Contents

1 Canonical quantization – conformal gauge

String theory in Mink. space in the **conf. gauge** is[1] FREE 2D CFT + CONSTRAINTS

$$S = \frac{T}{2}\int_\Sigma d^2\sigma[(\partial_\tau X)^2 - (\partial_\sigma X)^2] \tag{1.1}$$

$$(L_n - \delta_{n,0})|\widetilde{ph}\rangle = 0 = (\tilde{L}_n - \delta_{n,0})|\widetilde{ph}\rangle \tag{1.2}$$

Set of constraints on physical states: naively $T_{++}|\widetilde{ph}\rangle = T_{--}|\widetilde{ph}\rangle = 0$, but this is not good.

$$\text{Canonical momenta} \qquad (p_X)_\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\tau X^\mu)} = T\partial_\tau X_\mu \tag{1.3}$$

$$\text{CCR}: \qquad [X^\mu(\tau,\sigma),(p_X)^\nu(\tau,\sigma')] = i\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{1.4}$$

$$\text{Open strings:} \qquad [\alpha^\mu_m,\alpha^\nu_n] = m\delta_{m,-n}\eta^{\mu\nu}, \quad [x^\mu,p^\nu] = i\eta^{\mu\nu},\ \alpha_0\equiv 2p \tag{1.5}$$

$$\text{Closed string:} \qquad \text{as above + the same for } \tilde{\alpha}\ \ \&\ \ [\alpha,\tilde{\alpha}] = 0,,\ \alpha_0 \equiv p. \tag{1.6}$$

1.1 Free field space of states.

Cartan subalgebra i.e. maximal set of the hermitian commuting operators is e.g. $\boxed{\alpha_0}$.
$|q\rangle = e^{iqx}|0\rangle,\ p^\mu|q\rangle = q^\mu|q\rangle$

$$\alpha^\mu_m|p\rangle = 0, \quad m>0, \qquad \alpha_0|p\rangle = 2p|p\rangle \tag{1.7}$$

(unphysical) space of states

$$\left\{\prod_i \alpha^{\mu_i}_{-m_i}|p\rangle, \quad m_i>0\right\} \tag{1.8}$$

Some of the states has negative norms: e.g. $(m>0)$

$$|\alpha^0_{-m}|p\rangle|^2 = \langle p|\alpha^0_m\alpha^0_{-m}|p\rangle \stackrel{CCR}{=} -m\langle p||p\rangle$$

Similarly for Ⓒ . $\left\{\prod_i \alpha^{\mu_i}_{-m_i}\prod_i \tilde{\alpha}^{\nu_i}_{-n_i}|p\rangle, \quad n_i,m_i>0\right\}$

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[1] In the Old Covariant Quantization (OCQ) one does not need ghosts. Virasoro is anomalous.

1.2 Constraints

1.2.1 VIRASORO ALGEBRA. According to sec.**??** for all type of strings:

$$L_0 = \alpha_0^2/2 + \sum_{n>0} \alpha_{-n}\alpha_n, \quad L_m = \tfrac{1}{2}\sum_n \alpha_{m-n}\alpha_n, \quad L_{-m} = (L_m)^\dagger \tag{1.9}$$

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n} \tag{1.10}$$

Notice that

1.a. all operators must be $\mathfrak{normal\ ordered}$: non-trivial ordering occures only for: $L_0 \to$ leads to the constant a below in constraints (1.11).

1.b. normal ordering is essential to get the central term in $[L_m, L_n]$.

1.c. L_m generates symmetries (see App.A..

1.d. $L_0 = \frac{1}{2\pi}\int d\sigma T_{00}$ is sth-like the energy.

(Below: $m_i > 0$).

1.3 $\widetilde{\mathcal{H}_{\mathbf{ph}}} = \mathcal{H}_{\mathbf{ph}} \oplus$ $\mathfrak{null}$

For $\boxed{\sim}$ **.** All the physical states respect (in agreement with (1.10)) :

$$L_m|\widetilde{ph}\rangle = 0, \quad m>0, \quad (L_0 - a)|\widetilde{ph}\rangle = 0, \ \text{(mass-shell)} \tag{1.11}$$

$$\{|\widetilde{ph}\rangle\} \equiv \mathcal{H}_{\widetilde{ph}} \tag{1.12}$$

All states in $\mathcal{H}_{\widetilde{ph}}$ have non-negative norm.

1.a. ♣ *Among* $\{|\widetilde{ph}\rangle\}$ *there are still zero-norm states called null states which should be removed by some farther constraints i.e. by* $n \cdot \xi = 0$ *for some* n $\to$ *below.!!!!* ♣

1.b. For some a ($a \leq 1$) and c ($c \leq 26$) Virasoro constraints cut out all unphysical, negative norm states from the unconstraint Hilbert space.

1.c. For the critical case $a = 1, c = 26$ the spectrum consists of the same states as for the l-c gauge. This is the CRITICAL STRING.

$\boxed{\star}$ *derive a=1 from l-c quantization. It is important fo the fermionic string too* [].

1.d. No consistent interaction has been found for the non critical case.

1.e. There also are physical (?) operators corresponding to closed strings -HOW TO SEE IT ????

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$$(L_0 - \tilde{L}_0)|\widetilde{ph}\rangle = 0, \quad (L_0 + \tilde{L}_0 - 2a)|\widetilde{ph}\rangle = (L_0 - a)|\widetilde{ph}\rangle = 0 \tag{1.13}$$

1.3.1 SPURIOUS AND NULL STATES : $|spur\rangle = \prod_i L_{-m_i}|any\rangle_s$

1.a. For all $\{|\widetilde{ph}\rangle\}$: $\langle spur|\widetilde{ph}\rangle = 0$ There are no other states with the above property.

1.b. $\mathfrak{null}$ states $\{|null\rangle\} = \{|spur\rangle\} \cap \{|\widetilde{ph}\rangle\}$: All $\{|null\rangle\}$ are orthogonal to each other, because they are spurious and physical.

$$|any\rangle \supset |\widetilde{ph}\rangle \supset |null\rangle$$

$|\widetilde{ph}\rangle \to |\widetilde{ph}\rangle' = |\widetilde{ph}\rangle + |null\rangle$ is a $\to$ gauge transformation; [Pol.I p.123]. It does not change any sclar product with other physical states – it is also property of the interaction.

1.4 Physical states $\mathcal{P}_{\mathcal{H}}$

♣ *Here it is defined to be free from negative norm states. One gets physical Hilbert $\mathcal{H}$ space (of positive norm states) after eliminating all the $|null\rangle$ states from $\mathcal{P}_{\mathcal{H}}$.* ♣

- L_0 are mass shell relations
- L_n gives perpendicularity of polarization tensors.
- $|null\rangle$ gives gauge-like freedom

$(L_0 - a)\prod_i \alpha^{\mu_i}_{-m_i}|0,p\rangle = (\alpha_2^2/2 + \sum_i m_i - a)\prod_i \alpha^{\mu_i}_{-m_i}|0,p\rangle$ respect constraints iff

$$(\alpha_2^2/2 + \sum_i m_i - a) = 0 \quad \text{mass-shell} \tag{1.14}$$

1.4.1 TECHNICAL TOOL

- Level eigenstates $\hat{N}|state\rangle = N|state\rangle$
- $(L_0 = \alpha_0^2/2 + N), \ \to N = \sum_{n=1}\alpha_{-n}\alpha_n$
- $\hat{N}\prod_i \alpha^{\mu_i}_{-m_i}|0,p\rangle = (\sum_i m_i)\prod_i \alpha^{\mu_i}_{-m_i}|0,p\rangle$
  $\hat{N}\prod_i \alpha^{\mu_i}_{-m_i}|0,p\rangle = [\hat{N}, \prod_i \alpha^{\mu_i}_{-m_i}]|0,p\rangle = (\sum_i m_i)\prod_i \alpha^{\mu_i}_{-m_i}|0,p\rangle$
- $\hat{N}|any\rangle_s = s|any\rangle_s \to \hat{N}\prod_i L_{-m_i}|any\rangle = (\sum_i m_i + s)\prod_i L_{-m_i}|any\rangle$
  $\hat{N}\prod_i L_{-m_i}|any\rangle = [\hat{N}, \prod_i L_{-m_i}]|any\rangle_s + \prod_i L_{-m_i}\hat{N}|any\rangle_s = (s + \sum_i m_i)\prod_i L_{-m_i}|any\rangle$

- At the given level there are only finite number of the non-trivial constraints. Some constraints are trivial: we use ($[L_n, \alpha_{-m}] = m\alpha_{n-m}, \quad n > 0$) then $L_n \prod_i \alpha^{\mu_i}_{-m_i}|0,p\rangle = [L_n, \prod_i \alpha^{\mu_i}_{-m_i}]|0,p\rangle \equiv 0, \quad iff \quad n > sup_{(i,j)}(m_i, m_j)$ The state at level N is a linear combinaton of the above that in general

$$L_n|N\rangle = 0 \text{ trivially for } n > N \tag{1.15}$$

1.4.2 N=0 $|p\rangle \longrightarrow \alpha_0^2/2 = a, \rightarrow$ tachyon for a=1.

1.4.3 N=1

$$\alpha_0^2/2 = a - 1 \tag{1.16}$$

$$L_1 \xi_\mu \alpha^\mu_{-1}|0,p\rangle = p^\mu \xi_\mu |0,p\rangle, \rightarrow \ p^\mu \xi_\mu(p) = 0 \tag{1.17}$$

Null state: $L_{-1}|0,p\rangle = p_\mu \alpha^\mu_{-1}|0,p\rangle$. This state has $\xi_\mu = p_\mu$ thus is physiacal for $p^2 = 0 \longrightarrow$ gauge symmetry only for $a = 1$.

1.4.4 N=2 $\xi_{\mu_1,..} \prod_i \alpha^{\mu_i}_{-m_i}|0,p\rangle +$ have the same momentum as $\prod_i \alpha^{\mu_i}_{-m_i}|0,p\rangle \longrightarrow$ gauge symmetry

1.5 Effective fields

mass shell condition $(p^2+m^2)|p\rangle = 0$. Is equivalent to free field $(\partial^2 - m^2)\phi(x) = 0$ which has decomposition

$$\phi(x){=}\int \frac{d\!\!{}^-{}^3p}{2\omega_p}[a(\vec{p})e^{-ipx} + a^\dagger(\vec{p})e^{ipx}]|_{p_0=\omega_p} \tag{1.18}$$

thus

$$|p\rangle = a(p)|0\rangle \tag{1.19}$$

$$(\partial^2 - m^2)\Phi_a(x) = 0 \ \stackrel{?}{\rightarrow} \ \mathcal{L} \sim \phi(x)(\partial^2 - m^2)\phi(x) \tag{1.20}$$

We see that there is a correspondence between string quanta and states of a QFT.

N=1

$\xi^i_\mu \alpha^\mu_{-1}|0,p\rangle \leftrightarrow A_\mu(x)$ massless vector particle

$$A_\mu(x) = \int dp[\sum_{i=1}^{3} \xi^i_\mu(p)a^i(p)e^{-ipx} + h.c],\ p_0 = |\vec{p}| \tag{1.21}$$

in the gauge $\partial_\mu A_\mu = 0$ $(p^\mu \xi^i_\mu = 0)$, $\xi^i_\mu \alpha^\mu_{-1}|0,p\rangle = a^i(p)|0\rangle$ Null states $\rightarrow$ residual gauge symmetry: at massless level only

$$\delta A_\mu(x) = \partial_\mu \zeta(x), \quad \partial^2\zeta(x) = 0 \tag{1.22}$$

The state is physical if $p^2 = 0$ i.e. $\partial^2\zeta(x) = 0$.

Effective action: U(1) gauge theory with the gague choice $\partial_\mu A_\mu = 0$ we get exactly free Lagrangian.

$$\mathcal{L} = -\frac{1}{4}(F_{\mu\nu})^2 = -\tfrac{1}{2}\,(\partial_\mu A_\nu)^2 + \tfrac{1}{2}\,(\partial_\mu A_\nu)(\partial_\nu A_\mu) = -\tfrac{1}{2}\,(\partial_\mu A_\nu)^2 + \tfrac{1}{2}\,(\partial_\mu A_\mu)^2 \tag{1.23}$$

Residual gauge symm. is $\delta A_\mu = \partial_\mu \zeta$, $\delta\mathcal{L} = -\partial_\mu A_\nu \partial_\mu \partial_\nu \zeta$ – integrating by parts ∂_ν we get invariance. This gauge choice has residual gauge symmetry as in (1.22). The residual gauge symmetry gauges away the $p_\mu \hat{a}$ part. In field theory one farther fixes the gauge choice by choosing a vector $n^\mu,\ n\cdot p = -1, n^2 = 0$ and imposing $n \cdot \xi^i(p) = 0$.

Counting of the degrees of freedom (D): D-2 polarizations of gauge bosons.

1.6 ⓒ

$(L_0 - \tilde{L}_0)\prod_i \alpha^{\mu_i}_{-m_i}\prod_k \tilde{\alpha}^{\nu_k}_{-n_k}|0,p\rangle = 0, \rightarrow \sum_i m_i = \sum_k n_k$

Level 1. $\xi_{\mu\nu}\alpha^{\mu}_{-1}\tilde{\alpha}^{\nu}_{-1}|0,p\rangle$. $L_1, \tilde{L}_1 \rightarrow p^{\mu}\xi_{\mu\nu} = p^{\nu}\xi_{\mu\nu} = 0$ one of these $2d$ equations is not independent $p^{\mu}\xi_{\mu\nu}p^{\nu} = 0$; null= $\xi_{\mu\nu} = p_{\mu}\zeta_{\nu} + \tilde{\zeta}_{\mu}p_{\nu}$ iff $p^{\mu}\zeta_{\mu} = p^{\mu}\tilde{\zeta}_{\mu} = 0$; $\zeta_{\mu} = \tilde{\zeta}_{\mu} = p_{\mu}$ gives the same null state. Counting: $d^2 - (2d-1) - (2(d-1)-1) = d^2 - 4d + 4$. GOOD. Easier: take $p = (p, p, 0, ...)$ i.e. $p^+ = p$ all other =0; then $p^+\xi_{+\nu} = p^+\xi_{\mu+} = 0$ (eq. for ξ_{++} is the same) – (2d-1) constraints; null: $p^+\zeta_+ = p^+\tilde{\zeta}_+ = 0$ i.e. $\zeta_+ = \tilde{\zeta}_+ = 0$ – (2d-2) null but one p_-p_- is common.

Effective theory

N=1:masssless states: graviton $h_{\mu\nu}$, $B_{\mu\nu}$, dilaton ϕ.

The action:

$$\frac{1}{2\kappa^2}\int\sqrt{-G}\left(R - \tfrac{1}{2}(\partial\phi)^2 - \frac{1}{12}H^2\right) \tag{1.24}$$

Exapnsion around flat background: $g_{\mu\nu} = \eta_{\mu\nu} + \kappa_0 h_{\mu\nu}$. The linearized part of the graviton interaction is given by the following Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_{\lambda}h^{\lambda}_{\mu}\partial^{\mu}h^{\nu}_{\nu} + \frac{1}{2}\partial_{\lambda}h^{\lambda}_{\mu}\partial_{\nu}h^{\nu\mu} - \frac{1}{4}\partial_{\lambda}h_{\mu\nu}\partial^{\lambda}h^{\mu\nu} + \frac{1}{4}\partial_{\lambda}h^{\mu}_{\mu}\partial^{\lambda}h^{\nu}_{\nu} \tag{1.25}$$

For $h^{\mu}_{\mu} = 0, \partial^{\mu}h_{\mu\nu} = 0$ we get

$$\mathcal{L} = -\frac{1}{4}\partial_{\lambda}h_{\mu\nu}\partial^{\lambda}h^{\mu\nu} \tag{1.26}$$

($\frac{1}{4}$ is becuase the metric tensor is symmetric) with the gauge gauge invariance

$$h_{\mu\nu} \longrightarrow h_{\mu\nu} + \partial_{\mu}\xi_{\nu} + \partial_{\nu}\xi_{\mu}. \tag{1.27}$$

Antisymmetric tensors We have $-\frac{1}{12}H^2$ where H is a field strenght for B fields. In general:

$$\begin{aligned} A_k &= \frac{1}{k!}A_{\mu_1\dots\mu_k}dx^{\mu_1}\dots dx^{\mu_k} \qquad (1.28)\\ F_n &\stackrel{\text{df}}{=} d(A_{(n-1)}) = \frac{1}{(n-1)!}\partial_{\mu_n}A_{\mu_1\dots\mu_{(n-1)}}dx^{\mu_n}\wedge dx^{\mu_1}\dots dx^{\mu_{(n-1)}} = \frac{1}{n!}F_{\mu_1\dots\mu_n}dx^{\mu_1}\dots dx^{\mu_n} \end{aligned}$$

Thus: $F_{\mu_1\dots\mu_n} = n\partial_{[\mu_1}A_{\mu_2\dots\mu_n]}$, [2]

$F_{0\dots(n-1)} = \partial_0 A_{1\dots(n-1)} + \sum_{cycles}\text{sign}(cycle)\partial_{\mu_1}A_{\mu_2\dots\mu_n}$ (altogether d-terms)

e.g. $F_{12} = \partial_1 A_2 - \partial_2 A_1,\ H_{123} = \partial_1 B_{23} + \partial_2 B_{31} + \partial_3 B_{12}$. So

$$-\frac{1}{12}H^2_{\mu\nu\rho} = -\frac{1}{12}(\partial_{\mu}B_{\nu\rho} + cycl.)^2 = -\frac{1}{4}(\partial_{\mu}B_{\nu\rho})^2 - \frac{1}{2}\partial_{\mu}B_{\nu\rho}\partial_{\nu}B_{\rho\mu} - \frac{1}{2}\partial_{\mu}B_{\nu\rho}\partial_{\rho}B_{\mu\nu} + one\ more$$

[2] $[a_1\dots.a_n] \stackrel{\text{df}}{=} \frac{1}{n!}\sum_{\text{perm}}\text{sign(permutation)}\ a_1\dots.a_n$.

$\boxed{\star}$ *Analize null states for the massive modes* []

The spectrum is ghost free for a=1 and d=26 or $a \leq 1$ and $d \leq 25$ GSWI,85.

1.7 Spectra

Closed oriented - CO, Closed unoriented - CUO, Open oriented - OO, Open unoriented - OUO.

$closed$	CO	CUO	$open$	OO	OUO
$m^2 = -\frac{d-2}{12}$			$m^2 = -\frac{d-2}{48}$		
$\|0,p\rangle$	T	T	$\|0,p\rangle$	T	T
$m^2 = 2 - \frac{d-2}{12}$			$m^2 = \frac{1}{2} - \frac{d-2}{48}$		
$\alpha_{-1}^{(\mu}\tilde{\alpha}_{-1}^{\nu)}\|0,p\rangle$	$G_{\mu\nu}$	$G_{\mu\nu}$	$\alpha_{-1}^{\mu}\|0,p\rangle$	A_μ	
$\alpha_{-1}^{\mu}(\tilde{\alpha}_{-1})_\mu\|0,p\rangle$	ϕ	ϕ			
$\alpha_{-1}^{[\mu}\tilde{\alpha}_{-1}^{\nu]}\|0,p\rangle$	$B_{\mu\nu}$				

(1.29)

1.8 Appendices

A. Quantum symmetries

Classically $\partial_a\ J^a = 0 \to \partial_t Q = 0$ *so in* **Hamilton** *formalism* $Q = Q(p,q)$ *and* $\partial_t Q = 0, \to \{Q, H\}_{PB} = 0$. *Thus* Q *is generator of symmetry (see below). In quantum theory* $[Q, H] = 0$. *Transformation laws under the symmetry for fields are generated by the charge because the commutator is a differentiation i.e. for* $H = A\phi B$ *we have* $[Q, H] = [Q, A\phi B] = [Q, A]\phi B + A[Q, \phi]B + A\phi[Q, B]$. *We set:*

$$\phi' = e^{i\epsilon Q}\phi e^{-i\epsilon Q} \Rightarrow \delta_\epsilon \phi = i\epsilon[Q, \phi] \tag{1.30}$$

*In our case from (**??**) we get* $\epsilon Q = \frac{1}{2\pi}\int \epsilon(\tau)T(\tau) = \sum \epsilon_m L_m$. *For the scalar field* $\delta_{\epsilon^+_m} X = \epsilon^+_m e^{im\sigma^+}\partial_+ X$ *thus* $[L_m, \alpha_n] = -(m+n)\alpha_{m+n}$ *gives normalization of* L_n.

A) L-C QUANTIZATION *We solve constraints, thus we should work only with physical states, thus there should not be ghosts.*

$\boxed{\sim}$

1.a. *solution for* $X^+ = x^+_0 + 4p^+\tau$ *is conformally equivalent to the most general solution* ($p^+ > 0$). *Thus we can farther fix the gauge choosing* X^+.

1.b. *We analize* $T_{++} = 0 \equiv 2p^+\partial_+ X^- = (\partial_+ X^\perp)^2$. *Its zero mode*

$$2p^+ 2p^- = (2p^\perp)^2 + \sum \alpha_n \alpha_m \tag{1.31}$$

must be normaled ordered and it gives mass condition. The nonzero modes give relation between X^- *and* $X^\perp$ *oscilations* $\to$ *not interesting.*

Normal ordering influences the CR for generators of the $\mathfrak{Lorentz\ group}$ *(**??**) (which looks like broken here) so there might be* $\mathfrak{anomaly}$. *The vanishing of the anomaly leads to the critical string (above).*

1.c. *the Hilbert space is* $\mathcal{H}_{l-c} = \bigoplus_{\{i,p\}} (\prod_i \alpha^{j_i}_{m_i})|p^+, p^\perp\rangle, \quad m_i < 0$

1.d. *the masses of states we read from the zeroth component of (**??**). Normal ordering is important. We denote*

$$N = \sum_{n=1}^{\infty} \alpha^\perp_{-n}\alpha^\perp_n, \qquad \sum_{n=1}^{\infty} n = -\frac{1}{12}, \quad \alpha_0 = 2p \tag{1.32}$$

$$2p^+ 2p^- = 2N + (a^\perp_0)^2 - \frac{(d-2)}{12} \to M^2 = \tfrac{1}{2}\,(N - \frac{(d-2)}{24}) \tag{1.33}$$

c . *As above*

1.a. $X^+ = x^+_0 + 2p^+\tau$ *is conformally equivalent to the most general solution* ($p^+ > 0$).

1.b. *The above choice determine* X^- *by solving* $T_{++} = T_{--} = 0$ *to give* $p^+\partial_+ X^- = (\partial_+ X^\perp)^2, \quad p^+\partial_- X^- = (\partial_- X^\perp)^2$

1.c. *there is two constraints left over because the above constraint have the same l.h.s for the zeroth Fourier component:* $N = \tilde{N}$ *and mass shell* $M^2 = 2N - \frac{(d-2)}{12}$. *The Hilbert space is*

$$\mathcal{H}_{l-c} = \bigoplus_{\{i,j,p\}} (\prod_i \tilde{\alpha}_{n_i}^{k_i} \prod_j \alpha_{m_j}^{k_j}) |p^+, p^\perp\rangle, \quad m_i < 0 \tag{1.34}$$

1.d. *for* $a = 1, c = 26$ *we have* $\mathcal{H}_{OCQ} = \mathcal{H}_{l-c}$.

B) NORMAL ORDERING CONSTANTS a *Contribution from a single field to $-a$ is*

$$-\Delta a = \pm\tfrac{1}{2}\sum_{m=0}^{\infty}(m+\alpha) = \pm\tfrac{1}{2}\,\zeta(-1,\alpha) = -(\pm)(\alpha^2-\alpha+\tfrac{1}{6})/4, \quad \begin{cases} + \textit{ bosons} \\ - \textit{ fermions} \end{cases} \tag{1.35}$$

where $\alpha > 0$ depends on b.c..

For SST ψ's follow the pattern of X (mod 1/2) due to superconformal symmetry (s-conf. parameter ϵ can be periodic (R sector) or anti-per. (NS sector). CLARIFY.←(?) which 2 directions do not enter a ?

- *ψ in **R sector** has the same α as bosons, thus total $a = 0$, number of massless states depends on number of ψ_0's. In extreme case there can be no ψ_0's and only $|0\rangle_R$ massless, which moreover can be projectd out by GSO.*
- *ψ **in NS sector** has α shifterd by $\frac{1}{2}$*
  For D-D, N-N b.c.
  - *bosons have $\alpha = 0 \to -\Delta a = -\frac{1}{24}$,fermions have $\alpha = \frac{1}{2} \to -\Delta a = -\frac{1}{48}$*
  - *sum=$-\Delta a = -\frac{1}{16}$*

  For N-D b.c.
  - *bosons have $\alpha = \frac{1}{2} \to -\Delta a = \frac{1}{48}$ fermions have $\alpha = 0 \to -\Delta a = \frac{1}{24}$. In this case the ψ_0's appears !!!!!*
  - *sum=$-\Delta a = \frac{1}{16}$*
- *In the case # of N-D=4 (preserving Susy) one has (we removed 2 D-D directions from a, why ←(?))*

$$-a_R = -a_{NS} = 0 \tag{1.36}$$

  there appears massless spinor reps in NS sector !!!!! - contributions from 4 NN or DD cancels that from 4 ND (in NS sector).
- *For **two D-branes rotated** by θ in X^i, X^{i+1} ($Z^i = X^i + iX^{i+1}$) we got $-\Delta a = 2(\pm)\frac{1}{2}\sum_{m=0}^{\infty}(m+\alpha)$ with $\alpha = \theta/\pi$.[3] For one complex boson and fermion in NS (as above)*

$$-\Delta a = \sum_{m=0}^{\infty}(m+\alpha) - \sum_{m=0}^{\infty}(m+\alpha+\tfrac{1}{2}) = \zeta(-1,\alpha) - \zeta(-1,\alpha+\tfrac{1}{2}) = -\tfrac{1}{8} + \tfrac{1}{2}\alpha \tag{1.37}$$

Generalized Riemann function or Hutwitz function

$\zeta(-1) = -\frac{1}{12}$, $\zeta(0) = -\frac{1}{2}$

$$\frac{1}{2}\sum_{n=1}^{\infty} n = \frac{1}{2}\,\zeta(-1) = -\frac{1}{24}, \quad \frac{1}{2}\sum_{r=0}^{\infty}(r+\tfrac{1}{2})^{-s} = -\tfrac{1}{2}\,(2^s-1)\zeta(s)|_{s=-1} = \frac{1}{48} \tag{1.38}$$

[3] see arXiv:hep-th/9606139v2 and 0606001 for more detailed presentation

$\zeta(s, \frac{1}{2}) \equiv \sum_{r=0}^{\infty}(r + \frac{1}{2})^{-s} = 2^s \sum_{r=0}^{\infty}(2r + 1)^{-s} = 2^s(\sum_{r=0}^{\infty}(r + 1)^{-s} - \sum_{r=1}^{\infty}(2r)^{-s}) = (2^s - 1)\zeta(s)$.

$\sum_{r=0}^{\infty}(r + \alpha)^{-s} = \zeta(s, \alpha)$ *is so-called Generalized Riemann function or Hutwitz function.* $\zeta(-1, \alpha) = -(\alpha^2 - \alpha + \frac{1}{6})/2$.

C) UNORIENTED STRINGS *We can get unoriented strings requiring that states are invariant under* Ω *(below). This reduces number of states.*
Ⓒ *:* $\sigma \to 2\pi - \sigma, \Rightarrow \Omega : \alpha_n \to \tilde{\alpha}_n$
$\boxed{\sim}$ *:* $\sigma \to \pi - \sigma, \Rightarrow \Omega : \alpha_n \to (-1)^n \alpha_n$ *Also the only consistent (with the scattering amplitudes) choice is invariance of tachyon under the reflection* $\Omega|0, p\rangle = 0$. *For unorineted strings one needs:* $\Omega|0, p\rangle = |0, p\rangle$