Contents

1 Classical free string

1.0 Path

$\boxed{\star}$ ✠ *variational principle* []

1.a. **Nambu-Goto** action and its interpretation: Sec. 1.1.

$$S_{NG} = -T \int d^2\sigma \sqrt{-\det(g_{ab}^{ind})}, \qquad T = \frac{1}{2\pi\alpha'} = [\text{tension} = \frac{mass}{length}] \tag{1.1}$$

• N-G action is classically equivalent to the **Polyakov action**.

$$S = -\frac{T}{2} \int_\Sigma d^2\sigma^2 \sqrt{-\det(g)} g^{ab} \partial_a X_m \partial_b X^m, \quad \text{in metric } (-1, 1 \ldots) \tag{1.2}$$

g_{ab} are Lagrange multiplyers: $\delta/\delta g_{ab} \longrightarrow$ constraints.

Introduce **energy momentum tensor** T_{ab} which $= 0$ by g_{ab} e.o.m.

$$T_{ab} \equiv -\frac{4\pi}{\sqrt{-g}} \frac{\delta S}{\delta g^{ab}} = 2\pi T(\partial_a X \partial_b X - \tfrac{1}{2} g_{ab} g^{cd} \partial_c X \partial_d X) \overset{eom}{=} 0 \tag{1.3}$$

$$\Rightarrow g_{ab} = \lambda(\sigma) \partial_a X \partial_b X = \lambda(\sigma) g_{ab}^{ind} \tag{1.4}$$

1.b. **local symmetries**: Sec.1.2. •rep., •global(=local) Weyl symmetry $\rightarrow$ $\boxed{T_a^a = 0}$ without eq.m.

gauge fixing $g_{ab} = \eta_{ab}$

1.c. **Classical eqs of motion** Sec.1.3

- $T_{ab} = 0 \rightarrow T_{--} = T_{++} = 0$ (l.c. coords)
- for X:

$$0 = \frac{\delta S}{\delta X} = -\partial^a\left(\frac{\partial \mathcal{L}}{\partial(\partial_a X)}\right) = \partial_a \partial^a X = 4\partial_+ \partial_- X \tag{1.5}$$

B.C.:

– © $X(\sigma + 2\pi, \tau) = X(\sigma, \tau)$

– $\boxed{\sim}$ $\sigma \in (0, \pi)$. For any component of X:

$$\partial_\sigma X^i = 0 \qquad (\sigma = 0, \sigma = \pi) \quad \text{Neumann}$$

$$\delta X^i = 0 \quad \Rightarrow \quad X^i = c^i(\sigma) \qquad (\sigma = 0, \sigma = \pi) \quad \text{Dirichlet: break transl.}$$

• solutions to free e.o.m.

1.d. **conf.symmetry, charges and constraints** (?)$\longrightarrow$ Sec.1.4: L_m, $\tilde{L}_m$

1.1 String action

1.1.1 NON-RELATIVISTIC STRING .

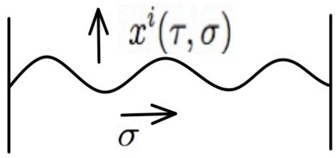

Fig.11: small transverse oscillations ($i = 1, \dots, D-2$)

$$\mathcal{L} \approx \frac{1}{2}T(\dot{x}_i^2 - x_i'^2)$$

world surface coordinates $x^i(\tau,\sigma)$ (here $\tau \equiv x^0 = t$), "transverse" string coordinates $\dot{x}_i = \frac{\partial x_i}{\partial t}$, $x_i' = \frac{\partial x_i}{\partial \sigma}$. Center of mass: $x_i(\tau,\sigma) = \overline{x}_i(\tau) + \dots$

$$S \to \frac{1}{2}TL \int d\tau\, \dot{\bar{x}}_i^2 + \dots = \frac{m}{2}\int d\tau\, \dot{\bar{x}}_i^2 + \dots \tag{1.6}$$

1.1.2 RELATIVISTIC STRING Basic principles:

- Relativistic space-time invariance (Poincaré group) $x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$
- No internal structure, i.e. no longitudinal oscillations $\to$ 2-d reparametrization invariance $\Longleftrightarrow$ relativistic generalization: $x^i \to x^\mu$

  Nambu-Goto action

  $$S = -T\int d\tau \int d\sigma \sqrt{(\dot{x}x')^2 - \dot{x}^2 x'^2} \tag{1.7}$$

  Motion in space-time: $\{x^\mu(\tau,\sigma)\}$, (τ,σ) – world-surface coordinates

  $$\dot{x}x' \equiv \dot{x}^\mu x'^\nu \eta_{\mu\nu}, \qquad \dot{x}^2 \equiv \dot{x}^\mu x'^\nu \eta_{\mu\nu} \tag{1.8}$$

1.1.3 MEANING OF THE ACTION : area of world surface

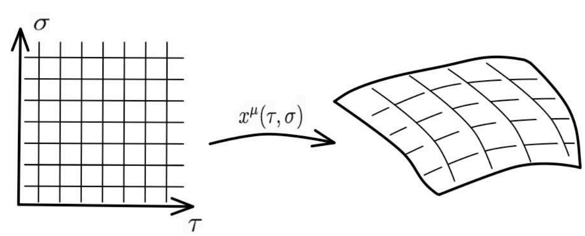

Fig.12: Embedding of (τ,σ)-plane into space-time

Induced metric on world surface $(a, b = 0, 1)$

$$ds^2 = \eta_{\mu\nu}dx^\mu dx^\nu = h_{ab}(\xi)d\xi^a d\xi^b \tag{1.9}$$

$$h_{ab} = \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} \tag{1.10}$$

$$h_{ab} = \begin{pmatrix} \dot{x}^2 & \dot{x}x' \\ \dot{x}x' & x'^2 \end{pmatrix}, \qquad \det h_{ab} = \dot{x}^2 x'^2 - (\dot{x}x')^2 \tag{1.11}$$

Thus Nambu action reads

$$S = -T \int d^2\xi\, \sqrt{-h}, \qquad h = \det h_{ab} \tag{1.12}$$

Euclidean signature ($\eta_{\mu\nu} \to \delta_{\mu\nu}$)

$$\tau \to i\tau_E, \qquad ds^2 = -d\tau^2 + d\sigma^2 \to d\tau_E^2 + d\sigma^2 \tag{1.13}$$

$$S_E = -iS_M = T \int d\tau_E \int d\sigma \sqrt{\dot{x}^2 x'^2 - (\dot{x}x')^2} \tag{1.14}$$

Element of the area:

$$d(\text{Area}) = |\dot{x}||x'| \sin\alpha\, d\sigma d\tau_E$$

$$\cos\alpha = \frac{\dot{x}x'}{|\dot{x}||x'|}\,, \qquad \sin\alpha = \sqrt{1-\cos^2\alpha}$$

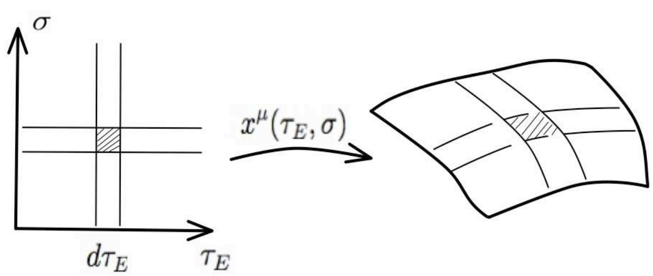

Fig.13: Element of area

$$(\text{Area})_E = \int d\tau_E d\sigma \sqrt{|\dot{x}|^2|x'|^2} \sqrt{1 - \frac{(\dot{x}x')^2}{|\dot{x}|^2|x'|^2}} = \int d\tau_E d\sigma \sqrt{\dot{x}^2 x'^2 - (\dot{x}x')^2} \tag{1.15}$$

Static gauge:

$$x^0 = \tau, \quad x^{D-1} = \sigma, \quad x^i(\tau,\sigma) = \text{transverse coordinates}, \quad i = 1, \ldots, D-2$$

$$h_{ab} = \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu} = \eta_{ab} + \partial_a x^i \partial_b x^i \tag{1.16}$$

Expand action assuming $|\partial_a x^i| \ll 1$ (small oscillations):

$$S = -T \int d^2\xi \sqrt{-\det(\eta_{ab} + \partial_a x^i \partial_b x^i)} \tag{1.17}$$

$$\det\eta_{ab} = -1$$

$$\det(\eta_{ab} + \partial_a x^i \partial_b x^i) = \det\eta_{ab} \det(\delta^c_d + \partial_d x^i \partial_b x^i \eta^{bc})$$

$$\simeq -(1 + \partial_a x^i \partial_b x^i \eta^{ab} + \ldots)$$

$$S = -T \int d^2\xi \big(1 + \frac{1}{2}\partial^a x^i \partial_a x^i\big) + \mathcal{O}((\partial x)^4)$$
$$\simeq -m \int d\tau + \frac{1}{2} T \int d\tau \int_0^L d\sigma \big[(\dot{x}^i)^2 - (x'^i)^2\big] + \mathcal{O}((\partial x)^4)$$

Equation for small oscillations:

$\ddot{x}_i - x''_i = 0 \longrightarrow$ wave equation in 1-d: transverse waves on the string

1.1.4 RELATION TO PARTICLE ACTION Nambu action describes string as collection of particles moving in direction transverse to the string.

Indeed, use static gauge for τ only ($x^0 = \tau$) and start with:

$$S = -T \int d\tau \int dl \sqrt{1 - V_\perp^2}, \tag{1.18}$$

where

$$dl \equiv d\sigma \sqrt{|x'|^2}, \qquad V_\perp^n x'^n = 0 \tag{1.19}$$

and $x \equiv (x^n(\tau, \sigma)),\ n = 1, \ldots, D-1$. Solution of $V_\perp^n x'_n = 0$:

$$V_\perp^m = P^{mn} \dot{x}^n \ , \qquad P^{mn} = \delta^{mn} - \frac{x'^m x'^n}{|x'|^2}, \qquad P^{mn} x'^n = 0 \tag{1.20}$$

$$V_\perp^2 \equiv V_\perp^n V_\perp^n = P^{nm} P^{nk} \dot{x}^m \dot{x}^k = \dot{x}^2 - \frac{(\dot{x}x')^2}{x'^2} \tag{1.21}$$

$$S = -T \int d\tau \int dl \sqrt{1 - V_\perp^2} = -T \int d\tau \int d\sigma \sqrt{x'^2 + (\dot{x}x')^2 - \dot{x}^2 x'^2} \tag{1.22}$$

This action (in $x^0 = \tau$ gauge) is invariant under $\sigma \to f(\sigma)$.

This coincides with the Nambu action in the "incomplete" static gauge $x^0 = \tau$:

$$-\mathrm{det} h_{ab} = -(\dot{x}_\mu \dot{x}^\mu)(x^\nu x'_\nu) + (\dot{x}^\mu x'_\mu)^2 = (x'_n)^2 - (x'_n)^2 (\dot{x}_m)^2 + (\dot{x}_n x'_n)^2$$

The Nambu action thus describes a collection of particles "coupled" by the constraint that they should move transversely to the profile of the string (i.e. there should be no longitudinal motions).

1.2 Local symmetries and gauge fixing

- reparam. $\tau' = f(\tau, \sigma), \qquad \sigma' = g(\tau, \sigma)$

$$\begin{aligned} Diff(\Sigma) \ : X'(\sigma) &= X(\sigma' = \sigma + \epsilon(\sigma)), \\ g'_{ab}(\sigma) &= \partial_a \sigma'^c \, \partial_b \sigma'^d \, g_{cd}(\sigma') \end{aligned} \tag{1.23}$$

2-d world-volume (local) — reparametrizations of world-surface

- Weyl

$$Weyl: g'_{ab}(\sigma) = \lambda(\sigma) g_{ab}(\sigma) \tag{1.24}$$

Reps and Weyl scales the volume element: $\sqrt{-g} \to \sqrt{-g}\ |\det(\partial\sigma'/\partial\sigma)|$, or $\to \sqrt{-g}\ \lambda(\sigma)$ but Weyl makes the volume of Σ unphysical.

- Counting degrees of freedom [1]

- Gauge fixing, l-c. coordinates.

  - in N-G where one fixes $g^{ind}_{++} = g^{ind}_{--} = 0$. One can solve the constraints going to the unitary gauge (e.g. light-cone)

  In Polyakov we use local symmetries to fix g_{ab}

  - geometry of the space $Met(\Sigma)$ e.g. $Met(S^2) = Weyl(S^2) \times Diff(S^2)$
  - "perfect" gauge fixing is hard: usually there is either a residual (global) remnant of the g.symmetry or gauge fixing is too rigid. One can go to "physical" (unitary) gauges but these break Lorentz invariance (not nice).
  - , unlike gauge choice can not give singular volume form: $\mathfrak{conformal\ gauge}$: $g_{ab} = \lambda(\sigma)\delta_{ab}$. Then:

$$S_P = \frac{T}{2}\int_\Sigma d^2\sigma[(\partial_\tau X)^2 - (\partial_\sigma X)^2] \tag{1.25}$$

  but we have to remamber about eq.m. for the metric $T_{ab} = 0$. There are only two of them.

$$g_{--} = 0, \quad g_{++} = 0, \quad g_{-+} = \tfrac{1}{2} \tag{1.26}$$

$$S = \frac{1}{2\pi}\int d^2\sigma \partial_+ X \partial_- X \tag{1.27}$$

$$T_{-+} \equiv 0, \quad T_{++} = \tfrac{1}{2}\,\partial_+ X \partial_+ X, \quad T_{--} = \tfrac{1}{2}\,\partial_- X \partial_- X \tag{1.28}$$

  THE EQ.M. $T_{--} = T_{++} = 0$ PUTS CONSTRAINTS ON SOLUTIONS.

[1] Polyakov: metric $\to$ 3, $x \to d$. Local symmetries: reps $\to$ -2, constraints from gravity eq.m. $\to$ -2, Weyl$\to$ -1. Altogethere $= d-2$. Nambu-Goto: $x \to d$. Local symmetries: reps $\to$ -2. Altogethere $= d-2$.

1.3 Free fields - classical solutions

$$S=\int_0^T d\tau\int_0^L d\sigma\frac{1}{2}\left(\partial_a X\right)^2,\qquad =\int_0^{T\pi/L} d\tau\int_0^\pi d\sigma\frac{1}{2}\left(\partial_a X\right)^2 \tag{1.29}$$

$$\partial_a\partial^a X=4\partial_+\partial_- X=0\ \Rightarrow X=f(\tau+\sigma)+g(\tau-\sigma) \tag{1.30}$$

Minkowski solutions depends on bc.

1.a. © : $\partial_+X=\partial_+f(\sigma^+)$ and $\partial_-X=\partial_-g(\sigma^-)$ are periodic because X is.

$$f(\sigma^+)=c+p\sigma^++\sum_{m\in\mathbb{Z}-\{0\}}\alpha_m e^{-im\sigma^+},\quad g(\sigma^-)=c'+\overline{p}\sigma^-+\sum_{m\in\mathbb{Z}-\{0\}}\tilde{\alpha}_m e^{-im\sigma^-} \tag{1.31}$$

Periodicity of X enforces $p=\overline{p},\ (\alpha_0=p)$ thus:

$$X(\sigma^+,\sigma^-)=x+2\alpha_0\tau+i\sum_{m\in\mathbb{Z}-\{0\}}\frac{1}{m}\left(\alpha_m e^{-im\sigma^+}+\tilde{\alpha}_m e^{-im\sigma^-}\right) \tag{1.32}$$

1.b. NN $\boxed{\sim}$ $\sigma\in(0,\pi)$ i.e. Neumann bc at $\sigma=0,\pi$. In addition we renormalize the momentum $p\to 2p$ due to half the interation region: $p_{phys}=\int d\sigma\Pi_X$. We want $p_{phys}=p$. $0=\partial_\sigma X\to f'(\tau)=g'(\tau)\to f=g+const$ from $\sigma=0$ and $f'(\tau+\pi)=f'(\tau-\pi)$ thus $f'(\tau)=2p+\sum_{m\in\mathbb{Z}-\{0\}}\alpha_m\, e^{-im\tau}$ thus ($\alpha_0=2p$)

$$X(\sigma^+,\sigma^-)=x+2\alpha_0\tau+i\sum_{m\in\mathbb{Z}-\{0\}}\frac{1}{m}\alpha_m(e^{-im\sigma^+}+e^{-im\sigma^-}) \tag{1.33}$$

1.c. DD $\boxed{\sim}$: we take $X(\tau,\sigma=0)=0,\ X(\tau,\sigma=\pi)=d$. At $\sigma=0$, $f'(\tau)=-g'(\tau)\to f=-g+const$, at $\sigma=\pi$ as above (no momentum), thus

$$X(\sigma^+,\sigma^-)=x+\frac{d\,\sigma}{\pi}+i\sum_{m\in\mathbb{Z}-\{0\}}\frac{1}{m}\alpha_m(e^{-im\sigma^+}-e^{-im\sigma^-}) \tag{1.34}$$

1.d. ND $\boxed{\sim}$: N for $\sigma=0$ and D for $\sigma=\pi$. At $\sigma=0$ $f=g+const$ but at $\sigma=\pi$ $f'(\tau+\pi)=-f'(\tau-\pi)$ (antiperiodic - expansion in half integers).

$$X(\sigma^+,\sigma^-)=x+i\sum_{m=\mathbf{Z}+1/2}\frac{1}{m}\alpha_m(e^{-im\sigma^+}+e^{-im\sigma^-}) \tag{1.35}$$

1.e. **reality** of X gives

$$\alpha_{-m}=\alpha_m^*$$

1.4 Conserved charges

1.4.1 CONFORMAL SYMMETRY Conformal symmetry is a property of some 2d action regardless of applicaition to string theory (no constraints are involved). It is a kind of global symmetry $\rightarrow$ conserved charges. [2]

$$X(\sigma^+,\sigma^-) \rightarrow X'(\sigma^+,\sigma^-) = X(\sigma'^+(\sigma^+),\sigma'^-(\sigma^-)) \tag{1.36}$$

We need to assure that Σ is unchanged: $\sigma'^{\pm}(\sigma^{\pm}) = \sigma^{\pm} + \epsilon^{\pm}(\sigma^{\pm})$.
For ⓒ $\epsilon^{\pm}(\sigma^{\pm})$ must be 2π periodic (see App).
For $\boxed{\sim}$ $\epsilon^-(x) = \epsilon^+(x) \equiv \epsilon(x)$ and $\epsilon(x) = \epsilon(x+2\pi)$:
Proof:$2\delta\sigma = \delta\sigma'^+ - \delta\sigma'^- = \epsilon^+ - \epsilon^- = 0$ for boundaries: $\Rightarrow \sigma = 0$: $\epsilon^-(\tau) = \epsilon^+(\tau)$ (i.e. $\epsilon^{\sigma}(\tau) = 0$); $\sigma = \pi, \rightarrow \ \epsilon^+(\tau+\pi) = \epsilon^+(\tau-\pi)$.$\square$
so there is half of the symmetries compare to the closed string case.

- it is a remnant of the reparameterization invariance
- conserved charges – used in quantization below.
  ♣ *the conserved charges are basically the same as constraints: this property holds because the gravity sets constraints while charges follow from remants of the reparameterization invariance.* ♣
- we can use it to further fixing $\rightarrow$ $light - cone\ gauge$.

1.4.2 CHARGES Derive chares by Noether procedure. $\partial_a T^{ab} = 0$. By direct inspection we also find $T^a{}_a = 0$. In the conforaml gauge by direct inspection

$$T^a{}_a = 0 \equiv T_{+-} = 0, \quad \partial_- T_{++} = 0,\ \partial_+ T_{--} = 0 \tag{1.37}$$

The first equality hold as a tauthology and it is a consecquance of the (global)Weyl invariance; the second and the thirs equality follows from conformal symmetries of the action.

For ⓒ we define charges

$$L_n = \frac{1}{2\pi}\int d\sigma e^{in\sigma^+} T_{++}, \quad \tilde{L}_n = \frac{1}{2\pi}\int_0^{2\pi} d\sigma e^{in\sigma^-} T_{--} \tag{1.38}$$

which are time-independent $\partial_\tau L_n = \partial_\tau \tilde{L}_n = 0$.

For $\boxed{\sim}$

In the presence of boundary $\partial\Sigma$ we get conservation of J^a_i only if there is no flow of J^a_i through

[2] For properties of f, g see below

$\partial\Sigma$. We shall require $n^a J^a_i|_{\partial\Sigma} = 0$ where n^a is a vector $\perp$ to the boundary. by direct inspection $T_{++}(\sigma^+) = T_{--}(\sigma^-)|_{brzeg}$ i.e. for $\sigma = 0$: $T_{++}(\tau) = T_{--}(\tau)$ – only one independent e-m tensor; for $\sigma = \pi$: $T_{++}(\tau + \pi) = T_{--}(\tau - \pi)$ i.e. e-m tensor is 2π–periodc.

$$L_n = \frac{1}{2\pi}\int_0^{2\pi} d\sigma e^{in\sigma^+} T_{++}(\sigma^+) \leftarrow (?) \text{ : normaliz}$$

are τ-independent i.e. $L_n = \frac{1}{2\pi}\int_0^{2\pi} d\sigma e^{in\sigma} T_{++}(\sigma) = \frac{1}{2\pi}\int_0^{2\pi} d\sigma e^{in\sigma} T_{--}(\sigma)$ We have only one set of charges.

One can express those charges by modes $\alpha, \tilde{\alpha}$.

1.5 light – cone gauge

as total gauge fixing

OPEN: $X^+ = x_0^+ + 4p^+\tau$ CLOSED: $X^+ = x_0^+ + 2p^+\tau$ is conformally equivalent to the most general solution ($p^+ > 0$). Thus we can farther fix the gauge choosing X^+. The above choice determine X^- by solving $T_{ab} = 0$ (see below).

1.6 Summery

String functional $S_{N-G} \sim \; Area$. It is classically equivalent to S_P if one solves eq.m. of 2d gravity.

$$0 = T_{ab} = -\frac{4\pi}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} \tag{1.39}$$

Only 2 of the above eqations are non-trivial $T_{--} = T_{++} = 0$.

S_P has enought gauge symmetries (reps.+ Weyl) to gauge away the 2d gravity $g_{ab} = \eta_{ab}$. In this case classically string theory is defined by the action

$$S = 2T \int d^2\sigma \partial_+ X \partial_- X \tag{1.40}$$

and the constraints $T_{--} = T_{++} = 0$. The above action as conformal symmetry which is a remant of the reps.inv. The corresponding conserved currents and charges:

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$$\partial_- T_{++} = 0, \; \partial_+ T_{--} = 0 \tag{1.41}$$

$$L_n = \frac{1}{2\pi}\int d\sigma e^{in\sigma^+} T_{++}, \quad \tilde{L}_n \tag{1.42}$$

$$\boxed{S = 2T \textstyle\int d^2\sigma \partial_+ X \partial_- X, \qquad L_n = \tilde{L}_n = 0} \tag{1.43}$$

$\boxed{\sim}$ b.c. gives $T_{++} = T_{--}|_{\sigma=0,\pi}$ (see App.)

$$\partial_- T_{++} + \partial_+ T_{--} \equiv \partial_a T^{a0} = 0 \tag{1.44}$$

$$\partial_-(T_{++}e^{in\sigma^+}) + \partial_+(T_{--}e^{in\sigma^-}) = 0 \tag{1.45}$$

$$\begin{aligned} L_m &= \frac{1}{2\pi}\int_0^\pi d\sigma \left(e^{im\sigma}T_{++}(\sigma) + e^{-im\sigma}T_{--}(-\sigma)\right) \\ &\stackrel{T_{--}=T_{++}}{=} \frac{1}{2\pi}[\int_0^\pi d\sigma e^{im\sigma}T_{++}(\sigma) - \int_0^{-\pi} e^{im\sigma}T_{++}(\sigma)] \\ &\stackrel{2\pi \ period.}{=} \frac{1}{2\pi}\int_0^{2\pi} T_{++}(\sigma)e^{im\sigma}. \qquad L_{-m} = \overline{L_m} \end{aligned} \tag{1.46}$$

1.7 Appendices

A. Global symmetries and charges for strings

G group of global continuous symmetries of a theory : $g = e^{i\omega^A T^A}$. [3] *the symmetry means that*

$$S = \int d\sigma \mathcal{L}(X'(\sigma), \partial_a X'(\sigma)) \stackrel{cond.}{=} \int d\sigma \mathcal{L}(\, X(\sigma), \partial_a \, X(\sigma)) \tag{1.47}$$

TO EACH T^A CORRESPONDS A CONSERVED CURRENT $\partial_a J^a_A = 0$. THIS IMPLIES EXISTENCE OF TIME INDEPENDENT CHARGES $Q_A \equiv \int_V J^0_A(\sigma)$.

A) BOUNDARY $\partial\Sigma$. *In the presence of boundary $\partial\Sigma$ we get conservation of J^a_i only if there is no flow of J^a_i throught $\partial\Sigma$. We shall require $n^a J^a_i|_{\partial\Sigma} = 0$ where n^a is a vector $\perp$ to the boundary (see (**??**)).* ♣ Do more boundary as in the section **??**. ♣

B) APPLICATION TO STRINGS *introduce $\mathfrak{light} - \mathfrak{cone}$ coordinates ($\eta_{+-} = -\frac{1}{2}$)*

$$S = -\frac{1}{8\pi}\int_\Sigma d^2\sigma^2 \partial_a X_\mu \partial^a X^\mu = \frac{1}{2\pi}\int_\Sigma d^2\sigma^2 \partial_+ X^\mu \partial_- X_\mu \tag{1.48}$$

1.a. *Target space shift: $X \to X + \epsilon$ defines target momentum p (equals canonical momentum (**??**) at this case).*

$$J^a_\mu = -T\partial^a X_\mu \Rightarrow Q^\mu = T\int d\sigma \partial_\tau X^\mu \stackrel{\text{df}}{=} p^\mu \tag{1.49}$$

and rotations $\delta X^\mu = i\epsilon^{\alpha\beta}(t_{\alpha\beta})^\mu{}_\nu X^\nu$ defines the target Lorentz generators.

$$\begin{aligned} (J^a)_{\alpha\beta} &= -i\, T\partial^a X_\mu (t_{\alpha\beta})^\mu{}_\nu X^\nu \Rightarrow \\ Q_{\alpha\beta} &= iT\int d\sigma \partial_\tau X^\mu (t_{\alpha\beta})^\mu{}_\nu X^\nu \stackrel{\text{df}}{=} M_{\alpha\beta} \end{aligned} \tag{1.50}$$

[3] It is not true that $\int \frac{\delta\mathcal{L}}{\delta X}\delta X = 0$ vanishes for arbitrary δX on eq.m. One needs $\delta X = 0$ on time boundaries which can not be imposed with global symmetries (charge flow).

1.b. © *world-sheet shift* $\delta_i \sigma^a \omega^i = \omega^a \Rightarrow$ *gives currents*

$$T_{ab} = -2\pi \leq \eta_{ab}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial^a X)}\partial_b X \mathrm{Re} \quad \textit{and} \;\; \partial^a T_{ab} = 0 \tag{1.51}$$

together with $\partial_a T^{ab} = 0$ *leads to* $\partial_- T_{++} = 0 = \partial_+ T_{--}$ *(see below). For free string* $T_{ab} = \frac{1}{2}[\partial_a X \partial_b X - \frac{1}{2}(\partial_a X)^2]$.

$\boxed{\sim}$ *as above but* $b = 0$ *only:* $\Rightarrow \partial_a T^{a0} = 0$. *But* ${T^a}_0 = {T^a}_+ + {T^a}_- \Rightarrow \partial_+(T_{-+} + T_{--}) + \partial_-(T_{++} + T_{+-}) = 0$, *due to* $T_{-+} = 0 \Rightarrow \partial_+ T_{--} + \partial_- T_{++} = 0$. ♣ For open string the b.c. breaks half of it. ♣ [4]

C) CONFORMAL SYMMETRY *is the remnant of the reparameterization invariance, follows from* $T^a_a = 0$ *(above) and in 2d leads to infinite number of conserved charges. All reparam. must preserve* Σ.

*The action (**??**) has big group of symmetries:– conformal symmetries*[5]

$$\sigma^+ \to \sigma'^+ = f(\sigma^+), \; \sigma^- \to \sigma'^- = g(\sigma^-) \tag{1.52}$$

$$S \to \int_\Sigma d^2\sigma \partial_+ X(f(\sigma^+), g(\sigma^-)) \partial_- X(f(\sigma^+), g(\sigma^-)) = \int_\Sigma \frac{df dg}{2} \partial_f X(f,g) \partial_g X(f,g) = S$$

We need to assure that Σ *is unchanged: let* ($f^+ = f, f^- = g$) $f^\pm(\sigma^\pm) = \sigma^\pm + \epsilon^\pm(\sigma^\pm)$.

© $\epsilon^\pm(\sigma^\pm)$ *must be* 2π *periodic. Proof:* $(a)\; \sigma'(\sigma + 2\pi, \tau) \overset{\text{con}}{=} \sigma'(\sigma,\tau) + 2\pi,\; (b)\tau'(\sigma + 2\pi, \tau) = \tau'(\sigma, \tau)$. *(a)*$\Rightarrow$ $\epsilon^+(\sigma^+ + 2\pi) - \epsilon^-(\sigma^- - 2\pi) = \epsilon^+(\sigma^+) - \epsilon^-(\sigma^-)$, *(b)*$\Rightarrow$ $\epsilon^+(\sigma^+ + 2\pi) + \epsilon^-(\sigma^- - 2\pi) = \epsilon^+(\sigma^+) + \epsilon^-(\sigma^-)$. □

$\boxed{\sim}$ $\epsilon_-(x) = \epsilon_+(x) \equiv \epsilon(x)$ *and* $\epsilon(x) = \epsilon(x + 2\pi)$*: Proof:* $2\delta\sigma = \delta\sigma'^+ - \delta\sigma'^- = \epsilon^+ - \epsilon^- = 0$ *for* $\sigma = 0, \pi$*: to* $\sigma = 0$*:* $\epsilon_-(\tau) = \epsilon_+(\tau)$ *(i.e.* $\epsilon^\sigma(\tau) = 0$*);* $\sigma = \pi, \; \to \; \epsilon_+(\tau + \pi) = \epsilon_+(\tau - \pi)$. □

so there is half of the symmetries *compare to the closed string case.*

Infinite number of global symmetries generated by Fourier comopnents of $\sigma^\pm$*:* $\epsilon^\pm(\sigma^\pm) = \sum_n \epsilon^\pm_n \, e^{in\sigma^+}$.

D) CHARGES

$$T_{++} = \sum e^{-in\sigma^+} L_n, \qquad L_n = \frac{1}{2\pi}\int d\sigma e^{in\sigma^+} T_{++}, \; \partial_\tau L_n = 0, \; \overline{L_n} = L_{-n} \tag{1.53}$$

$$T_{--} = \sum e^{-in\sigma^-} \tilde{L}_n \qquad \tilde{L}_n = \frac{1}{2\pi}\int d\sigma e^{in\sigma^-} T_{--} \tag{1.54}$$

© . *Derivation: using* $f = \sigma^+ + \epsilon^+(\sigma^+, \sigma^-)$ *we calculate* δS

$$S + \delta S = \int \partial_+ X(f, \sigma^-) \partial_- X(f, \sigma^-)$$

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[4] NO Scale symmetry in Minkowski variables: $\delta_i \sigma^a \omega^i = \omega \sigma^a$

[5] This is tha same as $X(\sigma^+, \sigma^-) \to X'(\sigma^+, \sigma^-) = X(f(\sigma^+), g(\sigma^-))$.

Then $\partial_+ X(f,\sigma^-) = \partial_+ f \partial_f X(f,\sigma^-)$, $\partial_- X(f,\sigma^-) = \partial_- f \partial_f X(f,\sigma^-) + \partial_- X$, *also* $df \wedge d\sigma^- = \partial_+ f d\sigma^+ \wedge d\sigma^-$ *thus*

$$\delta S = \int \partial_- f \partial_f X \partial_+ X = \int \partial_- \epsilon^+ \partial_+ X \partial_+ X + O(\epsilon^2)$$

Thus the conserved current is: T_{++} *(on eq.m.).*

For ⓒ and $\boxed{\sim}$ *We are looking for* $(\partial_-\epsilon^+,\ \partial_+\epsilon^-)$

$$\begin{aligned} 0 = \delta S &= \int \partial_+ X(\sigma^+ + \epsilon^+, \sigma^- + \epsilon^-)\partial_- X(\sigma^+ + \epsilon^+, \sigma^- + \epsilon^-) - S \\ &\to \int (\partial_+\epsilon^- \partial_- X \partial_- X + \partial_+ X(\partial_-\epsilon^+ \partial_+ X)) \end{aligned} \tag{1.55}$$

Using $\epsilon^+ = \sum \epsilon^+_n(\sigma^-)e^{im\sigma^+}$, $\epsilon^- = \sum \epsilon^-_n(\sigma^+)e^{im\sigma^-}$

$$ⓒ \quad \xrightarrow{\epsilon^-=0} \partial_-(T_{++}e^{im\sigma^+}) = 0 \tag{1.56}$$

$$ⓒ \quad \xrightarrow{\epsilon^+=0} \partial_+(T_{--}e^{im\sigma^-}) = 0 \tag{1.57}$$

$$\boxed{\sim} \quad \xrightarrow{\epsilon^+=\epsilon^-} \partial_-(T_{++}e^{im\sigma^+}) + \partial_+(T_{--}e^{im\sigma^-}) = 0 \tag{1.58}$$

$\boxed{\sim}$ *We got the above is boudary terms (of the integration by parts) vanishes. These are* $\epsilon(\partial_- X\partial_- X - \partial_+ X \partial_+ X)|_0^\pi = 0$ *This we require* $T_{--} = T_{++}|_{\sigma=0,\pi}$.

$\boxed{\sim}$ *The above follows also from tanslational invariance in* τ: $\partial_a T^{a\tau} = 0$ *and* $T_{+-} = 0$.*Maybe this is the best.*

$\boxed{\sim}$ *We define*

$$L_m \overset{\text{df}}{=} \frac{1}{2\pi}\int_0^\pi d\sigma \left(e^{im\sigma^+}T_{++} + e^{im\sigma^-}T_{--}\right) \tag{1.59}$$

Also $T_{\sigma\tau}|_{\sigma=0,\sigma=\pi} = 0$ *(see* **(??)***), i.e.* $T_{++} = T_{--}|_{\sigma=0,\sigma=\pi}$. *For free string* $T_{++}(\sigma^+), T_{--}(\sigma^-)$, *thus argueing as above we get* $T_{++}(x) = T_{--}(x)$ *and* $T_{++}(x) = T_{++}(x+2\pi)$ *so (we can choose* $\tau = 0$ *in (1.59)). Time independence*[6] [7] .

$$\begin{aligned} L_m &= \frac{1}{2\pi}\int_0^\pi d\sigma \left(e^{im\sigma}T_{++}(\sigma) + e^{-im\sigma}T_{--}(-\sigma)\right) \\ &\overset{T_{--}=T_{++}}{=} \frac{1}{2\pi}\left[\int_0^\pi d\sigma e^{im\sigma}T_{++}(\sigma) - \int_0^{-\pi} e^{im\sigma}T_{++}(\sigma)\right] \\ &\overset{2\pi\ period.}{=} \frac{1}{2\pi}\int_0^{2\pi} T_{++}(\sigma)e^{im\sigma}. \qquad L_{-m} = \overline{L_m} \end{aligned} \tag{1.60}$$

E) MOMENTUM AND SHIFT SYMMETRY $\quad X' = X + t(\sigma)$

$$\delta S \sim \int (\partial_a t \partial^a X) \tag{1.61}$$

thus the conserved current is $\partial_a X$ *thus* $p = \int \partial_\tau X$ *is the consedrved charge.*

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[6] $\partial_\tau(1.59) = \frac{1}{2\pi}\int_0^\pi d\sigma \left(\partial_\sigma(e^{im\sigma^+}T_{++} - e^{im\sigma^-}T_{--})\right) = 0$ due to $T_{++}(x) = T_{--}(x)$ and $T_{++}(x) = T_{++}(x+2\pi)$.

[7] $\epsilon(\tau)T(\tau)$ generates repar. of $\partial\Sigma|_{\sigma=0}$. We have $\epsilon(\tau)T(\tau) = \sum_{m\in Z}\epsilon_m L_m$ from the above.