Contents

1 Superstrings

As we shall see the SUPERSTRING is quite complex object consisting of several sectors (Hilbert spaces) of the following supersymmetric action.

$$S = \frac{1}{4\pi}\int(\frac{1}{\alpha'}\partial_a X^\mu \partial_a X_\mu + i\overline{\psi}^\mu \gamma^a \partial_a \psi^\mu) \tag{1.1}$$

1.1 Classical theory

1.1.1 ANTI-COMMUTING ψ'S 2d Minkowski $(-1,1)$ Spinors

$$\begin{aligned}
\{\gamma^a, \gamma^b\} &= -2\eta^{ab} = -\mathrm{diag}(-1,1)^{ab} && (1.2)\\
\psi_M &= C(\gamma^0)^T\psi^*,\ CC^\dagger = 1 && (1.3)\\
\overline{\psi}_M \gamma^a \partial_a \psi_M &= \overline{\psi}\gamma^a\partial_a\psi \Rightarrow C^\dagger\gamma^a C = \pm\gamma^{aT} && (1.4)
\end{aligned}$$

2d: $\gamma^0 = \sigma^2, \quad \gamma^1 = i\sigma^1, \quad C = -\gamma^0, \quad C^\dagger\gamma^a C = -\gamma^{aT}$

$$\begin{aligned}
\psi_M &= \gamma^0\gamma^0\psi^* = \psi^*, \quad \psi_M = \psi \Rightarrow \psi = \begin{pmatrix}\psi_- \\ \psi_+\end{pmatrix} && (1.5)\\
S &= \frac{1}{4\pi}\int d^2\sigma\, \overline{\psi}\gamma^a\partial_a\psi = \frac{1}{2\pi}\int d^2\sigma\, i(\psi_-\partial_+\psi_- + \psi_+\partial_-\psi_+) && (1.6)
\end{aligned}$$

The above has (also for arbitry backgroud) $Z_2 \times Z_2$ symmetry[1] with generators which we shall call $(-1)^F$ (for the left Z_2).

$$\psi_+ \to -\psi_+, \quad \psi_- \to -\psi_- \tag{1.7}$$

1.1.2 FINAL ACTION

$$\boxed{S = \frac{1}{2\pi}\int(\partial_+ X\partial_- X + i\psi_-\partial_+\psi_- + i\psi_+\partial_-\psi_+)} \tag{1.8}$$

[1] Because ψ's do not have any direct physical meaning (contrary to X which has clear meaning for some limiting cases) we could have $\psi^\mu(\sigma+2\pi) = t^{\mu\nu}\psi^\nu(\sigma)$. But t must commute with target Lorentz generators and be symmery of S thus in this case it must be ± 1 for all μ.

1.2 Eq.m. and b.c.

$$\partial_+\partial_- X = 0, \quad \partial_+\phi_- = 0, \quad \partial_-\psi_+ = 0 \tag{1.9}$$

ⓒ : We can consider the folowing R: (Ramond), NS (Neveu-Schwarz) boundary conditions (we suppress here $\pm$)

$$\begin{aligned}(R) \quad \psi(\sigma+2\pi) &= \psi(\sigma) \to \; r \in \mathbf{Z} & (1.10)\\ (NS) \quad \psi(\sigma+2\pi) &= -\psi(\sigma) \to \; r \in \mathbf{Z}+\tfrac{1}{2} & (1.11)\end{aligned}$$

♣ *NS sector can be called twisted sector: $\psi(\sigma+2\pi) = g\,\psi(\sigma)$ for any g in global symmetry group of the action.* ♣

Thus eq.m. $\partial_-\psi_+ = 0, \; \partial_+\psi_- = 0$ has solutions

$$\psi_+^\mu = \sum_r \psi_r^\mu e^{-ir\sigma^+}, \; \psi_-^\mu = \sum_r \tilde{\psi}_r^\mu e^{-ir\sigma^-} \qquad r \in Z\;(R), \; r \in Z+\tfrac{1}{2}\;(NS) \tag{1.12}$$

$\boxed{\sim}$ Derivation of the eq.m. from (**??**) gives the following boundary contributions $\int d\tau \int_0^\pi d\sigma \partial_\sigma(-\psi_+\delta\psi_+ + \psi_-\delta\psi_-)$ yielding

$$\psi_+\delta\psi_+ - \psi_-\delta\psi_- = 0 \text{ for } x = 0, \pi \tag{1.13}$$

Possible b.c.

1.a. $\sigma = 0$, we set $\delta\psi_+ = \delta\psi_-$ what is always possible by the Z_2 symmetry (**??**). This gives $\psi_+(\tau,0) = \psi_-(\tau,0)$.
Mode expansion (**??**) $\sum_r \psi_r^\mu e^{-ir\tau} = \sum_{r'} \tilde{\psi}_{r'}^\mu e^{-ir'\tau}$ sets $\{r\} = \{r'\}$ and $\psi_r = \tilde{\psi}_r$. Thus ther is only one fermion ψ so that $\psi_\pm(\sigma^\pm) = \psi(\sigma^\pm)$ and

1.b. $\sigma = \pi$, $\delta\psi_+ = \pm\delta\psi_-$ i.e.

$$\begin{aligned}\psi(\tau+\pi) &= \psi(\tau-\pi) && \text{(R)} && \Rightarrow \; r \in Z\\ \psi(\tau+\pi) &= -\psi(\tau-\pi) && \text{(NS)} && \Rightarrow \; r \in Z+\tfrac{1}{2}\end{aligned} \tag{1.14}$$

Reality $\quad \psi_r^+ = \psi_{-r}, \quad r \neq 0$ and CCR dictate creation-anihilation splitting. There is problem with ψ_0 which is real – see $\to$.

1.3 NS/R in general

- One chooses NS/R b.c. the same for all of the space components.
- R: $\psi's$ follow the mode expansion of X fields. It always gives anti-commuting fields.
  NS: $\psi's$ have 1/2 shift.
- $a_R = 0$ but a_{NS} varies for different situations.
- GSO changes (how ???)

$\boxed{\star}$ *Spectra of intersecting branes/instantons.* []

1.4 Symmetries

1.4.1 CONFORMAL SYMMETRY

$$X \to X(f(\sigma^+), \sigma^-),\ \psi_+ \to (\partial_+ f)^{1/2}\psi_+(f(\sigma^+), \sigma^-),\ \psi_- \to \psi_-(f(\sigma^+), \sigma^-) \tag{1.15}$$

Current from terms linear in $\partial_- f(\sigma^+, \sigma^-)$ from $\partial_- = \partial_- f \partial_+ + \partial_-$

$$\boxed{T_{++} = \tfrac{1}{2}(\partial_+ X \partial_+ X + i\psi_+ \partial_+ \psi_+)} \tag{1.16}$$

1.4.2 SUPERCONFORMAL SYMMETRY Global (1,1) SUSY is part of it

$$\delta X = i\epsilon^+(\sigma^+)\psi_+,\ \delta\psi_+ = -\epsilon^+(\sigma^+)\partial_+ X \tag{1.17}$$

Current from terms linear in $\partial_- \epsilon^+(\sigma^+, \sigma^-)$ from $\partial_- X$ and $\partial_- \psi_+$ gives

$$\boxed{G_{++} = i\psi_+ \partial_+ X} \tag{1.18}$$

For $\boxed{\sim}$ $\epsilon^+ = \pm\epsilon^-$ i.e. on the boundary $\epsilon^\sigma = 0$.????

1.5 Quantization of ψ^μ

$\pi_+ = \frac{i}{4\pi}\psi_+$ are constraints, thus we introduce

$$F = \pi_+ - \frac{i}{4\pi}\psi_+$$

we calculate $\{F, F\} = 1/2\pi$ [2] (2nd class constraints (see e.g. Lust-Theisen)) thus we need Dirac bracket to define CCR [3]

$$\{\psi, \pi\}_D = \{\psi, \pi\} - \{\psi, F\}\{F, F\}^{-1}\{F, \pi\} = \frac{i}{2} \tag{1.19}$$

[2] This anti-commutator here would be classically Poisson bracket. $\{a, b\}_{a-PB} \stackrel{\text{df}}{=} \partial_q a\, \partial_\pi b + \partial_q b\, \partial_\pi a$ thus $\{F, F\}_{a-PB} = -i/(2\pi)$.

[3] It gives $\{\psi, F\}_D = \{F, \pi\}_D = 0$

what gives

$$\{\psi^\mu_+(\tau,\sigma),\psi^\nu_+(\tau,\sigma')\} = 2\pi\delta(\sigma-\sigma')\eta^{\mu\nu} \tag{1.20}$$

1.a. $\{\psi^\mu_r,\psi^\nu_s\} = \delta_{r+s,0}\eta^{\mu\nu}$, $\psi^+_r = \psi_{-r}$. For $r \neq 0$ this defines creation-anihilation operators: $\psi_r|0\rangle = 0,\ r>0$.

1.b. For $r=0$,

$$\boxed{\{\psi^\mu_0,\psi^\nu_0\} = \eta^{\mu\nu}} \tag{1.21}$$

we get Dirac algebra – for the construction of the relevant Fock space: out of 2n real (for simplicity, **Euclidean**) ψ_0's we form n complex $b^i = \frac{1}{\sqrt{2}}(\psi^i_0 + i\psi^{i+n}_0)$, $b^{\bar{i}} = \frac{1}{\sqrt{2}}(\psi^i_0 - i\psi^{i+n}_0)$ They respect $\{b^i, b^{\bar{j}}\} = \delta^{i\bar{j}}$. We define a state $|\Omega>$, $b_i|\Omega>=0$. Fock space $\mathcal{S} = \oplus^{2m}_{n=0} \quad b_{\bar{i}_1}...b_{\bar{i}_n}|\Omega>$, its dimension $\dim_{\mathbb{C}}(\mathcal{S}) = 2^m =$ dim. of Dirac spinor rep. We then split into eigenspace of $(-1)^F$ of $\dim_{\mathbb{R}} = 2^m$, $\mathcal{S} = \mathcal{S}_+ \oplus \mathcal{S}_-$ where are .

$$\mathcal{S}_+ = \oplus^{2m}_{n=0\ even} \quad b_{\bar{i}_1}...b_{\bar{i}_n}|\Omega>, \quad \mathcal{S}_- = \oplus^{2m-1}_{n=1\ odd} \quad b_{\bar{i}_1}...b_{\bar{i}_n}|\Omega> \tag{1.22}$$

In other words (1.21) has the following reps[4]

$$\sqrt{2}\psi^\mu_0 = \Gamma^\mu. \tag{1.23}$$

Notice that (as always) $\Gamma^\mu : (\mathcal{S}_+,\mathcal{S}_-) \to (\mathcal{S}_-,\mathcal{S}_+)$. DO carefully the Dirac algebra: change notation to $\{\Gamma,\Gamma\} = 2\eta$ and then write down the Γ's expicitely.

D=10. In (1,9) Minkowski space these Weyl spinors can be real (Majorana) so $\dim_{\mathbb{R}}(S_\pm) = 16$. More notation:

$$\mathcal{S}_+ \ni S_+ = \xi_\alpha|\alpha\rangle, \quad \mathcal{S}_- \ni S_- = \xi_{\dot\alpha}|\dot\alpha\rangle. \tag{1.24}$$

We shall see that all these states has the same level i.e. the same mass shall condition.

1.6 Constrains

$(R)\ r \in \mathbf{Z},\ (NS)\ r \in \mathbf{Z}+\frac{1}{2}$.

$$\begin{aligned} L_m &= \frac{1}{2}\sum_m : \alpha_{m-n}\cdot\alpha_m : + \frac{1}{4}\sum_r (2r-m) : \psi_{m-r}\cdot\psi_r : \\ G_r &= \sum_n \alpha_n \cdot \psi_{r-n}. \end{aligned} \tag{1.25}$$

[4] The other reps $= \Gamma^{\mu\,T},\ \Gamma^{\mu\,*},\ \Gamma^{\mu\,+}$ are equivalent (which← (?)[]) or related by outer automorphism of the Lie algebra (which is **charge conjugation**).

Correspondingly, the Virasoro algebra is enlarged, with the non–zero (anti) commutators being

$$\begin{aligned}
[L_m, L_n] &= (m-n)L_{m+n} + \frac{c}{12}A(m)\delta_{m+n} \\
\{G_r, G_s\} &= 2L_{r+s} + \frac{c}{12}B(r)\delta_{r+s} \\
[L_m, G_r] &= \frac{1}{2}(m-2r)G_{m+r}\ ,
\end{aligned} \tag{1.26}$$

where $A_{NS} = (m^3 - m),\ B_{NS} = (4r^2 - 1),\ A_R = m^3,\ B_R = 4r^2$. For the free strings in d-dim Minkowski $c = d + d/2$. Shifting in R sector $L_0 \to\ L_0 + \frac{c}{24}$ we get the same algebra as for NS: N=1 SCA.

1.6.1 Virasoro constraints

$$(NS) \qquad (L_m - a_{NS}\delta_{m,0})|phys\rangle = 0, \qquad G_m|phys\rangle = 0\ \ m > 0 \tag{1.27}$$

$$(R) \qquad (L_m - a_R\delta_{m,0})|phys\rangle = 0, \qquad G_m|phys\rangle = 0\ \ m \geq 0 \tag{1.28}$$

and tilde for closed strings

$$a_{NS} = 1/2, \qquad a_R = 0 \tag{1.29}$$

♣ *Calculate a for arbitrary number of D and N b.c.* ♣

$$\boxed{G_0 = i\psi_0\alpha_0 + ... \sim i\sqrt{2}p_\mu\Gamma^\mu + ... \qquad \text{(Generalized Dirac operator)}} \tag{1.30}$$

$$(NS) \quad L_0|phys\rangle = 0 \to\ (p^2 + m^2)|phys\rangle = 0 \to\ (\partial_x^2 - m^2)\phi(x) = 0 \tag{1.31}$$

$$(R) \quad G_0|phys\rangle = 0 \to\ (\gamma^\mu p_\mu + ...)|phys\rangle = 0, \quad \text{(Generalized Dirac eq.)} \tag{1.32}$$

In (RR) sector we deal with forms[5]

1.7 GSO projection

$Z_2 \times Z_2$ for © and Z_2 for $\boxed{\sim}$ commute with SCA. (Left) Fermion number F is defined mod 2.

$$F = \sum_{m>0} \psi^\mu_{-m}\psi^\mu_m + (R\ only)\ b^{\bar{i}} b^i + c \tag{1.33}$$

$$[F, X] = 0,\ (-1)^F\psi(\sigma)(-1)^F = -\psi(\sigma) \tag{1.34}$$

[5] thus $\psi^\mu_0 p_\mu = \overline{\psi}^i\bar{p}_i + \psi^i p_i$. $\overline{\psi}^i : \Omega^n \to\ \Omega^{n+1},\ \psi^i : \Omega^n \to\ \Omega^{n-1}$ thus they represents $d,\ d^\dagger$ (Hodge). All together it leads to $(d + d^\dagger)F_n = 0$. We have $(d + d^\dagger) = \nabla$ according to S-Virasoro. Do it: use gamma identities $\to$?????.

For closed strings we have Hodge decomposition: $\psi^\mu_0 p_\mu = \partial + \partial^\dagger,\ \tilde{\psi}^\mu_0 p_\mu = \overline{\partial} + \overline{\partial}^\dagger$.

$(-1)^F = e^{i\pi F} = e^{-i\pi F}$ is well defined [6] . We can take physical states to be its eigenstates because it leaves invariant the constraints: $(-1)^F|phys\rangle = \pm|phys\rangle, \quad (-1)^{\tilde{F}}|phys\rangle = \pm|phys\rangle$. We take [7]

$$(-1)^F|\alpha\rangle_R = |\alpha\rangle_R \in S_+, \quad (-1)^F|\dot{\alpha}\rangle_R = -|\dot{\alpha}\rangle_R \in S_- \tag{1.35}$$

For the NS vacuum we have

$$(-1)^F|0\rangle_{NS} = -|0\rangle_{NS} \tag{1.36}$$

(-1) in (NS) comes from ghosts.

♣ *This is reasonable because colliding two massless spinors of the same chirality we get odd forms of $(-1)^F = 1$ so e.g. NS vector and there is such a state for $m^2 = 0$ with the above choice. With the opposite chice one would get $(-1)^F = 1$ vector at massive level !!! NO SUSY !!!* ♣

One can project the spectrum to states with $(-1)^F = 1$ and this is preserved by the interaction. **There is SUSY.** ♣ *Show it by counting states.* ♣

1.8 $\boxed{\sim}$

1.8.1 NS masses

$$(NS-),\ |0\rangle \rightarrow\ 2p^2 = \frac{1}{2}\,, \quad (NS+),\ \xi_\mu\psi^\mu_{-1/2}|0\rangle \rightarrow \qquad m^2 = 0, ... \tag{1.37}$$

Virasoro constraints: for ξ_μ only from $G_{1/2}$. It gives $p^\mu\xi_\mu = 0$.

Null states exists for ξ_μ and provide gauge symmetry.

$$\chi(p)G_{-1/2}|0> \sim \chi(p)p_\mu\phi^\mu_{-1/2}|0> \tag{1.38}$$

corresponds to $\delta A_\mu = \partial_\mu\chi(p)$.

1.8.2 R

$$(R+),\ \xi_\alpha|\alpha\rangle, \quad (R-),\ \xi_{\dot{\alpha}}|\dot{\alpha}\rangle, \quad p^2 = 0 \tag{1.39}$$

Virasoro constraints: for $\xi_\alpha,\ \xi_{\dot{\alpha}}$ only from $G_0 = i\alpha_0\cdot\psi_0 + ...$ By (1.21) we have [8]

$$\psi^\mu_0|\alpha\rangle = \frac{1}{\sqrt{2}}\Gamma^\mu_{\dot{\beta}\alpha}|\dot{\beta}\rangle \tag{1.40}$$

[6] Notice that in (R) $(-1)^F$ generalizes the γ_5. $(-1)^F\prod_{k=1}^s b^{\bar{i}_k}|0\rangle = (-1)^s\prod_{k=1}^s b^{\bar{i}_k}|0\rangle$. Also $[F,\psi_m] = -\text{sign}(m)\psi_m$.

[7] This fixes the irrelevant constant c in (R) in (1.33). It is irrelevant because it switches the chirality of spinors.

[8] One can check that $\overline{\Gamma}\psi^\mu_0|\alpha\rangle = -\psi^\mu_0|\alpha\rangle$ i.e. it is proportional to $|\dot{\beta}\rangle$.

Thus

$$G_0\xi_\alpha|\alpha\rangle = 0 = p_\mu\xi_\alpha\Gamma^\mu_{\dot\beta\alpha}|\dot\beta\rangle, \Rightarrow \boxed{p_\mu\Gamma^\mu_{\dot\beta\alpha}\xi_\alpha(p) = 0, \qquad \text{Dirac eq.}} \tag{1.41}$$

NO null states.

$$\begin{array}{c|c|c|l} NS+ & \xi\cdot\psi_{-1/2}|0\rangle & m^2=0 & \text{(odd \# of fermions)}|0\rangle \\ NS- & |0\rangle & m^2=-1/4 & \text{(even \# of fermions)}|0\rangle \\ R+ & S_+=|\alpha\rangle & m^2=0 & \text{(even \# of fermions)}|\alpha\rangle \\ & & & \text{(odd \# of fermions } \psi_r)|\dot\alpha\rangle \\ R- & S_-=|\dot\alpha\rangle & m^2=0 & (\alpha\leftrightarrow\dot\alpha \text{ in above}) \end{array} \tag{1.42}$$

The rightmost states are massive.

Take $(-1)^F = 1$ massless states $\xi\cdot\psi_{-1/2}|0\rangle,\ S_+ = |\alpha\rangle$. In vector state there are $10-2=8$ degrees of freedom, in spinor field there are $16-8=8$ d.o.f. (-8 due to Dirac eq.).

1.9 D-branes at angles

for 4 N-D BC the GSO changes sign !!! Polchinski II, p.169.

1.10 ©

Formally we have 4^2 possible sectors

$$(NS\pm, NS\pm), (NS\pm, R\pm), (R\pm, NS\pm), (R\pm, R\pm) \tag{1.43}$$

but some must be eliminated by $L_0 = \tilde{L}_0$ e.g. $NS-$ can apear only with $NS-$.

Sectors

- $(NS-, NS-)$ contains tachyon
- $(NS+, NS+)$ contains string gravity
- (NS, R) contains superpartners of gravity $\Psi^\mu_\alpha = |\alpha, p, R\rangle \otimes \tilde\psi^\mu_{-1}|0,p\rangle$ decomposes to χ^μ_α and $\lambda_{\dot\alpha} = (\Gamma_\mu)_{\dot\alpha}{}^\beta\Psi^\mu_\beta$. For critical string gravitino has null state:

$$G_{-1/2}|0> \otimes|\alpha\rangle \sim p_\mu\psi^\mu_{-1/2}|0> \otimes|\alpha\rangle \tag{1.44}$$

  corresponds to $\delta\chi^\mu_\mu = \partial_\mu\chi$. Local SUSY
- (R, R) contains form (gauge fields)

1.10.1 (R,R) We decompose tensor product of spinors $S_+ \otimes S_+ \to \xi_{\alpha\beta}|\alpha\rangle \otimes |\beta\rangle$ or $S_+ \otimes S_-$ into irrps of the Lorentz group. We need Clebsch-Gordon coeffs (CGS).

Denote $\mu_1 ... \mu_k \equiv [k]$ Notice that $(\Gamma^0 = C = C^T = -C^+)$ $\bar{S}'\Gamma^{[k]} S = S'^T C\Gamma^{[k]} S$ and $S'^T \Gamma^{[k]} C^+ S$ are irreps and both equivalent $\leftarrow$ (?)[] (see ...) thus $\Gamma^{[k]} C^+$ are CGC of $S \otimes S = anti-symmetric\ tensors$. Similarly we can use CGC: $\overline{\Gamma}\Gamma^{[k]} C^+ = (-)^{k(k+1)/2} \epsilon^{[k]}{}_{[10-k]} \Gamma^{[10-k]}/((10-k)!)$.

We denote $F_{[10-k]}\Gamma^{[10-k]}C^+ = (-)^{k(k+1)/2} * F_{[10-k]}(\overline{\Gamma}\Gamma^{[k]}C^+) \in \Lambda^{10-k}M$ (forms)

$$\xi_{\alpha\bullet}|\alpha\rangle \otimes |\bullet\rangle = \sum_k (F_k)_{\alpha\bullet}|\alpha\rangle \otimes |\bullet\rangle = \sum_{k\leq 10/2} (F_{[k]}\Gamma^{[k]}C^+ - (-)^{k(k+1)/2} * F_{[10-k]}\overline{\Gamma}\Gamma^{[k]}C^+)$$

($-$ is conventional)

By $(-1)^F|\alpha\rangle = \overline{\Gamma}_{\beta\alpha}|\beta\rangle = |\alpha\rangle$ thus $F_{[k]} = -(-)^{k(k+1)/2} * F_{[10-k]}$.

$$\xi_{\alpha\bullet}|\alpha\rangle \otimes |\bullet\rangle = \sum_{k\leq 10/2} F_{[k]}(1+\overline{\Gamma})\Gamma^{[k]}C^+ \tag{1.45}$$

- $(R+, R+)$. k is **odd**. $F_5 = *F_5$. $S_+ \otimes S_+ = F_1 \oplus F_3 \oplus F_5^+$.
  Counting: $S_+ \otimes S_+$: $lhs = 2^4 \cdot 2^4$, $rhs = 10 + \frac{10\cdot9\cdot8}{3!} + \frac{1}{2}\frac{10\cdot9\cdot8\cdot7\cdot6}{5!} = 10 + 120 + 126 = 256$.
- $(R+, R-)$. k is **even**. $S_- \otimes S_+ = F_0 \oplus F_2 \oplus F_4$.
  $S_- \otimes S_+$: $lhs = 2^4 \cdot 2^4 = 256 = rhs = 1 + \frac{10\cdot9}{2} + \frac{10\cdot9\cdot8\cdot7}{4!} = 1 + 45 + 210$.

1.10.2 CONSTRAINTS By (1.30) we get (picking only terms without $\overline{\Gamma}$)

$$\begin{aligned} 0 &= G_0 \sum F_k = \sum p_\mu \Gamma^\mu \cdot F_k = \sum p \wedge F_k - (p, F_k)_{k-1} \\ 0 &= \tilde{G}_0 \sum F_k = \sum p_\mu F_k \cdot \Gamma^\mu = \sum (-1)^k (p \wedge F_k + (p, F_k)_{k-1}) \quad [] \end{aligned}$$ [9]

(we used (1.67)). Thus all coefficients of the k-form must vanish i.e. $(p \wedge F_k) - (p, F_{k+2}) = 0$ and $(p \wedge F_k) + (p, F_{k+2}) = 0$

$$\boxed{p_{[\mu} F_{\mu_1 ... \mu_k]} = 0 = p^{\mu_1} F_{\mu_1 ... \mu_k}} \tag{1.46}$$

1.10.3 SUMMERY OF THE SPECTRA

We suppress momentum p from indexing the states.

sector			#
$(NS+, NS+)$	$\psi^\mu_{-1}\tilde{\psi}^\nu_{-1}\|0\rangle$	$G_{\mu\nu}, B_{\mu\nu}, \phi$	
$(R+, R+)$	$\|\alpha\rangle \otimes \|\beta\rangle$	$F_1 = dC_0, F_3 = dC_2, F_5^+ = dC_4$	
$(R+, R-)$	$\|\alpha\rangle \otimes \|\dot{\beta}\rangle$	$F_2 = dC_1,\ F_4 = dC_3$	
$(R+, NS+)$	$\|\alpha\rangle \otimes \tilde{\psi}^\mu_{-1}\|0\rangle$	χ^α_μ chirality $-$ $\lambda_{\dot{\alpha}}$ chirality $+$	
$(NS+, R-)$	$\psi^\mu_{-1}\|0\rangle \otimes \|\dot{\beta}\rangle$	$\chi^{\dot{\alpha}}_\mu$ chirality $+$ λ_α chirality $-$	
$(NS-, NS-)$	$\|0\rangle,\ m^2 = -1$	T	

Moreover we need more then one sector to produce modular invariant theory.

1.10.4 CONSISTENT CLOSED SUPERSTRINGS

type	GSO	sectors
IIA	$(-)^{F_L}$	$(NS+, NS+),\ (NS+, R-),\ (R+, NS+),\ (R+, R-)$
IIB	$(-)^{F_L}$	$(NS+, NS+),\ (NS+, R+),\ (R+, NS+),\ (R+, R+)$
$0A$	?	$(NS+, NS+),\ (NS-, NS-),\ (R+, R-),\ (R-, R+)$
$0B$	$(-)^{F_L+F_R}$	$(NS+, NS+),\ (NS-, NS-),\ (R+, R+),\ (R-, R-)$

(1.47)

1.10.5 CLOSED $IB = IIB/\Omega$

$$\Omega: \quad \begin{aligned} \sigma &\to 2\pi - \sigma \\ \alpha_n &\to \tilde{\alpha}_n \\ \psi_r &\to (-)^{2r}\tilde{\psi}_r \\ |\alpha\rangle &\leftrightarrow \widetilde{|\alpha\rangle} \end{aligned} \tag{1.48}$$

Invariant states:

$$G_{\mu\nu},\ \phi \tag{1.49}$$

$$C_2 \tag{1.50}$$

$$\chi^{\mu}_{\alpha},\ \lambda_{\dot{\alpha}} \tag{1.51}$$

(1.49): one must take into account that NS vacuum carries the fermion number (1.36). (1.50): only $\Gamma^{\mu\nu}$ (M. -l.c. = 8d E) is a-symmetric matrix due to (1) C=1 for M-W E-spinors (2) $(\Gamma^{\mu\nu})^T = C\Gamma^{\nu\mu}C^\dagger = -\Gamma^{\mu\nu}$. Thus from a-commuting of spinor states $|\alpha\rangle$, $\widetilde{|\beta\rangle}$ we get Ω-invatiant state $(\Gamma^{\mu\nu})^{\alpha\beta}|\alpha\rangle \otimes \widetilde{|\beta\rangle}$. (1.51): symmetric combination of IIB (NS+,R+), (R+,NS+).

1.11 Open I

Consists of closed $IB = IIB/\Omega$ and open unoriented superstring

1.11.1 OPEN UNORIENTED

$$\Omega: \quad \begin{aligned} \psi(\sigma^+) &\to \psi^{\Omega}(\sigma^+) = \psi(\sigma^- + \pi) \\ \psi(\sigma^-) &\to \psi^{\Omega}(\sigma^-) = \psi(\sigma^+ - \pi) \end{aligned} \tag{1.52}$$

It follows that

$$\Omega: \quad \begin{aligned} (R) &\to (R) \\ (NS) &\to (NS) \end{aligned} \tag{1.53}$$

so $\psi^{\Omega}(\sigma^+) = \pm\psi^{\Omega}(\sigma^-)|_{\sigma=0}$, and $\psi^{\Omega}(\sigma^+) = \psi^{\Omega}(\sigma^-)|_{\sigma=\pi}$ thus in order to get to the original situation we have to make Z_2 transform in the (NS) sector. Thus

$$\Omega: \ \psi_r = (-1)^{2r}\psi_r = \begin{cases} \psi_r & (R) \\ -\psi_r & (NS) \end{cases} \tag{1.54}$$

The spectrum is modified only in the (NS) sector: only (even # of ψ_r)$|0\rangle$ survaive so e.g. the U(1) gauge boson is lost.

1.11.2 CHAN-PATON FACTORS

1.12 Appendices

Appendix

M. Canonical quantization of fermions.

Out of 2n real (Euclidean) ψ's we form n complex

$$b^i = \frac{1}{\sqrt{2}}(\psi^i + i\psi^{i+n}),\ b^{\bar{i}} = \frac{1}{\sqrt{2}}(\psi^i - i\psi^{i+n}) \tag{1.55}$$

These are independent variables and $\frac{i}{2}\overline{\psi}\,\partial_\tau\psi = ib^{\bar{i}}\,\partial_\tau b^i \Rightarrow \pi_b^{\bar{i}} = ib^{\bar{i}}$
CCR

$$\{b^i, b^{\bar{i}}\} = \delta^{i\bar{i}},\ \{b^i, b^j\} = 0,\ \{b^{\bar{i}}, b^{\bar{j}}\} = 0 \Rightarrow \{\psi^\mu, \psi^\nu\} = \eta^{\mu\nu} \tag{1.56}$$

We define a state $|\Omega>$, $b_i|\Omega>= 0$
Then

$$\begin{array}{ll} \oplus_{n=0\ even}^{2m} & b_{\bar{i}_1}...b_{\bar{i}_n}|\Omega> \textit{rep.of one chirality} \\ \oplus_{n=1\ odd}^{2m-1} & b_{\bar{i}_1}...b_{\bar{i}_n}|\Omega> \textit{rep. of another chirality} \end{array} \tag{1.57}$$

$$\begin{aligned} \psi^\mu &= \frac{i}{2}(b_i + b_{\bar{i}}),\ \textit{for}\ \mu = i = 1, ...m \\ \psi^\mu &= \frac{i}{2}i(b_i - b_{\bar{i}}),\ \textit{for}\ \mu = m+i = m+1, ...2m \\ \{\psi^\mu, \psi^\nu\} &= \delta_{\mu\nu} \end{aligned} \tag{1.58}$$

In this algebra the Dirac matrices are $\Gamma^\mu = \sqrt{2}\psi^\mu$.

N. RR sector

- $|\alpha, p, R\rangle \otimes |\beta, p, R\rangle$ *or* $|\alpha, p, R\rangle \otimes |\dot{\beta}, p, R\rangle$ *plus (undoted* $\leftrightarrow$ *doted) which are equivalent.*
- $\{\Gamma^\mu, \Gamma^\nu\} \overset{M}{=} -2\eta^{\mu\nu} \overset{E}{=} 2\eta^{\mu\nu}$ *so* $(\Gamma^\mu)^\dagger \overset{M}{=} -\eta^{\mu\nu}(\Gamma^\nu) \overset{E}{=} \eta^{\mu\nu}(\Gamma^\nu)$
- $\Gamma^{\mu_1,...\mu_k} \overset{\mathrm{df}}{=} \frac{1}{k!}\sum_{permut} sgn(\mu_1, ...\mu_k)\Gamma^{\mu_1}...\Gamma^{\mu_k}$
- *Lorentz generators:* $M^{\mu\nu} = \frac{i}{2}\Gamma^{\mu\nu}$.
- *For even dimensions there are 2 Weyl reps* $S_\pm$*;* $\psi_\alpha \in S_-$, $\psi_{\dot{\alpha}} \in S_+$ *thus we can do projections of* $\Gamma^{\mu\nu}$ *on appropriate spaces using*

$$\overline{\Gamma}S_\pm = \pm S_\pm, \quad \{\overline{\Gamma}, \Gamma^\mu\} = 0, \quad \overline{\Gamma}^\dagger = \overline{\Gamma} \tag{1.59}$$

- $\Gamma^{\mu T}$ *also respect (N.). (?) This is an equivalent rep thus there is a unitary operation* C

$$C\Gamma^\mu C^\dagger = \pm\Gamma^{\mu T} \tag{1.60}$$

C is a nontrivial metric on $S_\pm$ i.e. $\psi^T C\psi$ is an Lorentz invartiant due to $(M^{\mu\nu})^T = -CM^{\mu\nu}C^\dagger$.

- *The charge (Marojana) conjugate spinor*

$$\psi_c \overset{\text{df}}{=} C\overline{\psi}^T \overset{M}{=} C(\Gamma^0)^T\psi^* \overset{E}{=} C\psi^* \tag{1.61}$$

transforms the same way as ψ. Moreover ψ is an eigenstate of $\overline{\Gamma}$ then also ψ_c [10]

$$\overline{\Gamma}(S_+)_c = \text{sign}(\cdot) \cdot (S_+)_c \tag{1.62}$$

- *$\overline{\psi}\,\psi$ is an Lorentz invatiant due to $(M^{\mu\nu})^\dagger[\Gamma^0] = [\Gamma^0]M^{\mu\nu}$.*

- *$\overline{\psi}\,\Gamma^\mu\psi$ and $\psi C\Gamma^\mu\psi$ are Lorentz vectors*
Dirac operator $V_\mu\Gamma^\mu : S_\pm \longrightarrow \ S_\mp$ (V is a vector) due to (1.59)
$V_\mu C\Gamma^\mu : S_\pm \longrightarrow \ \overline{S_\mp}$

- *Majorana-Weyl spinors in M space (GSW vol.I,p.200,288)*

  - *we take $(\Gamma^\mu)^\dagger = -\eta^{\mu\nu}\Gamma^\nu$, moreover for M-W we can take $(\Gamma^\mu)^T = -\eta^{\mu\nu}\Gamma^\nu$.*
  - *$C = -\Gamma^0$*
  $$C\Gamma^\mu C^\dagger = -\Gamma^{\mu T},\ \Psi_M = \Psi_M^* \tag{1.63}$$
  - *RR fields $S_+C\Gamma^{\mu_1\cdots\mu_k}S_+$ for odd k, $S_-C\Gamma^{\mu_1\cdots\mu_k}S_+$ for even k (dut to (1.59)) is anitsymmetric tensor of the rank k (we can change $(+\to\ -)$ above)*
  - *Due to (1.59) and the Hodge duality $\overline{\Gamma}\Gamma^{\mu_1\cdots\mu_k} \sim \epsilon^{\mu_1\cdots\mu_k}{}_{\mu_{k+1}\cdots\mu_d}\Gamma^{\mu_{k+1}\cdots\mu_d}$ the above tensors are linearly independent only for $k \le [d/2]$. For $k = d/2$ the tensor is (a-) self-dual for for S_+ (S_-).*

O. Some identities

A) CONVENTIONS *we take for M-W according to (1.63)*

$$(\Gamma^\mu)^\dagger = -\eta^{\mu\nu}, \quad \Gamma^\nu(\Gamma^\mu)^T = -\eta^{\mu\nu}\Gamma^\nu, \tag{1.64}$$
$$C\Gamma^\mu C^\dagger = -\Gamma^{\mu T}, \quad C = -\Gamma^0,\ \Psi_M = \Psi_M^* \tag{1.65}$$

B) DIRAC ALGEBRA IDENTITIES

$$\Gamma^{\mu_1,\cdots\mu_k} \overset{\text{df}}{=} \frac{1}{k!}\sum_{\sigma\in S_k} sgn(\sigma)\Gamma^{\sigma(\mu_1)}\ldots\Gamma^{\sigma(\mu_k)} \tag{1.66}$$

The following holds [[?] what convention ??(N.)]

$$\Gamma^{\mu_1\cdots\mu_k}\Gamma^\nu = \Gamma^{\mu_1\cdots\mu_k,\nu} - k\ \Gamma^{[\mu_1\cdots\mu_{k-1}}g^{\mu_k]\nu} = (-1)^k(\Gamma^{\nu\mu_1\cdots\mu_k} + k\ g^{\nu[\mu_1}\Gamma^{\mu_2\cdots\mu_k]}) \tag{1.67}$$
$$\Gamma^\nu\Gamma^{\mu_1\cdots\mu_k} = \Gamma^{\nu\mu_1\cdots\mu_k} - k\ g^{\nu[\mu_1}\Gamma^{\mu_2\cdots\mu_k]} \tag{1.68}$$

[10] d-dim of the space,
$lhs = \overline{\Gamma}C(\Gamma^0)^T(\overline{\Gamma}^T)^2(S_+) = \overline{\Gamma}C(\Gamma^0)^T\overline{\Gamma}^T(S_+) = -\overline{\Gamma}C\overline{\Gamma}^T(\Gamma^0)^T(S_+) = -(-1)^{d(d-1)/2}(\text{sign}(1.60))^d(S_+)_c$.
For E space the overal sign is changed.

c) FORMS

1.a. $A_k = \frac{1}{k!} A_{\mu_1...\mu_k} dx^{\mu_1}...dx^{\mu_k}$

1.b. $F_n \stackrel{\mathrm{df}}{=} d(A_{(n-1)}) = \frac{1}{(n-1)!}\partial_{\mu_n} A_{\mu_1...\mu_{(n-1)}} dx^{\mu_n} \wedge dx^{\mu_1}...dx^{\mu_{(n-1)}}$ *(as forms)*
and $F_n = \frac{1}{n!} F_{\mu_1...\mu_n} dx^{\mu_1}...dx^{\mu_n}$. *Thus:* $F_{\mu_1...\mu_n} = n\partial_{[\mu_1} A_{\mu_2...\mu_n]}$, [11]
$F_{0...(n-1)} = \partial_0 A_{1...(n-1)} + \sum_{cycles} \mathrm{sign}(cycle)\partial_{\mu_1} A_{\mu_2...\mu_n}$ *(altogether d-terms)*
e.g. $F_{12} = \partial_1 A_2 - \partial_2 A_1,\ F_{123} = \partial_1 A_{23} + \partial_2 A_{31} + \partial_3 A_{12}$.

1.c. *Hodge star* [12]

1.d. $[a_1....a_n] \stackrel{\mathrm{df}}{=} \frac{1}{n!}$(*all sign permutations of* $a_1....a_n$)

$$* \, dx^{\mu_1} \wedge ...dx^{\mu_k} \stackrel{\mathrm{df}}{=} \frac{\sqrt{|g|}}{(d-k)!} \epsilon^{\mu_1...\mu_k}{}_{\nu_1...\nu_{d-k}} \, dx^{\nu_1} \wedge ...dx^{\nu_{(d-k)}} \tag{1.69}$$

$$*^2 \, a_k = \mathrm{sign}(\det(g)) \, (-)^{k(d-k)} a_k \tag{1.70}$$

$$a \wedge *b = (-)^{k(d-k)} * a \wedge b = \frac{1}{k!}\sqrt{|g|} a_{\mu_1,...\mu_k} b^{\mu_1,...\mu_k} dV = \frac{1}{k!}(a \cdot b) \tag{1.71}$$

1.e. *kinetic terms*

$$\mathcal{L} = -\frac{1}{2} F_n \wedge *F_n = -\frac{1}{2n!} F_n \cdot F_n = \frac{1}{2}\sum_{diff...} [(F_{0...})^2 - (space\ indices)] \tag{1.72}$$

[11] $[a_1....a_n] \stackrel{\mathrm{df}}{=} \frac{1}{n!}\sum_{\mathrm{perm}} \mathrm{sign(permutation)}\ a_1....a_n$.

[12] $\epsilon_{\mu_1...\mu_k} \stackrel{\mathrm{df}}{=} \mathrm{sign}(\mu_1...\mu_k),\ \ \epsilon^{\mu_1...\mu_k} = \mathrm{sign}(\det(g))\sqrt{|\det(g)|}$ for even perm.