Laplacian on an interval

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0.1 Some series

Proposition 0.1

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n-\alpha)^2-\omega^2} = \frac{\pi\sin(2\omega\pi)}{2\omega\sin(\alpha-\omega)\pi\sin(\alpha+\omega)\pi}, \tag{0.1}$$

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2-\omega^2} = -\frac{\pi\cos(\omega\pi)}{\omega\sin\alpha\pi}. \tag{0.2}$$

Proof. $f(z) := \frac{1}{(z-\alpha)^2-\omega^2}$ is meromorphic on $\mathbb{C}$, has a finite number of poles and $\lim_{z\to\infty} zf(z) = 0$. Hence one can use the method of Prob. 4.6.1. J. Krzyż, "Zbiór zadań z funkcji analitycznych" involving integrating $\cot(\pi z)f(z)$ on a big square:

$$0 = \sum_{n=-\infty}^{\infty} \frac{2\mathrm{i}}{(n-\alpha)^2-\omega^2} \tag{0.3}$$

$$+ 2\pi\mathrm{i}\mathrm{Res}\frac{\cot(\pi z)}{(z-\alpha)^2-\omega^2}\Big|_{z=\alpha+\omega} + +2\pi\mathrm{i}\mathrm{Res}\frac{\cot(\pi z)}{(z-\alpha)^2-\omega^2}\Big|_{z=\alpha-\omega} \tag{0.4}$$

$$= \sum_{n=-\infty}^{\infty} \frac{2\mathrm{i}}{(n-\alpha)^2-\omega^2} + 2\pi\mathrm{i}\frac{\cot\pi(\omega+\alpha)}{2\omega} - 2\pi\mathrm{i}\frac{\cot\pi(\omega-\alpha)}{2\omega} \tag{0.5}$$

□

Proposition 0.2 *For* $x \in [0, 2\pi]$,

$$\sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}(n-\alpha)x}}{(n-\alpha)^2-\omega^2} = \frac{\pi\big(\sin(2\omega\pi-\omega x) + \mathrm{e}^{-2\mathrm{i}\alpha\pi}\sin(\omega x)\big)}{2\omega\sin((\alpha-\omega)\pi)\sin((\alpha+\omega)\pi)}. \tag{0.6}$$

Proof. Set

$$f(x) = f_{\omega,\alpha}(x) := \sum_{n=-\infty}^{\infty} \frac{\mathrm{e}^{\mathrm{i}(n-\alpha)x}}{(n-\alpha)^2 - \omega^2}.$$

We have

$$(\partial_x^2 + \omega^2) f(x) = \sum_{n=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}(n-\alpha)x} = 2\pi \mathrm{e}^{-\mathrm{i}\alpha x} \sum_{m=-\infty}^{\infty} \delta(x - 2\pi m)), \tag{0.7}$$

In particular, (0.7) is zero in $]0, 2\pi[$. Hence $f(x) = a_+ \mathrm{e}^{\mathrm{i}\omega x} + a_+ \mathrm{e}^{-\mathrm{i}\omega x}$ there. Now

$$\begin{aligned} f(0) &= a_+ + a_- \qquad , \\ f(2\pi) &= a_+ \mathrm{e}^{\mathrm{i}\omega 2\pi} + a_- \mathrm{e}^{-\mathrm{i}\omega 2\pi} = \mathrm{e}^{-\mathrm{i}\alpha 2\pi} f(0). \end{aligned}$$

Hence

$$a_- = \frac{\mathrm{e}^{2\mathrm{i}\omega\pi} - \mathrm{e}^{-2\mathrm{i}\alpha\pi}}{\mathrm{e}^{2\mathrm{i}\omega\pi} - \mathrm{e}^{-2\mathrm{i}\omega\pi}} f(0), \tag{0.8}$$

$$a_+ = \frac{-\mathrm{e}^{-2\mathrm{i}\omega\pi} + \mathrm{e}^{-2\mathrm{i}\alpha\pi}}{\mathrm{e}^{2\mathrm{i}\omega\pi} - \mathrm{e}^{-2\mathrm{i}\omega\pi}} f(0), \tag{0.9}$$

$$f(x) = \frac{\big(\sin(2\omega\pi - \omega x) + \mathrm{e}^{-2\mathrm{i}\alpha\pi} \sin(\omega x)\big)}{\sin(2\omega\pi)} f(0) \tag{0.10}$$

$$= \frac{\pi\big(\sin(2\omega\pi - \omega x) + \mathrm{e}^{-2\mathrm{i}\alpha\pi} \sin(\omega x)\big)}{2\omega \sin((\alpha-\omega)\pi) \sin((\alpha+\omega)\pi)}. \tag{0.11}$$

□

0.2 Laplacian on an interval with twisted boundary conditions

Let H_κ be $-\partial_x^2$ on smooth functions on $[0, \pi]$ satisfying

$$\mathrm{e}^{\mathrm{i}\pi\kappa} f(0) = f(\pi), \quad \mathrm{e}^{\mathrm{i}\pi\kappa} f'(0) = f'(\pi). \tag{0.12}$$

Then H_κ defines a self-adjoint operator. From its eigenvectors one can form an o.n. basis

$$e_n(x) = \frac{1}{\sqrt{\pi}} \mathrm{e}^{\mathrm{i}(2n+\kappa)x}, \quad H_\kappa e_n = (2n+\kappa)^2 e_n, \quad n \in \mathbb{Z}. \tag{0.13}$$

Hence

$$\mathrm{sp} H_\kappa = \mathrm{sp}_\mathrm{p} H_\kappa = \{(2n+\kappa)^2 \;:\; n = 0, 1, 2, \dots\}.$$

The collowing cases are especially important:

(1) periodic $\kappa = 0$;

(2) antiperiodic, for $\kappa = 1$.

Set

$$R_\kappa(\omega^2,x,y)=(\omega^2-H_\kappa)^{-1}(x,y)$$

Equation

$$(\partial_x^2+\omega^2)R_\kappa(\omega^2,x,y)=\delta(x-y)$$

is solved by

$$R_\kappa(\omega^2,x,y)=\begin{cases}a_-\mathrm{e}^{\mathrm{i}x\omega}+b_-\mathrm{e}^{-\mathrm{i}x\omega}, & x<y;\\ a_+\mathrm{e}^{\mathrm{i}x\omega}+b_+\mathrm{e}^{-\mathrm{i}x\omega}, & x>y.\end{cases} \tag{0.14}$$

Let $y^\pm=y$, where we use the left- resp. right-sided limit. We get

$$R_\kappa(\omega^2,y^+,y)-R_\kappa(\omega^2,y^-,y)=0, \tag{0.15}$$

$$\partial_xR_\kappa(\omega^2,y^+,y)-\partial_xR_\kappa(\omega^2,y^-,y)=1, \tag{0.16}$$

$$\mathrm{e}^{\mathrm{i}\kappa\pi}R_\kappa(\omega^2,0,y)=R_\kappa(\omega^2,\pi,y), \tag{0.17}$$

$$\mathrm{e}^{\mathrm{i}\kappa\pi}\partial_xR_\kappa(\omega^2,0,y)=\partial_xR_\kappa(\omega^2,\pi,y). \tag{0.18}$$

We have 4 equations with 4 unknowns. According to W. Ciszewski this is solved by

$$R_\kappa(\omega^2,x,y)=\frac{\mathrm{i}}{2\omega}\begin{cases}\frac{\mathrm{e}^{\mathrm{i}\omega(y-x)}}{\mathrm{e}^{\mathrm{i}\pi(\omega+\kappa)}-1}-\frac{\mathrm{e}^{-\mathrm{i}\omega(y-x)}}{\mathrm{e}^{\mathrm{i}\pi(-\omega+\kappa)}-1}, & x<y;\\ \frac{\mathrm{e}^{\mathrm{i}\omega(x-y)}}{\mathrm{e}^{\mathrm{i}\pi(\omega-\kappa)}-1}-\frac{\mathrm{e}^{-\mathrm{i}\omega(y-x)}}{\mathrm{e}^{\mathrm{i}\pi(-\omega-\kappa)}-1}, & x>y.\end{cases} \tag{0.19}$$

Problem. Check (0.19) using (0.6) and

$$R_\kappa(\omega^2,x,y)=\sum_{n\in\mathbb{Z}}\frac{\mathrm{e}^{\mathrm{i}(2n+\kappa)(x-y)}}{\pi(\omega^2-(2n+\kappa)^2)}. \tag{0.20}$$

0.3 Laplacian on an interval with Dirichlet and Neumann boundary conditions

Problem. Using (0.6) and (0.21),

$$R_{\mathrm{D}}(\omega^2)(x,y)=\sum_{n=1}^{\infty}\frac{2\sin(xn)\sin(yn)}{\pi(\omega^2-n^2)}, \tag{0.21}$$

check the following formula for the integral kernel of the resolvent of the Dirichlet Laplacian:

$$\begin{aligned}R_{\mathrm{D}}(\omega^2)(x,y)\;&=\;\frac{\sin\omega x\sin\omega(y-\pi)\theta(y-x)}{\omega\sin\omega\pi}\\ &\quad+\frac{\sin\omega(x-\pi)\sin\omega y\theta(x-y)}{\omega\sin\omega\pi}\end{aligned}$$

Problem. Using (0.6) and (0.22)

$$R_{\mathrm{N}}(\omega^2)(x,y)=\frac{1}{\pi\omega^2}+\sum_{n=1}^{\infty}\frac{2\cos(xn)\cos(yn)}{\pi(\omega^2-n^2)}, \tag{0.22}$$

check the following formula for the integral kernel of the resolvent of the Neumann Laplacian:

$$R_{\mathrm{N}}(\omega^2)(x,y) = \frac{\cos\omega x\cos\omega(y-\pi)\theta(y-x)}{\omega\sin\omega\pi} + \frac{\cos\omega(x-\pi)\cos\omega y\theta(x-y)}{\omega\sin\omega\pi}.$$