EXCITATION SPECTRUM

OF INTERACTING QUANTUM GASES

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I will describe arguments indicating that homogeneous interacting quantum gas, both Bosonic and Fermionic, may have energy–momentum spectrum with an interesting shape. This can be used to explain physical phenomena: superfluidity and supeconductivity at zero temperature. Some of the arguments that I will describe are heuristic and go back to old ideas of Bogoliubov and Bardeen-Cooper-Schrieffer. There will be also, however, some rigous recent results.

Contents

1 HOMOGENEOUS BOSE GAS

n identical bosonic particles are described by the Hilbert space

$$\mathcal{H}_n := L^2_{\mathrm{s}}\Big((\mathbb{R}^d)^n\Big) = \otimes^n_{\mathrm{s}} L^2(\mathbb{R}^d),$$

the Schrödinger Hamiltonian

$$H_n = -\sum_{i=1}^{n} \Delta_i + \lambda \sum_{1\le i<j\le n} V(\mathrm{x}_i - \mathrm{x}_j)$$

and the momentum $P_n := -\sum_{i=1}^{n} \mathrm{i}\partial_{\mathrm{x}_i}$. We have $P_n H_n = H_n P_n$, which expresses the translational invariance of our system.

The potential V is a real function on $\mathbb{R}^d$ that decays at infinity and satisfies $V(\mathrm{x}) = V(-\mathrm{x})$.

We enclose these particles in a box of size L with fixed density $\rho := \frac{n}{L^d}$ and n large. Instead of the more physical Dirichlet boundary condtions, to keep translational invariance we impose the periodic boundary conditions, replacing the original V by the periodized potential

$$V^L(\mathrm{x}) \; := \sum_{n\in\mathbb{Z}^d} V(\mathrm{x}+Ln) = \frac{1}{L^d} \sum_{\mathrm{p}\in(2\pi/L)\mathbb{Z}^d} \mathrm{e}^{\mathrm{ipx}} \hat{V}(\mathrm{p}),$$

well defined on the torus $[-L/2, L/2[^d$. (Note that above we used the Poisson summation formula).

The original Hilbert space is replaced by

$$\mathcal{H}_n^L := L^2_{\mathrm{s}}\Big(\big([-L/2, L/2[^d\big)^n\Big) = \otimes_{\mathrm{s}}^n \big(L^2([-L/2, L/2[^d)\big).$$

We have a new Hamiltonian

$$H_n^L = -\sum_{i=1}^{n} \Delta_i^L + \lambda \sum_{1\le i<j\le n} V^L(\mathrm{x}_i - \mathrm{x}_j)$$

and a new momentum $P_n^L := -\sum_{i=1}^{n} \mathrm{i}\partial_{\mathrm{x}_i}^L$.

Because of the periodic boundary conditions we still have

$P_n^L H_n^L = H_n^L P_n^L$. In the sequel we drop the superscript L.

We prefer to work in the momentum representation, where the Hilbert space is $\mathcal{H}_n = l^2_{\mathrm{s}}\Big(\big(\frac{2\pi}{L}\mathbb{Z}^d\big)^n\Big)$, the Hamiltonian and the momentum are

$$\begin{aligned} H_n &= \sum_{i=1}^{n}\sum_{\mathrm{p}} \mathrm{p}^2|p)_i(p|_i \\ &\quad +\frac{\lambda}{L^d}\sum_{1\le i<j\le n}\ \sum_{\mathrm{p}'+\mathrm{k}'=\mathrm{k}+\mathrm{p}} \hat{V}(\mathrm{p}'-\mathrm{p})|\mathrm{p}')_i|\mathrm{k}')_j(\mathrm{k}|_j(\mathrm{p}|_i. \\ P_n &= \sum_{i=1}^{n}\sum_{\mathrm{p}} \mathrm{p}|\mathrm{p})_i(\mathrm{p}|_i. \end{aligned}$$

Consider all n at once by introducing the Bosonic Fock space

$$\mathcal{H} := \bigoplus_{n=0}^{\infty} \mathcal{H}_n = \Gamma_{\mathrm{s}}\Big(l^2\big(\frac{2\pi}{L}\mathbb{Z}^d\big)\Big).$$

The Hamiltonian and the momentum in second quantized notation are

$$\begin{aligned} H := \bigoplus_{n=0}^{\infty} H_n \;&=\; \sum_{\mathrm{p}} \mathrm{p}^2 a_{\mathrm{p}}^* a_{\mathrm{p}} \\ &\quad + \frac{\lambda}{2L^d} \sum_{\mathrm{p,q,k}} \hat{V}(\mathrm{k}) a_{\mathrm{p+k}}^* a_{\mathrm{q-k}}^* a_{\mathrm{q}} a_{\mathrm{p}}, \\ P := \bigoplus_{n=0}^{\infty} P_n \;&=\; \sum_{\mathrm{p}} \mathrm{p} a_{\mathrm{p}}^* a_{\mathrm{p}}. \end{aligned}$$

Above we use the standard formalism of second quantization involving the creation and annihilation operators a^*_{p}, a_{p} satisfying the canonical commutation relations

$$[a_{\mathrm{p}}, a_{\mathrm{k}}] = [a^*_{\mathrm{p}}, a^*_{\mathrm{k}}] = 0, \quad [a_{\mathrm{p}}, a^*_{\mathrm{k}}] = \delta_{\mathrm{p},\mathrm{k}}.$$

Note in particular that we have the number operator

$$N := \sum_{\mathrm{p}} a^*_{\mathrm{p}} a_{\mathrm{p}}.$$

2 EXCITATION SPECTRUM AND CRITICAL VELOCITY

Thus homogeneous Bose gas is described by a family of commuting self-adjoint operators (H, P), where $P = (P_1, \ldots, P_d)$. We can define its energy-momentum spectrum

$$\operatorname{spec}(H, P) \subset \begin{cases} \mathbb{R} \times \mathbb{R}^d, & L = \infty, \\ \mathbb{R} \times \frac{2\pi}{L}\mathbb{Z}^d, & L < \infty. \end{cases}$$

By general arguments the momentum of the ground state is zero. Let E denote the ground state energy of H. The excitation spectrum can be defined as

$$\operatorname{spec}(H-E,P)\backslash\{(0,0)\},$$

Note that

$$H = \begin{cases} \int_{\mathbb{R}^d}^{\oplus} H(\mathrm{k}) \mathrm{dk}, & L = \infty, \\ \bigoplus_{\mathrm{k} \in \frac{2\pi}{L}\mathbb{Z}^d} H(\mathrm{k}), & L < \infty. \end{cases}$$

We are especially interested in the infimum of the excitation spectrum

$$\varepsilon(\mathrm{k}) := \inf \operatorname{spec} \big(H(\mathrm{k}) - E \big), \quad \mathrm{k} \neq 0,$$

$$\varepsilon(0) := \inf \big(\operatorname{spec} (H(0) - E) \backslash \{0\} \big).$$

Introduce the critical velocity and the energy gap

$$\begin{aligned} c_{\rm cr} &:= \inf \frac{\varepsilon(\mathrm{k})}{|\mathrm{k}|}, \\ \varepsilon_{\rm gap} &:= \inf \big(\mathrm{spec}\,(H-E)\backslash\{0\}\big) = \inf \varepsilon(\mathrm{k}). \end{aligned}$$

In the limit $L \to \infty$, we can also try to define the phonon velocity

$$c_{\rm ph} := \lim_{\mathrm{k}\to 0} \frac{\varepsilon(\mathrm{k})}{|\mathrm{k}|}.$$

We will argue that Bose gas with repulsive interaction in thermodynamic limit has positive critical velocity, well defined positive phonon velocity and a zero energy gap.

3 LANDAU'S ARGUMENT FOR SUPERFLUIDITY

Suppose that our system is described with (H, P) with critical velocity $c_{\rm cr}$. We add to H a perturbation u travelling at a speed w:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t = \Big(H + \lambda \sum_{i=1}^{n} u(x_i - \mathrm{w}t)\Big)\Psi_t.$$

We go to the moving frame:

$$\Psi_t^{\mathrm{w}}(x_1, \dots, x_n) := \Psi_t(x_1 - \mathrm{w}t, \dots, x_n - \mathrm{w}t).$$

We obtain a Schrödinger equation with a time-independent Hamiltonian

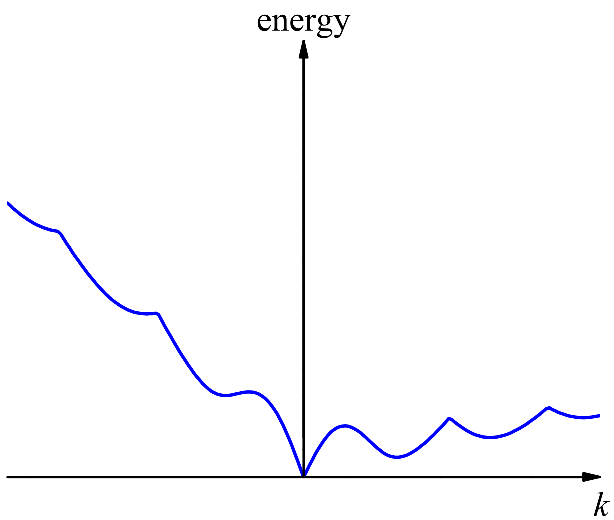
$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Psi_t^{\mathrm{w}} = \Big(H - \mathrm{w}P + \lambda\sum_{i=1}^{n} u(x_i)\Big)\Psi_t^{\mathrm{w}}.$$

Let Ψ_{gr} be the ground state of H. Is it stable against a travelling perturbation? We need to consider the tilted Hamiltonian $H - \mathrm{w}P$.

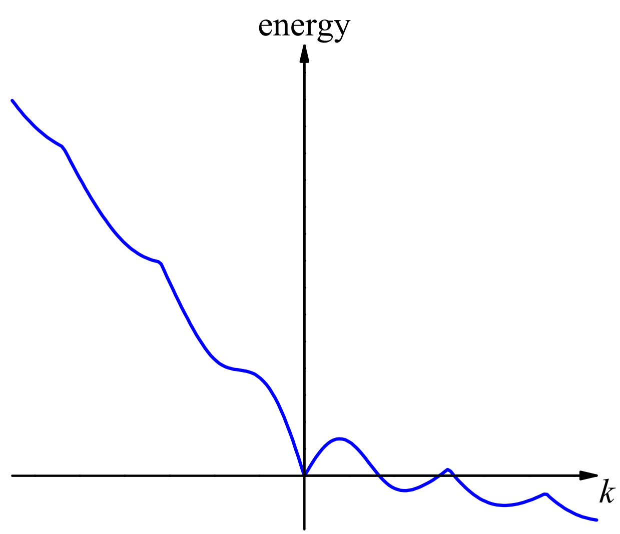
If $|\mathrm{w}| < c_{\mathrm{cr}}$, then $H - \mathrm{w}P \geq E$ and Ψ_{gr} is still a ground state of $H - \mathrm{w}P$. So Ψ_{gr} is stable.

If $|\mathrm{w}| > c_{\mathrm{cr}}$, then $H - \mathrm{w}P$ is unbounded from below. So Ψ_{gr} is not stable any more.

Bose gas travelling slower than critical velocity



Bose gas travelling faster than critical velocity



4 QUADRATIC HAMILTONIANS

In many situations we try to describe physical particles in terms of quasiparticles. This roughly means that the Hamiltonian and total momentum are

$$
\begin{aligned}
H &= \int_{\mathbb{R}^d} \omega(\mathrm{k}) a_{\mathrm{k}}^* a_{\mathrm{k}} \mathrm{dk}, \\
P &= \int_{\mathbb{R}^d} \mathrm{k} a_{\mathrm{k}}^* a_{\mathrm{k}} \mathrm{dk}
\end{aligned}
$$

for a function ω called the elementary excitation spectrum or the dispersion relation. The excitation spectrum of such a system cannot have an arbitrary shape. In particular, its infimum must be subadditive–it equals the subadditive hull of ω.

We say that a function $\mathbb{R}^d \ni \mathrm{k} \mapsto \epsilon(\mathrm{k}) \in \mathbb{R}$ is subadditive iff

$$\epsilon(\mathrm{k}_1 + \mathrm{k}_2) \leq \epsilon(\mathrm{k}_1) + \epsilon(\mathrm{k}_2), \quad \mathrm{k}_1, \mathrm{k}_2 \in \mathbb{R}^d.$$

Let $\mathbb{R}^d \ni \mathrm{k} \mapsto \omega(\mathrm{k}) \in \mathbb{R}$ be another function. We define the subbadditive hull of ω to be

$$\epsilon(\mathrm{k}) := \inf\{\omega(\mathrm{k}_1)+\cdots+\omega(\mathrm{k}_n) \ : \ \mathrm{k}_1+\cdots+\mathrm{k}_n = \mathrm{k}, \ n = 1, 2, \dots\}.$$

Clearly, the subadditive hull is always subadditive.

Proposition Let f be an increasing concave function with $f(0) \geq 0$. Then $f(|\mathrm{k}|)$ is subadditive.

Proposition Let ε_0 be subadditive and $\varepsilon_0 \leq \omega$. Let ε be the subadditive hull of ω. Then $\varepsilon_0 \leq \varepsilon$.

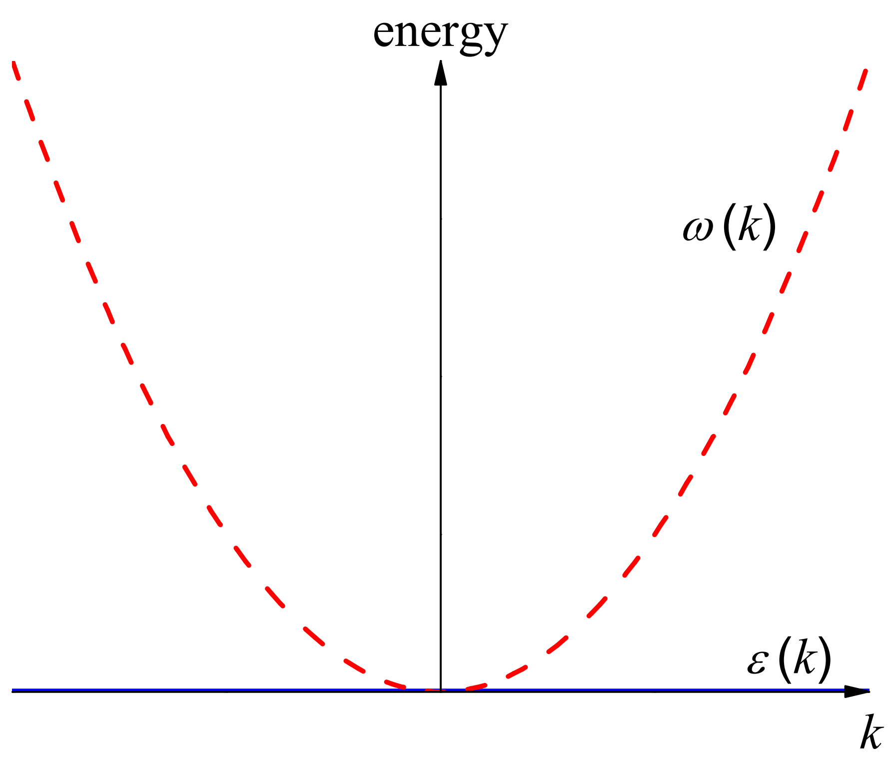
Proposition Suppose that ω satisfies

$$\inf \frac{\omega(\mathrm{k})}{|\mathrm{k}|} = c_{\mathrm{cr}} \geq 0,$$

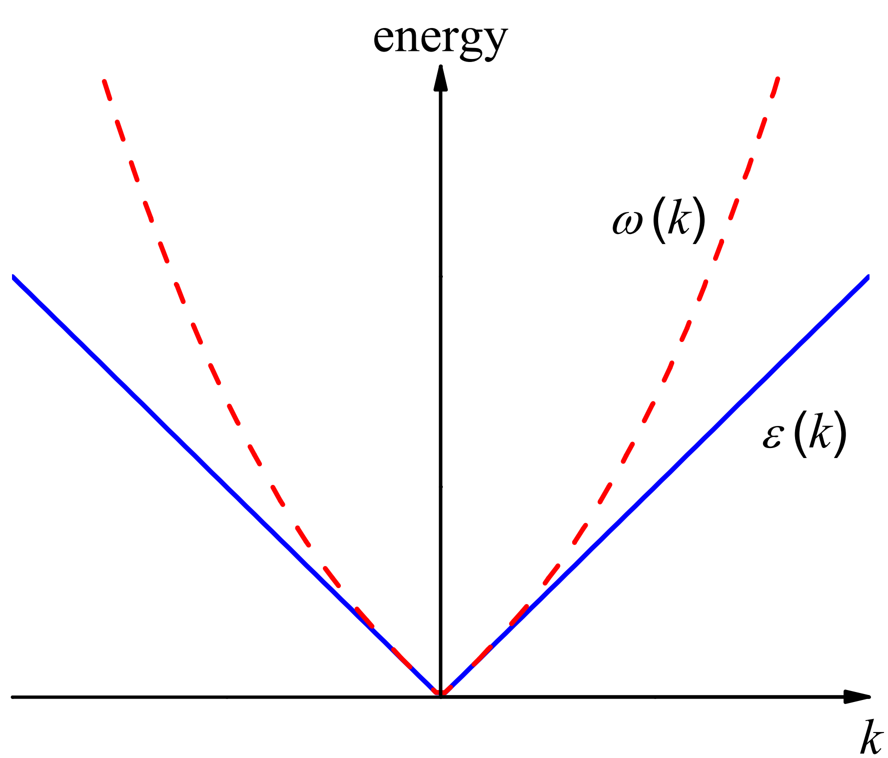
Let ε be the subadditive hull of ω. Then ε also satisfies

$$\inf \frac{\varepsilon(\mathrm{k})}{|\mathrm{k}|} = c_{\mathrm{cr}}.$$

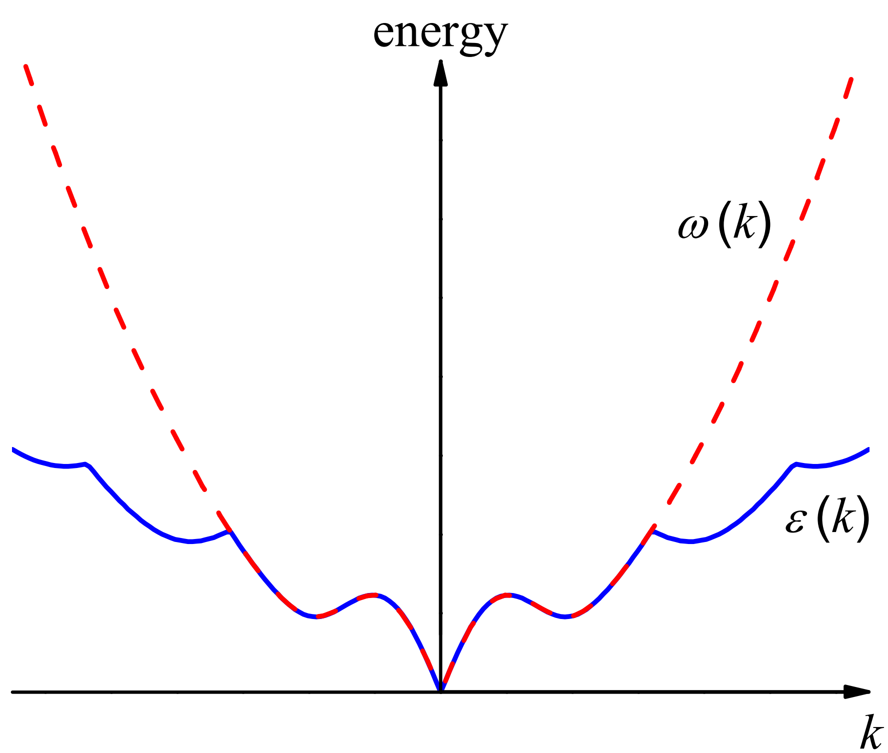
Excitation spectrum of free Bose gas



Hypothethic excitation spectrum of interacting Bose gas with no "rotons"



Hypothethic excitation spectrum of interacting Bose gas with “rotons”



5 BOGOLIUBOV'S ARGUMENT

We consider Bose gas with repulsive potential, more precisely,

$$\hat{V} \geq 0, \qquad V \geq 0.$$

We expect that most particles will be spread evenly over the whole box staying in the zeroth mode, so that $N \simeq N_0 := a_0^* a_0$. (The Bose statistics does not prohibit to occupy the same state).

Following the arguments of N. N. Bogoliubov from 1947, we drop all terms in the Hamiltonian involving more than two creation/annihilation operators of a nonzero mode. We obtain

$$
\begin{aligned}
H \;\simeq\; & \frac{\lambda \hat{V}(0)}{2L^d} a_0^* a_0^* a_0 a_0 + \sum_{\mathrm{k}\neq 0} \Big(\mathrm{k}^2 + N_0 \frac{\lambda}{L^d} \big(\hat{V}(\mathrm{k}) + \hat{V}(0) \big) \Big) a_{\mathrm{k}}^* a_{\mathrm{k}} \\
& + \sum_{\mathrm{k}\neq 0} \frac{\lambda}{2L^d} \hat{V}(\mathrm{k}) \Big(a_0^* a_0^* a_{\mathrm{k}} a_{-\mathrm{k}} + a_{\mathrm{k}}^* a_{-\mathrm{k}}^* a_0 a_0 \Big).
\end{aligned}
$$

Using $\rho = \frac{N}{L^d}$, we obtain

$$
\begin{aligned}
H &\approx \frac{\lambda \hat{V}(0)\rho}{2}(N-1) + H_{\mathrm{Bog}} + R, \\
H_{\mathrm{Bog}} &:= \sum_{\mathrm{k}\neq 0} \big(\mathrm{k}^2 + \lambda\rho\hat{V}(\mathrm{k})\big) a_{\mathrm{k}}^* a_{\mathrm{k}} \\
&\quad + \frac{1}{2}\sum_{\mathrm{k}\neq 0} \lambda\rho\hat{V}(\mathrm{k})\big(a_{\mathrm{k}}^* a_{-\mathrm{k}}^* + a_{\mathrm{k}} a_{-\mathrm{k}}\big), \\
R &= -\frac{\lambda \hat{V}(0)}{2L^d}(N-N_0)(N-N_0-1) \\
&\quad + \sum_{\mathrm{k}\neq 0} \frac{\lambda}{2L^d}\hat{V}(\mathrm{k})\Big((a_0^* a_0^* - N) a_{\mathrm{k}} a_{-\mathrm{k}} + a_{\mathrm{k}}^* a_{-\mathrm{k}}^* (a_0 a_0 - N)\Big).
\end{aligned}
$$

We look for a Bogoliubov transformation, a linear transformation of creation/annihilation operators

$$\tilde{a}_{\mathrm{p}} := c_{\mathrm{p}} a_{\mathrm{p}} + s_{\mathrm{p}} a^*_{-\mathrm{p}}, \quad \mathrm{p} \neq 0,$$

preserving the commutation relations, that diagonalizes the quadratic Hamiltonian H_{Bog}:

$$\begin{aligned} H_{\mathrm{Bog}} &= E_{\mathrm{Bog}} + \sum_{\mathrm{p}\neq 0} \omega(\mathrm{p}) \tilde{a}^*_{\mathrm{p}} \tilde{a}_{\mathrm{p}}, \\ P_{\mathrm{Bog}} &= \sum_{\mathrm{p}\neq 0} \mathrm{p} \tilde{a}^*_{\mathrm{p}} \tilde{a}_{\mathrm{p}}, \end{aligned}$$

This is realized by $c_{\mathrm{p}} = \cosh \beta_{\mathrm{p}}$, $s_{\mathrm{p}} = \sinh \beta_{\mathrm{p}}$, where

$$\tanh(\beta_{\mathrm{p}}) := \frac{|\mathrm{p}|^2 + \lambda\rho\hat{V}(\mathrm{p}) - |\mathrm{p}|\sqrt{|\mathrm{p}|^2 + 2\lambda\rho\hat{V}(\mathrm{p})}}{\lambda\rho\hat{V}(\mathrm{p})},$$

with the Bogoliubov energy

$$E_{\mathrm{Bog}} := -\frac{1}{2}\sum_{\mathrm{p}\neq 0}\left(|\mathrm{p}|^2 + \lambda\rho\hat{V}(\mathrm{p}) - |\mathrm{p}|\sqrt{|\mathrm{p}|^2 + 2\lambda\rho\hat{V}(\mathrm{p})}\right)$$

and the Bogoliubov dispersion relation

$$\omega(\mathrm{p}) = |\mathrm{p}|\sqrt{|\mathrm{p}|^2 + 2\lambda\rho\hat{V}(\mathrm{p})}.$$

The Bogoliubov dispersion relation depends on λ and $\rho := \frac{n}{L^d}$ only through $\lambda\rho$. It is therefore natural to set $\lambda := \rho^{-1}$, which we will do in what follows. Thus the initial Hamiltonian becomes

$$H = \bigoplus_{n=0}^{\infty} H_n = \sum_{\mathrm{p}} \mathrm{p}^2 a_{\mathrm{p}}^* a_{\mathrm{p}} + \frac{1}{2N} \sum_{\mathrm{p,q,k}} \hat{V}(\mathrm{k}) a_{\mathrm{p+k}}^* a_{\mathrm{q-k}}^* a_{\mathrm{q}} a_{\mathrm{p}}.$$

The Bogoliubov Hamiltonian depends on L only through the choice of the lattice spacing $\frac{2\pi}{L}$.

We expect that the low energy part of the excitation spectra of H_n and H_{Bog} are close to one another for large n, hoping that then $n - n_0 \to 0$. We expect some kind of uniformity wrt L.

Note that formally we can even take the limit $L \to \infty$ obtaining

$$\begin{aligned} H_{\mathrm{Bog}} - E_{\mathrm{Bog}} &= (2\pi)^{-d} \int \omega(\mathrm{p}) \tilde{a}_{\mathrm{p}}^* \tilde{a}_{\mathrm{p}} \mathrm{dp}, \\ P &= (2\pi)^{-d} \int \mathrm{p} \tilde{a}_{\mathrm{p}}^* \tilde{a}_{\mathrm{p}} \mathrm{dp}. \end{aligned}$$

(For finite L) set

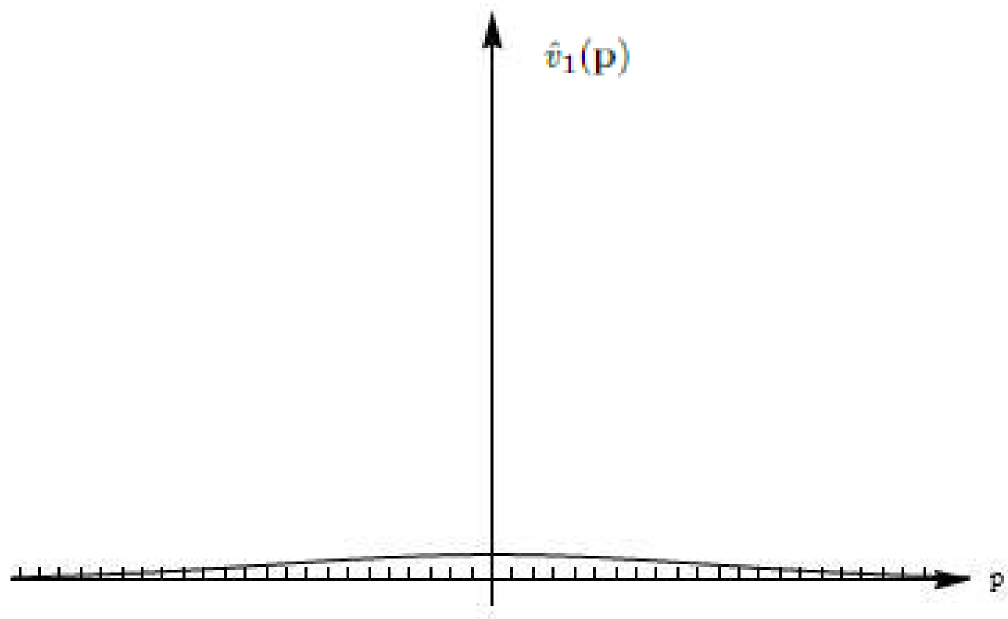
$$U = \exp\Big(\sum_{\mathrm{p}\neq 0} \frac{\beta_{\mathrm{p}}}{2}\left(a_{\mathrm{p}}^* a_{-\mathrm{p}}^* - a_{\mathrm{p}} a_{-\mathrm{p}}\right)\Big).$$

Then U is unitary and

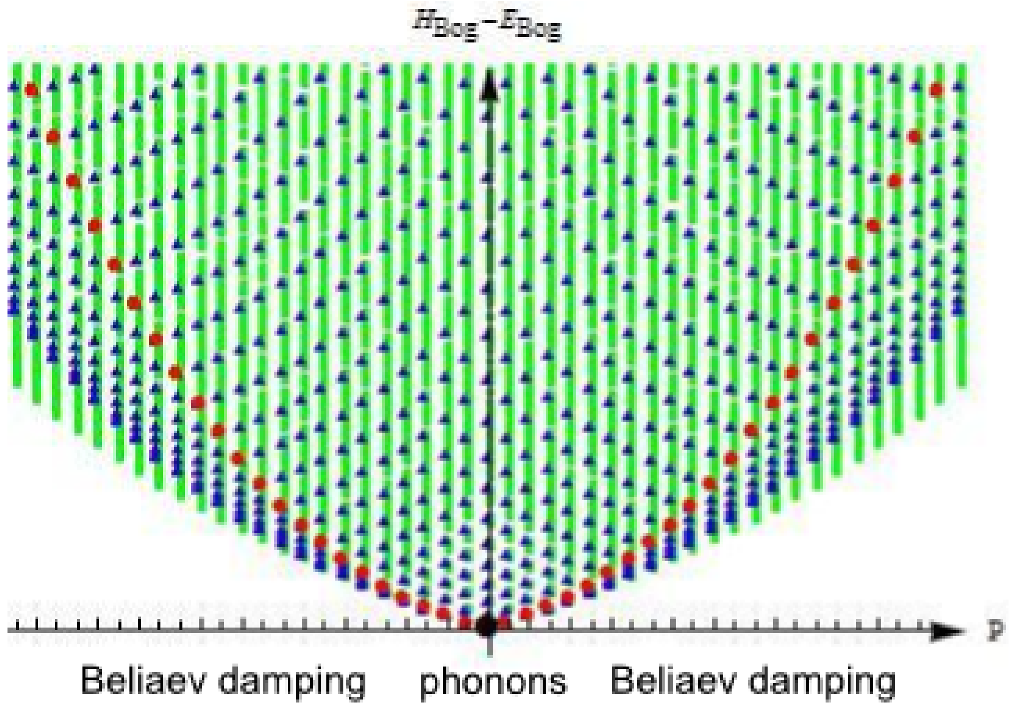
$$\begin{aligned}
\tilde{a}_{\mathrm{p}} &= U^* a_{\mathrm{p}} U, \\
\tilde{a}_{\mathrm{p}}^* &= U^* a_{\mathrm{p}}^* U, \\
H_{\mathrm{Bog}} &= E_{\mathrm{Bog}} + U^* \sum_{\mathrm{p}\neq 0} \omega(\mathrm{p}) a_{\mathrm{p}}^* a_{\mathrm{p}} U, \\
P &= U^* \sum_{\mathrm{p}\neq 0} \mathrm{p}\, a_{\mathrm{p}}^* a_{\mathrm{p}} U.
\end{aligned}$$

The excitation spectrum of H_{Bog} is given by

$$\begin{aligned}
&\operatorname{spec}\big(H_{\mathrm{Bog}} - E_{\mathrm{Bog}}, P\big) \backslash \{(0,0)\} \\
&= \Big\{ \Big(\sum_{i=1}^{j} \omega(\mathrm{p}_i),\ \sum_{i=1}^{j} \mathrm{p}_i \Big) \ :\ \mathrm{p}_1, \ldots, \mathrm{p}_j \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{0\}, \quad j = 1, 2, \ldots \Big\}.
\end{aligned}$$

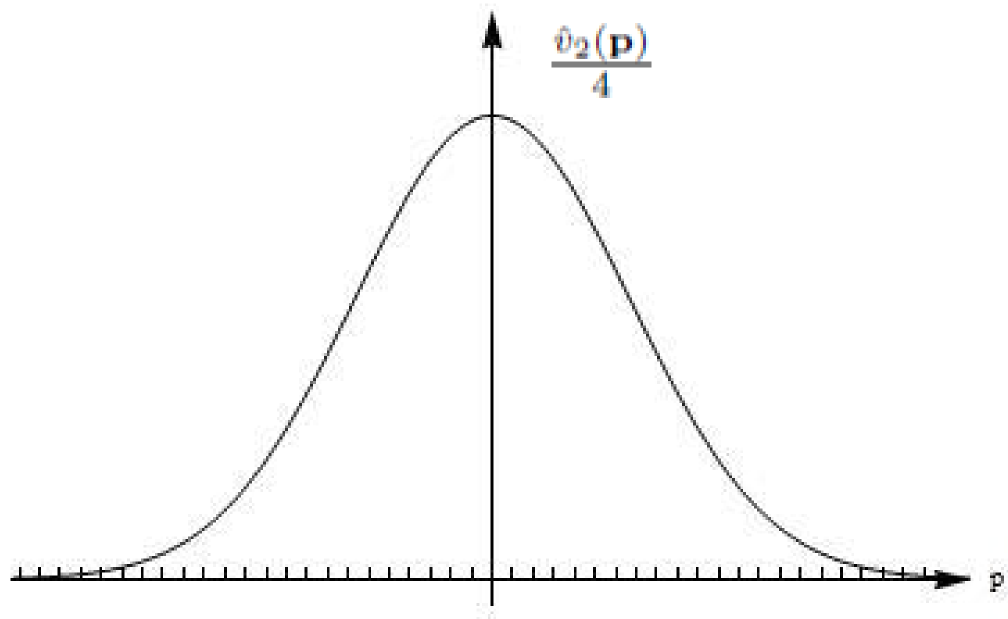


$$\hat{v}_1(\mathrm{p}) = \frac{\mathrm{e}^{-\mathrm{p}^2/5}}{10}$$

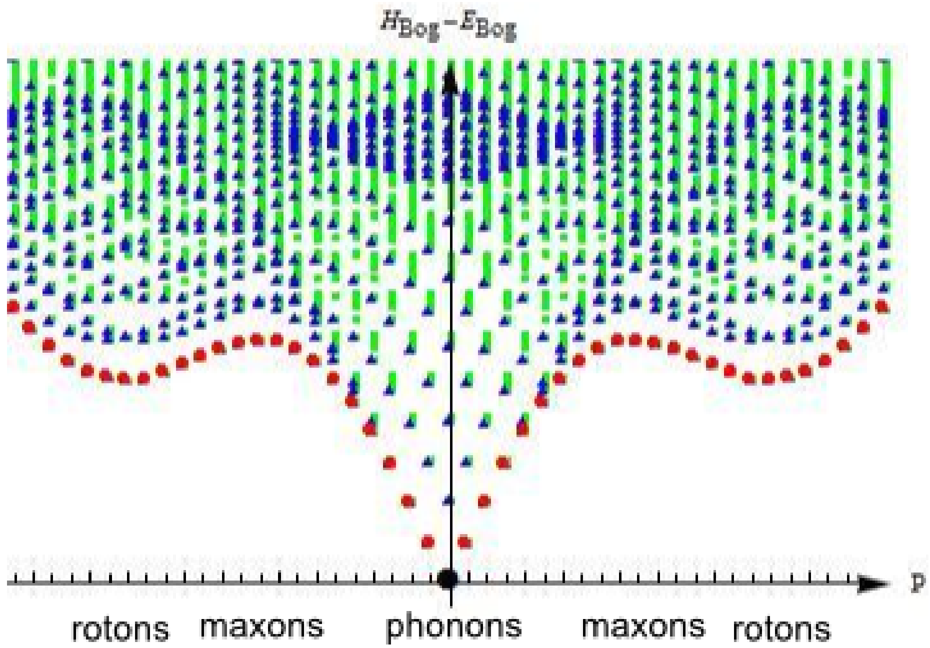


Excitation spectrum of 1-dimensional homogeneous Bose gas

with potential v_1 in the Bogoliubov approximation.



$$\hat{v}_2(\mathrm{p}) = \frac{15\mathrm{e}^{-\mathrm{p}^2/2}}{2}$$



Excitation spectrum of 1-dimensional homogeneous Bose gas

with potential v_2 in the Bogoliubov approximation.

6 RIGOROUS RESULTS ON EXCITATION SPECTRUM OF INTERACTING BOSONS

Jan Dereziński and Marcin Napiórkowski: On the excitation spectrum of interacting bosons in the infinite-volume mean-field limit, Annales Henri Poincare, DOI: 10.1007/s00023-013-0302-4

Our main result says that for large n and not too large L the low energy part of the excitation spectrum of H_n is well approximated by the low energy part of the excitation spectrum of H_{Bog}.

Note that we cannot make L go to infinity arbitrarily fast as $n \to \infty$. In particular, when we want to use arguments based on the weak coupling, we should assume $\lambda^{-1} = \rho = \frac{n}{L^d} \to \infty$.

Before we describe our result let us introduce some notation.

Let A be a bounded from below self-adjoint operator with only discrete spectrum. We define

$$\overrightarrow{\mathrm{sp}}(A) := (a_1, a_2, \dots),$$

where $a_1, a_2, \dots$ are the eigenvalues of A in the increasing order. If $\dim \mathcal{H} = n$, then we set $a_{n+1} = a_{n+2} = \cdots = \infty$.

Excitation energies of the n-body Hamiltonian.

If $\mathrm{p} \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{0\}$, set

$$\big(K_n^1(\mathrm{p}), K_n^2(\mathrm{p}), \dots\big) := \overrightarrow{\mathrm{sp}}\big(H_n(\mathrm{p}) - E_n\big).$$

The lowest eigenvalue of $H_n(0) - E_n$ is 0 by general arguments.

Set

$$\big(0, K_n^1(0), K_n^2(0), \dots\big) := \overrightarrow{\mathrm{sp}}\big(H_n(0) - E_n\big).$$

Bogoliubov excitation energies.

If $\mathrm{p} \in \frac{2\pi}{L}\mathbb{Z}^d \backslash \{0\}$, set

$$\big(K^1_{\mathrm{Bog}}(\mathrm{p}), K^2_{\mathrm{Bog}}(\mathrm{p}), \dots\big) := \overrightarrow{\mathrm{sp}}\big(H_{\mathrm{Bog}}(\mathrm{p}) - E_{\mathrm{Bog}}\big).$$

The lowest eigenvalue of $H_{\mathrm{Bog}}(0) - E_{\mathrm{Bog}}$ is obviously 0. Set

$$\big(0, K^1_{\mathrm{Bog}}(0), K^2_{\mathrm{Bog}}(0), \dots\big) := \overrightarrow{\mathrm{sp}}\big(H_{\mathrm{Bog}}(0) - E_{\mathrm{Bog}}\big).$$

Besides the assumptions on V that we already mentioned

$$\hat{V} \geq 0, \qquad V \geq 0$$

we add technical assumptions

$$\int V(\mathrm{x})\mathrm{dx} < \infty,$$

$$\hat{V}(\mathrm{p}) \leq C(1 + |\mathrm{p}|)^{-\mu}, \quad \mu > d.$$

Upper bound. Let $c > 0$. Then there exists C such that if $L^{2d+2} \le cn$, then

$$E_n \ge \frac{1}{2}\hat{v}(0)(n-1) + E_{\mathrm{Bog}} - Cn^{-1/2}L^{2d+3}.$$

If in addition $K_n^j(\mathrm{p}) \le cnL^{-d-2}$, then

$$\begin{aligned} E_n + K_n^j(\mathrm{p}) \;\ge\; & \frac{1}{2}\hat{v}(0)(n-1) + E_{\mathrm{Bog}} + K_{\mathrm{Bog}}^j(\mathrm{p}) \\ & -Cn^{-1/2}L^{d/2+3}\big(K_n^j(\mathrm{p}) + L^d\big)^{3/2}. \end{aligned}$$

Lower bound. Let $c > 0$. Then there exists $c_1 > 0$ and C such that if $L^{2d+1} \leq cn$, $L^{d+1} \leq c_1 n$, then

$$E_n \leq \frac{1}{2}\hat{v}(0)(n-1) + E_{\mathrm{Bog}} + Cn^{-1/2}L^{2d+3/2}.$$

If in addition $K^j_{\mathrm{Bog}}(\mathrm{p}) \leq cnL^{-d-2}$ and $K^j_{\mathrm{Bog}}(\mathrm{p}) \leq c_1 nL^{-2}$, then

$$\begin{aligned} E_n + K^j_n(\mathrm{p}) \ &\leq \ \frac{1}{2}\hat{v}(0)(n-1) + E_{\mathrm{Bog}} + K^j_{\mathrm{Bog}}(\mathrm{p}) \\ &\quad + Cn^{-1/2}L^{d/2+3}(K^j_{\mathrm{Bog}}(\mathrm{p}) + L^{d-1})^{3/2}. \end{aligned}$$

Special case of this theorem with $L = 1$ was proven by R. Seiringer. Mimicking his proof gives big error terms for large L: they are of the order $n^{-1/2}\exp(L^{d/2})$. To get better error estimates we need to use additional ideas.

Basic tools of the proof:

Consequence of the min-max principle:

$$A \leq B \ \text{ implies } \overrightarrow{\mathrm{sp}}(A) \leq \overrightarrow{\mathrm{sp}}(B).$$

Rayleigh-Ritz principle:

$$\overrightarrow{\mathrm{sp}}(A) \leq \overrightarrow{\mathrm{sp}}\Big(P_{\mathcal{K}} A P_{\mathcal{K}}\Big|_{\mathcal{K}}\Big).$$

It is impossible to apply the Raileigh-Ritz principle directly, because the physical Hamiltonian H_n acts on the physical space $\mathcal{H}_n$ and the Bogoliubov Hamiltonian H_{Bog} acts on the Fock space

$$\mathcal{H}_{\mathrm{Bog}} := \Gamma_{\mathrm{s}}\Big(l^2\big(\frac{2\pi}{L}\mathbb{Z}^d\backslash\{0\}\big)\Big).$$

These spaces are incomparable – neither is contained in the other.

Introduce the operator of the number of particles outside of the zeroth mode

$$N^> := \sum_{\mathrm{p}\neq 0} a_{\mathrm{p}}^* a_{\mathrm{p}}.$$

We want to use the fact that on low energy states $N^>$ is small.

The exponential property of Fock spaces says

$$\Gamma_{\rm s}(\mathcal{Z}_1 \oplus \mathcal{Z}_2) \simeq \Gamma_{\rm s}(\mathcal{Z}_1) \otimes \Gamma_{\rm s}(\mathcal{Z}_2).$$

We have

$$l^2\Big(\frac{2\pi}{L}\mathbb{Z}^d\Big) \simeq \mathbb{C} \oplus l^2\Big(\frac{2\pi}{L}\mathbb{Z}^d \backslash \{0\}\Big)$$

Thus $\mathcal{H} \simeq \Gamma_{\rm s}(\mathbb{C}) \otimes \mathcal{H}_{\rm Bog}$. Embed the space of zero modes $\Gamma_{\rm s}(\mathbb{C}) = l^2(\{0, 1, \dots\})$ in a larger space $l^2(\mathbb{Z})$. Thus we obtain the extended Hilbert space

$$\mathcal{H}^{\textsf{ext}} := l^2(\mathbb{Z}) \otimes \mathcal{H}_{\rm Bog}$$

The operator N_0 extends to an operator $N_0^{\rm ext}$. Similarly, N extends to $N^{\rm ext} = N_0^{\rm ext} + N^{>}$. The space $\mathcal{H}$ sits in $\mathcal{H}^{\rm ext}$:

$$\begin{aligned} \mathcal{H} &= \mathbb{1}_{[0,\infty[}(N_0^{\rm ext})\mathcal{H}^{\rm ext}, \\ \mathcal{H}_n &= \mathbb{1}_n(N^{\rm ext})\mathbb{1}_{[0,\infty[}(N_0^{\rm ext})\mathcal{H}^{\rm ext}. \end{aligned}$$

For any value of n there is a copy of $\mathcal{H}_{\rm Bog}$ in $\mathcal{H}^{\rm ext}$:

$$\mathcal{H}_{\rm Bog} \simeq \mathcal{H}_n^{\rm ext} := \mathbb{1}_n(N^{\rm ext})\mathcal{H}^{\rm ext}.$$

We have also a unitary operator

$$U|n_0\rangle \otimes \Psi^{>} = |n_0 - 1\rangle \otimes \Psi^{>}.$$

We now define for $\mathrm{p} \neq 0$ the following operator on $\mathcal{H}^{\mathsf{ext}}$:

$$b_{\mathrm{p}} := a_{\mathrm{p}} U^*.$$

Operators b_{p} and b_{k}^* satisfy the same CCR as a_{p} and a_{k}^*.

Let us repeat Bogoliubov's heuristic argument:

$$
\begin{aligned}
H \;\simeq\;& \frac{\hat{V}(0)}{2N} a_0^* a_0^* a_0 a_0 + \sum_{\mathrm{p}\neq 0} \Big(\mathrm{p}^2 + \frac{N_0}{N}\big(\hat{V}(\mathrm{p}) + \hat{V}(0)\big)\Big) a_{\mathrm{p}}^* a_{\mathrm{p}} \\
&+ \sum_{\mathrm{p}\neq 0} \frac{1}{2N} \hat{V}(\mathrm{p}) \Big(a_0^* a_0^* a_{\mathrm{p}} a_{-\mathrm{p}} + a_0 a_0 a_{\mathrm{p}}^* a_{-\mathrm{p}}^* \Big) \\
=\;& \frac{\hat{V}(0)}{2N} N_0(N_0 - 1) + \sum_{\mathrm{p}\neq 0} \Big(\mathrm{p}^2 + \frac{N_0}{N}\big(\hat{V}(\mathrm{p}) + \hat{V}(0)\big)\Big) b_{\mathrm{p}}^* b_{\mathrm{p}} \\
&+ \sum_{\mathrm{p}\neq 0} \frac{1}{2N} \hat{V}(\mathrm{p}) \Big(\sqrt{N_0(N_0-1)}\, b_{\mathrm{p}} b_{-\mathrm{p}} + b_{\mathrm{p}}^* b_{-\mathrm{p}}^* \sqrt{N_0(N_0-1)} \Big) \\
\simeq\;& \frac{\hat{V}(0)}{2}(N-1) + \sum_{\mathrm{p}\neq 0} \big(\mathrm{p}^2 + \hat{V}(\mathrm{p}) \big) b_{\mathrm{p}}^* b_{\mathrm{p}} \\
&+ \sum_{\mathrm{p}\neq 0} \frac{\hat{V}(\mathrm{p})}{2} \big(b_{\mathrm{p}}^* b_{-\mathrm{p}}^* + b_{\mathrm{p}} b_{-\mathrm{p}} \big)
\end{aligned}
$$

In the actual proof we use an estimating Hamiltonian on $\mathcal{H}_n$

$$
\begin{aligned}
H_{n,\epsilon} \ :=\ & \frac{1}{2}\hat{V}(0)(n-1) + \sum_{\mathrm{p}\neq 0} \big(|\mathrm{p}|^2 + \hat{V}(\mathrm{p})\big) a_{\mathrm{p}}^* a_{\mathrm{p}} \\
& + \frac{1}{2n} \sum_{\mathrm{p}\neq 0} \hat{V}(\mathrm{p}) \Big(a_0^* a_0^* a_{\mathrm{p}} a_{-\mathrm{p}} + a_{\mathrm{p}}^* a_{-\mathrm{p}}^* a_0 a_0 \Big) \\
& - \frac{1}{n} \sum_{\mathrm{p}\neq 0} \big(\hat{V}(\mathrm{p}) + \frac{\hat{V}(0)}{2}\big) a_{\mathrm{p}}^* a_{\mathrm{p}} N^{>} + \frac{\hat{V}(0)}{2n} N^{>} \\
+ \frac{\epsilon}{n} \sum_{\mathrm{p}\neq 0} & \big(\hat{V}(\mathrm{p}) + \hat{V}(0)\big) a_{\mathrm{p}}^* a_{\mathrm{p}} N_0 + +(1+\epsilon^{-1}) \frac{1}{2n} V(0) L^d N^{>}(N^{>}-1)
\end{aligned}
$$

$$
H_n \geq H_{n,-\epsilon}, \quad 0 < \epsilon \leq 1; \quad H_n \leq H_{n,\epsilon}, \quad 0 < \epsilon.
$$

Extended estimating Hamiltonian on $\mathcal{H}_n^{\mathrm{ext}}$

$$\begin{aligned} H_{n,\epsilon}^{\mathrm{ext}} \;:=\; & \frac{1}{2}\hat{V}(0)(n-1) + \sum_{\mathrm{p}\neq 0}\big(|\mathrm{p}|^2 + \hat{V}(\mathrm{p})\big) b_{\mathrm{p}}^* b_{\mathrm{p}} \\ & + \frac{1}{2}\sum_{\mathrm{p}\neq 0}\hat{V}(\mathrm{p})\Big(\frac{\sqrt{(N_0^{\mathrm{ext}}-1)N_0^{\mathrm{ext}}}}{n} b_{\mathrm{p}} b_{-\mathrm{p}} + \mathrm{hc}\Big) \\ & - \frac{1}{n}\sum_{\mathrm{p}\neq 0}\big(\hat{V}(\mathrm{p}) + \frac{\hat{V}(0)}{2}\big) b_{\mathrm{p}}^* b_{\mathrm{p}} N^{>} + \frac{\hat{V}(0)}{2n} N^{>} \\ & + \frac{\epsilon}{n}\sum_{\mathrm{p}\neq 0}\big(\hat{V}(\mathrm{p}) + \hat{V}(0)\big) b_{\mathrm{p}}^* b_{\mathrm{p}} N_0^{\mathrm{ext}} \\ & + (1+\epsilon^{-1})\frac{1}{2n} V(0) L^d N^{>}(N^{>}-1). \end{aligned}$$

$H_{n,\epsilon}^{\mathrm{ext}}$ preserves $\mathcal{H}_n$ and restricted to $\mathcal{H}_n$ coincides with $H_{n,\epsilon}$.

$$\sum_{\mathrm{p}\neq 0}\left(|\mathrm{p}|^2+\hat{V}(\mathrm{p})\right)b_{\mathrm{p}}^*b_{\mathrm{p}}+\frac{1}{2}\sum_{\mathrm{p}\neq 0}\hat{V}(\mathrm{p})\Big(b_{\mathrm{p}}b_{-\mathrm{p}}+b_{\mathrm{p}}^*b_{-\mathrm{p}}^*\Big).$$

preserves $\mathcal{H}_n^{\mathrm{ext}}$. Its restriction to $\mathcal{H}_n^{\mathrm{ext}}$ will be denoted $H_{\mathrm{Bog},n}$.

Clearly, $H_{\mathrm{Bog},n}$ is unitarily equivalent to H_{Bog}.

$$
\begin{aligned}
H^{\mathrm{ext}}_{n,\epsilon} &= \frac{1}{2}\hat{V}(0)(n-1) + H_{\mathrm{Bog},n} + R_{n,\epsilon}, \\
R_{n,\epsilon} &:= \frac{1}{2}\sum_{\mathrm{p}\neq 0}\hat{V}(\mathrm{p})\Big(\Big(\frac{\sqrt{(N_0^{\mathrm{ext}}-1)N_0^{\mathrm{ext}}}}{n} - 1\Big)b_{\mathrm{p}}b_{-\mathrm{p}} + \mathrm{hc}\Big) \\
&\quad -\frac{1}{n}\sum_{\mathrm{p}\neq 0}\big(\hat{V}(\mathrm{p}) + \frac{\hat{V}(0)}{2}\big)b_{\mathrm{p}}^{*}b_{\mathrm{p}}N^{>} + \frac{\hat{V}(0)}{2n}N^{>} \\
+\frac{\epsilon}{n}&\sum_{\mathrm{p}\neq 0}\big(\hat{V}(\mathrm{p}) + \hat{V}(0)\big)b_{\mathrm{p}}^{*}b_{\mathrm{p}}N_0^{\mathrm{ext}} + (1+\epsilon^{-1})\frac{1}{2n}V(0)L^{d}N^{>}(N^{>}-1).
\end{aligned}
$$

Proof of lower bound. We use the inclusion $\mathcal{H}_n \subset \mathcal{H}_n^{\mathrm{ext}}$. For brevity set

$$\mathbb{1}_\kappa^n := \mathbb{1}_{[0,\kappa]}(H_n - E_n).$$

For $0 < \epsilon \leq 1$,

$$\mathbb{1}_\kappa^n H_n \mathbb{1}_\kappa^n \geq \mathbb{1}_\kappa^n \left(\frac{1}{2}\hat{V}(0)(n-1) + H_{\mathrm{Bog},n} + R_{n,-\epsilon} \right) \mathbb{1}_\kappa^n.$$

Hence,

$$\overrightarrow{\mathrm{sp}}\Big(\mathbb{1}_\kappa^n H_n \mathbb{1}_\kappa^n\Big) \geq \frac{1}{2}\hat{V}(0)(n-1) + \overrightarrow{\mathrm{sp}}\Big(H_{\mathrm{Bog}}\Big) - \|R_{n,-\epsilon}\|.$$

Proof of upper bound. Let $G \in C^\infty([0,\infty[)$, $G \geq 0$,

$$G(s) = \begin{cases} 1, & \text{if } s \in [0, \frac{1}{3}] \\ 0, & \text{if } s \in [1, \infty[. \end{cases}$$

For brevity, we set $\mathbb{1}_\kappa^{\mathrm{Bog}} := \mathbb{1}_{[0,\kappa]}(H_{\mathrm{Bog},n} - E_{\mathrm{Bog}})$. We define

$$Z_\kappa := \left(\mathbb{1}_\kappa^{\mathrm{Bog}} G(N^>/n)^2 \mathbb{1}_\kappa^{\mathrm{Bog}}\right)^{-1/2} \mathbb{1}_\kappa^{\mathrm{Bog}} G(N^>/n).$$

Z_κ is a partial isometry with initial space $\mathrm{Ran}(G(N^>/n)\mathbb{1}_\kappa^{\mathrm{Bog}}) \subset \mathcal{H}$ and final space $\mathrm{Ran}(\mathbb{1}_\kappa^{\mathrm{Bog}}) \subset \mathcal{H}_n^{\mathrm{ext}}$.

$$\overrightarrow{\mathrm{sp}} H_n \ \le \ \overrightarrow{\mathrm{sp}}\left(Z_\kappa^* Z_\kappa H_n Z_\kappa^* Z_\kappa \Big|_{\mathrm{Ran} Z_\kappa^*} \right) = \overrightarrow{\mathrm{sp}}\left(Z_\kappa H_n Z_\kappa^* \Big|_{\mathrm{Ran} \mathbb{1}_\kappa^{\mathrm{Bog}}} \right).$$

$$\begin{aligned} Z_\kappa H_n Z_\kappa^* \ &\le \ Z_\kappa H_{n,\epsilon} Z_\kappa^* \\ &= \ \frac{1}{2}\hat{V}(0)(n-1)\mathbb{1}_\kappa^{\mathrm{Bog}} + H_{\mathrm{Bog}} \mathbb{1}_\kappa^{\mathrm{Bog}} \\ &\quad + Z_\kappa (H_{\mathrm{Bog}} - E_{\mathrm{Bog}}) Z_\kappa^* - (H_{\mathrm{Bog}} - E_{\mathrm{Bog}}) \mathbb{1}_\kappa^{\mathrm{Bog}} \\ &\quad + Z_\kappa R_{n,\epsilon} Z_\kappa^*. \end{aligned}$$

Therefore,

$$\begin{aligned}
\overrightarrow{\mathrm{sp}}(H_n) &\le \overrightarrow{\mathrm{sp}}\big(Z_\kappa H_{n,\epsilon} Z_\kappa^*\big) \\
&= \frac{1}{2}\hat{V}(0)(n-1) + \overrightarrow{\mathrm{sp}}\Big(H_{\mathrm{Bog}} \mathbb{1}_\kappa^{\mathrm{Bog}}\Big) \\
&\quad + \Big\| Z_\kappa (H_{\mathrm{Bog}} - E_{\mathrm{Bog}}) Z_\kappa^* - (H_{\mathrm{Bog}} - E_{\mathrm{Bog}}) \mathbb{1}_\kappa^{\mathrm{Bog}} \Big\| \\
&\quad + \Big\| Z_\kappa R_{n,\epsilon} Z_\kappa^* \Big\|.
\end{aligned}$$

7 FINITE VOLUME EFFECTS

For $\mathrm{w} \in \frac{2\pi}{L}\mathbb{Z}^d$ we define the boost operator in the direction of w:

$$U(\mathrm{w}) := \exp\left(\mathrm{i}\sum_{i=1}^{n} \mathrm{x}_i \mathrm{w}\right).$$

We easily compute

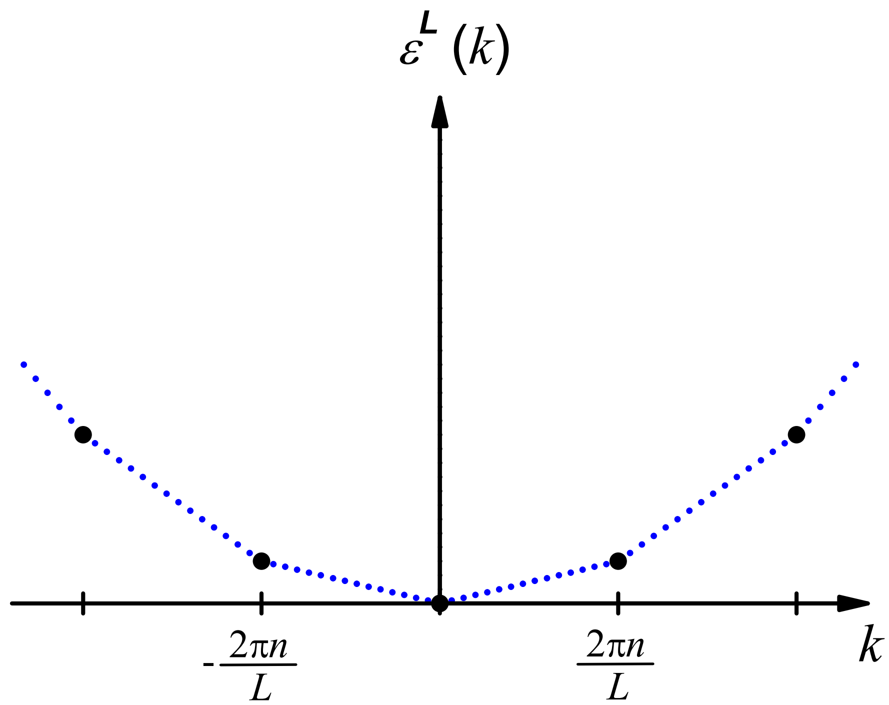
$$\begin{aligned} U^*(\mathrm{w}) P^n U(\mathrm{w}) &= P^n + \mathrm{w}n, \\ U^*(\mathrm{w}) \left(H^n - \frac{1}{n}(P^n)^2\right) U(\mathrm{w}) &= H^n - \frac{1}{n}(P^n)^2 \end{aligned}$$

Hence

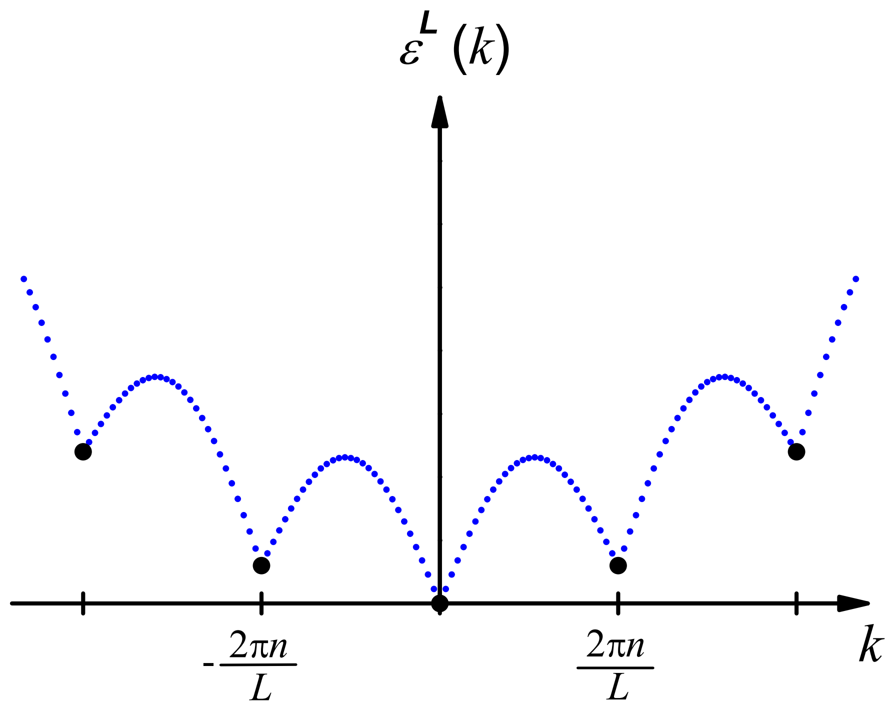
$$\operatorname{spec} H(\mathrm{p} + n\mathrm{w}) - \frac{(\mathrm{p} + n\mathrm{w})^2}{n} = \operatorname{spec} H(\mathrm{p}) - \frac{\mathrm{p}^2}{n}.$$

Excitation spectrum of free Bose gas

in finite volume



Excitation spectrum of interacting Bose gas in finite volume



In dimension $d = 1$ in the limit $L \to \infty$ we have

$\epsilon(\mathrm{k} + 2\pi\rho) = \epsilon(\mathrm{k})$, because

$$(H^{L,n} - E)\Phi \;=\; 0,$$

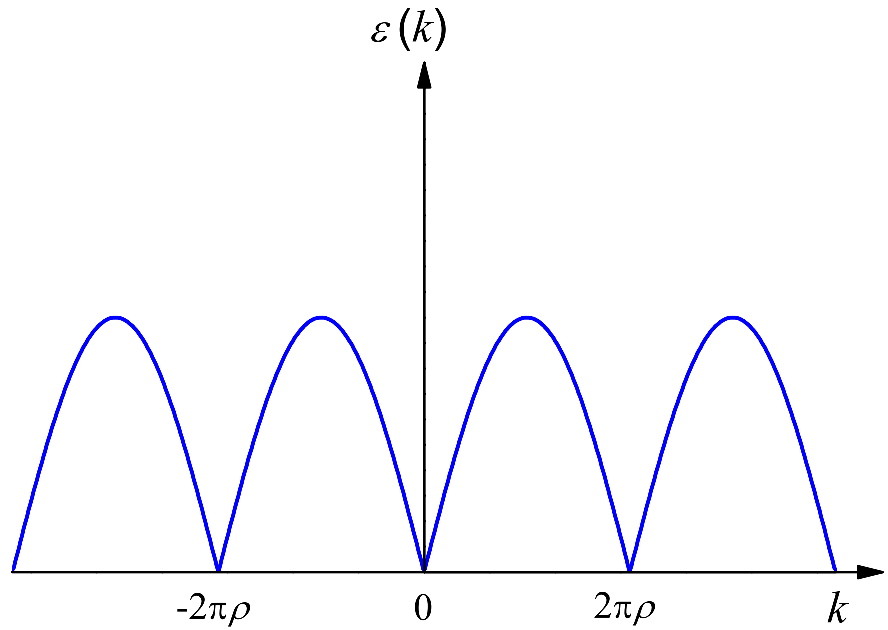
$$(P^{L,n} - \mathrm{k})\Phi \;=\; 0,$$

with $U = U(\frac{2\pi}{L})$, implies

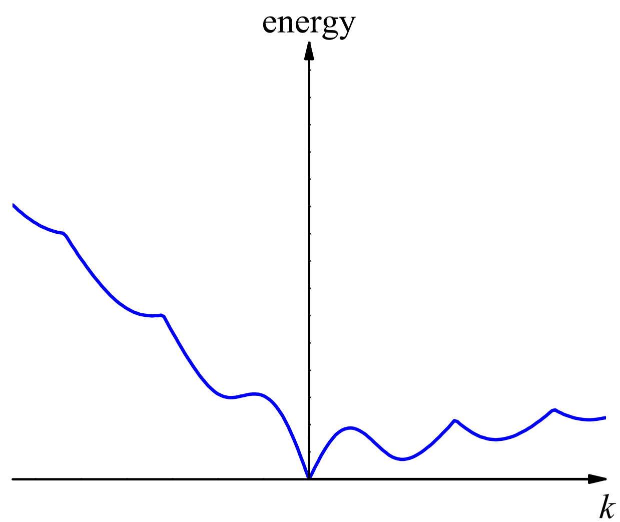
$$(H^{L,n} - E)U\Phi \;=\; \frac{1}{L}(2\pi\mathrm{k} + 2\pi^2\rho)U\Phi \to 0,$$

$$(P^{L,n} - \mathrm{k} - 2\pi\rho)U\Phi \;=\; 0.$$

Excitation spectrum of 1-dimensional interacting Bose gas

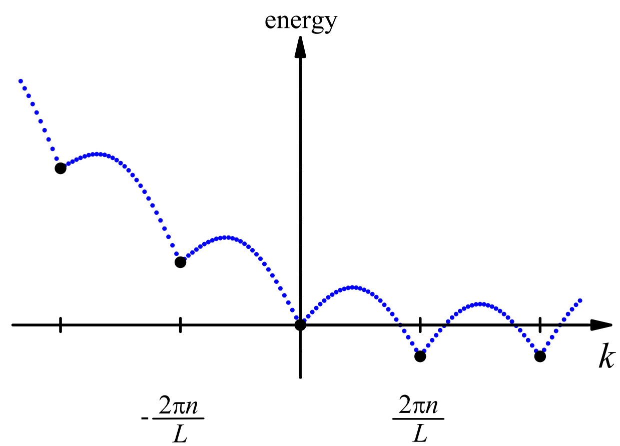


In Landau's argument we gave the following picture of the tilted Hamiltonian:



In finite volume it is incorrect.

Travelling Bose gas in finite volume



Define the global critical velocity

$$c_{\mathrm{cr}}^{L,n} := \inf_{|\mathrm{k}|} \frac{\epsilon^{L,n}(\mathrm{k})}{|\mathrm{k}|}$$

If $|w| < c_{\mathrm{cr}}^{L,n}$, then the ground state of $H^{L,n}$ remains the ground state of the "tilted Hamiltonian", hence it is stable.

For the free Bose gas we have $c_{\mathrm{cr}}^{L,n} = \frac{\pi}{L} > 0$. In general, $c_{\mathrm{cr}}^{L,n} \leq \frac{\pi}{L}$. Hence the global critical velocity is very small and vanishes in the thermodynamic limit.

Define the restricted critical velocity below the momentum R as

$$c^{L,n}_{\mathrm{cr},R} := \inf\left\{\frac{\epsilon^{L,n}(\mathrm{k})}{|\mathrm{k}|} \quad \mathrm{k} \neq 0, \quad |\mathrm{k}| < R\right\}.$$

We expect that for repulsive potentials

$$c^{\rho}_{\mathrm{cr},R} := \lim_{L\to\infty} c^{L,n}_{\mathrm{cr},R}, \quad \frac{n}{L^d} = \rho,$$

exists and, in dimension $d \geq 2$, we have $c^{\rho}_{\mathrm{cr}} := \liminf_{R\to\infty} c^{\rho}_{\mathrm{cr},R} > 0$.

This may imply the metastability against travelling perturbations travelling at a speed smaller than c^{ρ}_{cr}.

8 GRAND-CANONICAL APPROACH

Consider the symmetric Fock space $\Gamma_{\rm s}\left(L^2([L/2, L/2]^d)\right)$ and the (canonical) Hamiltonian H with $\lambda = 1$. For a chemical potential $\mu > 0$, we define the grand-canonical Hamiltonian

$$\begin{aligned} H_\mu &:= H - \mu N \\ &= \sum_{\rm p} ({\rm p}^2 - \mu) a^*_{\rm p} a_{\rm p} \\ &\quad + \frac{1}{2L^d} \sum_{\rm p,q,k} \hat{V}({\rm k}) a^*_{\rm p+k} a^*_{\rm q-k} a_{\rm q} a_{\rm p}. \end{aligned}$$

If E_μ is the ground state energy of H_μ, then it is realized in the sector n satisfying

$$\partial_\mu E_\mu = -n.$$

In what follows we drop the subscript μ.

For $\alpha \in \mathbb{C}$, we define the displacement or Weyl operator of the zeroth mode: $W_\alpha := \mathrm{e}^{-\alpha a_0^* + \overline{\alpha} a_0}$. Let $\Omega_\alpha := W_\alpha \Omega$ be the corresponding coherent vector. Note that $P\Omega_\alpha = 0$. The expectation of the Hamiltonian in Ω_α is

$$(\Omega_\alpha | H \Omega_\alpha) = -\mu |\alpha|^2 + \frac{\hat{V}(0)}{2L^d} |\alpha|^4.$$

It is minimized for $\alpha = \mathrm{e}^{\mathrm{i}\tau} \frac{\sqrt{L^d \mu}}{\sqrt{\hat{V}(0)}}$, where τ is an arbitrary phase.

We apply the Bogoliubov translation to the zero mode of H by $W(\alpha)$. This means making the substitution

$$a_0 = \tilde{a}_0 + \alpha,\ \ a_0^* = \tilde{a}_0^* + \overline{\alpha},$$

$$a_{\mathrm{k}} = \tilde{a}_{\mathrm{k}}, \qquad a_{\mathrm{k}}^* = \tilde{a}_{\mathrm{k}}^*, \qquad \mathrm{k} \neq 0.$$

Note that

$$\tilde{a}_{\mathrm{k}} = W_\alpha^* a_{\mathrm{k}} W_\alpha, \quad \tilde{a}_{\mathrm{k}}^* = W_\alpha^* a_{\mathrm{k}}^* W_\alpha,$$

and thus the operators with and without tildes satisfy the same commutation relations. We drop the tildes.

Translated Hamiltonian

$$\begin{aligned}
H \; := \; & -L^d \frac{\mu^2}{2\hat{V}(0)} \\
& + \sum_{\mathrm{k}} \left(\frac{1}{2}\mathrm{k}^2 + \hat{V}(\mathrm{k}) \frac{\mu}{\hat{V}(0)} \right) a_{\mathrm{k}}^* a_{\mathrm{k}} \\
& + \sum_{\mathrm{k}} \hat{V}(\mathrm{k}) \frac{\mu}{2\hat{V}(0)} \left(\mathrm{e}^{-\mathrm{i}2\tau} a_{\mathrm{k}} a_{-\mathrm{k}} + \mathrm{e}^{\mathrm{i}2\tau} a_{\mathrm{k}}^* a_{-\mathrm{k}}^* \right) \\
& + \sum_{\mathrm{k},\mathrm{k}'} \frac{\hat{V}(\mathrm{k})\sqrt{\mu}}{\sqrt{\hat{V}(0) L^d}} (\mathrm{e}^{-\mathrm{i}\tau} a_{\mathrm{k}+\mathrm{k}'}^* a_{\mathrm{k}} a_{\mathrm{k}'} + \mathrm{e}^{\mathrm{i}\tau} a_{\mathrm{k}}^* a_{\mathrm{k}'}^* a_{\mathrm{k}+\mathrm{k}'}) \\
& + \sum_{\mathrm{k}_1+\mathrm{k}_2=\mathrm{k}_3+\mathrm{k}_4} \frac{\hat{V}(\mathrm{k}_2 - \mathrm{k}_3)}{2L^d} a_{\mathrm{k}_1}^* a_{\mathrm{k}_2}^* a_{\mathrm{k}_3} a_{\mathrm{k}_4}.
\end{aligned}$$

If we (temporarily) replace the potential $V(\mathrm{x})$ with $\lambda V(\mathrm{x})$, where λ is a (small) positive constant, the translated Hamiltonian can be rewritten as

$$H^{\lambda} = \lambda^{-1} H_{-1} + H_0 + \sqrt{\lambda} H_{\frac{1}{2}} + \lambda H_1.$$

Thus the 3rd and 4th terms are in some sense small, which suggests dropping them.

Thus

$$H \approx -L^d \frac{\mu^2}{2\hat{V}(0)} + \mu(\mathrm{e}^{\mathrm{i}\tau} a_0^* + \mathrm{e}^{-\mathrm{i}\tau} a_0)^2 + H_{\mathrm{Bog}},$$

where

$$\begin{aligned} H_{\mathrm{Bog}} \;=\; & \sum_{\mathrm{k}\neq 0} \Big(\frac{1}{2}\mathrm{k}^2 + \hat{V}(\mathrm{k}) \frac{\mu}{\hat{V}(0)}\Big) a_{\mathrm{k}}^* a_{\mathrm{k}} \\ & + \sum_{\mathrm{k}\neq 0} \hat{V}(\mathrm{k}) \frac{\mu}{2\hat{V}(0)} \left(\mathrm{e}^{-\mathrm{i}2\tau} a_{\mathrm{k}} a_{-\mathrm{k}} + \mathrm{e}^{\mathrm{i}2\tau} a_{\mathrm{k}}^* a_{-\mathrm{k}}^*\right) \end{aligned}$$

Then we proceed as before with the Bogoliubov energy

$$E_{\mathrm{Bog}} := -\frac{1}{2}\sum_{\mathrm{p}\neq 0}\left(|\mathrm{p}|^2 + \mu\frac{\hat{V}(\mathrm{p})}{\hat{V}(0)} - |\mathrm{p}|\sqrt{|\mathrm{p}|^2 + 2\mu\frac{\hat{V}(\mathrm{p})}{\hat{V}(0)}}\right)$$

and the Bogoliubov dispersion relation

$$\omega(\mathrm{p}) = |\mathrm{p}|\sqrt{|\mathrm{p}|^2 + 2\mu\frac{\hat{V}(\mathrm{p})}{\hat{V}(0)}}.$$

Note that the grand-canonical Hamiltonian H_μ is invariant wrt the $U(1)$ symmetry $\mathrm{e}^{\mathrm{i}\tau N}$. The parameter α has an arbitrary phase. Thus we broke the symmetry when translating the Hamiltonian.

The zero mode is not a harmonic oscillator – it has continuous spectrum and it can be interpreted as a kind of a Goldstone mode.

9 IMPROVING BOGOLIUBOV APPROXIMATION

Let $\alpha \in \mathbb{C}$ and $\frac{2\pi}{L}\mathbb{Z}^d \ni \mathrm{k} \mapsto \theta_{\mathrm{k}} \in \mathbb{C}$ be a sequence with $\theta_{\mathrm{k}} = \theta_{-\mathrm{k}}$.

Set

$$U_\theta := \prod_{\mathrm{k}} \mathrm{e}^{-\frac{1}{2}\theta_{\mathrm{k}} a_{\mathrm{k}}^* a_{-\mathrm{k}}^* + \frac{1}{2}\overline{\theta}_{\mathrm{k}} a_{\mathrm{k}} a_{-\mathrm{k}}}$$

Recall that $W_\alpha := \mathrm{e}^{-\alpha a_0^* + \overline{\alpha} a_0}$. Then $U_{\alpha,\theta} := U_\theta W_\alpha$ is the general form of a Bogoliubov transformation commuting with momentum.

Let Ω denote the vacuum vector. $\Psi_{\alpha,\theta} := U^*_{\alpha,\theta}\Omega$ is the general form of a squeezed vector of zero momentum. We are looking for α, θ such that

$$(\Psi_{\alpha,\theta}|H\Psi_{\alpha,\theta}) \quad (*)$$

attains the minimum. $(*)$ is equal to

$$(\Omega|U_{\alpha,\theta}HU^*_{\alpha,\theta}\Omega).$$

Therefore, to find $(*)$ it is enough to compute the Bogoliubov-rotated Hamiltonian $U_{\alpha,\theta}HU^*_{\alpha,\theta}$ and transform it to the Wick ordered form.

This can be done by noting that

$$\begin{aligned} U_{\alpha,\theta} a_{\mathrm{k}}^* U_{\alpha,\theta}^* &= c_{\mathrm{k}} a_{\mathrm{k}}^* - \overline{s}_{\mathrm{k}} a_{-\mathrm{k}} + \delta_{0,\mathrm{k}} \overline{\alpha}, \\ U_{\alpha,\theta} a_{\mathrm{k}} U_{\alpha,\theta}^* &= c_{\mathrm{k}} a_{\mathrm{k}} - s_{\mathrm{k}} a_{-\mathrm{k}}^* + \delta_{0,\mathrm{k}} \alpha, \end{aligned}$$

where

$$c_{\mathrm{k}} := \cosh|\theta_{\mathrm{k}}|, \quad s_{\mathrm{k}} := -\frac{\theta_{\mathrm{k}}}{|\theta_{\mathrm{k}}|} \sinh|\theta_{\mathrm{k}}|.$$

and inserting this into H.

This is usually presented in a different but equivalent way: one introduces

$$b_{\mathrm{k}} := U_{\alpha,\theta}^{*} a_{\mathrm{k}} U_{\alpha,\theta}, \quad b_{\mathrm{k}}^{*} := U_{\alpha,\theta}^{*} a_{\mathrm{k}}^{*} U_{\alpha,\theta},$$

and one inserts

$$a_{\mathrm{k}}^{*} = c_{\mathrm{k}} b_{\mathrm{k}}^{*} - \overline{s}_{\mathrm{k}} b_{-\mathrm{k}} + \delta_{0,\mathrm{k}} \overline{\alpha},\ a_{\mathrm{k}} = c_{\mathrm{k}} b_{\mathrm{k}} - s_{\mathrm{k}} b_{-\mathrm{k}}^{*} + \delta_{0,\mathrm{k}} \alpha,$$

into the expression for the Hamiltonian.

$$
\begin{aligned}
H &= B + Cb_0^* + \overline{C}b_0 \\
&\quad + \frac{1}{2}\sum_{\mathrm{k}} O(\mathrm{k}) b_{\mathrm{k}}^* b_{-\mathrm{k}}^* + \frac{1}{2}\sum_{\mathrm{k}} \overline{O}(\mathrm{k}) b_{\mathrm{k}} b_{-\mathrm{k}} + \sum_{\mathrm{k}} D(\mathrm{k}) b_{\mathrm{k}}^* b_{\mathrm{k}} \\
&\quad + \text{terms higher order in } b\text{'s.}
\end{aligned}
$$

Clearly we have bound

$$E \ \leq \ (\Psi_{\alpha,\theta}|H\Psi_{\alpha,\theta}) = B,$$

Vectors $\Psi_{\alpha,\theta,\mathrm{k}} := U^*_{\alpha,\theta} a^*_{\mathrm{k}} \Omega$ have momentum k, that means

$$(P - \mathrm{k})\Psi_{\alpha,\theta,\mathrm{k}} = 0.$$

We can use $\Psi_{\alpha,\theta,\mathrm{k}}$ to obtain a variational upper bound for the infimum of energy-momentum spectrum:

$$E + \epsilon(\mathrm{k}) \ \leq \ (\Psi_{\alpha,\theta,\mathrm{k}}|H\Psi_{\alpha,\theta,\mathrm{k}}) = B + D(\mathrm{k}).$$

Recall that we look for the infimum of $(\Psi_{\alpha,\theta}|H\Psi_{\alpha,\theta}) = B$,

Computing the derivatives with respect to α and $\overline{\alpha}$ we obtain

$$C = c_0\partial_{\overline{\alpha}}B - s_0\partial_{\alpha}B$$

so that the condition

$$\partial_{\overline{\alpha}}B = \partial_{\alpha}B = 0$$

entails $C = 0$.

Computing the derivatives with respect to s and $\overline{s}$ we obtain

$$O(\mathrm{k}) = \left(-2c_{\mathrm{k}} + \frac{|s_{\mathrm{k}}|^2}{c_{\mathrm{k}}}\right)\partial_{\overline{s}_{\mathrm{k}}} B - \frac{s_{\mathrm{k}}^2}{c_{\mathrm{k}}}\partial_{s_{\mathrm{k}}} B.$$

Thus $\partial_{s_{\mathrm{k}}} B = \partial_{\overline{s}_{\mathrm{k}}} B = 0$ entails $O(\mathrm{k}) = 0$.

Instead of s_k, c_k, it is more convenient to use functions

$$\begin{aligned} S_\mathrm{k} &:= 2s_\mathrm{k}c_\mathrm{k}, \\ C_\mathrm{k} &:= c_\mathrm{k}^2 + |s_\mathrm{k}|^2. \end{aligned}$$

We will keep $\alpha = |\alpha|\mathrm{e}^{\mathrm{i}\tau}$ instead of μ as the parameter of the theory. We can later on express μ in terms of α^2:

$$\begin{aligned} \mu &= \frac{\hat{V}(0)}{L^d}|\alpha|^2 + \sum_{\mathrm{k}'} \frac{\hat{V}(0) + \hat{V}(\mathrm{k}')}{2L^d}(C_{\mathrm{k}'} - 1) - \mathrm{e}^{\mathrm{i}2\tau} \sum_{\mathrm{k}'} \frac{\hat{V}(\mathrm{k}')}{2L^d}\overline{S}_{\mathrm{k}'}, \\ \rho &= \frac{|\alpha|^2 + \sum_\mathrm{k} |s_\mathrm{k}|^2}{L^d}. \end{aligned}$$

We obtain a fixed point equation

$$
\begin{aligned}
D(\mathrm{k}) &= \sqrt{f_{\mathrm{k}}^2 - |g_{\mathrm{k}}|^2}, \\
S_{\mathrm{k}} &= \frac{g_{\mathrm{k}}}{D(\mathrm{k})}, \\
C_{\mathrm{k}} &= \frac{f_{\mathrm{k}}}{D_{\mathrm{k}}}, \\
f_{\mathrm{k}} :&= \frac{\mathrm{k}^2}{2} + |\alpha|^2 \frac{\hat{V}(\mathrm{k})}{L^d} \\
&\quad + \sum_{\mathrm{k}'} \frac{\hat{v}(\mathrm{k}' - \mathrm{k}) - \hat{V}(\mathrm{k}')}{2L^d} (C_{\mathrm{k}'} - 1) + \sum_{\mathrm{k}'} \frac{\hat{V}(\mathrm{k}')}{2L^d} \mathrm{e}^{\mathrm{i}2\tau} \overline{S}_{\mathrm{k}'}, \\
g_{\mathrm{k}} :&= |\alpha|^2 \mathrm{e}^{\mathrm{i}2\tau} \frac{\hat{V}(\mathrm{k})}{L^d} - \sum_{\mathrm{k}'} \frac{\hat{V}(\mathrm{k}' - \mathrm{k})}{2V} S_{\mathrm{k}'}.
\end{aligned}
$$

In the limit $L \to \infty$ one should take $\alpha = \sqrt{L^d \kappa}$, where κ has the interpretation of the density of the condensate. Then one could expect that S_{k} will converge to a function depending on $\mathrm{k} \in \mathbb{R}^d$ in a reasonable class and we can replace $\frac{1}{L^d}\sum\limits_{\mathrm{k}}$ by $\frac{1}{(2\pi)^d}\int \mathrm{dk}$.

In particular,

$$D(0) = \sqrt{\frac{\hat{V}(0)}{2L^d}\alpha^2 \sum_{\mathrm{k}} \frac{\hat{V}(\mathrm{k})}{L^d}\overline{S}_{\mathrm{k}}} \to \sqrt{\frac{\hat{V}(0)\kappa}{2(2\pi)^d}\int \hat{V}(\mathrm{k})\overline{S}_{\mathrm{k}}\mathrm{dk}}.$$

Thus we expect that $D(0) > 0$, which would mean that we have an energy gap in this approximation. It is believed that this is an artefact of the approach and that the true excitation spectrum of the Bose gas has no energy gap. Thus while we improved the approximation quantitatively, we made it worse qualitatively.

10 HOMOGENEOUS FERMI GAS

We consider fermions with spin $\frac{1}{2}$ described by the Hilbert space

$$\mathcal{H}_n := \otimes_{\mathrm{a}}^n \left(L^2(\mathbb{R}^d, \mathbb{C}^2) \right) .$$

We use the chemical potential from the beginning and we do not to assume the locality of interaction, so that the Hamiltonian is

$$H_n = -\sum_{i=1}^{n} \left(\Delta_i - \mu \right) + \lambda \sum_{1 \leq i < j \leq n} v_{ij}.$$

The interaction will be given by a 2-body operator

$$(v\Phi)_{i_1,i_2}(\mathrm{x}_1,\mathrm{x}_2) \;=\; \frac{1}{2}\int\int\Big(v(\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_3,\mathrm{x}_4)\Phi_{i_2,i_1}(\mathrm{x}_4,\mathrm{x}_3)$$
$$-v(\mathrm{x}_1,\mathrm{x}_2,\mathrm{x}_4,\mathrm{x}_3)\Phi_{i_1,i_2}(\mathrm{x}_3,\mathrm{x}_4)\Big)\mathrm{d}\mathrm{x}_3\mathrm{d}\mathrm{x}_4,$$

where $\Phi \in \otimes_{\mathrm{a}}^2\left(L^2(\mathbb{R}^d,\mathbb{C}^2)\right)$. We will assume that v is Hermitian, real and translation invariant:

$$
\begin{aligned}
& v(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4) = v(\mathrm{x}_2, \mathrm{x}_1, \mathrm{x}_4, \mathrm{x}_3) \\
= \; & v(\mathrm{x}_4, \mathrm{x}_3, \mathrm{x}_2, \mathrm{x}_1) = v(\mathrm{x}_1 + \mathrm{y}, \mathrm{x}_2 + \mathrm{y}, \mathrm{x}_3 + \mathrm{y}, \mathrm{x}_4 + \mathrm{y}) \\
= \; & (2\pi)^{-4d} \int \mathrm{e}^{\mathrm{i}\mathrm{k}_1\mathrm{x}_1 + \mathrm{i}\mathrm{k}_2\mathrm{x}_2 - \mathrm{i}\mathrm{k}_3\mathrm{x}_3 - \mathrm{i}\mathrm{k}_4\mathrm{x}_4} q(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4) \\
& \times \delta(\mathrm{k}_1 + \mathrm{k}_2 - \mathrm{k}_3 - \mathrm{k}_4) \mathrm{d}\mathrm{k}_1 \mathrm{d}\mathrm{k}_2 \mathrm{d}\mathrm{k}_3 \mathrm{d}\mathrm{k}_4,
\end{aligned}
$$

where q is a function defined on the subspace $\mathrm{k}_1 + \mathrm{k}_2 = \mathrm{k}_3 + \mathrm{k}_4$.

An example of interaction is a 2-body potential $V(\mathrm{x})$ such that $V(\mathrm{x}) = V(-\mathrm{x})$, which corresponds to

$$
\begin{aligned}
v(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4) &= V(\mathrm{x}_1 - \mathrm{x}_2)\delta(\mathrm{x}_1 - \mathrm{x}_4)\delta(\mathrm{x}_2 - \mathrm{x}_3), \\
q(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4) &= \int \mathrm{dp}\hat{V}(\mathrm{p})\delta(\mathrm{k}_1 - \mathrm{k}_4 - \mathrm{p})\delta(\mathrm{k}_2 - \mathrm{k}_3 + \mathrm{p}).
\end{aligned}
$$

Similarly, as before, we periodize the interaction

$$
\begin{aligned}
& v^L(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4) \\
= & \sum_{\mathrm{n}_1, \mathrm{n}_2, \mathrm{n}_3 \in \mathbb{Z}^d} v(\mathrm{x}_1 + \mathrm{n}_1 L, \mathrm{x}_2 + \mathrm{n}_2 L, \mathrm{x}_3 + \mathrm{n}_3 L, \mathrm{x}_4) \\
= & \frac{1}{L^{3d}} \sum_{\mathrm{k}_1 + \mathrm{k}_2 = \mathrm{k}_3 + \mathrm{k}_4} \mathrm{e}^{\mathrm{i}\mathrm{k}_1 \cdot \mathrm{x}_1 + \mathrm{i}\mathrm{k}_2 \mathrm{x}_2 - \mathrm{i}\mathrm{k}_3 \mathrm{x}_3 - \mathrm{i}\mathrm{k}_4 \mathrm{x}_4} q(\mathrm{k}_1, \mathrm{k}_2, \mathrm{k}_3, \mathrm{k}_4),
\end{aligned}
$$

where $\mathrm{k}_i \in \frac{2\pi}{L}\mathbb{Z}^d$.

The Hamiltonian

$$H^{L,n} = \sum_{1\le i\le n} \big(-\Delta_i^L - \mu\big) + \sum_{1\le i<j\le n} v_{ij}^L$$

acts on $\mathcal{H}^{n,L} := \otimes_{\mathrm{a}}^n \big(L^2([-L/2, L/2]^d, \mathbb{C}^2)\big)$. We drop the superscript L.

It is convenient to put all the n-particle spaces into a single Fock space

$$\bigoplus_{n=0}^{\infty} \mathcal{H}^n = \Gamma_{\mathrm{a}}\big(L^2([L/2, L/2]^d, \mathbb{C}^2)\big)$$

and rewrite the Hamiltonian and momentum in the language of 2nd quantization:

$$
\begin{aligned}
H &:= \bigoplus_{n=0}^{\infty} H^n \\
&= \sum_i \int a^*_{\mathrm{x},i} (\Delta_{\mathrm{x}} - \mu) a_{\mathrm{x},i_2} \mathrm{dx} \\
&\quad + \frac{1}{2} \sum_{i_1,i_2} \int \int a^*_{\mathrm{x}_1,i_1} a^*_{\mathrm{x}_2,i_2} v(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3, \mathrm{x}_4) a_{\mathrm{x}_3,i_2} a_{\mathrm{x}_4,i_1} \\
&\qquad \mathrm{dx}_1 \mathrm{dx}_2 \mathrm{dx}_3 \mathrm{dx}_4, \\
P &:= \bigoplus_{n=0}^{\infty} P^n = -\mathrm{i} \int a^*_{\mathrm{x},i} \nabla_{\mathrm{x}} a_{\mathrm{x},i} \mathrm{dx}.
\end{aligned}
$$

In the momentum representation,

$$
\begin{aligned}
H &= \sum_i \sum_{\mathrm{k}} (\mathrm{k}^2 - \mu) a^*_{\mathrm{k},i} a_{\mathrm{k},i} \\
&+ \frac{1}{2L^d} \sum_{i_1,i_2} \sum_{\mathrm{k}_1+\mathrm{k}_2=\mathrm{k}_3+\mathrm{k}_4} q(\mathrm{k}_1,\mathrm{k}_2,\mathrm{k}_3,\mathrm{k}_4) a^*_{\mathrm{k}_1,i_1} a^*_{\mathrm{k}_2,,i_2} a_{\mathrm{k}_3,i_2} a_{\mathrm{k}_4,i_1}, \\
P &= \sum_i \sum_{\mathrm{k}} \mathrm{k} a^*_{\mathrm{k},i} a_{\mathrm{k},i}.
\end{aligned}
$$

$H^\pm$ will denote the operator H restricted to the subspace $(-1)^N = \pm 1$.

11 EXCITATION SPECTRUM OF FERMI GAS

Consider first non-interacting Fermi gas in finite volume, where for simplicity we drop the spin:

$$H_{\mathrm{fr}}^{L} = \sum_{\mathrm{k}} (\mathrm{k}^2 - \mu) a^*(\mathrm{k}) a(\mathrm{k}),$$

$$P_{\mathrm{fr}}^{L} = \sum_{\mathrm{k}} \mathrm{k} a^*(\mathrm{k}) a(\mathrm{k}).$$

We introduce new creation/annihilation operators

$$b_{\mathrm{k}}^* : = a_{\mathrm{k}}^*, \ b_{\mathrm{k}} := a_{\mathrm{k}}, \ \mathrm{k}^2 > \mu,$$

$$b_{\mathrm{k}}^* : = a_{-\mathrm{k}}, \ b_{\mathrm{k}} := a_{-\mathrm{k}}^*, \ \mathrm{k}^2 \leq \mu.$$

Dropping the constant

$$E = \sum_{\mathrm{k}^2 \le \mu} (\mathrm{k}^2 - \mu)$$

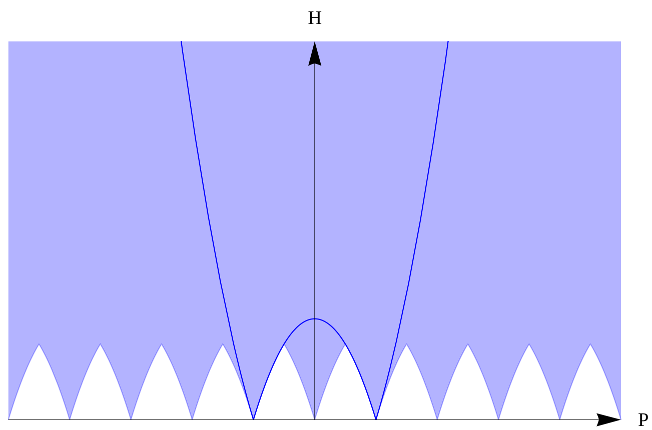
from the Hamiltonian and setting $\omega(\mathrm{k}) = |\mathrm{k}^2 - \mu|$, we obtain

$$\begin{aligned} H_{\mathrm{fr}}^L &= \sum_{\mathrm{k}} \omega(\mathrm{k}) b^*(\mathrm{k}) b(\mathrm{k}), \\ P_{\mathrm{fr}}^L &= \sum_{\mathrm{k}} \mathrm{k} b^*(\mathrm{k}) b(\mathrm{k}). \end{aligned}$$

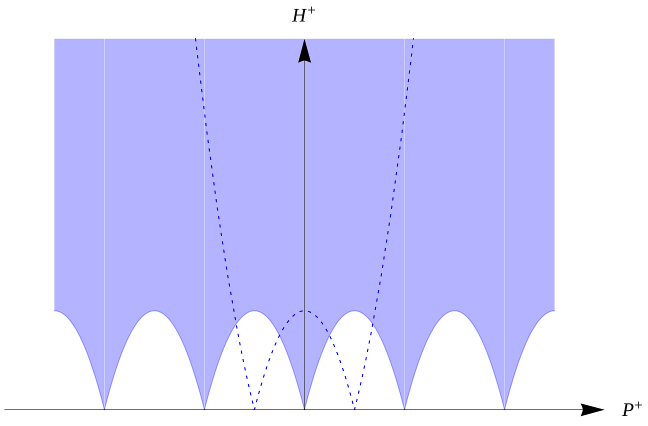
Performing formally the limit $L \to \infty$, we obtain

$$
\begin{aligned}
H_{\mathrm{fr}} &= \int \omega(\mathrm{k}) b^*(\mathrm{k}) b(\mathrm{k}) \mathrm{dk}, \\
P_{\mathrm{fr}} &= \int \mathrm{k} b^*(\mathrm{k}) b(\mathrm{k}) \mathrm{dk}.
\end{aligned}
$$

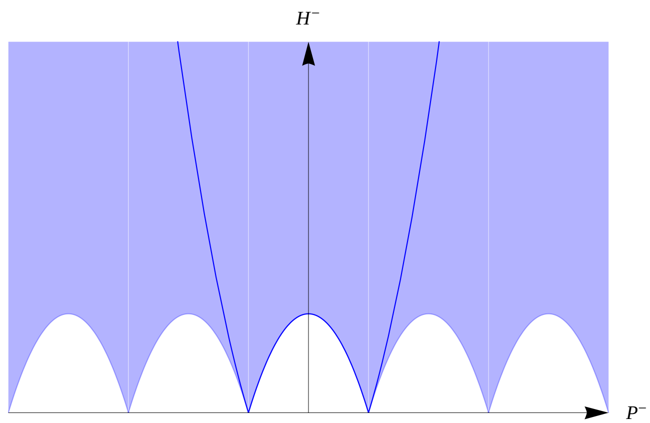
In dimension 1 its energy-momentum spectrum looks quite interesting:



$\operatorname{spec}(H, P)$ in the non-interacting case, $d = 1$.

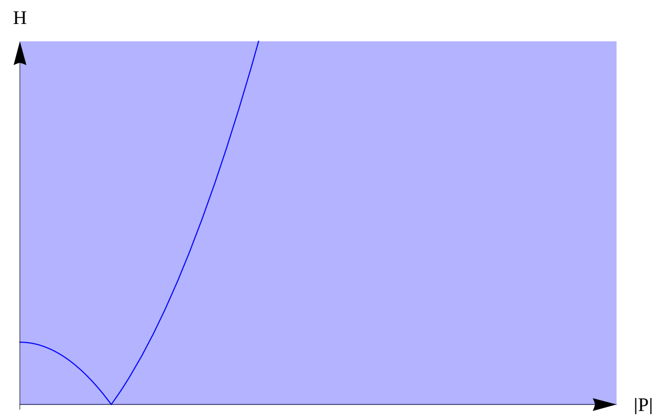


$\mathrm{spec}\,(H^+, P^+)$ in the non-interacting case, $d = 1$.



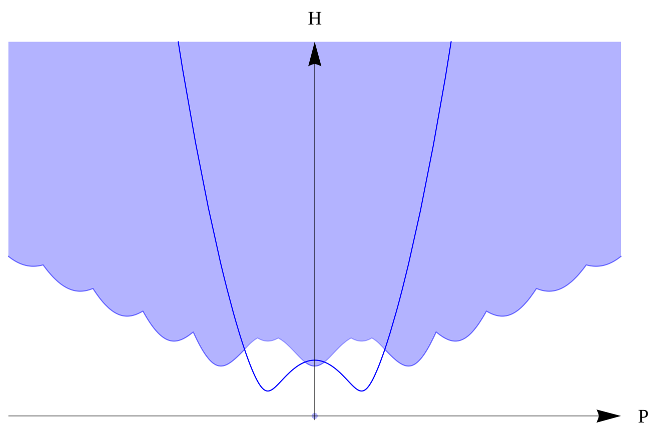
$\mathrm{spec}\,(H^-, P^-)$ in the non-interacting case, $d = 1$.

Clearly, for $d \geq 2$ the energy-momentum spectrum is rather boring:

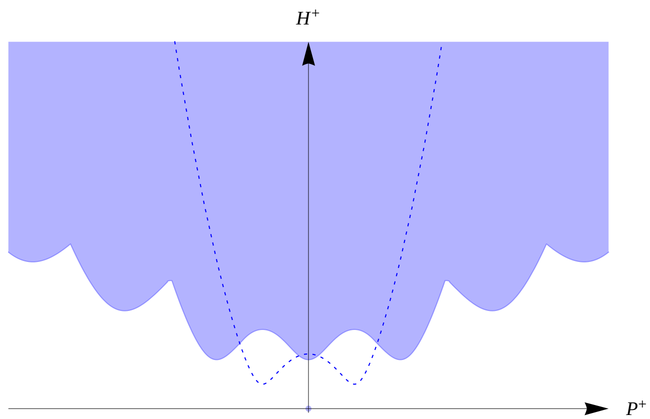


$\mathrm{spec}\,(H, P)$, $\mathrm{spec}\,(H^+, P^+)$, $\mathrm{spec}\,(H^-, P^-)$ in the non-interacting case, $d \geq 2$.

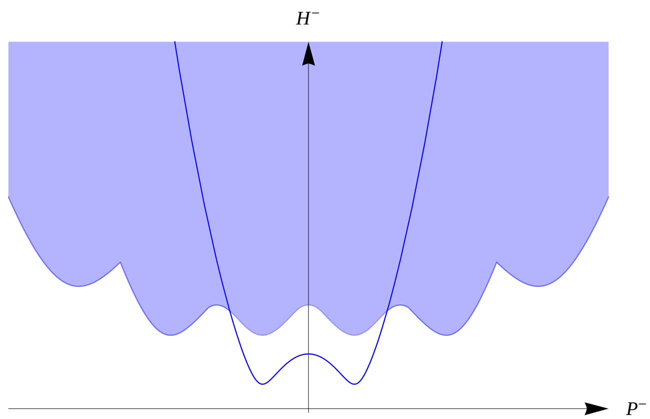
Suppose now that the dispersion relation is slightly modified, so that its minimum is stricly positive. For interacting Fermi gas, this is ideed suggested by the Hartree-Fock-Bogoliubov method with the Bardeen-Cooper-Schrieffer ansatz. Then the energy-momentum spectrum has an energy gap and the critical velocity is strictly positive! This can be used to explain superconductivity at zero temperature.



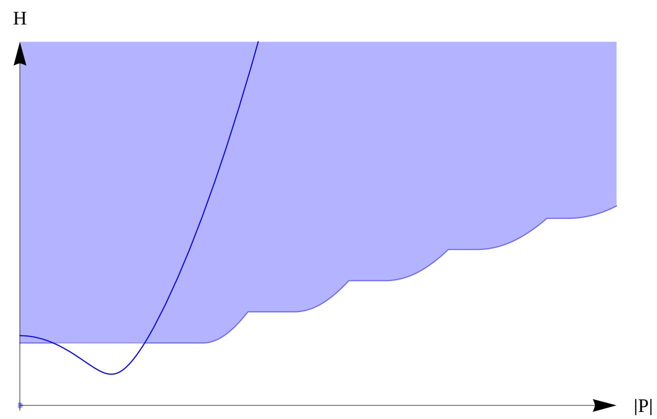
$\mathrm{spec}\,(H, P)$ in the interacting case, $d = 1$.



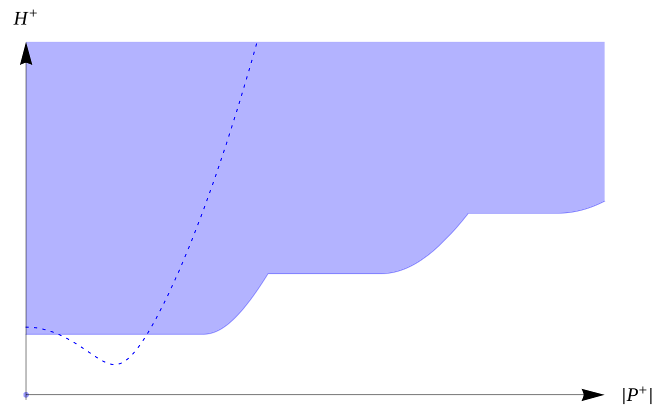
$\mathrm{spec}\,(H^{+}, P^{+})$ in the interacting case, $d = 1$.



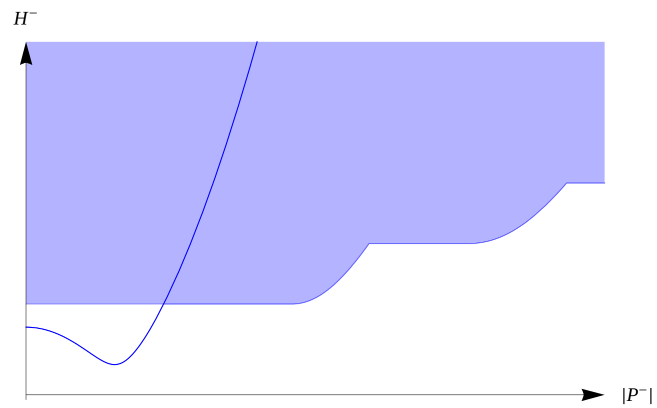
$\operatorname{spec}(H^-, P^-)$ in the interacting case, $d = 1$.



$\mathrm{spec}\,(H, P)$ in the interacting case, $d \geq 2$.



$\mathrm{spec}\,(H^+, P^+)$ in the interacting case, $d \geq 2$.



$\mathrm{spec}\,(H^-, P^-)$ in the interacting case, $d \geq 2$.

12 HFB APPROXIMATION WITH BCS ANSATZ

One can try to compute the excitation spectrum of the Fermi gas by approximate methods. We will use the Hartree-Fock-Bogoliubov approximation with the Bardeen-Cooper-Schrieffer ansatz. We start with a Bogoliubov rotation. For any k this corresponds to a substitution

$$a_{\mathrm{k}}^* = c_{\mathrm{k}} b_{\mathrm{k}}^* + s_{\mathrm{k}} b_{-\mathrm{k}}, \; a_{\mathrm{k}} = c_{\mathrm{k}} b_{\mathrm{k}} + s_{\mathrm{k}} b_{-\mathrm{k}}^*,$$

where c_{k} and s_{k} are matrices on $\mathbb{C}^2$

$$c_{\mathrm{k}} = \cos\theta_{\mathrm{k}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$s_{\mathrm{k}} = \sin\theta_{\mathrm{k}} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

For a sequence $\frac{2\pi}{L}\mathbb{Z}^d \ni \mathrm{k} \mapsto \theta_{\mathrm{k}}$ with values in matrices on $\mathbb{C}^2$ such that $\theta_{\mathrm{k}} = \theta_{-\mathrm{k}}$, set

$$U_\theta := \prod_{\mathrm{k}} \mathrm{e}^{-\frac{1}{2}\theta_{\mathrm{k}} a_{\mathrm{k}}^* a_{-\mathrm{k}}^* + \frac{1}{2}\theta_{\mathrm{k}}^* a_{\mathrm{k}} a_{-\mathrm{k}}}.$$

U_θ implements Bogoliubov rotations:

$$U_\theta^* a_{\mathrm{k}} U_\theta = b_{\mathrm{k}}, \qquad U_\theta^* a_{\mathrm{k}}^* U_\theta = b_{\mathrm{k}}^*,$$

and commutes with P.

Our Hamiltonian after the Bogoliubov rotation and the Wick ordering becomes

$$
\begin{aligned}
H &= B \\
&+ \frac{1}{2}\sum_{\mathrm{k}} O(\mathrm{k}) b^*_{\mathrm{k}} b^*_{-\mathrm{k}} + \frac{1}{2}\sum_{\mathrm{k}} \overline{O}(\mathrm{k}) b_{-\mathrm{k}} b_{\mathrm{k}} + \sum_{\mathrm{k}} D(\mathrm{k}) b^*_{\mathrm{k}} b_{\mathrm{k}} \\
&+ \text{terms higher order in } b\text{'s.}
\end{aligned}
$$

Let Ω denote the vacuum vector. Consider even fermionic Gaussian vectors of zero momentum of the form $\Omega_\theta := U_\theta^* \Omega$. We look for Ω_θ minimizing

$$(\Omega_\theta | H \Omega_\theta) = B.$$

For this it is enough to look for $O(\mathrm{k}) = 0$.

(Again, we use the Beliaev Theorem, see M.Napiórkowski, J.P.Solovej and J.D.)

If we choose the Bogoliubov transformation according to the minimization procedure, the Hamiltonian equals

$$H = B + \sum_{\mathrm{k}} D(\mathrm{k}) b^*_{\mathrm{k}} b_{\mathrm{k}} + \text{terms higher order in } b\text{'s}$$

with

$$\begin{aligned} B &= \sum_{\mathrm{k}} (\mathrm{k}^2 - \mu)(1 - \cos 2\theta_{\mathrm{k}}) \\ &\quad + \frac{1}{4L^d} \sum_{\mathrm{k},\mathrm{k}'} \alpha(\mathrm{k}, \mathrm{k}') \sin 2\theta_{\mathrm{k}} \sin 2\theta_{\mathrm{k}'} \\ &\quad + \frac{1}{4L^d} \sum_{\mathrm{k},\mathrm{k}'} \beta(\mathrm{k}, \mathrm{k}')(1 - \cos 2\theta_{\mathrm{k}})(1 - \cos 2\theta_{\mathrm{k}'}). \end{aligned}$$

Here,

$$\begin{aligned}\alpha(\mathrm{k},\mathrm{k}') &:= \frac{1}{2}\big(q(\mathrm{k},-\mathrm{k},-\mathrm{k}',\mathrm{k}') + q(-\mathrm{k},\mathrm{k},-\mathrm{k}',\mathrm{k}')\big),\\ \beta(\mathrm{k},\mathrm{k}') &= 2q(\mathrm{k},\mathrm{k}',\mathrm{k}',\mathrm{k}) - q(\mathrm{k}',\mathrm{k},\mathrm{k}',\mathrm{k}).\end{aligned}$$

In particular, in the case of local potentials we have

$$\begin{aligned}\alpha(\mathrm{k},\mathrm{k}') &:= \frac{1}{2}\big(\hat{V}(\mathrm{k}-\mathrm{k}') + \hat{V}(\mathrm{k}+\mathrm{k}')\big),\\ \beta(\mathrm{k},\mathrm{k}') &= 2\hat{V}(0) - \hat{V}(\mathrm{k}-\mathrm{k}').\end{aligned}$$

The condition $\partial_{\theta_k} B = 0$, or equivalently $O(k) = 0$, has many solutions. We can have

$$\sin 2\theta_k = 0, \quad \cos 2\theta_k = \pm 1,$$

They correspond to Slater determinants and have a fixed number of particles. The solution of this kind minimizing B, is called the normal or Hartree-Fock solution.

Under some conditions the global minimum of B is reached by a non-normal configuration satisfying

$$\sin 2\theta_{\mathrm{k}} = -\frac{\delta(\mathrm{k})}{\sqrt{\delta^2(\mathrm{k}) + \xi^2(\mathrm{k})}}, \quad \cos 2\theta_{\mathrm{k}} = \frac{\xi(\mathrm{k})}{\sqrt{\delta^2(\mathrm{k}) + \xi^2(\mathrm{k})}},$$

where

$$\begin{aligned} \delta(\mathrm{k}) &= \frac{1}{2L^d} \sum_{\mathrm{k}'} \alpha(\mathrm{k}, \mathrm{k}') \sin 2\theta_{\mathrm{k}'}, \\ \xi(\mathrm{k}) &= \mathrm{k}^2 - \mu + \frac{1}{2L^d} \sum_{\mathrm{k}'} \beta(\mathrm{k}, \mathrm{k}')(1 - \cos 2\theta_{\mathrm{k}'}), \end{aligned}$$

and at least some of $\sin 2\theta_{\mathrm{k}}$ are different from 0. It is sometimes called a superconducting solution.

For a superconducting solution we get

$$D(\mathrm{k}) = \sqrt{\xi^2(\mathrm{k}) + \delta^2(\mathrm{k})} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus we obtain a positive dispersion relation. One can expect that it is strictly positive, since otherwise the two functions δ and ξ would have a coinciding zero, which seems unlikely. Thus we expect that the dispersion relation $D(\mathrm{k})$ has a positive energy gap.

Conditions guaranteeing that a superconducting solution minimizes the energy should involve some kind of negative definiteness of the quadratic form α – this is what we vaguely indicated by saying that the interaction is attractive. Indeed, multiply the definition of $\delta(\mathrm{k})$ with $\sin 2\theta_{\mathrm{k}}$ and sum it up over k. We then obtain

$$
\begin{aligned}
&\sum_{\mathrm{k}} \sin^2 2\theta_{\mathrm{k}} \sqrt{\delta^2(\mathrm{k}) + \xi^2(\mathrm{k})} \\
= \ & -\frac{1}{2L^d} \sum_{\mathrm{k},\mathrm{k}'} \sin 2\theta_{\mathrm{k}} \alpha(\mathrm{k},\mathrm{k}') \sin 2\theta_{\mathrm{k}'}.
\end{aligned}
$$

The left hand side is positive. This means that the quadratic form given by the kernel $\alpha(\mathrm{k},\mathrm{k}')$ has to be negative at least at the vector given by $\sin 2\theta_{\mathrm{k}}$.

13 SOME CONJECTURES

Study of quantum gases is much easier if we stay in a fixed finite volume or consider an external confining potential. However, it seems that thermodynamic limit, that is $L \to \infty$ leads to important simplifications, and some properties are visible only in this limit:

- To define the physical critical velocity in dimension $d \geq 2$ one needs to consider thermodynamic limit.
- Only in thermodynamic limit the momentum is a continuous variable and one can ask about analytic continuation of correlation functions onto the nonphysical sheet of the complex plane.
- In thermodynamic limit one can expect a description in terms of essentially independent quasiparticles.

In our presentation we stick to the following set-up: we fix the interaction and we manipulate only with the coupling constant λ, the number of particles n, the chemical potential μ and the size of the box L. In particular, we do not scale the potential. In the literature, both physical and mathematical, it is common to scale the potential so that in some sense it approaches a zero range interaction. There exists a number of rigorous results in this formulation, especially about the ground state energy per volume in dimension 3.

With that set-up detailed information about the potential is not needed. Typically the only parameter that remains relevant is the scattering length. A similar point of view is often used in the physics literature devoted to superfluidity and superconductivity. In this case, for instance for bosons, one obtains a dispersion relation that depends on a single parameter and has the form $\omega(\mathrm{p}) = |\mathrm{p}|\sqrt{\mathrm{p}^2 + 2\mu}$.

The approach that involves fixing a potential has its drawbacks and one can criticize its physical relevance – in particular, typical physical potentials have a hard core, so the "weak coupling approach" seems rather inappropriate. However, with this approach one can obtain various shapes of the quasiparticle dispersion relations. In particular, they may have "rotons".

Clearly, the limit $L \to \infty$ is very difficult to control. In the result of Dereziński-Napiórkowski, that I described, simultaneously with increasing L one has to increase the density and decrease the coupling constant in order to obtain a meaningful result.

Nevertheless, based on heuristic arguments, I would expect that the excitation spectrum has a limit as $L \to \infty$. Let me formulate some conjectures. In all these conjectures I fix an interaction and a chemical potential μ.

Note that in these conjectures I am pretty vague about what we mean by the convergence of a family of subsets in $\mathbb{R} \times \frac{2\pi}{L}\mathbb{Z}^d$ and their convergence to a subset of $\mathbb{R} \times \mathbb{R}^d$. Intuitively, the meaning should be clear – the precise mathematical formulation will be left open.

Conjecture about the Bose gas in thermodynamic limit. For a large class of repulsive interactions V the following holds.

(1) There exists a function ω on $\mathbb{R}^d$ such that $\mathrm{spec}\,(H^L, P^L)$ as $L \to \infty$ converges to

$$\mathrm{spec}\Big(\int \omega(\mathrm{p}) a_{\mathrm{p}}^* a_{\mathrm{p}} \mathrm{dp}\,,\int \mathrm{p} a_{\mathrm{p}}^* a_{\mathrm{p}} \mathrm{dp}\Big).$$

(2) ω has no energy gap (its infimum is zero), it has a positive critical velocity and a well defined positive phonon velocity.

(3) Replace V with λV. Then we can choose ω so that for $\lambda \to 0$ it converges to the Bogoliubov dispersion relation.

Conjecture about the Fermi gas in thermodynamic limit. For a large class of attractive interactions v, the following holds.

(1) There exists a function ω on $\mathbb{R}^d$ such that $\mathrm{spec}\,(H^L, P^L, (-1)^N)$ as $L \to \infty$ converges to

$$\mathrm{spec}\Big(\int \omega(\mathrm{p}) a_{\mathrm{p}}^* a_{\mathrm{p}} \mathrm{dp}\,, \int \mathrm{p} a_{\mathrm{p}}^* a_{\mathrm{p}} \mathrm{dp}, (-1)^N\Big).$$

(2) ω has an energy gap (its infimum is strictly positive) and it has a positive critical velocity.

Note that these conjectures say that the asymptotic shape of the excitation spectrum is rather special. In particular, the infimum of the excitation spectrum for $L \to \infty$ should converge to a subadditive function.

In the fermionic case, these conjectures involve the fermionic parity. which plays an important role in fermionic systems. For examole, it is well known that nuclei have rather different properties depending on whether they have an even or odd number of nucleons.