recurrence relations for ladders of solutions. The involution of the quadric (1.5) $(t, s, z) \mapsto (s, t, z)$ generates a "Weyl symmetry" of Miller's Lie algebra. It leads to a discrete symmetry of hypergeometric class equations.

We devote Section 3 to the properties of hypergeometric class equations and functions which follow directly from the action of root operators and Weyl symmetry of Miller's Lie algebra: the *basic recurrence relations* and the *basic symmetry*. Thanks to the recurrence relations, once we know a solution of a certain hypergeometric class equation, we know its solutions for a whole ladder of parameters labelled by $n \in \mathbb{Z}$.

Certain ladders are special. One of them contains solutions which can be expressed in terms of elementary functions. We call it the *Chebyshev ladder*, because it contains the well-known Chebyshev polynomials. It is described in Subsection 3.6. See also [DGR] Subsec. 3.3 and 4.5, [Poole], Sec. 31.

Another ladder of solutions consists of polynomials. In the convention that we adopted, this ladder is descending—its elements are zero for $n > 0$. We devote to this ladder the whole Section 7. The basic symmetry produces from the polynomial ladder an ascending ladder, which needs no separate discussion.

For the Hermite equation the polynomial σ does not have a zero. For other types, that is for equations reducible to the $_2F_1$, $_1F_1$, $_2F_0$ and $_0F_1$ types, σ has at least one zero. We devote Section 4 to these

equations, where without loss of generality we assume $\sigma(0) = 0$. We can then write a formal power series $F(\sigma, \kappa, \omega; z)$ which solves the equation

$$\big(\mathcal{H}(\sigma, \kappa) + \omega\big)F = 0, \tag{1.9}$$

normalized to be 1 at zero. This power series is convergent if $\sigma'(0) \neq 0$, that is for $_2F_1$, $_1F_1$, and $_0F_1$. If $\sigma'(0) = 0$, that is for $_2F_0$, it either terminates and is a polynomial, or is divergent. Nevertheless, even in the divergent case this series is asymptotic to a well-defined function $_2F_0$, which we separately discuss in Appendix A. Thus we obtain a unification of 4 types of hypergeometric functions in a single function $F(\sigma, \kappa, \omega; z)$, which we call the *unified hypergeometric function*. It depends meromorphically on 5 complex parameters (two parameters for σ, two for κ and one for ω).

If $\sigma(0) = 0$ and $\sigma'(0) \neq 0$, it is natural to assume $\sigma'(0) = 1$ and to normalize the unified hypergeometric function by dividing it by $\Gamma(m+1)$, where $m = \frac{\kappa(0)}{\sigma'(0)}$. This is sometimes called *Olver's normalization*. Thus we obtain the *Olver normalized unified hypergeometric function* $\mathbf{F}(\sigma, \kappa, \omega; z)$, which depends analytically on 4 complex parameters (one parameter for σ, two for κ and one for ω).

If $\sigma(0) = 0$ and $\sigma'(0) \neq 0$, the hypergeometric class has an additional discrete symmetry, which we call the *power symmetry*. This transformation involves gauging with the function x^m, that is, $(\cdot) \mapsto x^m(\cdot)x^{-m}$. (In the literature this transformation is sometimes called an *F-homotopy*.)

If $\sigma(0) = 0$, and $\sigma'' \neq 0$ or $\kappa' \neq 0$, that is for the $_2F_1$, $_1F_1$ and $_2F_0$ equations, we have yet another discrete symmetry, which we call the *inversion symmetry*. For the $_2F_1$ equation it involves the change of the variable $z \mapsto \frac{1}{z}$ and the transformed equation is still of the form $_2F_1$. In the confluent cases it interchanges the $_1F_1$ and $_2F_0$ equations, and it involves $z \to -\frac{1}{z}$. It is however not defined for the $_0F_1$ equation.

In Section 5 we describe a unified approach to representations of hypergeometric class functions in terms of integrals of elementary functions. The $_2F_1$, $_1F_1$, $_2F_0$ and Hermite functions can be represented as *Euler integrals*, that is,

$$\int_\gamma p(s)(s-z)^{-n-1}\mathrm{d}s, \tag{1.10}$$

where p is an elementary function, $n \in \mathbb{C}$ and γ is an appropriate contour. $_1F_1$, $_0F_1$ and Hermite functions possess representations in the form of the *Laplace integral*

$$\int_\gamma q(s)\mathrm{e}^{zs}\mathrm{d}s, \tag{1.11}$$

where q is an elementary function and γ is an appropriate contour.

As we mentioned above, some ladders of solutions of hypergeometric class equations consist of polynomials. These polynomials include the famous *classical orthogonal polynomials* (sometimes called the *very classical orthogonal polynomials*). They allow for a very elegant unified treatment, which includes not only recurrence relations, but also generating functions, the famous Rodrigues formula and the orthogonality relations. All of this is briefly described in Section 7.

As we mentioned above, if $\sigma(0) = 0$ and $\sigma'(0) \neq 0$, that is in the cases equivalent to types $_2F_1$, $_1F_1$, and $_0F_1$, we have the power symmetry. Because of that, these equations have two linearly independent solutions with a distinct behavior at zero: the unified hypergeometric function, which is analytic at zero, and another solution behaving as $\sim z^m$, where $m = \frac{\kappa(0)}{\sigma'(0)}$. The linear independence breaks down for $m \in \mathbb{Z}$, when both solutions are proportional to one another. This case is called *degenerate* and is discussed in Section 8. The Olver normalizzed unified hypergeometric function in this case has an additional integral representation and an elegant generating function.

The present work should be used only as an "invitation" to hypergeometric class functions. It leaves out many of their properties, which are difficult to describe in a unified way. For instance, as we mentioned above, various types of hypergeometric class functions possess additional recurrence relations and

additional symmetries. However only those derived directly from Miller's Lie algebra can be described in a fully unified fashion.

Almost all our discussion is algebraic, without functional analysis. We are aware that it is natural to view hypergeometric class operators as closed (or self-adjoint) Sturm-Liouville operators on appropriate weighted L^2 spaces. It would be also interesting to consider representations of Miller's Lie algebra in Hilbert spaces. This would, however, require breaking our discussion into separate types. The only place where we use some elements of functional analysis is Section 7 about classical orthogonal polynomials. We show how to view them as eigenfunctions of certain self-adjoint Sturm-Liouville operators on weighted L^2 spaces with appropriate boundary conditions—this can be done in a rather unified fashion.

The literature on hypergeometric class equations is very large. Usually, each type is considered separately, without an attempt of a unified treatment. Let us list some of the more famous treatises about these equations: [EMOT, Ho, MOS, MF, R, Wa, WW]

Under the name "equations of the hypergeometric type" they were considered in a unified way in the book by Nikiforov–Uvarov [NU]. This book was one of the two main inspirations for our article. A part of the material of our work is adapted from [NU], notably the material of Subsections 3.5, 5.1, 7.2, 7.6.

We use also ideas of Miller, notably in Section 2. As we mentioned above, Miller's Lie algebra was defined in [M1]. Miller was an early champion of the use of Lie algebras in the theory of special functions. To my knowledge, he was the first to note that various types of hypergeometric class admit a larger symmetry algebra. This topic was further developed in [De2].

There are many works that contain elements of a unified theory of hypergeometric class equations, the idea of recurrence relations and factorizations. Among early ones let us mention the classic work of Infeld and Hull [IH] and of Truesdell [Tr]. Later treatments include [CKS, SHD].

More complete analysis of various types hypergeometric class equations, including the Lie-algebraic interpretation of their recurrence relations and discrete symmetries, can be found in the literature, notably in the works of Miller, and also in [De1, De2].

The family of hypergeometric class polynomials that form an orthogonal basis in an appropriate weighted Hilbert space consists essentially of Jacobi, Laguerre and Hermite polynomials, often called *classical orthogonal polynomials*. It has an especially large literature, e.g. [NU, R]. In the more recent literature the name "classical orthogonal polynomials" is sometimes given to a broader family, given by the so-called Askey scheme. Some authors proposed to use the name *very classical orthogonal polynomials* for the family consisting of Jacobi, Laguerre and Hermite polynomials.

2 Miller's Lie algebra

In this section we introduce a certain two-parameter family of 4-dimensional Lie algebras, which to our knowledge was first introduced by Willard Miller. We will call it *Miller's Lie algebra* and denote by $m_{\alpha,\beta}$ with $\alpha, \beta \in \mathbb{C}$. We will also describe its *Casimir operator* $\mathcal{C}_{\alpha,\beta}$, that is, a quadratic expression in elements of $m_{\alpha,\beta}$, which in any representation commutes with the whole Lie algebra. In other words, $\mathcal{C}_{\alpha,\beta}$ belongs to the center of the enveloping algebra of $m_{\alpha,\beta}$. We will also describe a family of representations of $m_{\alpha,\beta}$ by 1st order differential operators on certain 2nd degree surfaces in $\mathbb{C}^3$.

In appropriate coordinates the eigenvalue equation for the Casimir operator equation will have the form of a hypergeometric class equation. A number of properties of hypergeometric class equations will have a simple interpretation in terms of properties of Miller's Lie algebra. They include the basic symmetry, basic factorizations and basic recurrence relations. They will be described in the next section.

2.1 Three low-dimensional Lie algebras

Let us first introduce three low-dimensional complex Lie algebras.

1. $sl(2, \mathbb{C})$. It is spanned by N, A_+, A_- satisfying the commutation relations

$$
\begin{aligned}
[N, A_\pm] &= \pm A_\pm, \\
[A_+, A_-] &= 2N.
\end{aligned} \tag{2.1}
$$

The operator

$$
\begin{aligned}
\mathcal{C} :=& A_- A_+ + N(N+1) & (2.2) \\
=& A_+ A_- + N(N-1) & (2.3) \\
=& \frac{1}{2}(A_- A_+ + A_+ A_-) + N^2 & (2.4)
\end{aligned}
$$

commutes with all elements of $sl(2, \mathbb{C})$. It is often called the *Casimir (operator)*.

Here is a typical representation of $sl(2, \mathbb{C})$:

$$
\begin{aligned}
N :=& x\partial_x - y\partial_y, & (2.5) \\
A_+ :=& x\partial_y, \quad A_- := y\partial_x. & (2.6)
\end{aligned}
$$

2. The so-called *complex oscillator Lie algebra*, denoted $osc(\mathbb{C})$. It is spanned by $N, A_+, A_-, \mathbb{1}$ satisfying the commutation relations

$$
\begin{aligned}
[N, A_\pm] &= \pm A_\pm, \\
[A_+, A_-] &= \mathbb{1}, \\
[\mathbb{1}, A_\pm] &= [\mathbb{1}, N] = 0.
\end{aligned} \tag{2.7}
$$

The following operator commutes with $osc(\mathbb{C})$, and will be called the *Casimir* of $osc(\mathbb{C})$:

$$
\begin{aligned}
\mathcal{C} :=& A_- A_+ + N + \tfrac{1}{2} & (2.8) \\
=& A_+ A_- + N - \tfrac{1}{2} & (2.9) \\
=& \frac{1}{2}(A_- A_+ + A_+ A_-) + N. & (2.10)
\end{aligned}
$$

Here is a typical representation of $osc(\mathbb{C})$:

$$
\begin{aligned}
N :=& \frac{1}{2}(-\partial_x^2 + x^2), & (2.11) \\
A_\pm :=& \frac{1}{\sqrt{2}}(x \mp \partial_x). & (2.12)
\end{aligned}
$$

Note that N is the quantum harmonic oscillator, which justifies the name of this Lie algebra.

3. The Lie algebra of *Euclidean movements of the plane*, denoted $\mathbb{C}^2 \rtimes so(2, \mathbb{C})$. It is spanned by N, A_+, A_- satisfying the commutation relations

$$
\begin{aligned}
[N, A_\pm] &= \pm A_\pm, \\
[A_+, A_-] &= 0.
\end{aligned} \tag{2.13}
$$

The following operator commutes with $\mathbb{C}^2 \rtimes so(2, \mathbb{C})$, and will be called the *Casimir* of $\mathbb{C}^2 \rtimes so(2, \mathbb{C})$:

$$
\mathcal{C} := A_- A_+ \tag{2.14}
$$

Here is a typical representation of $\mathbb{C}^2 \rtimes so(2, \mathbb{C})$:

$$
\begin{aligned}
N :=& x\partial_y - y\partial_x, & (2.15) \\
A_\pm :=& \partial_x \pm i\partial_y. & (2.16)
\end{aligned}
$$

Thus N is the generator of rotations of the plane and $A_\pm$ generate translations.

2.2 The family of Lie algebras introduced by W. Miller

The three Lie algebras introduced in the previous subsection can be joined in a single family depending on two complex parameters α, β. This family will be denoted by $m_{\alpha,\beta}$. It was introduced by Willard Miller in [M1]. We will call it *Miller's Lie algebra*.

$m_{\alpha,\beta}$ is defined as the complex Lie algebra spanned by $N, A_+, A_-, \mathbb{1}$ satisfying the commutation relations

$$
\begin{aligned}
[N, A_\pm] &= \pm A_\pm, \\
[A_+, A_-] &= 2\alpha N + \beta \mathbb{1}, \\
[\mathbb{1}, A_\pm] &= [\mathbb{1}, N] = 0.
\end{aligned}
\tag{2.17}
$$

It is easy to see that

$$
\begin{aligned}
m_{\alpha,\beta} &\simeq sl(2,\mathbb{C}) \oplus \mathbb{C}, & \alpha \neq 0; \tag{2.18} \\
m_{\alpha,\beta} &\simeq osc(\mathbb{C}), & \alpha = 0, \quad \beta \neq 0; \tag{2.19} \\
m_{\alpha,\beta} &\simeq \mathbb{C}^2 \rtimes so(2,\mathbb{C}) \oplus \mathbb{C}, & \alpha = 0, \quad \beta = 0. \tag{2.20}
\end{aligned}
$$

Define the linear map $\varepsilon : m_{\alpha,\beta} \to m_{\alpha,-\beta}$ by

$$
\varepsilon(N) := -N, \quad \varepsilon(A_\pm) := A_\mp, \quad \varepsilon(\mathbb{1}) := \mathbb{1}.
\tag{2.21}
$$

Then ε is an isomorphism. The identity automorphism together with ε form a group isomorphic to $\mathbb{Z}_2$, which will be called the *Weyl group* of $m_{\alpha,\beta}$.

Similarly, for $n \in \mathbb{C}$, the linear map $\pi_n : m_{\alpha,\beta} \to m_{\alpha,\beta-2n\alpha}$ given by

$$
\pi_n(N) := N + n\mathbb{1}, \quad \pi_n(A_\pm) := A_\pm, \quad \pi_n(\mathbb{1}) := \mathbb{1},
\tag{2.22}
$$

is an isomorphism.

2.3 Casimir

Consider a representation of $m_{\alpha,\beta}$ on a vector space $\mathcal{Z}$. Following Miller [M1], introduce the following operator $\mathcal{C}$, called the *Casimir of $m_{\alpha,\beta}$*:

$$
\begin{aligned}
\mathcal{C}_{\alpha,\beta} = \mathcal{C} :=& A_- A_+ + \alpha N(N+1) + \beta(N + \tfrac{1}{2}) \tag{2.23} \\
=& A_+ A_- + \alpha N(N-1) + \beta(N - \tfrac{1}{2}) \tag{2.24} \\
=& \frac{1}{2}(A_- A_+ + A_+ A_-) + \alpha N^2 + \beta N. \tag{2.25}
\end{aligned}
$$

As Miller noted, $\mathcal{C}$ commutes with the whole Lie algebra:

$$
N\mathcal{C} = \mathcal{C}N, \qquad A_\pm \mathcal{C} = \mathcal{C}A_\pm.
\tag{2.26}
$$

Extend the isomorphisms ε and π_n defined in (2.21) and (2.22) to the algebra of operators on $\mathcal{Z}$. Then

$$
\begin{aligned}
\varepsilon(\mathcal{C}_{\alpha,\beta}) &= \mathcal{C}_{\alpha,-\beta}, \tag{2.27} \\
\pi_n(\mathcal{C}_{\alpha,\beta}) &= \mathcal{C}_{\alpha,\beta+2n\alpha} + (\alpha n^2 + \beta n)\mathbb{1}. \tag{2.28}
\end{aligned}
$$

2.4   Ladders

By a *two-sided ladder* we mean a subset of $\mathbb{C}$ of the form $n_0 + \mathbb{Z}$, where $n_0 \in \mathbb{C}$. A subset of the form $n_0 + \mathbb{N}_0$ or $n_0 - \mathbb{N}_0$ will be called a *one-sided ladder* (*ascending* or *descending*). A subset of the form $\{n_0, n_0 + 1, \ldots, n_0 + n\}$ for some $n \in \mathbb{N}_0$ will be called a *finite ladder*.

It is easy to see that if the representation of $m_{\alpha,\beta}$ on $\mathcal{Z}$ is irreducible and N possesses an eigenvalue, then the spectrum of N is a ladder. In fact this follows from

$$Nv = nv \quad \Rightarrow \quad NA_{\pm}v = (n \pm 1)A_{\pm}v. \tag{2.29}$$

For $n \in \mathbb{C}$ we define

$$\mathcal{Z}^n := \{v \in \mathcal{Z} \ : \quad Nv = nv\}. \tag{2.30}$$

Let $\mathcal{C}^n$, resp. $A_{\pm}^{n \pm \frac{1}{2}}$, be the operator $\mathcal{C}$, resp. $A_{\pm}$, restricted to $\mathcal{Z}^n$. (2.26) can be rewritten as

$$A_{\pm}^{n \pm \frac{1}{2}} \mathcal{C}^n = \mathcal{C}^{n \pm 1} A_{\pm}^{n \pm \frac{1}{2}}. \tag{2.31}$$

An irreducible representation with a one-sided or finite ladder with the lowest, resp. highest element 0 will be called a *lowest*, resp. *heighest weight representation*. It follows from (2.23), resp. (2.24) that

$$\mathcal{C}^0 = \frac{\beta}{2} \quad \text{for highest weight representations}, \tag{2.32}$$

$$\mathcal{C}^0 = -\frac{\beta}{2} \quad \text{for lowest weight representations}. \tag{2.33}$$

2.5   Representation by 1st order differential operators

Let σ be a polynomial of degree ≤ 2 and κ of degree ≤ 1. Consider $\mathbb{C}^3$ with elements denoted by (t, s, z). Define the operators

$$N := t\partial_t - s\partial_s,$$
$$A_+ := t\partial_z + \sigma'(z)\partial_s, \tag{2.34}$$
$$A_- := s\partial_z + \sigma'(z)\partial_t + \frac{\kappa(z)}{t}.$$

$A_+, A_-, N, \mathbb{1}$ span a Lie algebra with commutation relations

$$[N, A_+] = A_+,$$
$$[N, A_-] = -A_-, \tag{2.35}$$
$$[A_+, A_-] = \sigma'' N + \kappa' \mathbb{1}.$$

Thus it is a representation of Miller's Lie algebra $m_{\alpha,\beta}$ with

$$\alpha = \frac{\sigma''}{2}, \quad \beta = \kappa'. \tag{2.36}$$

The operators (2.34) commute with the multiplication by $\sigma(z) - ts$. Therefore, we can restrict them to analytic functions on Ω, the universal cover of the manifold

$$\left\{ (s, t, z) \in \mathbb{C}^3 \ : \ \sigma(z) - ts = 0, \ t \neq 0, \ s \neq 0 \right\}. \tag{2.37}$$

Let $\mathcal{A}(\Omega)$ denote the space of analytic functions on Ω. The Lie algebra $m_{\alpha,\beta}$ represented by (2.35) acting on $\mathcal{A}(\Omega)$ will be denoted $m(\sigma, \kappa)$.

By (2.25), the Casimir of $m(\sigma, \kappa)$ is following differential operator on $\mathcal{A}(\Omega)$:

$$
\begin{aligned}
\mathcal{C} = \quad & st\partial_z^2 + \sigma'(z)(s\partial_s + t\partial_t + 1)\partial_z + \kappa(z)\partial_z \\
& + \tfrac{\sigma''}{2}\left((t\partial_t)^2 + (s\partial_s)^2 - 2t\partial_t s\partial_s + t\partial_t + s\partial_s\right) \\
& + \tfrac{(\sigma'(z))^2}{ts}t\partial_t s\partial_s + \tfrac{\kappa(z)\sigma'(z)}{ts}s\partial_s + \kappa'(t\partial_t - s\partial_s + \tfrac{1}{2}).
\end{aligned}
\tag{2.38}
$$

Clearly, $\mathcal{C}$ commutes with $m(\sigma, \kappa)$.

Introduce new coordinates on $\mathbb{C}^3$:

$$
v = \sigma(z) - ts, \qquad w = t, \qquad \check{z} = z.
\tag{2.39}
$$

The inverse transformation is

$$
t = w, \qquad s = \tfrac{\sigma(z) - v}{w}, \qquad z = \check{z}.
$$

We have

$$
\partial_z = \partial_{\check{z}} + \sigma'(z)\partial_v, \qquad \partial_t = \partial_w - s\partial_v, \qquad \partial_s = -t\partial_v
$$

Clearly, (2.37) is $v = 0$. Therefore the operators (2.34) in new coordinates, after restricting to the surface (2.37) (and dropping "checks"), are

$$
\begin{aligned}
N &= w\partial_w, \\
A_+ &= w\partial_z, \\
A_- &= \tfrac{1}{w}\left(\sigma(z)\partial_z + \sigma'(z)w\partial_w + \kappa(z)\right).
\end{aligned}
\tag{2.40}
$$

The Casimir operator is

$$
\begin{aligned}
\mathcal{C} &= A_- A_+ + \tfrac{\sigma''}{2}N(N+1) + \kappa'(N + \tfrac{1}{2}) \\
&= \sigma(z)\partial_z^2 + (\kappa(z) + \sigma'(z)(w\partial_w + 1))\partial_z + \tfrac{\sigma''}{2}w\partial_w(w\partial_w + 1) + \kappa'(w\partial_w + \tfrac{1}{2}).
\end{aligned}
$$

Consistently with (2.30), for $n \in \mathbb{C}$ let

$$
\mathcal{Z}^n := \{g \in \mathcal{A}(\Omega) \ : \quad Ng = ng\}.
$$

As in the previous subsection, $\mathcal{C}^n$, resp. $A_\pm^{n \pm \frac{1}{2}}$ denote the operator $\mathcal{C}$, resp. $A_\pm$ restricted to $\mathcal{Z}^n$. Clearly, in the coordinates (2.39)

$$
\mathcal{Z}^n = \{w^n F(z) \ : \quad F \in \mathcal{A}(\Theta)\},
$$

where Θ is the universal covering of $\mathbb{C}\backslash\{\text{zeros of } \sigma\}$.

We have

$$
\mathcal{C}^n + \omega := \sigma(z)\partial_z^2 + (\kappa(z) + \sigma'(z)(n+1))\partial_z + \frac{\sigma''}{2}n(n+1) + \kappa'(n + \tfrac{1}{2}) + \omega.
\tag{2.41}
$$

$$
A_+^{n+\frac{1}{2}} := \partial_z, \qquad A_-^{n-\frac{1}{2}} := \sigma(z)\partial_z + \kappa(z) + n\sigma'(z).
$$

2.6 Implementation of isomorphisms

Let us go back to $\mathbb{C}^3$ in the coordinates s, t, z. An implementation of the isomorphism π_n defined in (2.22) is simple:

$$
\pi_n g(\cdot) = t^{-n} \cdot t^n.
\tag{2.42}
$$

Let us now describe an implementation of the isomorphism ε. Let $\rho(z)$ solve

$$(\sigma(z)\partial_z - \kappa(z))\rho(z) = 0, \tag{2.43}$$

(which defines $\rho(z)$ up to a coefficient). We introduce the following transformation on $\mathcal{A}(\Omega)$:

$$Tg(t,s,z) := \rho(z)g(s,t,z).$$

Theorem 2.1

$$TNT^{-1} = -N, \tag{2.44}$$

$$TA_+T^{-1} = A_-, \tag{2.45}$$

$$TA_-T^{-1} = A_+. \tag{2.46}$$

$$T\mathcal{C}T^{-1} = \mathcal{C}. \tag{2.47}$$

where on the left we have the operators from $m(\sigma, \kappa)$ and on the right from $m(\sigma, -\kappa)$.

Proof. (2.44) is immediate.

Consider $g(t,s,z) \in \mathcal{A}(\Omega)$. Then

$$T^{-1}g(t,s,z) \quad = \rho^{-1}(z)g(t,s,z),$$

$$A_+T^{-1}g(t,s,z) \quad = (t\rho^{-1}(z)\partial_z - \frac{\kappa(z)}{\sigma(z)}t\rho^{-1}(z) + \sigma'(z)\rho^{-1}(z)\partial_s)g(s,t,z)$$

$$= (t\rho^{-1}(z)\partial_z - \frac{\kappa(z)}{s}\rho^{-1}(z) + \sigma'(z)\rho^{-1}(z)\partial_s)g(t,z)$$

$$TA_+T^{-1}g(t,s,z) \quad = (s\partial_z - \frac{\kappa(z)}{t} + \sigma'(z)\partial_t)g(t,s,z).$$

This proves (2.45). The proof of (2.46) is similar.

To show (2.47) we use the second line of (2.38), (2.44), (2.45) and (2.46). $\qquad\qquad\square$

In the coordinates (2.39) the symmetry equals

$$Tg(w,z) = \rho(z)g(\tfrac{\sigma(z)}{w}, z) \ .$$

Let T^n be T restricted to $\mathcal{Z}^n$. Clearly,

$$\mathcal{C}^nT^n = \mathcal{C}^{-n}T^{-n}. \tag{2.48}$$

We easily see that in the coordinates (2.39), for all $F \in \mathcal{A}(\Omega)$,

$$T^nF(z) = \rho(z)\sigma^n(z)F(z).$$

3 Basic properties of hypergeometric class equations

In this section we introduce hypergeometric class equations. They will be presented as the eigenvalue equation of a certain second order differential operator $\mathcal{H}(\sigma, \kappa)$. We discuss a number of properties of these equations that can be described in a unified way: the basic symmetry, basic factorization and basic recurrence relations.

The operator $\mathcal{H}(\sigma, \kappa)$ is essentially the Casimir $\mathcal{C}$ of Miller's Lie algebra at $N = 0$, and all its properties discussed in this section follow from the properties of Miller's Lie algebra analyzed in the previous section. However, this section can be read independently.

3.1 Remark about notation

Let a, b, c be complex functions. Instead of saying that we consider the equation

$$\Big(a(z)\partial_z^2 + b(z)\partial_z + c(z)\Big)f(z) = 0, \tag{3.1}$$

we will usually say that the equation is *given by the operator*

$$\mathcal{A} := a(z)\partial_z^2 + b(z)\partial_z + c(z) \tag{3.2}$$

Instead of $\mathcal{A}$, we will sometimes use the notation $\mathcal{A}(z, \partial_z)$ to indicate the variable that is used in the given operator. This is useful when we consider a change of variables.

3.2 Parametrization of hypergeometric class operators

As was described in the introduction, the main topic of the paper are equations given by operators of the form

$$\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta, \tag{3.3}$$

where σ, τ, η satisfy the conditions (1.2). However, the parametrization of these equations with σ, τ, η is not always convenient. We will usually prefer to use a different parametrization, described below.

Let $\sigma(z)$, $\kappa(z), \omega$ be polynomials with

$$\deg\sigma \leq 2, \ \deg\kappa \leq 1, \ \deg\omega = 0 \quad \text{(in other words } \omega \in \mathbb{C}\text{)}. \tag{3.4}$$

Let us define the differential operator

$$\begin{aligned}
\mathcal{H}(\sigma, \kappa) &:= \sigma(z)\partial_z^2 + \big(\sigma'(z) + \kappa(z)\big)\partial_z + \tfrac{1}{2}\kappa' \\
&= \partial_z \sigma(z)\partial_z + \tfrac{1}{2}\big(\partial_z \kappa(z) + \kappa(z)\partial_z\big).
\end{aligned} \tag{3.5}$$

Clearly, the class of operators (3.3) coincides with the class of operators

$$\mathcal{H}(\sigma, \kappa) + \omega \tag{3.6}$$

Note that $\mathcal{H}(\sigma, \kappa)$ coincides with $\mathcal{C}^0$, the Casimir operator for Miller's Lie algebra restricted to the subspace $N = 0$, expressed in the variable z, see (2.41).

3.3 Basic symmetry

Recall from (3.16) that $\rho(z)$ is defined as a solution of

$$(\sigma(z)\partial_z - \kappa(z))\rho(z) = 0. \tag{3.7}$$

We have the identity

$$\mathcal{H}(\sigma, \kappa) = \rho^{-1}(z)\partial_z \sigma(z)\rho(z)\partial_z + \tfrac{1}{2}\kappa'. \tag{3.8}$$

The following theorem describes a certain symmetry of the entire family of hypergeometric class equations. We will call it the *basic symmetry*.

Theorem 3.1 *We have*

$$\rho(z)\mathcal{H}(\sigma, \kappa)\rho^{-1}(z) = \mathcal{H}(\sigma, -\kappa). \tag{3.9}$$

Hence,

$$\big(\mathcal{H}(\sigma, \kappa) + \omega\big)F = 0 \ \Rightarrow \ \big(\mathcal{H}(\sigma, -\kappa) + \omega\big)\rho F = 0. \tag{3.10}$$

Proof. Using $\sigma(z)\partial_z\rho^{-1}(z) = -\kappa(z)\rho^{-1}(z)$, we obtain

$$\rho(z)\mathcal{H}(\sigma,\kappa)\rho^{-1}(z) = \partial_z\sigma(z)\rho(z)\partial_z\rho^{-1}(z) + \frac{1}{2}\kappa' \tag{3.11}$$

$$= \sigma(z)\partial_z^2 + (\sigma'(z) - \kappa(z))\partial_z - \frac{1}{2}\kappa' = \mathcal{H}(\sigma,-\kappa). \tag{3.12}$$

$\square$

Applying twice the basic symmetry we get the identity. Hence we obtain a group of symmetries of the hypergeometric class isomorphic to $\mathbb{Z}_2$.

Note that the basic symmetry of Theorem 3.1 corresponds to the symmetry ε of Miller's Lie algebra, see (2.21) and Theorem 2.1.

3.4 Basic pair of factorizations

It is often useful to represent a 2nd order operator as a product of two 1st order operators plus a constant. In this section we describe a pair of such factorizations of hypergeometric class operators, which can be easily formulated in a unified way. It is convenient to formulate them for a family indexed by $n \in \mathbb{C}$. These factorizations lead to recurrence relations for hypergeometric class functions.

Fix σ, a polynomial of degree ≤ 2, κ_0, a polynomial of degree ≤ 1 and $\omega_0 \in \mathbb{C}$. We set

$$\kappa_n(z) := n\sigma'(z) + \kappa_0(z), \tag{3.13}$$

$$\omega_n := n^2\frac{\sigma''}{2} + n\kappa_0' + \omega_0. \tag{3.14}$$

Note that $\mathcal{H}(\sigma,\kappa_n) + \omega_n$ coincides with $\mathcal{C}^n + \omega_0$, where $\mathcal{C}^n$ is the Casimir for σ,κ_0 restricted to $N = n$, see (2.41).

Theorem 3.2 *(1) Factorization properties*

$$\mathcal{H}(\sigma,\kappa_n) + \omega_n = \Big(\sigma(z)\partial_z + \kappa_{n+1}(z)\Big)\partial_z + n(n+1)\frac{\sigma''}{2} + (n+\tfrac{1}{2})\kappa_0' + \omega_0$$

$$= \partial_z\Big(\sigma(z)\partial_z + \kappa_n(z)\Big) + n(n-1)\frac{\sigma''}{2} + (n-\tfrac{1}{2})\kappa_0' + \omega_0.$$

(2) Transmutation properties

$$\partial_z\Big(\mathcal{H}(\sigma,\kappa_n) + \omega_n\Big) = \Big(\mathcal{H}(\sigma,\kappa_{n+1}) + \omega_{n+1}\Big)\partial_z,$$

$$\Big(\sigma(z)\partial_z + \kappa_{n+1}(z)\Big)\Big(\mathcal{H}(\sigma,\kappa_{n+1}) + \omega_{n+1}\Big) = \Big(\mathcal{H}(\sigma,\kappa_n) + \omega_n\Big)\Big(\sigma(z)\partial_z + \kappa_{n+1}(z)\Big).$$

(3) Recurrence relations

$$\Big(\mathcal{H}(\sigma,\kappa_n) + \omega_n\Big)F = 0 \quad\Rightarrow\quad \Big(\mathcal{H}(\sigma,\kappa_{n+1}) + \omega_{n+1}\Big)\partial_z F = 0,$$

$$\Big(\mathcal{H}(\sigma,\kappa_{n+1}) + \omega_{n+1}\Big)F = 0 \quad\Rightarrow\quad \Big(\mathcal{H}(\sigma,\kappa_n) + \omega_n\Big)\Big(\sigma(z)\partial_z + \kappa_{n+1}(z)\Big)F = 0.$$

The argument used in the proof of the implication (1)$\Rightarrow$(2) is typical for the so-called *supersymmetric quantum mechanics* and is described in the following lemma:

Lemma 3.3 *Suppose that for $n \in \mathbb{C}$, $\mathcal{A}_+^{n+\frac{1}{2}}$, $\mathcal{A}_-^{n-\frac{1}{2}}$, $\mathcal{H}^n$ are operators and $\eta_{n\pm\frac{1}{2}}$ are numbers satisfying*

$$\begin{aligned}
\mathcal{H}^n &= \mathcal{A}_+^{n+\frac{1}{2}} \mathcal{A}_-^{n-\frac{1}{2}} + \eta_{n-\frac{1}{2}}, \\
&= \mathcal{A}_-^{n-\frac{1}{2}} \mathcal{A}_+^{n+\frac{1}{2}} + \eta_{n+\frac{1}{2}}.
\end{aligned}$$

Then

$$\begin{aligned}
\mathcal{A}_-^{n-\frac{1}{2}} \mathcal{H}^n &= \mathcal{H}^{n-1} \mathcal{A}_-^{n-\frac{1}{2}}, \\
\mathcal{A}_+^{n+\frac{1}{2}} \mathcal{H}^n &= \mathcal{H}^{n+1} \mathcal{A}_+^{n+\frac{1}{2}}.
\end{aligned}$$

Proof of Theorem 3.2 (1) follows by a direct computation. (1) implies (2) by Lemma 3.3 if we set

$$\begin{aligned}
\mathcal{A}_+^{n+\frac{1}{2}} &= \partial_z, \quad \mathcal{A}_-^{n-\frac{1}{2}} = \sigma(z)\partial_z + \kappa_n(z), \quad \eta_{n+\frac{1}{2}} = n(n+1)\tfrac{\sigma''}{2} + (n+\tfrac{1}{2})\kappa' + \omega, \\
\mathcal{H}^n &:= \mathcal{H}(\sigma, \kappa_n) + \omega_n.
\end{aligned}$$
$$(3.15)$$

(2) easily implies (3). $\qquad\square$

3.5 Ladders of solutions

Let κ_n, ω_n be as in (3.13) and (3.14). Let ρ_0 solve (3.8) for σ_0, κ_0, that is,

$$\big(\sigma_0(z)\partial_z - \kappa_0(z)\big)\rho_0(z) = 0. \tag{3.16}$$

Suppose we have a solution

$$\Big(\mathcal{H}(\sigma, \kappa_0) + \omega_0\Big)f = 0. \tag{3.17}$$

Then from f we can construct a two-sided ladder of solutions. More precisely, for any $n \in \mathbb{N}_0$

$$\mathcal{H}(\sigma, \kappa_n) + \omega_n \quad \text{annihilates} \quad \partial_z^n f, \tag{3.18}$$
$$\mathcal{H}(\sigma, \kappa_{-n}) + \omega_{-n} \quad \text{annihilates} \quad \sigma^n \rho_0^{-1} \partial_z^n \rho_0 f. \tag{3.19}$$

To see (3.19), we note that

$$\sigma(z)\partial_z + \kappa_{-j}(z) = \sigma(z)^{j+1}\rho_0(z)^{-1}\partial_z^j \sigma(z)^{-j}\rho_0(z), \tag{3.20}$$
$$\text{hence } \big(\sigma(z)\partial_z + \kappa_{-n}(z)\big) \cdots \big(\sigma(z)\partial_z + \kappa(z)\big) = \sigma(z)^n \rho_0(z)^{-1}\partial_z^n \rho_0(z). \tag{3.21}$$

Consider now special cases $\omega_0 = \mp\frac{\kappa_0'}{2}$:

$$\mathcal{H}(\sigma, \kappa_0) - \frac{\kappa_0'}{2} = \sigma(z)\partial_z^2 + \big(\sigma'(z) + \kappa_0(z)\big)\partial_z, \tag{3.22}$$

$$\mathcal{H}(\sigma, \kappa_0) + \frac{\kappa_0'}{2} = \sigma(z)\partial_z^2 + \big(\sigma'(z) + \kappa_0(z)\big)\partial_z + \kappa_0'. \tag{3.23}$$

They have elementary solutions:

$$\mathcal{H}(\sigma, \kappa_0) - \frac{\kappa_0'}{2} \qquad \text{annihilates } 1, \tag{3.24}$$

$$\mathcal{H}(\sigma, \kappa_0) + \frac{\kappa_0'}{2} \qquad \text{annihilates } \rho_0^{-1}. \tag{3.25}$$

(3.24) is obvious. To see (3.25) we differentiate

$$(\sigma(z)\partial_z + \kappa_0(z))\rho_0^{-1}(z) = 0,$$

obtaining

$$0 = (\sigma(z)\partial_z^2 + (\sigma'(z) + \kappa_0(z))\partial_z + \kappa_0')\rho_0^{-1}(z) \tag{3.26}$$

Alternatively, we can derive (3.25) from (3.24) by the basic symmetry (3.10). Indeed, (3.24) and (3.10) imply that $\mathcal{H}(\sigma, -\kappa_0) - \frac{\kappa_0'}{2}$ annihilates ρ_0. Then we switch the sign of κ_0, which corresponds to replacing ρ_0 with ρ_0^{-1}.

The special solutions (3.24) and (3.25) lead to a pair of one-sided ladders: for any $n \in \mathbb{N}_0$

$$\mathcal{H}(\sigma, -n\sigma' + \kappa_0) - (n + \tfrac{1}{2})\kappa_0' + n^2\frac{\sigma''}{2} \quad \text{annihilates} \quad \sigma^n \rho_0^{-1}\partial_z^n \rho_0 \tag{3.27}$$

$$\mathcal{H}(\sigma, n\sigma' + \kappa_0) + (n + \tfrac{1}{2})\kappa_0' + n^2\frac{\sigma''}{2} \quad \text{annihilates} \quad \partial_z^n \rho_0^{-1}. \tag{3.28}$$

(3.27) consists of polynomials, which we will analyze in more detail in Section 7. The ladder (3.28) consists of functions of the form $\rho_0^{-1}\sigma^{-n}P_n$, where $P_n := \sigma^n \rho_0 \partial_z^n \rho_0^{-1}$ are polynomials.

3.6 Chebyshev ladder

Let us fix σ, as usual a polynomial of degree ≤ 2, and $\omega \in \mathbb{C}$. The equations given by the following two operators can be easily solved in elementary functions:

$$\mathcal{H}\Big(\sigma, -\frac{\sigma'}{2}\Big) + \omega + \frac{\sigma''}{4} = \sqrt{\sigma(z)}\partial_z\sqrt{\sigma(z)}\partial_z + \omega \tag{3.29}$$

$$= \sigma(z)\partial_z^2 + \frac{\sigma'(z)}{2}\partial_z + \omega, \tag{3.30}$$

$$\mathcal{H}\Big(\sigma, \frac{\sigma'}{2}\Big) + \omega + \frac{\sigma''}{4} = \partial_z\sqrt{\sigma(z)}\partial_z\sqrt{\sigma(z)} + \omega \tag{3.31}$$

$$= \sigma(z)\partial_z^2 + \frac{3\sigma'(z)}{2}\partial_z + \frac{\sigma''}{2} + \omega. \tag{3.32}$$

In fact, set

$$y(z) = \int_0^z \frac{\mathrm{d}x}{\sqrt{\sigma(x)}}, \tag{3.33}$$

which solves the equation $\frac{\mathrm{d}y}{\mathrm{d}z} = \frac{1}{\sqrt{\sigma(z)}}$. Then

$$\mathcal{H}\Big(\sigma, -\frac{\sigma'}{2}\Big) + \omega + \frac{\sigma''}{4} \quad \text{annihilates} \quad \Big(A\sin\big(\omega y(z)\big) + B\cos\big(\omega y(z)\big)\Big), \tag{3.34}$$

$$\mathcal{H}\Big(\sigma, \frac{\sigma'}{2}\Big) + \omega + \frac{\sigma''}{4} \quad \text{annihilates} \quad \frac{1}{\sqrt{\sigma(z)}}\Big(A\sin\big(\omega y(z)\big) + B\cos\big(\omega y(z)\big)\Big). \tag{3.35}$$

We can embed the parameters of (3.29) and (3.31) into a single ladder as follows. We set $\kappa_0 = 0$ and $\omega_0 = \omega + \frac{\sigma''}{8}$, so that

$$\kappa_m = m\sigma', \quad \omega_m = \omega + \Big(m^2 + \frac{1}{4}\Big)\frac{\sigma''}{2}. \tag{3.36}$$

then

$$(3.29) = \mathcal{H}\big(\sigma, \kappa_{-\frac{1}{2}}\big) + \omega_{-\frac{1}{2}}, \tag{3.37}$$

$$(3.31) = \mathcal{H}\big(\sigma, \kappa_{\frac{1}{2}}\big) + \omega_{\frac{1}{2}}. \tag{3.38}$$

By using the recurrence relations (3.18) and (3.19) we obtain elementary solutions for

$$\mathcal{H}(\sigma, \kappa_m) + \omega_m = \sigma(z)\partial_z^2 + (m+1)\sigma'(z)\partial_z + \Big(m + \frac{1}{2}\Big)^2 \frac{\sigma''}{2} + \omega \tag{3.39}$$

with $m \in \mathbb{Z} + \frac{1}{2}$.

Remark 3.4 *Chebyshev ladders are very special, since κ is determined by σ. The only two nontrivial hypergeometric types where they appear are $_2F_1$ and F_1. However, they are quite important. One can note that Bessel functions of half-integer parameters are elementary functions because they correspond to the Chebyshev ladder of the F_1 type.*

4 Singular point

Throughout this section we assume that

$$\sigma(0) = 0. \tag{4.1}$$

We will discuss solutions of hypergeometric class equations given by power series around 0. We will also describe some symmetries which exist if (4.1) holds.

4.1 Solutions around a singular point

(4.1) usually means that 0 is a singular point of the equation (3.6). In fact, rewriting the equation (3.6) as

$$\frac{1}{\sigma(z)}\big(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \omega\big)f(z) = \Big(\partial_z^2 + \frac{\tau(z)}{\sigma(z)}\partial_z + \frac{\omega}{\sigma(z)}\Big)f(x) = 0 \tag{4.2}$$

we see that $\frac{\tau(z)}{\sigma(z)}$ or $\frac{\omega}{\sigma(z)}$ will usually have a singularity at 0. Straightforward calculations (known under the name of the *Frobenius method* [WW]) lead then to the following result

Theorem 4.1 *There exists a unique formal power series $F(z)$ solving*

$$\Big(\mathcal{H}(\sigma, \kappa) + \omega\Big)F(z) = 0, \quad F(0) = 1. \tag{4.3}$$

It is equal to

$$F(\sigma, \kappa, \omega; z) = \sum_{n=0}^{\infty} \frac{\Pi_{j=0}^{n-1}(\omega + (j + \frac{1}{2})\kappa' + j(j+1)\frac{\sigma''}{2})}{\Pi_{j=0}^{n-1}(\kappa(0) + (j+1)\sigma'(0))n!}(-z)^n \tag{4.4}$$

If $\sigma'(0) = 0$ and $\sigma'' \neq 0$, then $\sigma(z)$ has a double root at zero and then the series (4.4) either terminates and defines a polynomial, or is divergent. However, it is always asymptotic to one of the solutions of (3.6) defined on $\mathbb{C}\backslash[0, \infty[$ (see Section 6.3 and Appendix A). In all other cases the series (4.4) has a nonzero radius of convergence. The function $F(\sigma, \kappa, \omega; z)$ given by (4.4) will be called the *unified hypergeometric function*. It depends meromorphically on $\sigma'(0), \sigma'', \kappa, \omega$.

Recall that κ_n and ω_n were defined in (3.13) and (3.14). The basic pair of recurrence relations is as follows:

$$\partial_z F(\sigma, \kappa_n, \omega_n; z) = -\frac{\omega_n + \frac{1}{2}\kappa_n'}{\kappa_{n+1}(0)}F(\sigma, \kappa_{n+1}, \omega_{n+1}; z), \tag{4.5}$$

$$(\sigma(z)\partial_z + \kappa_{n+1}(z))F(\sigma, \kappa_{n+1}, \omega_{n+1}; z) = \kappa_{n+1}(0)F(\sigma, \kappa_n, \omega_n; z). \tag{4.6}$$

4.2 Olver's normalization

Assume now $\sigma'(0) \neq 0$. We set $m := \frac{\kappa(0)}{\sigma'(0)}$. Without sacrificing the generality we can suppose additionally that $\sigma'(0) = 1$. Thus

$$\sigma(z) = z + \frac{\sigma''}{2}z^2, \qquad \kappa(z) = m + \kappa'z. \tag{4.7}$$

0 is a regular singular point and $0, -m$ are its indices, see [WW] or Appendix B.

In this case often instead of the function (4.4) it is more convenient to use the function

$$\mathbf{F}(\sigma, \kappa, \omega; z) := \frac{F(\sigma, \kappa, \omega; z)}{\Gamma(1+m)} = \sum_{n=0}^{\infty} \frac{\Pi_{j=0}^{n-1}(\omega + (j + \frac{1}{2})\kappa' + j(j+1)\frac{\sigma''}{2})}{\Gamma(m+n+1)n!}(-z)^n. \tag{4.8}$$

Note that $\mathbf{F}$ is holomorphic in σ'', κ, ω. Dividing by $\Gamma(1+m)$, as in (4.8), is sometimes called *Olver's normalization*, since it was made popular by Frank Olver's textbook [Ol2]. Here is the basic pair of recurrence relations for the *Olver normalized unified hypergeometric function*:

$$\partial_z\mathbf{F}(\sigma, \kappa_n, \omega_n; z) = -\left(\omega_n + \frac{1}{2}\kappa_n'\right)\mathbf{F}(\sigma, \kappa_{n+1}, \omega_{n+1}; z), \tag{4.9}$$

$$(\sigma(z)\partial_z + \kappa_{n+1}(z))\,\mathbf{F}(\sigma, \kappa_{n+1}, \omega_{n+1}; z) = \mathbf{F}(\sigma, \kappa_n, \omega_n; z). \tag{4.10}$$

4.3 Power symmetry

We keep the assumptions of Subsection 4.2. Let us introduce a certain transformation of the parameters. We keep σ the same, and the transformed κ and ω are

$$\kappa^\circ(z) := -m + (\kappa' - m\sigma'')z, \quad \omega^\circ := \omega - m\kappa' + m^2\frac{\sigma''}{2}. \tag{4.11}$$

Consequently, $m^\circ = -m$.

Theorem 4.2

$$z^m\Big(\mathcal{H}(\sigma, \kappa) + \omega\Big)z^{-m} = \mathcal{H}(\sigma, \kappa^\circ) + \omega^\circ. \tag{4.12}$$

Proof. Using

$$z^m\partial_z z^{-m} = \partial_z - \frac{m}{z}, \qquad z^m\partial_z^2 z^{-m} = \partial_z^2 - \frac{2m}{z}\partial_z + \frac{m(m+1)}{z^2} \tag{4.13}$$

we compute

$$z^m\Big(\mathcal{H}(\sigma, \kappa) + \omega\Big)z^{-m} \tag{4.14}$$

$$= z^m\Big(\Big(z + \frac{\sigma''}{2}z^2\Big)\partial_z^2 + \big(1 + m + (\sigma'' + \kappa')z\big)\partial_z + \frac{\kappa'}{2} + \omega\Big)z^{-m} \tag{4.15}$$

$$= \Big(z + \frac{\sigma''}{2}z^2\Big)\partial_z^2 + \Big(1 - m + \big((1-m)\sigma'' + \kappa'\big)z\Big)\partial_z + \frac{\sigma''}{2}m(m-1) - \kappa'(m - \tfrac{1}{2}) + \omega \tag{4.16}$$

$$= \mathcal{H}(\sigma, \kappa^\circ) + \omega^\circ. \tag{4.17}$$

Note that $\kappa^{\circ\circ} = \kappa$ and $\omega^{\circ\circ} = \omega$. Thus we obtain a $\mathbb{Z}_2$ symmetry of the hypergeometric class. It is different from the basic symmetry of Thm 3.1.

As a consequence of (4.12), both $\mathbf{F}(\sigma, \kappa, \omega; z)$ and $z^{-m}\mathbf{F}(\sigma, \kappa^\circ, \omega^\circ; z)$ are annihilated by $\mathcal{H}(\sigma, \kappa) + \omega$. If $m \notin \mathbb{Z}$ they form a basis of solutions of the corresponding equation. The situation for $m \in \mathbb{Z}$ will be discussed in Section 8 about the degenerate case.

4.4 Inversion symmetry

We still assume $\sigma(0) = 0$. Besides, we suppose that

$$\sigma'' \neq 0 \quad \text{or} \quad \kappa' \neq 0. \tag{4.18}$$

Suppose that $\zeta \in \mathbb{C}$ solves the equation

$$\frac{\sigma''}{2}\zeta^2 + (\sigma'' + \kappa')\zeta + \frac{\kappa}{2} + \omega = 0. \tag{4.19}$$

Note that (4.18) guarantees that (4.19) has a solution. Set

$$\sigma^\triangle := \frac{\sigma''}{2}w - \sigma'(0)w^2, \tag{4.20}$$

$$\kappa^\triangle = \big(\kappa(0) + 2(\zeta + 1)\sigma'(0)\big)w - \sigma''(1 + \zeta) - \kappa', \tag{4.21}$$

$$\omega^\triangle := -\sigma'(0)(1 + \zeta)^2 - \kappa(0)(\zeta + \tfrac{1}{2}). \tag{4.22}$$

Theorem 4.3 *Consider the substitution* $w = -\frac{1}{z}$. *Then we have*

$$-z^{-\zeta+1}\Big(\mathcal{H}(\sigma, \kappa; z, \partial_z) + \omega\Big)z^\zeta = \mathcal{H}(\sigma^\triangle, \kappa^\triangle; w, \partial_w) + \omega^\triangle \tag{4.23}$$

Hence

$$\mathcal{H}(\sigma, \kappa) + \omega \quad \text{annihilates} \quad z^{-\zeta}\mathbf{F}(\sigma^\triangle, \kappa^\triangle, \omega^\triangle; -z^{-1}). \tag{4.24}$$

Proof. Indeed, using $\partial_z = w^2\partial_w$ and $\partial_z^2 = w^4\partial_w^2 + 2w^3\partial_w$, we obtain

$$-z\Big(\mathcal{H}(\sigma, \kappa; z, \partial_z) + \omega\Big) \tag{4.25}$$

$$=\sigma(-w^{-1})w^3\partial_w^2 + \Big(2\sigma(-w^{-1})w^2 + \sigma'(-w^{-1})w + \kappa(-w^{-1})w\Big)\partial_w + \Big(\frac{\kappa'}{2} + \omega\Big)w^{-1}. \tag{4.26}$$

Then we use $w^\zeta\partial_w w^{-\zeta} = \partial_w - \frac{\zeta}{w}$ and $w^\zeta\partial_w^2 w^{-\zeta} = \partial_w^2 - \frac{2\zeta}{w}\partial_w + \frac{(\zeta+\zeta^2)}{w^2}$ to obtain

$$w^\zeta(4.26)w^{-\zeta} \tag{4.27}$$

$$=\Big(\frac{\sigma''}{2}w - \sigma'(0)w^2\Big)\partial_w^2 + \Big(\big(\kappa(0) + 2\zeta\sigma'(0)\big)w - \frac{\sigma''}{2} - \zeta\sigma'' - \kappa'\Big)\partial_w \tag{4.28}$$

$$- \sigma'(0)(\zeta + \zeta^2) - \kappa(0)\zeta + \Big(\frac{\kappa'}{2} + \omega + \frac{\sigma''}{2}(2\zeta + \zeta^2) + \kappa'\zeta\Big)w^{-1} \tag{4.29}$$

$$=\sigma^\triangle(w)\partial_w^2 + \big(\sigma^{\triangle\prime}(w) + \kappa^\triangle(w)\big)\partial_w + \frac{\kappa^{\triangle\prime}}{2} + \omega^\triangle. \tag{4.30}$$

$$\square$$

Note that $\sigma^{\triangle\triangle} = \sigma$, $\kappa^{\triangle\triangle} = \kappa$, $\omega^{\triangle\triangle} = \omega$. Hence the inversion symmetry generates a group isomorphic to $\mathbb{Z}_2$ acting on the hypergeometric class.

Remark 4.4 *It is easy to see that the substitution* $z \mapsto \frac{1}{w}$ *also generates a symmetry of hypergeometric class equations. This follows immediately from Theorem 4.3 and the trivial fact that* $z \mapsto -z$ *leads to a symmetry of the hypergeometric class as well.*

5 Integral representations

Hypergeometric class functions possess useful integral representations. Most of them have the form of an Euler transform of an elementary function. In Subsection 5.1 we will show how to derive these representations in a unified way. Note that typically for a given equation one can choose various contours of integration, obtaining various solutions.

Some important integral represenations have a different form—they can be viewed as Laplace transforms of certain elementary functions. They will be described in Subsection 5.2.

In this section we will often deal with multivalued functions $s \mapsto f(s)$ defined on a certain Riemann surface. These functions are analytically continued along a certain curve γ contained in this Riemann surface. We will use the notation

$$f(s)\Big|_{s_0}^{s_1} = f(s_1) - f(s_0), \tag{5.1}$$

where s_1 and s_2 are the endpoints of the curve γ.

As usual, we fix σ, κ_0, and for $n \in \mathbb{C}$, as in (3.13), we set

$$\kappa_n(z) := n\sigma'(z) + \kappa_0(z).$$

In this section $\omega_0 = \frac{\kappa_0'}{2}$, so that according to (3.14),

$$\omega_n := n^2\frac{\sigma''}{2} + (n + \tfrac{1}{2})\kappa_0'. \tag{5.2}$$

5.1 Euler transforms

As usual, we assume that

$$(\sigma(z)\partial_z - \kappa_0(z))\rho_0(z) = 0. \tag{5.3}$$

Theorem 5.1 *Let $n \in \mathbb{C}$. Suppose that the curve $[0, 1] \ni \tau \overset{\gamma}{\mapsto} s(\tau)$ satisfies*

$$\sigma(s)(s - z)^{-n-2}\rho_0^{-1}(s)\Big|_{s(0)}^{s(1)} = 0 \tag{5.4}$$

and

$$f_n(z) := \int_\gamma (s - z)^{-n-1}\rho_0^{-1}(s)\mathrm{d}s \tag{5.5}$$

is well-defined. Then f_n is annihillated by

$$\mathcal{H}(\sigma, \kappa_n) + \omega_n \tag{5.6}$$

$$=\sigma(z)\partial_z^2 + (\kappa_0(z) + (n + 1)\sigma'(z))\partial_z + \kappa_0'(n + 1) + \frac{\sigma''}{2}n(n + 1). \tag{5.7}$$

Besides, if f_{n+1} is also well-defined, then

$$\partial_z f_n(z) = (n + 1)f_{n+1}(z), \tag{5.8}$$

$$(\sigma(z)\partial_z + \kappa_{n+1}(z)) f_{n+1}(z) = -\left(\kappa_0' + \frac{\sigma''}{2}n\right) f_n(z). \tag{5.9}$$

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Proof. For simplicity, we will assume that both f_n and f_{n+1} are well defined. (5.8) is obvious. Let us prove (5.9). Using the Taylor expansion of the polynomials $\sigma(z)$ and $\kappa(z)$ we obtain

$$\Big(\sigma(z)\partial_z + \kappa_0(z) + (n+1)\sigma'(z)\Big)f_{n+1}(z) \tag{5.10}$$

$$=(n+2)\int_\gamma \Big(\sigma(s) + \sigma'(s)(z-s) + \frac{\sigma''}{2}(z-s)^2\Big)(s-z)^{-n-3}\rho_0(s)^{-1}\mathrm{d}s \tag{5.11}$$

$$+\int_\gamma \Big(\kappa_0(s) + (z-s)\kappa' + (n+1)\sigma'(s) + (n+1)(z-s)\sigma''\Big)(s-z)^{-n-2}\rho_0(s)^{-1}\mathrm{d}s \tag{5.12}$$

$$=-\Big(n\frac{\sigma''}{2} + \kappa_0'\Big)\int_\gamma (s-z)^{-n-1}\rho_0(s)^{-1}\mathrm{d}s \tag{5.13}$$

$$+\int_\gamma \Big((n+2)\sigma(s)(s-z)^{-n-3} - \sigma'(s)(s-z)^{-n-2} + \kappa_0(s)(s-z)^{-n-2}\Big)\rho_0(s)^{-1}\mathrm{d}s \tag{5.14}$$

$$=-\Big(n\frac{\sigma''}{2} + \kappa_0'\Big)f_n(z) \tag{5.15}$$

$$+\int_\gamma \Big(\partial_s\sigma(s)(s-z)^{-n-2}\rho_0(s)^{-1}\Big)\mathrm{d}s \tag{5.16}$$

$$+\int_\gamma (s-z)^{-n-2}\Big(\sigma(s)\partial\rho_0(s)^{-1} + \kappa_0(s)\rho_0(s)^{-1}\Big)\mathrm{d}s. \tag{5.17}$$

(5.16) vanishes because of (5.4). (5.17) is zero by (5.3). $\qquad\square$

It is sometimes useful to consider differently normalized Euler integrals:

$$\mathbf{f}_n(z) := \frac{f_n(z)}{\Gamma(n+1)} = \frac{1}{\Gamma(n+1)}\int_\gamma (s-z)^{-n-1}\rho_0^{-1}(s)\mathrm{d}s \tag{5.18}$$

Then the recurrence relations (5.8) and (5.9) are modified:

$$\begin{aligned}
\partial_z\mathbf{f}_n(z) &= \mathbf{f}_{n+1}(z), \\
(\sigma(z)\partial_z + \kappa_{n+1}(z))\,\mathbf{f}_{n+1}(z) &= -(n+1)\Big(\kappa_0' + \tfrac{\sigma''}{2}n\Big)\mathbf{f}_n(z).
\end{aligned} \tag{5.19}$$

5.2 Laplace integrals

Suppose that $\sigma'' = 0$. We still assume (5.2), which now can be rewritten as

$$\omega_n = (n + \tfrac{1}{2})\kappa_0'. \tag{5.20}$$

Let

$$\Big(\sigma(-\partial_s)s + \kappa_0(-\partial_s)\Big)\delta_0(s) = 0, \tag{5.21}$$

which defines up to a coefficient an elementary function δ_0.

Theorem 5.2 *Assume that $[0,1] \ni \tau \overset{\gamma}{\mapsto} s(\tau)$ satisfies*

$$(s^{n+2}\sigma' + s^{n+1}\kappa')\delta_0(s)\mathrm{e}^{sz}\Big|_{s(0)}^{s(1)} = 0, \tag{5.22}$$

and the following integral exists:

$$g_n(z) = \int_\gamma \delta_0(s)s^n\mathrm{e}^{zs}\mathrm{d}s. \tag{5.23}$$

Then g_n is annihillated by

$$\mathcal{H}(\sigma, \kappa_n) + \omega_n \tag{5.24}$$

$$= \sigma(z)\partial_z^2 + (\sigma'(z)(n+1) + \kappa_0(z))\partial_z + (n+1)\kappa_0'. \tag{5.25}$$

Besides, if also g_{n+1} is well defined, then

$$\partial_z g_n(z) = g_{n+1}, \tag{5.26}$$

$$(\sigma(z)\partial_z + \kappa_{n+1}(z))g_{n+1}(z) = -(n+1)\kappa_0' g_n(z). \tag{5.27}$$

Proof. For simplicity, we will assume that both g_n and g_{n+1} are well defined, and we will prove the recurrence relations, which directly imply (5.25).

(5.26) is obvious. Let us show (5.27):

$$(\sigma(z)\partial_z + (n+1)\sigma' + \kappa_0(z))g_{n+1} \tag{5.28}$$

$$= \int_\gamma \Big(\sigma'\delta_0(s)s^{n+2}z + \sigma(0)\delta_0(s)s^{n+2} + (n+1)\sigma'\delta_0(s)s^{n+1}\Big)e^{zs}\mathrm{d}s \tag{5.29}$$

$$+ \int_\gamma \Big(\kappa_0'\delta_0(s)s^{n+1}z + \kappa_0(0)\delta_0(s)s^{n+1}\Big)e^{zs}\mathrm{d}s \tag{5.30}$$

$$= \int_\gamma \mathrm{d}s\partial_s\Big(\sigma'\delta_0(s)s^{n+2}e^{zs} + \kappa_0'\delta_0(s)s^{n+2}e^{zs}\Big) \tag{5.31}$$

$$+ \int_\gamma \Big(-\sigma's\delta_0'(s) - \sigma'\delta_0(s) - \kappa_0'\delta_0'(s) + \sigma(0)s\delta_0(s) + \kappa_0(0)\delta_0(s)\Big)s^{n+1}e^{zs}\mathrm{d}s \tag{5.32}$$

$$- (n+1)\kappa_0' \int \delta_0(s)s^n e^{zs}\mathrm{d}s. \tag{5.33}$$

Now (5.31) vanishes because of (5.22) and (5.32) due to (5.21).   □

Sometimes it will be more convenient to present the integral represention (5.23) in a different form, with the variable s replaced with $t = \frac{1}{s}$:

$$g_n(z) = \int_{\tilde\gamma} \delta_0(t^{-1})t^{-n+2}e^{\frac{z}{t}}\mathrm{d}t. \tag{5.34}$$

where the contour $\tilde\gamma$ is obtaned by the change of variable and reversing the orientation.

6 Applications case by case

Hypergeometric class operators can be divided into several types. By an affine transformation $z \mapsto az+b$ and division by a constant, an operator of each type can be reduced to its normal form. There are 5 nontrivial types, with normal forms listed in the introduction. For instance, if σ has two distinct roots, then the equation belongs to the $_2F_1$ type, if it has a double root, it belongs to the $_2F_0$ type, etc. The hypergometric class contains also 4 "trivial types", which can be solved in an elementary way. All these 9 types are listed in the table in Appendix B.

Strictly speaking, this table is devoted to types of the Riemann class, which is larger than the hypergeometric class. Therefore, this table contains 10 types. It includes one additional type: the Airy operator, which cannot be reduced to the hypergeometric class. However, all the 9 other types of the Riemann class are represented in the hypergeometric class.

In this section we describe the properties of all five nontrivial types of the hypergeometric class that are direct consequences of the unified theory, discussed in the previous sections. We try to follow the same pattern. Here is the list of items that we will give for each hypergeometric type, if available:

- Operator that generates the equation, that is $\mathcal{H}(\sigma, \kappa) + \omega$, expressed in the traditional form.

- Parameters $\sigma(z), \kappa(z), \omega$; see (3.4) and (3.5).

- Weight $\rho(z)$, see (3.16) or (3.7).

- Basic symmetry, see (3.9).

- Power symmetry (if available), see (4.12).

- Inversion symmetry (if available), see (4.24).

- Theorem about integral representations, see Theorems 5.1 and 5.2.

- Standard solution annihillated by $\mathcal{H}(\sigma, \kappa) + \omega$ (typically a special case of the unified hypergeometric function (4.4), but for the Hermite equation we need to make an exception).

- Olver normalized standard solution (if available), see (4.8).

- Basic pair of recurrence relations for standard solutions, see (4.5)-(4.6) or (4.9)-(4.10).

- Integral representations of standard solution (obtained by an application of Theorems 5.1 or 5.2).

- Chebyshev solutions (if available), see Subsection 3.6.

We will often use the *Pochhammer symbol*

$$(a)_j := a(a+1)\cdots(a+j-1), \qquad a \in \mathbb{C}, \quad j \in \mathbb{N}_0. \tag{6.1}$$

6.1 The $_2F_1$ equation

$_2F_1$ **operator:**

$$\mathcal{F}(a,b;c) := z(1-z)\partial_z^2 + \big(c - (a+b+1)z\big)\partial_z - ab. \tag{6.2}$$

Parameters:

$$\sigma(z) = z(1-z), \quad \kappa(z) = c - 1 - (a+b-1)z, \quad \omega = -\big(a - \tfrac{1}{2}\big)\big(b - \tfrac{1}{2}\big) - \tfrac{1}{4}.$$

Weight:

$$\rho(z) = z^{c-1}(z-1)^{a+b-c}.$$

Basic symmetry:

$$z^{c-1}(z-1)^{a+b-c}\mathcal{F}(a,b;c)z^{1-c}(z-1)^{-a-b+c} = \mathcal{F}(1-b, 1-a; 2-c).$$

Power symmetry:

$$z^{c-1}\mathcal{F}(a,b;c)z^{1-c} = \mathcal{F}(b+1-c, a+1-c; 2-c).$$

Inversion symmetry:

$$(-z)^{1+a}\mathcal{F}(a,b;c;z,\partial_z)(-z)^{-a} = \mathcal{F}(a, a-c+1; a-b+1; w, \partial_w), \qquad w = z^{-1}.$$

Theorem 6.1 (about integral representations) *Let $[0,1] \ni \tau \overset{\gamma}{\mapsto} t(\tau)$ satisfy*

$$t^{b-c+1}(1-t)^{c-a}(t-z)^{-b-1}\Big|_{\gamma(0)}^{\gamma(1)} = 0.$$

Then

$$\mathcal{F}(a,b;c) \quad \text{annihilates} \quad \int_\gamma t^{b-c}(1-t)^{c-a-1}(t-z)^{-b}\mathrm{d}t. \tag{6.3}$$

Proof. We check that

$$\mathcal{F}(a,b;c)t^{b-c}(1-t)^{c-a-1}(t-z)^{-b} = -b\partial_t t^{b-c+1}(1-t)^{c-a}(t-z)^{-b-1}. \tag{6.4}$$

$\square$

$_2F_1$ **function:** For $|z| < 1$ defined by the following power series, then analytically extended:

$$F(a,b;c;z) = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{(c)_j} \frac{z^j}{j!}, \quad |z| < 1.$$

Olver normalized $_2F_1$ function:

$$\mathbf{F}(a,b;c;z) := \frac{F(a,b,c,z)}{\Gamma(c)} = \sum_{j=0}^{\infty} \frac{(a)_j (b)_j}{\Gamma(c+j)} \frac{z^j}{j!}.$$

Integral representation:

$$\int_1^{\infty} t^{b-c}(t-1)^{c-a-1}(t-z)^{-b}\mathrm{d}t \tag{6.5}$$
$$= \Gamma(a)\Gamma(c-a)\mathbf{F}(a,b;c;z), \quad \mathrm{Re}(c-a) > 0, \ \mathrm{Re}\,a > 0, \quad z \notin [1,\infty[.$$

Basic pair of recurrence relations:

$$\partial_z \mathbf{F}(a,b;c;z) = ab\mathbf{F}(a+1,b+1;c+1;z), \tag{6.6}$$

$$\Big(z(1-z)\partial_z + \big(c - (a+b+1)z\big)\Big)\mathbf{F}(a+1,b+1;c+1;z) = \mathbf{F}(a,b;c;z). \tag{6.7}$$

Chebyshev solutions for $k = 0, 1, \ldots$:

$$\mathbf{F}\Big(1+k+\lambda, 1+k-\lambda; \frac{3}{2}+k; \frac{1-w}{2}\Big) = \frac{(1-w^2)^{\frac{1}{2}+k}}{2\sqrt{\pi}(-2)^k}\partial_w^k \frac{\big(w+\mathrm{i}\sqrt{1-w^2}\big)^\lambda + \big(w-\mathrm{i}\sqrt{1-w^2}\big)^\lambda}{\sqrt{1-w^2}}, \tag{6.8}$$

$$\mathbf{F}\Big(-k+\lambda, -k-\lambda; \frac{1}{2}-k; \frac{1-w}{2}\Big) = \frac{2^k}{\mathrm{i}\sqrt{\pi}(\lambda-k)_{2k+1}}\partial_w^k \frac{\big(w+\mathrm{i}\sqrt{1-w^2}\big)^\lambda - \big(w-\mathrm{i}\sqrt{1-w^2}\big)^\lambda}{\sqrt{1-w^2}}. \tag{6.9}$$

6.2 The $_1F_1$ equation

$_1F_1$ **operator**

$$\mathcal{F}(a;c) := z\partial_z^2 + (c-z)\partial_z - a. \tag{6.10}$$

Parameters

$$\sigma(z) = z, \quad \kappa(z) = c - 1 - z, \quad \omega = -a + \tfrac{1}{2}.$$

Weight:

$$\rho(z) = \mathrm{e}^{-z}z^{c-1}.$$

Theorem 6.2 (about integral representations)   *1. Let $[0,1] \ni \tau \overset{\gamma}{\mapsto} t(\tau)$ satisfy*

$$t^{a-c+1}e^t(t-z)^{-a-1}\Big|_{t(0)}^{t(1)} = 0.$$

Then

$$\mathcal{F}(a;c) \quad \text{annihilates} \quad \int_\gamma t^{a-c}e^t(t-z)^{-a}\mathrm{d}t. \tag{6.11}$$

2. Let $[0,1] \ni \tau \overset{\gamma}{\mapsto} t(\tau)$ satisfy

$$e^{\frac{z}{t}}t^{-c}(1-t)^{c-a}\Big|_{t(0)}^{t(1)} = 0.$$

Then

$$\mathcal{F}(a;c) \quad \text{annihilates} \quad \int_\gamma e^{\frac{z}{t}}t^{-c}(1-t)^{c-a-1}\mathrm{d}t. \tag{6.12}$$

Proof.

$$\mathcal{F}(a,c)\, t^{a-c}e^t(t-z)^{-a} = -a\partial_t t^{a-c+1}e^t(t-z)^{-a-1}, \tag{6.13}$$
$$\mathcal{F}(a,c)\, e^{\frac{z}{t}}t^{-c}(1-t)^{c-a-1} = -\partial_t e^{\frac{z}{t}}t^{-c}(1-t)^{c-a}. \tag{6.14}$$

$\square$

Basic symmetry:

$$z^{c-1}e^{-z}\mathcal{F}(a;c;z,\partial_z)z^{1-c}e^z = \mathcal{F}(1-a;2-c;w,\partial_w), \quad w = -z.$$

Power symmetry:

$$z^{c-1}\mathcal{F}(a;c)z^{1-c} = \mathcal{F}(1+a-c;2-c).$$

Inversion symmetry:

$$z^{a+1}\mathcal{F}(a;c;z,\partial_z)z^{-a} = \mathcal{F}(a,1+a-c;-;w,\partial_w), \qquad w = -z^{-1}.$$

(Thus the $_1F_1$ equation is equivalent to the $_2F_0$ equation.)

$_1F_1$ **function:**

$$F(a;c;z) := \sum_{j=0}^{\infty} \frac{(a)_j}{(c)_j} \frac{z^j}{j!}.$$

Olver normalized $_1F_1$ function

$$\mathbf{F}(a;c;z) := \frac{F(a;c;z)}{\Gamma(c)} = \sum_{j=0}^{\infty} \frac{(a)_j}{\Gamma(c+j)} \frac{z^j}{j!}.$$

Basic pair of recurrence relations:

$$\partial_z\mathbf{F}(a;c;z) = a\mathbf{F}(a+1;c+1;z), \tag{6.15}$$
$$\big(\partial_z + c - z)\big)\mathbf{F}(a+1;c+1;z) = \mathbf{F}(a;c;z). \tag{6.16}$$

Integral representations: for all parameters

$$\frac{1}{2\pi\mathrm{i}} \int_{]-\infty,(0,z)^+,-\infty[} t^{a-c}e^t(t-z)^{-a}\mathrm{d}t \quad = \quad \mathbf{F}(a;c;z);$$

for $\mathrm{Re}\,a > 0$, $\mathrm{Re}(c-a) > 0$

$$\int_{[1,+\infty[} e^{\frac{z}{t}}t^{-c}(t-1)^{c-a-1}\mathrm{d}t \quad = \quad \Gamma(a)\Gamma(c-a)\mathbf{F}(a;c;z).$$

6.3 The $_2F_0$ equation

The $_2F_0$ operator:
$$\mathcal{F}(a,b;-) := z^2\partial_z^2 + (-1 + (1+a+b)z)\partial_z + ab, \tag{6.17}$$

Parameters:
$$\sigma(z) = z^2, \quad \kappa(z) = -1 + (a+b-1)z, \quad \omega = \left(a - \tfrac{1}{2}\right)\left(b - \tfrac{1}{2}\right) + \tfrac{1}{4}.$$

Weight:
$$\rho(z) = z^{-1+a+b}e^{\frac{1}{z}}.$$

Basic symmetry:
$$z^{-1+a+b}e^{\frac{1}{z}}\mathcal{F}(a,b;-;z,\partial_z)z^{1-a-b}e^{-\frac{1}{z}} = \mathcal{F}(1-b,1-a;-;w,\partial_w), \quad w = -z.$$

Inversion symmetry:
$$z^{a+1}\mathcal{F}(a,b;-;z,\partial_z)z^{-a} = -\mathcal{F}(a;1+a-b;w,\partial_w), \quad w = -z^{-1}.$$

(Thus the $_2F_0$ equation is equivalent to the $_1F_1$ equation.)

Theorem 6.3 (about integral representations) *Let* $[0,1] \ni \tau \overset{\gamma}{\mapsto} t(\tau)$ *satisfy*

$$e^{-\frac{1}{t}}t^{b-a-1}(t-z)^{-b-1}\Big|_{t(0)}^{t(1)} = 0.$$

Then

$$\mathcal{F}(a,b;-) \quad annihilates \quad \int_\gamma e^{-\frac{1}{t}}t^{b-a-1}(t-z)^{-b}\mathrm{d}t. \tag{6.18}$$

A second integral representation is obtained if we interchange a and b.

Proof. We check that

$$\mathcal{F}(a,b;-)e^{-\frac{1}{t}}t^{b-a-1}(t-z)^{-b} = -b\partial_t e^{-\frac{1}{t}}t^{b-a-1}(t-z)^{-b-1}. \tag{6.19}$$

$\square$

$_2F_0$ **function:** It is defined for $z \in \mathbb{C}\backslash[0,+\infty[$,

$$F(a,b;-;z) := \lim_{c\to\infty} F(a,b;c;cz),$$

where $|\arg c| > \epsilon$, $\epsilon > 0$. It extends to an analytic function on the universal cover of $\mathbb{C}\backslash\{0\}$ with a branch point of an infinite order at 0. It is annihilated by $\mathcal{F}(a,b;-)$.

Asymptotic expansion:
$$F(a,b;-;z) \sim \sum_{j=0}^{\infty} \frac{(a)_j(b)_j}{j!}z^j, \quad |\arg z| > \epsilon.$$

Basic pair of recurrence relations:

$$\partial_z F(a,b;-;z) = abF(a+1,b+1;-;z) \tag{6.20}$$

$$\left(z^2\partial_z + \left(-1 + (1+a+b)z\right)\right)F(a+1,b+1;-;z) = F(a,b;-;z). \tag{6.21}$$

Integral representation for $\mathrm{Re}\,a > 0$:

$$\int_0^\infty e^{-\frac{1}{t}}t^{b-a-1}(t-z)^{-b}\mathrm{d}t = \Gamma(a)F(a,b;-;z), \quad z \notin [0,\infty[.$$

The function $_2F_0$ is in our opinion insufficiently known. Therefore, we devote Appendix A to a derivation of some of its properties.

Note that the equivalence of $_2F_0$ and $_1F_1$ equations is helpful in deriving properties of Subsection 6.2 from Subsection 6.3. For instance, the integral representation (6.18) can be deduced from the integral representation (6.12) and the inversion symmetry. However, the relationship between some of the properties is not so straightforward. For instance, the basic recurrence relations for $_2F_0$ equations do not follow directly from the basic recurrence relations for the $_1F_1$ equations. Besides $_2F_0$ equation does not possess the power symmetry, even though the $_1F_1$ equation has it. Instead, the $_2F_0$ equation is symmetric with respect to the parameter interchange $a \leftrightarrow b$.

6.4 The $_0F_1$ equation

$_0F_1$ **operator:**

$$\mathcal{F}(c; z, \partial_z) := z\partial_z^2 + c\partial_z - 1.$$

Parameters:

$$\sigma(z) = z, \quad \kappa(z) = c - 1, \quad \omega = -1.$$

Weight:

$$\rho(z) = z^{c-1}.$$

Basic symmetry = power symmetry:

$$z^{c-1}\mathcal{F}(c)z^{1-c} = \mathcal{F}(2 - c). \tag{6.22}$$

Theorem 6.4 (about integral representations) *Suppose that $[0,1] \ni t \mapsto \gamma(t)$ satisfies*

$$\left. e^t e^{\frac{z}{t}} t^{-c} \right|_{\gamma(0)}^{\gamma(1)} = 0.$$

Then

$$\mathcal{F}(c; z, \partial_z) \quad annihilates \quad \int_\gamma e^t e^{\frac{z}{t}} t^{-c} \mathrm{d}t. \tag{6.23}$$

Proof. We check that

$$\mathcal{F}(c)\, e^t e^{\frac{z}{t}} t^{-c} = -\partial_t e^t e^{\frac{z}{t}} t^{-c}. \tag{6.24}$$

$\square$

$_0F_1$ **function**

$$F(c; z) := \sum_{j=0}^{\infty} \frac{1}{(c)_j} \frac{z^j}{j!}.$$

Olver normalized $_0F_1$ function:

$$\mathbf{F}(c; z) := \frac{F(c; z)}{\Gamma(c)} = \sum_{j=0}^{\infty} \frac{1}{\Gamma(c+j)} \frac{z^j}{j!}.$$

Basic pair of recurrence relations:

$$\partial_z \mathbf{F}(c; z) = \mathbf{F}(c + 1; z), \tag{6.25}$$

$$(\partial_z + c)\mathbf{F}(c + 1; z) = \mathbf{F}(a; c; z). \tag{6.26}$$

Integral representation for all parameters:

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{]-\infty,0^+,-\infty[} \mathrm{e}^t \mathrm{e}^{\frac{z}{t}} t^{-c}\mathrm{d}t \quad = \quad \mathbf{F}(c;z), \qquad \mathrm{Re}\,z > 0.$$

Chebyshev solutions for $k = 0, 1, 2, \ldots$:

$$\mathbf{F}(\tfrac{3}{2}+k;z) = \frac{2^{2k}}{\sqrt{\pi}} \partial_z^k \frac{\sinh 2\sqrt{z}}{\sqrt{z}}, \tag{6.27}$$

$$\mathbf{F}(\tfrac{1}{2}-k;z) = \frac{z^{\frac{1}{2}+k}}{\sqrt{\pi}} \partial_z^k \frac{\cosh 2\sqrt{z}}{\sqrt{z}}. \tag{6.28}$$

6.5   Hermite equation

Hermite operator:

$$\mathcal{S}(a) := \partial_z^2 - 2z\partial_z - 2a. \tag{6.29}$$

Parameters:

$$\sigma(z) = 1, \quad \kappa(z) = -2z, \quad \omega = -2a+1.$$

Weight:

$$\rho(z) = \mathrm{e}^{-z^2}.$$

Basic symmetry:

$$\mathrm{e}^{-z^2} \mathcal{S}(a;z,\partial_z) \mathrm{e}^{z^2} = -\mathcal{S}(1-a;w,\partial_w), \qquad w = \pm \mathrm{i}z.$$

Theorem 6.5 (about integral representations)   *1. Let $[0,1] \ni t \mapsto \gamma(t)$ satisfy*

$$\mathrm{e}^{t^2}(t-z)^{-a-1}\Big|_{\gamma(0)}^{\gamma(1)} = 0.$$

Then

$$\mathcal{S}(a) \quad annihilates \quad \int_\gamma \mathrm{e}^{t^2}(t-z)^{-a}\mathrm{d}t. \tag{6.30}$$

2. Let $[0,1] \ni t \mapsto \gamma(t)$ satisfy

$$\mathrm{e}^{-t^2-2zt}t^a\Big|_{\gamma(0)}^{\gamma(1)} = 0.$$

Then

$$\mathcal{S}(a) \quad annihilates \quad \int_\gamma \mathrm{e}^{-t^2-2zt}t^{a-1}\mathrm{d}t. \tag{6.31}$$

Proof. We check that

$$\begin{aligned}
\mathcal{S}(a)\,\mathrm{e}^{t^2}(t-z)^{-a} &= -a\partial_t \mathrm{e}^{t^2}(t-z)^{-a-1}, \\
\mathcal{S}(a)\,\mathrm{e}^{-t^2-2zt}t^{a-1} &= -2\partial_t \mathrm{e}^{-t^2-2zt}t^a.
\end{aligned}$$

$\square$

Hermite function:

$$S(a;z) \quad := \quad z^{-a} F\Big(\frac{a}{2}, \frac{a+1}{2}; -; -z^{-2}\Big). \tag{6.32}$$

Basic pair of recurrence relations:

$$\partial_z S(a; z) = -aS(a + 1; z), \tag{6.33}$$

$$\big(\partial_z - 2z\big)S(a + 1; z) = S(a; z). \tag{6.34}$$

Integral representation for $z \notin\,]-\infty, 0]$. For $0 < \mathrm{Re}\,a$:

$$\int\limits_0^\infty \mathrm{e}^{-t^2 - 2tz} t^{a-1}\mathrm{d}t \;\; = \;\; 2^{-a}\Gamma(a)S(a; z);$$

and for all parameters:

$$-\mathrm{i}\int\limits_{]-\mathrm{i}\infty, z^-, \mathrm{i}\infty[} \mathrm{e}^{t^2}(z - t)^{-a}\mathrm{d}t \;\; = \;\; \sqrt{\pi}S(a; z).$$

6.6   Lie algebras of symmetries type by type

Miller's Lie algebra is responsible only for one pair of recurrence relations of hypergeometric class functions. Some of types from the hypergeometric class possess larger sets of recurrence relations, which were described e.g. in [De1, De2]. These recurrence relations can be interpreted in terms of roots of larger Lie algebras of symmetries, described e.g. in [De2].

Miller's Lie algebra possesses a Weyl group isomorphic to $\mathbb{Z}_2$. Some types of hypergeometric class equations possess larger groups of discrete symmetries.

In the following table we list all nontrivial types of hypergeometric class equations. We include the *Gegenbauer equation*

$$\Big((1 - w^2)\partial_w^2 - 2(1 + \alpha)w\partial_w + \lambda^2 - \big(\alpha + \tfrac{1}{2}\big)^2\Big)f(w) = 0, \tag{6.35}$$

treating it as a seperate type. Strictly speaking, the Gegenbauer type is contained in the $_2F_1$ type. In fact, the transformation $w \mapsto \frac{1-w}{2}$ transforms the Gegenbauer equation into a special case of the $_2F_1$ equation. The Gegenbauer equation has the special property of the *mirror symmetry* $w \mapsto -w$. In the remaining part of this paper there is no need to consider it separately, however it has special symmetry properties.

The second and third column are based on Section 2 of this paper. In particular, the second column can be compared with (2.18)–(2.20).

The fourth and fifth column are based on [De2]. They describe a more complete Lie algebra of symmetries and the corresponding group of discrete symmetries.

Equation	Miller's Lie algebra	surface Ω	Lie algebra [De2]	Discrete symmetries [De2]
$_2F_1$	$sl(2, \mathbb{C}) \oplus \mathbb{C}$	sphere	$so(6, \mathbb{C})$	symmetries of cube
$_1F_1$	$osc(\mathbb{C})$	paraboloid	$sch(2, \mathbb{C})$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$_2F_0$	$sl(2, \mathbb{C}) \oplus \mathbb{C}$	null quadric	$sch(2, \mathbb{C})$	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$_0F_1$	$\mathbb{C}^2 \rtimes so(2, \mathbb{C}) \oplus \mathbb{C}$	paraboloid	$\mathbb{C}^2 \rtimes so(2, \mathbb{C})$	$\mathbb{Z}_2$
Gegenbauer	$sl(2, \mathbb{C}) \oplus \mathbb{C}$	sphere	$so(5, \mathbb{C})$	symmetries of square
Hermite	$osc(\mathbb{C})$	cylinder	$sch(1, \mathbb{C})$	$\mathbb{Z}_4$.

Above, $sch(n, \mathbb{C})$ denotes the complex Schrödinger Lie algebra, the Lie algebra of generalized symmetries of the heat equation in dimension n—see e.g. [De2].

The group from the fifth column always contains various symmetries described in our paper. First of all, all types possess the basic symmetry, which is the generator of the Weyl group of Miller's Lie algebra. Some of the types possess additional symmetries.

The power symmetry is a property of the $_2F_1$, $_1F_1$ and F_1 equation. For the F_1 equation it coincides with the basic symmetry.

The inversion symmetry is a feature of the $_2F_1$ equation, and connects the $_1F_1$ equation with the $_2F_0$ equation.

The $_2F_1$ and $_2F_0$ equations are invariant wrt swapping a and b.

The Gegenbauer and Hermite equation are invariant wrt the symmetry $z \mapsto -z$.

The Lie algebra of the fourth column always contains the corresponding Miller's Lie algebra (possibly, without the trivial term $\oplus \mathbb{C}$). By applying discrete symmetries from the fifth column to Miller's Lie algebra, we can enlarge Miller's Lie algebra to a subalgebra of the Lie algebra from the fourth column.

7 Hypergeometric polynomials

Polynomial solutions of hypergeometric class equations will be called *hypergeometric class polynomials*. In this section we describe these solutions in detail.

We have already mentioned in (3.27) that there exist ladders of solutions of hypergeometric class equations consisting of polynomials. Note that according to our conventions, these ladders are *descending*. This is related to the usual convention for parameters of various types of hypergeometric functions: in order to get a polynomial of degree $n \in \mathbb{N}_0$, the parameter a takes the value $-n$.

In Subsections 7.1 and 7.2 we describe the algebraic theory of hypergeometric class polynomials. They are centered around the so-called (generalized) Rodrigues formula.

Under some conditions on σ and κ hypergeometric class polynomials can be viewed as eigenfunctions of a certain self-adjoint Sturm-Liouville operator acting on an appropriate weighted Hilbert space. Besides, they form an orthogonal basis of this Hilbert space. To explain this point of view, we devote Subsections 7.3, 7.4 and 7.5 to a few general remarks about orthogonal polynomials and Sturm-Liouville operators. In particular, we find some useful conditions for the Hermiticity of Sturm-Liouville operators. These conditions yield the weight function ρ, and also specify possible endpoints of the interval $]a, b[$.

In the remaining subsections we return to the theory of hypergeometric class polynomials. We are convinced that the unified approach to their theory, which we present, based partly on [NU], has a considerable pedagogical value.

7.1 2nd order differential operators with polynomial eigenfunctions

The following well-known and easy proposition shows that hypergeometric class operators have many polynomial eigenfunctions. Actually, it is the property that characterizes this class among all second order differential operators.

Proposition 7.1 *Let $\sigma(z)$, $\tau(z)$, $\mu(z)$ be arbitrary functions. Let $P_n(z)$, $n = 0, 1, 2$, be polynomials such that $\deg P_n = n$ and $\eta_n \in \mathbb{C}$. Suppose that*

$$\left(\sigma(z) \partial_z^2 + \tau(z) \partial_z + \mu(z) + \eta_n \right) P_n(z) = 0, \quad n = 0, 1, 2.$$

Then $\sigma(z)$ is a polynomial of degree at most 2, $\tau(z)$ is a polynomial of degree at most 1 and $\mu(z)$ a polynomial of degree 0 (a complex number).

Note that for differential operators from Prop. 7.1, by replacing $\mu + \eta_n$ with η_n, without limiting the generality we can assume that $\mu = 0$.

Let us describe some other simple facts about polynomial solutions of hypergeometric class equations:

Proposition 7.2 *Let $\deg \sigma \leq 2$, $\deg \tau \leq 1$, $\eta \in \mathbb{C}$,*

1. *Suppose that P is a polynomial of degree n solving*

$$\left(\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta\right) P(z) = 0. \tag{7.1}$$

   Then

$$n(n-1)\frac{\sigma''}{2} + n\tau' + \eta = 0. \tag{7.2}$$

2. *If*

$$k\frac{\sigma''}{2} + \tau' \neq 0, \quad k \in \mathbb{N}_0, \tag{7.3}$$

   then the space of polynomial solutions of (7.1) *is at most 1-dimensional.*

Proof. Differentiating n times (7.1) we obtain $(7.2) \times P^{(n)} = 0$. This implies 1.

Suppose the space of polynomial solutions of (7.1) is 2-dimensional. We can assume that the degrees of these solutions are $0 \leq n_1 < n_2$. By (7.2),

$$n_i(n_i - 1)\frac{\sigma''}{2} + n_i\tau' + \eta = 0, \qquad i = 1, 2. \tag{7.4}$$

Subtracting (7.4) for 2 and 1 and dividing by $n_1 - n_2$ we obtain

$$(n_1 + n_2 - 1)\frac{\sigma''}{2} + \tau' = 0. \tag{7.5}$$

Now possible values of $k := n_1 + n_2 - 1$ are $0, 1, 2, \ldots$. This contradicts (7.3). $\square$

7.2 Raising and lowering operators. Rodriguez formula

The following theorem shows how to construct hypergeometric class polynomials and describes some of their properties, which are easy to describe in a unified fashion. As usual, σ denotes a polynomial of degree ≤ 2, κ a polynomial of degree ≤ 1.

Theorem 7.3 1. *Suppose that $\omega \in \mathbb{C}$ and a polynomial of degree n solves*

$$0 = \left(\mathcal{H}(\sigma, \kappa) + \omega\right) P(z) \tag{7.6}$$

$$= \left(\sigma(z)\partial_z^2 + (\sigma'(z) + \kappa(z))\partial_z + \frac{\kappa'}{2} + \omega\right) P(z). \tag{7.7}$$

   Then

$$n(n+1)\frac{\sigma''}{2} + (n + \tfrac{1}{2})\kappa' + \omega = 0. \tag{7.8}$$

2. *Suppose that*

$$k\frac{\sigma''}{2} + \kappa' \neq 0, \quad k = 2, 3, \ldots. \tag{7.9}$$

   Then for any $\omega \in \mathbb{C}$ the space of polynomial solutions of (7.6) *is at most 1-dimensional.*

3. Define

$$P_n(\sigma, \kappa; z) := \frac{1}{n!}\rho^{-1}(z)\partial_z^n \sigma^n(z)\rho(z) \tag{7.10}$$

$$= \frac{1}{2\pi i}\rho^{-1}(z)\int_{[z^+]} \sigma^n(z+t)\rho(z+t)t^{-n-1}\mathrm{d}t, \tag{7.11}$$

where $[z^+]$ denotes a loop around z in the counterclockwise direction. Then $P_n(\sigma, \kappa; z)$ is a polynomial of degree n or less, and we have

$$0 = \left(\mathcal{H}(\sigma, \kappa) - n(n+1)\frac{\sigma''}{2} - (n+\frac{1}{2})\kappa'\right)P_n(\sigma, \kappa; z) \tag{7.12}$$

$$= \left(\sigma(z)\partial_z^2 + (\sigma'(z) + \kappa(z))\partial_z - n(n+1)\frac{\sigma''}{2} - n\kappa'\right)P_n(\sigma, \kappa; z).$$

Proof. 1. and 2. are just a reformulations of Proposition 2, where instead of parameters τ, η we use κ, ω.

To show 3. we note that the integral representation (7.11) satisfies the assumptions of Theorem 5.1 about Euler transforms. This representation implies both (7.12) and (7.10). $\qquad\square$

(7.10) is usually called the *Rodriguez formula*.

Consider the descending ladder starting at $\kappa_0(z) := \kappa(z)$, $\omega_0 = -\frac{\kappa'}{2}$, that is,

$$\kappa_{-n}(z) := \kappa(z) - n\sigma'(z), \tag{7.13}$$

$$\omega_{-n} := -(n+\tfrac{1}{2})\kappa' + n^2\frac{\sigma''}{2} = -\kappa'_{-n}(n+\tfrac{1}{2}) - n(n+1)\frac{\sigma''}{2}. \tag{7.14}$$

Proposition 7.4 *The polynomials $P_n(\sigma, \kappa_{-n}; z)$ satisfy the corresponding hypergeometric class equation*

$$\left(\mathcal{H}(\sigma, \kappa_{-n}) + \omega_{-n}\right)P_n(\sigma, \kappa_{-n}; z) = 0, \tag{7.15}$$

the recurrence relations

$$(\sigma(z)\partial_z + \kappa_{-n}(z))P_n(\sigma, \kappa_{-n}; z) = (n+1)P_{n+1}(\sigma, \kappa_{-n-1}; z), \tag{7.16}$$

$$\partial_z P_{n+1}(\sigma, \kappa_{-n-1}; z) = \left(n\frac{\sigma''}{2} + \kappa'_{-n}\right)p_n(\sigma, \kappa_{-n}; z); \tag{7.17}$$

and have a generating function

$$\frac{\rho(z + t\sigma(z))}{\rho(z)} = \sum_{n=0}^{\infty} t^n P_n(\sigma, \kappa_{-n}; z). \tag{7.18}$$

7.3   Orthogonal polynomials

Consider $-\infty \le a < b \le +\infty$. Suppose that $]a, b[\ni x \mapsto \rho(x)$ is a positive measurable function. Define the Hilbert space

$$L^2(]a, b[, \rho) := \left\{f \quad \text{measurable on} \quad]a, b[\mid \int_a^b \rho(x)|f(x)|^2\mathrm{d}x < \infty\right\}, \tag{7.19}$$

with the scalar product $\quad (f|g) := \int_a^b \overline{f(x)}g(x)\rho(x)\mathrm{d}x. \tag{7.20}$

Assume in addition

$$\int_a^b \rho(x)|x|^n \mathrm{d}x < \infty, \quad n \in \mathbb{N}. \tag{7.21}$$

Then the space of polynomials is contained in $L^2(]a, b[, \rho)$. Applying the Gram-Schmidt orthogonalization to the sequence $1, x, x^2, \ldots$ we can define an orthogonal family of polynomials $p_0, p_1, p_2, \ldots$ The following simple criterion is proven e.g. in [RS2]:

Theorem 7.5 *Suppose that for some $\epsilon > 0$*

$$\int_a^b \mathrm{e}^{\epsilon|x|} \rho(x) \mathrm{d}x < \infty. \tag{7.22}$$

Then polynomials are dense in $L^2(]a, b[, \rho)$. Therefore, the family $p_0, p_1, p_2, \ldots$ is an orthogonal basis of $L^2(]a, b[, \rho)$.

7.4   Hermiticity of Sturm-Liouville operators

In most of this paper we avoid using functional analysis. However, for hypergeometric class polynomials we will make an exception. In fact, often it is natural to view them as orthogonal bases consisting of eigenfunctions of certain self-adjoint operators.

Let us briefly recall some elements of the theory of operators on Hilbert spaces. Let $\mathcal{H}$ be a Hilbert space with the scalar product $(\cdot|\cdot)$. Let $\mathcal{A}$ be an operator on the domain $\mathcal{D} \subset \mathcal{H}$. We say that $\mathcal{A}$ is *Hermitian* on $\mathcal{D}$ in the sense of $\mathcal{H}$ if

$$(f|\mathcal{A}g) = (\mathcal{A}f|g), \quad f, g \in \mathcal{D}. \tag{7.23}$$

(Unfortunately, in most of the contemporary mathematics literature, instead of *Hermitian* the word *symmetric* is used, which is a confusing misnomer). We say that $\mathcal{A}$ is *self-adjoint* if it is Hermitian and its spectrum is real. We say that it is *essentially self-adjoint* if it has a unique self-adjoint extension.

Let $\mathcal{A}$ be Hermitian. Suppose that $f_i \in \mathcal{D}$, $i = 1, 2$, and

$$\mathcal{A}f_i = \eta_i f_i, \quad f_i \in \mathcal{D}, \quad i = 1, 2. \tag{7.24}$$

Then it is easy to see that η_i are real, and if $\eta_1 \neq \eta_2$, then $(f_i|f_j) = 0$. Therefore, eigenfunctions of Hermitian operators can be arranged into orthogonal families. If an operator is Hermitian and possesses an orthogonal basis of eigenvectors, then it is is essentially self-adjoint on (finite) linear combinations of its eigenvectors.

Second order differential operators on a segment of the real line are often called *Sturm-Liouville operators*. Let us make some remarks about the general theory of such operators, which will be useful in the analysis of orthogonality properties of hypegeometric class polynomials.

Consider $-\infty \leq a < b \leq +\infty$. Suppose that σ, ρ are functions on $]a, b[$ (not necessarily polynomials). Consider an operator of the form

$$\mathcal{A} := \sigma(x)\partial_x^2 + \tau(x)\partial_x \tag{7.25}$$

(Thus we consider an arbitrary Sturm-Liouville operator without the zeroth order term).

Let ρ satisfy

$$\sigma(x)\rho'(x) = (\tau(x) - \sigma'(x))\rho(x). \tag{7.26}$$

Often it is convenient to rewrite (7.25) in the form

$$\mathcal{A} = \rho(x)^{-1}\partial_x \rho(x)\sigma(x)\partial_x. \tag{7.27}$$

Suppose that σ, τ are real and ρ is positive. It is useful to interpret $\mathcal{A}$ as an (unbounded) operator on the Hilbert space $L^2(]a, b[, \rho)$. Indeed, the following easy theorem shows that $\mathcal{A}$ is Hermitian on smooth functions that vanish close to the endpoints of the interval.

Theorem 7.6 *Let*

$$\mathcal{D}_0 = \{f \in C^\infty(]a,b[) \ : \ f = 0 \ \text{ in a neighborhood of } \ a, b\}. \tag{7.28}$$

Then $\mathcal{A}$ is Hermitian on $\mathcal{D}_0$ in the sense of the Hilbert space $L^2(]a,b[,\rho)$.

7.5   Selecting endpoints for Sturm-Liouville operators

Unfortunately, the operator $\mathcal{A}$ is rarely essentially self-adjoint on (7.28). We will see that $\mathcal{A}$ is still Hermitian on functions that do not vanish near the endpoints, if these endpoints satisfy appropriate conditions:

Theorem 7.7 *Let $-\infty < a < b < +\infty$ and*

$$\sigma(a)\rho(a) = \sigma(b)\rho(b) = 0.$$

Then $\mathcal{A}$ is Hermitian on the domain $C^2([a,b])$ in the sense of the space $L^2(]a,b[,\rho)$.

Proof. Let $g, f \in C^2([a,b])$.

$$
\begin{aligned}
(g|\mathcal{A}f) &= \int_a^b \rho(x)\bar{g}(x)\rho(x)^{-1}\partial_x\sigma(x)\rho(x)\partial_x f(x)\mathrm{d}x \\
&= \int_a^b \bar{g}(x)\partial_x\sigma(x)\rho(x)\partial_x f(x)\mathrm{d}x \\
&= \left. \bar{g}(x)\rho(x)\sigma(x)f'(x)\right|_a^b - \int_a^b (\partial_x\bar{g}(x))\sigma(x)\rho(x)\partial_x f(x)\mathrm{d}x \\
&= \left. -\bar{g'}(x)\rho(x)\sigma(x)f(x)\right|_a^b + \int_a^b (\partial_x\rho(x)\sigma(x)\partial_x\bar{g}(x))f(x)\mathrm{d}x \\
&= \int_a^b \rho(x)(\rho(x)^{-1}\partial_x\sigma(\bar{x})\rho(x)\partial_x g(x))f(x)\mathrm{d}x = (\mathcal{A}g|f).
\end{aligned}
$$

$\square$

Analogously we prove the following fact:

Theorem 7.8 *Let*

$$\lim_{x\to-\infty}\sigma(x)\rho(x)|x|^n = \lim_{x\to+\infty}\sigma(x)\rho(x)|x|^n = 0, \quad n \in \mathbb{N}.$$

Then $\mathcal{A}$ is Hermitian on the domain consisting of polynomial functions in the sense of the Hilbert space $L^2(]-\infty,\infty[,\rho)$.

Obviously, statements similar to Theorems 7.7 and 7.8 hold if $a = -\infty$ and b is finite, or a is finite and $b = \infty$.

We will see later that in applications to hypergeometric class equations the conditions of Theorems 7.7 and 7.8 are often sufficient to guarantee the essential self-adjointness of the operator $\mathcal{A}$ on the set of polynomials.

7.6 Orthogonality of hypergeometric class polynomials

In some cases hypergeometric class polynomials can be interpreted as an orthogonal basis of a certain weighted Hilbert space. They are then often called *(very) classical orthogonal polynomials*.

Let us fix σ, κ and ρ satisfying (3.7). Let σ, κ be real and ρ positive. Choose an interval $]a, b[$, where $-\infty \leq a < b \leq +\infty$ satisfy the conditions of Theorem 7.7 or 7.8. By these theorems, the operator

$$\mathcal{H}(\sigma, \kappa) := \sigma(x)\partial_x^2 + \big(\kappa(x) + \sigma'(x)\big)\partial_x + \frac{\kappa'}{2} = \rho(x)^{-1}\partial_x\rho(x)\sigma(x)\partial_x + \frac{\kappa'}{2}. \tag{7.29}$$

is Hermitian on the space of polynomials in the sense of the Hilbert space $L^2(]a, b[, \rho)$. Therefore, polynomial eigenfunctions of $\mathcal{H}(\sigma, \kappa)$ are pairwise orthogonal (at least if the corresponding eigenvalues are distinct, which is generically the case, see (7.8)). The following theorem says more: we compute the square norm of the polynomials obtained from the Rodriguez formula.

Theorem 7.9

$$\int_a^b P_m(\sigma, \kappa; x) P_n(\sigma, \kappa; x) \rho(x)\mathrm{d}x = \frac{\delta_{mn}}{n!} \prod_{j=1}^{n} \Big(-\kappa' + j\frac{\sigma''}{2} \Big) \int_a^b \sigma^n(x)\rho(x)\mathrm{d}x. \tag{7.30}$$

Proof. It is enough to assume that $m \leq n$. Write $P_n(x)$ for $P_n(\sigma, \kappa; x)$.

$$\int_a^b P_m(x)P_n(x)\rho(x)\mathrm{d}x \tag{7.31}$$

$$= \frac{1}{n!} \int_a^b P_m(x)\partial_x^n \sigma^n(x)\rho(x)\mathrm{d}x \tag{7.32}$$

$$= \frac{(-1)^n}{n!} \int_a^b \big(\partial_x^n P_m(x)\big)\sigma^n(x)\rho(x)\mathrm{d}x. \tag{7.33}$$

This is zero if $m < n$ and for $m = n$ we use

$$\partial_x^n P_n(x) = \prod_{j=1}^{n} \Big(\kappa' - j\frac{\sigma''}{2} \Big). \tag{7.34}$$

$\square$

7.7 Review of types of hypergeometric class polynomials

In the remaining part of the section we review various types of hypergeometric class polynomials. We will discuss only the properties that follow directly from the general theory described in previous subsections. Here is the list of items that we will cover:

- The choice of σ, κ and the corresponding weight ρ.

- The Rodriguez formula (7.10) defining the polynomial and its expression in terms of hypergeometric class functions.

- The degree of the polynomial, see (7.2).

- The differential equation, see (7.12).

- The pair of recurrence relations that follow from the Rodriguez formula, see (7.16) and (7.17).

- The generating function related to the Rodriguez formula, see (7.18).

- The choice of endpoints for which the corresponding Sturm-Liouville operator is essentially self-adjoint on the space of polynomials (if applicable), see Theorems 7.7 , 7.8 and 7.5.

- The square norm (if applicable), see Theorem 7.9.

Hypergeometric class polynomials possess various useful features that will not be listed in the following subsections. For instance, they typically have additional recurrence relations and generating functions. We list only those that follow directly from the general theory described above.

7.8 Jacobi polynomials

Consider $\alpha, \beta \in \mathbb{C}$ and

$$\sigma(z) = 1 - z^2, \quad \kappa(z) = \alpha(1 - z) + \beta(1 + z), \quad \rho(z) = (1 - z)^\alpha (1 + z)^\beta. \tag{7.35}$$

For $n \in \{0, 1, \dots\}$ set

$$P_n^{\alpha,\beta}(x) = \frac{(-1)^n}{2^n n!}(1 - x)^{-\alpha}(1 + x)^{-\beta}\partial_x^n(1 - x)^{\alpha+n}(1 + x)^{\beta+n} \tag{7.36}$$

$$= \frac{(1 + \alpha)_n}{n!}\,_2F_1\Big(-n, n + \alpha + \beta + 1; \alpha + 1; \frac{1 - x}{2}\Big). \tag{7.37}$$

Then $P_n^{\alpha,\beta}$ is a polynomial of degree at most n. More precisely:

1. If $\alpha + \beta \notin \{-2n, \dots, -n - 1\}$, then $\deg P_n^{\alpha,\beta} = n$. It is then up to a coefficient the unique eigenfunction of the operator

$$\mathcal{H}(\sigma, \kappa) = (1 - x^2)\partial_x^2 + (\beta - \alpha - (\alpha + \beta + 2)x)\partial_x + \frac{\beta - \alpha}{2} \tag{7.38}$$

which is a polynomial of degree n.

2. If $\alpha + \beta \in \{-2n, \dots, -n - 1\}$, but $\alpha \notin \{-n, \dots, -1\}$ (or, equivalently, $\beta \notin \{-n, \dots, -1\}$), then $\deg P_n^{\alpha,\beta} = -\alpha - \beta - n - 1$.

3. If $\alpha + \beta \in \{-2n, \dots, -n - 1\}$, but $\alpha \in \{-n, \dots, -1\}$ (or, equivalently, $\beta \in \{-n, \dots, -1\}$), then $P_n^{\alpha,\beta} = 0$.

$P_n^{\alpha,\beta}$ satisfy the Jacobi equation, which is a slightly modified $_2F_1$ equation

$$\big((1 - x^2)\partial_x^2 + (\beta - \alpha - (\alpha + \beta + 2)x)\partial_x + n(n + \alpha + \beta + 1)\big)\,P_n^{\alpha,\beta}(x) = 0.$$

Recurrence relations:

$$\partial_x P_n^{\alpha,\beta}(x) = \frac{\alpha + \beta + n + 1}{2}P_{n-1}^{\alpha+1,\beta+1}, \tag{7.39}$$

$$-\frac{(1 - x^2)\partial_x + \beta - \alpha - (\alpha + \beta)x}{2}P_n^{\alpha,\beta}(x) = (n + 1)P_{n+1}^{\alpha-1,\beta-1}(x). \tag{7.40}$$

Generating function:

$$\sum_{n=0}^{\infty} P_n^{\alpha-n,\beta-n}(x)2^n t^n = \big(1 + t(1 + x)\big)^\alpha\big(1 - t(1 - x)\big)^\beta. \tag{7.41}$$

If $\alpha, \beta > -1$, then $\mathcal{H}(\sigma, \kappa)$ is self-adjoint on the space of polynomials in the sense of

$$L^2(]-1, 1[, (1-x)^\alpha (1+x)^\beta).$$

Jacobi polynomials are its eigenfunctions and form an orthogonal basis with the normalization

$$\int_{-1}^{1} P_n^{\alpha,\beta}(x)^2 (1-x)^\alpha (1+x)^\beta \mathrm{d}x = \frac{\Gamma(1+\alpha+n)\Gamma(1+\beta+n)2^{\alpha+\beta+1}}{(1+2n+\alpha+\beta)n!\Gamma(1+\alpha+\beta+n)}. \tag{7.42}$$

7.9 Laguerre polynomials

Consider $\alpha \in \mathbb{C}$ and

$$\sigma(z) = z, \quad \kappa(z) = \alpha - z, \quad \rho(z) = \mathrm{e}^{-z} z^\alpha. \tag{7.43}$$

For $n \in \mathbb{N}$ set

$$\begin{aligned}
L_n^\alpha(x) &= \frac{1}{n!} \mathrm{e}^x x^{-\alpha} \partial_x^n \mathrm{e}^{-x} x^{n+\alpha} \\
&= \frac{(1+\alpha)_n}{n!} {}_1F_1(-n; 1+\alpha; x).
\end{aligned}$$

Then L_n^α is a polynomial of degree n. It is a unique (up to a coefficient) eigenfunction of the operator

$$\mathcal{H}(\sigma, \kappa) = x\partial_x^2 + (\alpha + 1 - x)\partial_x - \frac{1}{2} \tag{7.44}$$

which is a polynomial of degree n. L_n^α satisfy the Laguerre equation, which is the ${}_1F_1$ equation with modified parameters:

$$\left(x\partial_x^2 + (\alpha + 1 - x)\partial_x + n\right) L_n^\alpha(x) = 0.$$

Recurrence relations:

$$\begin{aligned}
(x\partial_x + \alpha - x)L_n^\alpha(x) &= (n+1)L_{n+1}^{\alpha-1}(x), \tag{7.45} \\
\partial_x L_n^\alpha(x) &= -L_{n-1}^{\alpha+1}(x). \tag{7.46}
\end{aligned}$$

Generating function:

$$\mathrm{e}^{-tz}(1+t)^\alpha = \sum_{n=0}^{\infty} t^n L_n^{\alpha-n}(z). \tag{7.47}$$

If $\alpha > -1$, then $\mathcal{H}(\sigma, \kappa)$ is essentially self-adjoint on the space of polynomials in the sense of $L^2([0, \infty[, \mathrm{e}^{-x} x^\alpha)$. Laguerre polynomials are its eigenfunctions and form an orthonormal basis with the normalization

$$\int_0^\infty L_n^\alpha(x)^2 x^\alpha \mathrm{e}^{-x} \mathrm{d}x = \frac{\Gamma(1+\alpha+n)}{n!}.$$

7.10 Bessel polynomials

Consider $\theta \in \mathbb{C}$ and

$$\sigma(z) = z^2, \quad \kappa(z) = -1 + \theta z, \quad \rho(z) = \mathrm{e}^{-z^{-1}} z^\theta. \tag{7.48}$$

For $n = 0, 1, \ldots$ set

$$B_n^\theta(z) := \frac{1}{n!} z^{-\theta} \mathrm{e}^{z^{-1}} \partial_z^n \mathrm{e}^{-z^{-1}} z^{\theta+2n} \tag{7.49}$$

$$= \frac{1}{n!} {}_2F_0(-n, n+\theta+1; -; z) \tag{7.50}$$

$$= (-z)^n L_n^{-\theta-2n-1}(-z^{-1}). \tag{7.51}$$

Then B_n^θ is a polynomial of degree n. It is a unique (up to a coefficient) eigenfunction of the operator

$$\mathcal{H}(\sigma, \kappa) = z^2 \partial_z^2 + (-1 + (2 + \theta)z)\partial_z + \frac{\theta}{2} \tag{7.52}$$

which is a polynomial of degree n. B_n^θ satisfy the $_2F_0$ equation with adjusted parametes:

$$\left(z^2 \partial_z^2 + (-1 + (2 + \theta)z)\partial_z - \frac{1}{2}n(1 + \theta + n)\right)B_n^\theta(z) \quad = \quad 0.$$

Recurrence relations:

$$\partial_z B_n^\theta(z) \quad = \quad -(n + \theta + 1)B_{n-1}^{\theta+2}(z),$$
$$\left(z^2 \partial_z - 1 - \theta z\right)B_n^\theta(z) \quad = \quad -(n + 1)B_{n+1}^{\theta-2}(z).$$

Generating function:

$$(1 + tz)^\theta \exp\left(\frac{-t}{1 + tz}\right) = \sum_{n=0}^\infty t^n B_n^{\theta - 2n}(z). \tag{7.53}$$

Bessel polynomials do not form an orthogonal basis on any interval.

7.11 Hermite polynomials

Consider

$$\sigma(z) = 1, \quad \kappa(z) = -2z, \quad \rho(z) = e^{-z^2}. \tag{7.54}$$

For $n = 0, 1, \ldots$ set

$$H_n(x) = \frac{(-1)^n}{n!} e^{x^2} \partial_x^n e^{-x^2} \tag{7.55}$$

$$= \frac{2^n}{n!} S(-n; x), \tag{7.56}$$

where $S(a, x)$ is the Hermite function defined in (6.32). Then H_n is a polynomial of degree n and is (up to a multiplicative constant) the only eigenfunction of the operator

$$\mathcal{H}(\sigma, \kappa) = \partial_x^2 - 2x\partial_x - 1 \tag{7.57}$$

which is a polynomial of degree n. It satisfies the Hermite equation

$$(\partial_x^2 - 2x\partial_x + 2n)H_n(x) = 0.$$

Recurrence relations:

$$(-\partial_x + 2x)H_n(x) = (n + 1)H_{n+1}(x), \tag{7.58}$$
$$\partial_x H_n(x) = 2H_{n-1}(x). \tag{7.59}$$

Generating function:

$$\sum_{n=0}^\infty t^n H_n(x) = e^{2tx - t^2}. \tag{7.60}$$

The operator $\mathcal{H}(\sigma, \kappa)$ is essentially self-adjoint on the space of polynomials in the sense of $L^2(\mathbb{R}, e^{-x^2})$. Hermite polynomials are its eigenfunctions and form an orthogonal basis with the normalization

$$\int_{-\infty}^\infty H_n(x)^2 e^{-x^2} dx = \frac{\sqrt{\pi} 2^n}{n!}.$$

Remark 7.10 *The definition of Hermite polynomials that we gave is consistent with the generalized Rodrigues formula* (7.10)*. In the literature one can also find other conventions for Hermite polynomials, e.g.* $H_n(x) := (-1)^n e^{x^2} \partial_x^n e^{-x^2}$.

8 Degenerate case

If two indices of a regular-singular (also called Fuchsian) point of a differential equation differ by an integer, then the usual Frobenius method [WW] produces in general, up to a coefficient, only one solution. We call this case *degenerate*. In this section we will be discuss the degenerate case of hypergeometric class equations. Without limiting the generality, the regular-singular point will be 0.

8.1 Unified hypergeometric function in the degenerate case

Suppose that we are in the setting of Section 4.3. That means, $\sigma(0) = 0$, $\sigma'(0) = 1$, and we set $\kappa(0) = m$. Thus,

$$\sigma(z) = \frac{\sigma''}{2}z^2 + z, \quad \kappa(z) = m + \kappa'z, \quad \omega \tag{8.1}$$

are the parameters of the equation we consider, see (4.7). As in (4.11), we also introduce the transformed parameters:

$$\kappa^\circ(z) := -m + (\kappa' - m\sigma'')z, \quad \omega^\circ := \omega - m\kappa' + m^2\frac{\sigma''}{2}, \quad m^\circ = -m. \tag{8.2}$$

We assume in addition that m is an integer. Using

$$\frac{1}{\Gamma(m+1)} = \begin{cases} 0 & m = \dots, -2, -1; \\ \frac{1}{m!}, & m = 0, 1, \dots, \end{cases} \tag{8.3}$$

we obtain

$$\mathbf{F}(\sigma, \kappa, \omega; z) = \sum_{n=\max\{0,-m\}}^{\infty} \frac{\Pi_{j=0}^{n-1}(\omega + (j + \frac{1}{2})\kappa' + \frac{j(j+1)}{2}\sigma'')}{(n+m)!n!}(-z)^n. \tag{8.4}$$

This easily implies the identity

$$\mathbf{F}(\sigma, \kappa^\circ, \omega^\circ; z) = \prod_{j=0}^{m-1}\left(\omega - \kappa'(j + \frac{1}{2}) + \frac{\sigma''}{2}j(j+1)\right)(-z)^m \mathbf{F}(\sigma, \kappa, \omega; z), \quad m \in \mathbb{N}_0. \tag{8.5}$$

Therefore, for integer m the functions $\mathbf{F}(\sigma, \kappa, \omega; z)$ and $z^{-m}\mathbf{F}(\sigma, \kappa^\circ, \omega^\circ; z)$ are proportional to one another and do not form a basis of solutions.

8.2 The power-exponential function

In order to describe the theory of the degenerate case in a unified way, it will be convenient first to unify the power and exponential function in a single function. More precisely, let $a, \mu \in \mathbb{C}$. Consider the function

$$f(a, \mu; u) := \begin{cases} (1 + \mu u)^{a/\mu}, & \mu \neq 0, \\ e^{au}, & \mu = 0. \end{cases}$$

It is analytic for $u \neq -\mu^{-1}$. From now on we will write $(1 + \mu u)^{a/\mu}$ instead of $f(a, \mu; u)$.

Let us list some properties of this function.

Theorem 8.1 *(1)* $(1 + \mu u)^{a/\mu}$ *is the unique solution of*

$$\big((1 + \mu u)\partial_u - a\big)f(u) = 0, \tag{8.6}$$

$$f(0) = 1. \tag{8.7}$$

(2) For $|u| < |\mu|^{-1}$ we have

$$(1 + \mu u)^{a/\mu} = \sum_{n=0}^{\infty} \frac{a(a - \mu) \cdots (a - (n-1)\mu)}{n!} u^n,$$

(3)

$$(1 + \mu u)^{a_1/\mu}(1 + \mu u)^{a_2/\mu} = (1 + \mu u)^{(a_1+a_2)/\mu}.$$

8.3 Generating functions

Let us now fix $\sigma(z) = \frac{\sigma''}{2}z^2 + z$, as in (8.1). Let $a, b, \mu, \nu \in \mathbb{C}$. Consider two families of hypergeometric class operators

$$\left(\frac{\sigma''}{2}z^2 + z\right)\partial_z^2 + \left(m + 1 + \left(\frac{\sigma''}{2}(1 + m) - \mu b - \nu a\right)z\right)\partial_z - m\mu b - ab, \tag{8.8}$$

$$\left(\frac{\sigma''}{2}z^2 + z\right)\partial_z^2 + \left(m + 1 + \left(\frac{\sigma''}{2}(1 + m) - \mu b - \nu a\right)z\right)\partial_z - m\nu a - ab. \tag{8.9}$$

(Note that the parameters we introduced are redundant—three would suffice instead of four. However they help us to describe the degenerate case in a symmetric and unified way).

In terms of our standard parameters, the operators (8.8) and (8.9) can be written as

$$\mathcal{H}(\sigma, \kappa_m) + \omega_m, \tag{8.10}$$

$$\mathcal{H}(\sigma, \kappa_m) + \tilde{\omega}_m, \tag{8.11}$$

where

$$\kappa_m(z) := m + \left(\frac{\sigma''}{2}(m - 1) - \mu b - \nu a\right)z, \tag{8.12}$$

$$\omega_m := \frac{1}{2}(\mu b + \nu a) - ab - m\mu b - \frac{\sigma''}{4}(m - 1), \tag{8.13}$$

$$\tilde{\omega}_m := \frac{1}{2}(\mu b + \nu a) - ab - m\nu a - \frac{\sigma''}{4}(m - 1). \tag{8.14}$$

Note that

$$\kappa_m^\circ = \kappa_{-m}, \quad \omega_m^\circ = \tilde{\omega}_{-m}, \quad \tilde{\omega}_m^\circ = \omega_{-m}. \tag{8.15}$$

By $[(\alpha, \beta)^+]$ we will denote a counterclockwise loop around $\alpha, \beta \in \mathbb{C}$. We define

$$\Psi_m(z) := \frac{1}{2\pi i} \int_{[(0, -z\nu))^+]} (1 + \mu u)^{-\frac{a}{\mu}} \left(1 + \frac{\nu z}{u}\right)^{-\frac{b}{\nu}} u^{-m-1} du, \tag{8.16}$$

$$\tilde{\Psi}_m(z) := \frac{1}{2\pi i} \int_{[(0, -z\mu)^+]} \left(1 + \frac{\mu z}{v}\right)^{-\frac{a}{\mu}} (1 + \nu v)^{-\frac{b}{\nu}} v^{-m-1} dv. \tag{8.17}$$

Theorem 8.2 *1. For $|z\nu| < |u| < |\mu^{-1}|$ we have*

$$(1 + \mu u)^{-\frac{a}{\mu}} \left(1 + \frac{\nu z}{u}\right)^{-\frac{b}{\nu}} = \sum_{m \in \mathbb{Z}} u^m \Psi_m(z). \tag{8.18}$$

and for $|z\mu| < |u| < |\nu^{-1}|$ we have

$$\left(1 + \frac{\mu z}{v}\right)^{-\frac{a}{\mu}} (1 + \nu v)^{-\frac{b}{\nu}} = \sum_{m \in \mathbb{Z}} v^m \tilde{\Psi}_m(z). \tag{8.19}$$

2. $\Psi_m(z) = z^{-m}\tilde{\Psi}_{-m}(z)$.

3. For $m \geq 0$,

$$\Psi_m(0) = (-1)^m \frac{a(a+\mu)\cdots(a+\mu(m-1))}{m!}, \tag{8.20}$$

$$\tilde{\Psi}_m(0) = (-1)^m \frac{b(b+\nu)\cdots(b+(m-1)\nu)}{m!}. \tag{8.21}$$

4. The functions $\Psi_m(z)$ and $\tilde{\Psi}_m(z)$ satisfy the following hypergeometric class differential equations:

$$\left(z(1-\mu\nu z)\partial_z^2 + (m+1-(\mu\nu(1+m)+a\nu+b\mu)z)\partial_z - (m\mu b+ab)\right)\Psi_m(z) = 0, \tag{8.22}$$

$$\left(z(1-\mu\nu z)\partial_z^2 + (m+1-(\mu\nu(1+m)+a\nu+b\mu)z)\partial_z - (m\nu a+ab)\right)\tilde{\Psi}_m(z) = 0. \tag{8.23}$$

5. The functions $\Psi_m(z)$ and $\tilde{\Psi}_m(z)$ are proportional to the Olver normalized unified hypergeometric function:

$$\Psi_m(z) = (-1)^m a(a+\mu)\cdots(a+\mu(m-1))\mathbf{F}(\sigma,\kappa_m,\omega_m;z); \tag{8.24}$$

$$\tilde{\Psi}_m(z) = (-1)^m b(b+\nu)\cdots(b+(m-1)\nu)\mathbf{F}(\sigma,\kappa_m,\tilde{\omega}_m;z); \tag{8.25}$$

Proof. 1. follows immediately from the definitions (8.16), (8.17) and the Laurent expansion.

To show 2. we can rewrite (8.18) and (8.19) as

$$(1+\mu u)^{-\frac{a}{\mu}}(1+\nu v)^{-\frac{b}{\nu}} = \sum_{m\in\mathbb{Z}} u^m \Psi_m(uv), \tag{8.26}$$

$$(1+\mu u)^{-\frac{a}{\mu}}(1+\nu v)^{-\frac{b}{\nu}} = \sum_{m\in\mathbb{Z}} v^m \tilde{\Psi}_m(uv) = \sum_{m\in\mathbb{Z}} u^{-m}(uv)^m \tilde{\Psi}_m(uv). \tag{8.27}$$

3. follows by setting $v = 0$ in (8.26) and $u = 0$ in (8.27).

Let us show 4. We have the identity

$$0 = (\partial_u\partial_v - \mu\nu uv\partial_u\partial_v - \mu bu\partial_u - a\nu v\partial_v - ab)(1+\mu u)^{-\frac{a}{\mu}}(1+\nu v)^{-\frac{b}{\nu}}. \tag{8.28}$$

Besides,

$$(\partial_u\partial_v - \mu\nu uv\partial_u\partial_v - \mu bu\partial_u - a\nu v\partial_v - ab)\, u^m \Psi_m(uv)$$
$$= u^m \left(z(1-\mu\nu z)\partial_z^2 + (m+1-(\mu\nu+(m\mu+a)\nu+b\mu)z)\partial_z - (m\mu+a)b\right)\Psi_m(z).$$

Thus (8.28) together with (8.26) can be rewritten as

$$0 = \sum_{m\in\mathbb{Z}} u^m \left(z(1-\mu\nu z)\partial_z^2 + (m+1-(\mu\nu+(m\mu+a)\nu+b\mu)z)\partial_z - (m\mu+a)b\right)\Psi_m(z), \tag{8.29}$$

which implies (8.22).

5. follows from 3. and 4. $\qquad\square$

In the remaining part of the section we describe separately three types of degenerate hypergeometric class functions.

8.4   The $_2F_1$ function

For $m \in \mathbb{Z}$ we have

$$\mathbf{F}(a, b; 1 + m; z) = \sum_{n=\max(0,-m)} \frac{(a)_n (b)_n}{n!(m+n)!} z^n, \tag{8.30}$$

$$(a - m)_m (b - m)_m \mathbf{F}(a, b; 1 + m; z) = z^{-m} \mathbf{F}(a - m, b - m; 1 - m; z). \tag{8.31}$$

We have an integral representation and a generating function:

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{[(0,z)^+]} (1 - t)^{-a} \left(1 - \frac{z}{t}\right)^{-b} t^{-m-1} \mathrm{d}t = (a)_m \mathbf{F}(a + m, b; 1 + m; z), \tag{8.32}$$

$$(1 - t)^{-a} \left(1 - \frac{z}{t}\right)^{-b} = \sum_{m \in \mathbb{Z}} t^m (a)_m \mathbf{F}(a + m, b; 1 + m). \tag{8.33}$$

8.5   The $_1F_1$ function

If $m \in \mathbb{Z}$, we have

$$\mathbf{F}(a; 1 + m; z) = \sum_{n=\max(0,-m)} \frac{(a)_n}{n!(m+n)!} z^n, \tag{8.34}$$

$$(a - m)_m \mathbf{F}(a; 1 + m; z) = z^{-m} \mathbf{F}(a - m; 1 - m; z). \tag{8.35}$$

We have two integral representations and the corresponding generating functions:

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{[(z,0)^+]} \mathrm{e}^t \left(1 - \frac{z}{t}\right)^{-a} t^{-m-1} \mathrm{d}t = \mathbf{F}(a, 1 + m; z),$$

$$\frac{1}{2\pi \mathrm{i}} \int\limits_{[(0,1)^+]} \mathrm{e}^{\frac{z}{t}} (1 - t)^{-a} t^{-m-1} \mathrm{d}t = z^{-m} \mathbf{F}(a, 1 - m; z),$$

$$\mathrm{e}^t \left(1 - \frac{z}{t}\right)^{-a} = \sum_{m \in \mathbb{Z}} t^m \mathbf{F}(a; m; z)),$$

$$\mathrm{e}^{\frac{z}{t}} (1 - t)^{-a} = \sum_{m \in \mathbb{Z}} t^m z^{-m} \mathbf{F}(a; 1 - m; z).$$

8.6   The $_0F_1$ function

If $m \in \mathbb{Z}$, then

$$\mathbf{F}(1 + m; z) = \sum_{n=\max(0,-m)} \frac{1}{n!(m+n)!} z^n, \tag{8.36}$$

$$\mathbf{F}(1 + m; z) = z^{-m} \mathbf{F}(1 - m; z). \tag{8.37}$$

We have an integral representation, called the *Bessel formula*, and a generating function:

$$\frac{1}{2\pi i} \int_{[0^+]} e^{t+\frac{z}{t}} t^{-m-1} dt = \mathbf{F}(1+m; z),$$ (8.38)

$$e^t e^{\frac{z}{t}} = \sum_{m\in\mathbb{Z}} t^m \mathbf{F}(1+m; z).$$ (8.39)

A Function $_2F_0$

The function $_2F_0$ seems to be rarely discussed in the literature. For convenience of the reader in the following theorem we state and prove basic facts about this function.

Theorem A.1 *For $w \in \mathbb{C}\backslash[0, \infty[$, there exists the limit*

$$F(a, b; -; w) := \lim_{c\to+\infty} F(a, b; c; cw).$$ (A.1)

It defines a function analytically depending on $a, b \in \mathbb{C}$ and $w \in \mathbb{C}\backslash[0, \infty[$.
We have the following asymptotic expansion:

$$F(a, b; -; w) \sim \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n!} w^n.$$ (A.2)

More precisely, for any $\epsilon > 0$, n, there exists c_n such that

$$\left| F(a, b; -; w) - \sum_{j=0}^{n} \frac{(a)_j (b)_j}{j!} w^j \right| \le c_n |w|^{n+1}, \quad |\arg w| \ge \epsilon, \quad |w| < 1.$$ (A.3)

Moreover, for $\mathrm{Re}(a) > 0$ we have an integral representation

$$F(a, b; -; w) = \frac{1}{\Gamma(a)} \int_0^\infty e^{-t} t^{a-1} (1-wt)^{-b} dt.$$ (A.4)

Proof. Assume first that $\mathrm{Re}(a) > 0$. For $\mathrm{Re}(c-a) > 0$ and $w \in \mathbb{C}\backslash[c^{-1}, +\infty[$, inserting $t = cs^{-1}$ and $z = cw$ into (6.5), we obtain

$$F(a, b; c; cw) = \frac{\Gamma(c)c^{-a}}{\Gamma(a)\Gamma(c-a)} \int_0^c s^{a-1}(1-c^{-1}s)^{c-a-1}(1-ws)^{-b} ds.$$ (A.5)

Using $\lim_{c\to\infty} \frac{\Gamma(c)c^{-a}}{\Gamma(c-a)} = 1$ and the Lebesgue Dominated Convergence Theorem we see that (A.5) converges to the right hand side of (A.4). This proves that $F(a, b; -; w)$ is well defined for $\mathrm{Re}(a) > 0$.
Now let a be arbitrary. We have

$$\partial_z^n F(a, b; c; z) = \frac{(a)_n (b)_n}{(c)_n} F(a+n, b+n; c+n; z).$$ (A.6)

Using Taylor's formula with a remainder

$$f(z) = \sum_{j=0}^{n-1} \frac{f^{(j)}(0)z^j}{j!} + z^n \int_0^1 \frac{f^{(n)}(sz)n(1-s)^{n-1}}{n!} \mathrm{d}s,$$

we obtain

$$F(a,b;c;z) = \sum_{j=0}^{n-1} \frac{(b)_j(a)_j w^j}{j!} \tag{A.7}$$

$$+ \frac{w^n(b)_n(a)_n}{(n-1)!} \int_0^1 (1-s)^{n-1} \mathrm{d}s F(a+n,b+n;c;ws). \tag{A.8}$$

Now, choose n such that $\mathrm{Re}(a+n) > 0$. Then we can apply what we proved before to show the convergence

$$\lim_{c \to +\infty} F(a,b;c;cz) = \sum_{j=0}^{n-1} \frac{(b)_j(a)_j w^j}{j!} \tag{A.9}$$

$$+ \frac{w^n(b)_n(a)_n}{(n-1)!} \int_0^1 (1-s)^{n-1} \mathrm{d}s F(a+n,b+n;-;ws). \tag{A.10}$$

Clearly, (A.10) is $O(w^n)$, and n can be made arbitrarily large. □

If a or b is a negative integer, then the series (A.2) terminates and functions $_2F_0(a,b;-;\cdot)$ essentially coincide with Bessel polynomials, see Subsect. 7.10. Otherwise, $_2F_0(a,b;-;\cdot)$ have a logarithmic branch point at 0.

B   Riemann class

We hope that we convinced our reader that the hypergeometric class studied in this paper is a natural family of equations. In this appendix, following [DIL], we will describe a somewhat wider family, which appears in the literature in many sources, e.g. [SL, DIL, EEKS]. According to the terminology used in [DIL], directly inspired by and partly borrowed from the monograph of Slavyanov-Lay [SL], this family is called *Riemann class*.

Consider an equation given by the operator

$$\partial_z^2 + b(z)\partial_z + c(z), \tag{B.1}$$

where $p(z)$, $q(z)$ are rational functions. Let $z_0 \in \mathbb{C} \cup \{\infty\}$ be a singular point of (B.1).

Recall that $z_0 \in \mathbb{C}$ is called *regular-singular* or *Fuchsian* if $b(z) = \frac{p(z)}{z-z_0}$ and $c(z) = \frac{q(z)}{(z-z_0)^2}$ with p, q regular at z_0. The equation

$$\lambda(\lambda-1) + p(z_0)\lambda + q(z_0) = 0 \tag{B.2}$$

is called the *indicial equation of z_0* and its roots are called *indices* of z_0.

We say that ∞ is *regular-singular* or *Fuchsian* if $b(z) = \frac{p(z)}{z}$ and $c(z) = \frac{q(z)}{z^2}$ with p, q regular at ∞.

$$\lambda(\lambda+1) - p(\infty)\lambda + q(\infty) = 0 \tag{B.3}$$

is called the *indicial equation of ∞* and its roots are called *indices* of ∞.

It is easy to see that every equation having no more than 3 singular points in $\mathbb{C} \cup \{\infty\}$, all of them Fuchsian and at most 2 finite, is given by an operator of the form

$$\partial_z^2 + \left(\frac{a_1}{z-z_1} + \frac{a_2}{z-z_2}\right)\partial_z + \frac{b_1}{z-z_1} + \frac{b_2}{z-z_2} + \frac{c_1}{(z-z_1)^2} + \frac{c_2}{(z-z_2)^2} \quad \text{with} \quad b_1 + b_2 = 0, \tag{B.4}$$

where z_1, z_2 are distinct points in $\mathbb{C}$. Following [DIL], the family of equations (B.4) is called *the Riemann type*. (Another name, introduced in [SL], is M_2-type).

Each finite singularity has at least one index equal 0 if and only if $c_1 = c_2 = 0$. Such equations are given by operators

$$\partial_z^2 + \Big(\frac{a_1}{z-z_1} + \frac{a_2}{z-z_2}\Big)\partial_z + \frac{b_1}{z-z_1} + \frac{b_2}{z-z_2} \quad \text{with} \quad \sum_{j=1}^{n} b_j = 0. \tag{B.5}$$

Following [DIL], the family of equations given by (B.5) will be called *the grounded Riemann type*. It is easy to see that by gauging with power functions we can always transform a Riemann type equation into a grounded Riemann type equation.

We say that a differential equation belongs to the *Riemann class* (or the M_2-class) if it is given by

$$\partial_z^2 + \frac{\tau(z)}{\sigma(z)}\partial_z + \frac{\xi(z)}{\sigma(z)^2}, \tag{B.6}$$

where σ, τ, ξ are polynomials satisfying

$$\sigma \neq 0, \quad \deg \sigma \leq 2, \quad \deg \tau \leq 1, \quad \deg \xi \leq 2. \tag{B.7}$$

Thus the Riemann class comprises the Riemann type together with all its confluent cases.

We say that a differential equation belongs to the *grounded Riemann class* if it is given by

$$\partial_z^2 + \frac{\tau(z)}{\sigma(z)}\partial_z + \frac{\eta(z)}{\sigma(z)}, \tag{B.8}$$

where σ, τ, η are polynomials satisfying

$$\sigma \neq 0, \quad \deg \sigma \leq 2, \quad \deg \tau \leq 1, \quad \deg \eta = 0. \tag{B.9}$$

Thus the grounded Riemann class besides the grounded Riemann includes all its confluent cases.

The following proposition is proven in [DIL]:

Proposition B.1 *1. The Riemann type is contained in the Riemann class. An equation of the Riemann class is of the Riemann type iff σ possesses 2 distinct roots.*

2. The grounded Riemann type is the intersection of the grounded Riemann class and Riemann type.

In this paper instead of the *grounded Riemann class* we use the name *hypergeometric class*. We use this name mostly for brevity, besides the word "grounded" may sound bizarre to some readers. Furthermore, we prefer to multiply these equations by $\sigma(z)$, so that we consider operators of the form

$$\sigma(z)\partial_z^2 + \tau(z)\partial_z + \eta \tag{B.10}$$

with σ, τ, ν satisfying (B.9).

One can ask whether the unified theory presented in this paper can be extended to operators of the form

$$\sigma(z)\partial_z^2 + \tau(z)\partial_z + \frac{\xi(z)}{\sigma(z)}, \tag{B.11}$$

where σ, τ, ξ satisfy (B.7). In other words whether the unified theory presented in this paper can be extended to the full Riemann class. We do not know an answer to this question. However, let us remark that we do not gain much by considering the full Riemann class instead of the grounded Riemann class.

By a division by a constant, transformation $z \mapsto az + b$ and gauging with a power and exponential all operators from the Riemann type can be always reduced to an $_2F_1$ operator. More generally, all operators from the Riemann class can be reduced to one of normal forms listed in [SL] or in Subsect. 3.3 of [DIL]:

the $_2F_1$ operator	$z(1-z)\partial_z^2 + \big(c - (a+b+1)z\big)\partial_z - ab$;
the $_2F_0$ operator	$z^2\partial_z^2 + \big(-1 + (a+b+1)z\big)\partial_z + ab$;
the $_1F_1$ operator	$z\partial_z^2 + (c-z)\partial_z - a$;
the $_0F_1$ operator	$z\partial_z^2 + c\partial_z - 1$;
the Hermite operator	$\partial_z^2 - 2z\partial_z - 2a$;
the Airy operator	$\partial_z^2 + z$;
the Euler operator I	$z^2\partial_z^2 + cz\partial_z$;
the Euler operator II	$z\partial_z^2 + c\partial_z$;
the 1d Helmholtz operator	$\partial_z^2 + 1$;
the 1d Laplace operator	∂_z^2.

In this list the first five are the normal forms listed in the introduction and described in Section 6. The last four are trivial operators that can be solved in by elementary methods. They are also special cases of the grounded Riemann class.

The Airy equation is the only type within the Riemann class which does not belong to the grounded Riemann class and therefore is left out from the unified theory presented in this paper.

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