Thermodynamics of the Early Universe

- Fundamental Interactions
- Interaction Rates Γ_i
- Rudiments of Equilibrium Thermodynamics
- Entropy

Fundamental Interactions – the Standard Model

Introduction to the Standard Model: Experimental constraints

♠ Gauge symmetry: $SU(3)_C \times SU(2)_L \times U(1)_Y$

$$\mathcal{L} \supset -\frac{1}{4}\underbrace{F_a^{\mu\nu}F_{a\,\mu\nu}}_{SU(3)_C} - \frac{1}{4}\underbrace{W_i^{\mu\nu}W_{i\,\mu\nu}}_{SU(2)_L} - \frac{1}{4}\underbrace{B^{\mu\nu}B_{\mu\nu}}_{U(1)_Y}$$

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$G_a^\mu|_{a=1,\dots 8} \qquad\qquad W_\mu^\pm, Z_\mu, A_\mu$$

♠ The Higgs sector:

- The minimal choice $H = \begin{pmatrix} G^+ \\ (h+iG^0)/\sqrt{2} \end{pmatrix}$ necessary for $SU(2)_L \times U(1)_Y \to U(1)_{EM}$.

$$\mathcal{L} \supset (D_\mu H)^\dagger D^\mu H - V(H)$$

for $D_\mu \equiv \partial_\mu + igW_\mu^i T^i + ig'\frac{1}{2}YB_\mu$ and $V(H) = \mu^2|H|^2 + \lambda|H|^4 \quad \text{with} \quad Y_H = \frac{1}{2}$

- If $\mu^2 < 0$ then $\langle 0||H|^2|0\rangle = -\frac{1}{2}\frac{\mu^2}{\lambda} \equiv \frac{v^2}{2}$ (spontaneous symmetry breaking, the origin of mass)

- Boson masses: $m_h = \sqrt{2\lambda}v$, $m_{W^\pm} = \frac{1}{2}gv$ and $m_Z = m_W/c_W$, for $c_W \equiv \cos\theta_W = g/(g^2 + g'^{\,2})^{1/2}$

♠ Fermions

fermion	T	T_3	$\frac{1}{2}Y$	Q
$\nu_{i\,L}$	$\frac{1}{2}$	$+\frac{1}{2}$	$-\frac{1}{2}$	0
$l_{i\,L}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	-1
$u_{i\,L}$	$\frac{1}{2}$	$+\frac{1}{2}$	$\frac{1}{6}$	$\frac{2}{3}$
$d_{i\,L}$	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{6}$	$-\frac{1}{3}$
$l_{i\,R}$	0	0	-1	-1
$u_{i\,R}$	0	0	$\frac{2}{3}$	$\frac{2}{3}$
$d_{i\,R}$	0	0	$-\frac{1}{3}$	$-\frac{1}{3}$
$\nu_{i\,R}$	0	0	0	0

$i = 1, \ldots, N_f = 3$, $\psi_{L,R} \equiv \frac{1}{2}(1 \mp \gamma_5)\psi$ (parity violation), $Q = T_3 + \frac{1}{2}Y$
Neutrino masses:

- Dirac mass: $f_{ij}\, \bar{L}_{i\,L}\nu_{j\,R}\, \tilde{H} +$ H.c. for $\tilde{H} \equiv i\tau_2 H^\star$

- Majorana mass: $\frac{1}{2}M_{ij}\, \nu_{i\,R}\, \mathbf{C}\, \nu_{j\,R} +$ H.c.

Gauge transformations: $\psi(x) \to \exp\left\{-igT^i\theta_i(x) - ig'\frac{1}{2}Y\beta(x)\right\}\psi(x)$
Gauge interactions:

$$\mathcal{L} \supset \sum_{\psi} \bar{\psi} i\gamma^{\mu} D_{\mu}\psi \quad \text{for} \quad D_{\mu} \equiv \partial_{\mu} + igW_{\mu}^{i}T^{i} + ig'\frac{1}{2}YB_{\mu}$$

Yukawa interactions:

$$\mathcal{L} \supset -\sum_{i,j=1}^{3}\left(\tilde{\Gamma}_{ij}\bar{u}_{i\,R}\tilde{H}^{\dagger}Q_{j\,L} + \Gamma_{ij}\bar{d}_{i\,R}H^{\dagger}Q_{j\,L} + \text{H.c.}\right)$$

$$\Downarrow$$

$$\text{if } \langle H\rangle \neq 0 \text{ then } m_q \neq 0$$

$$\mathcal{L}_{\text{q mass}} = -\sum_{i,j=1}^{3}\left(\bar{u}_{i\,R}\mathcal{M}_{ij}^{u}u_{j\,L} + \bar{d}_{i\,R}\mathcal{M}_{ij}^{d}d_{j\,L} + \text{H.c.}\right)$$

for

$$\mathcal{M}_{ij}^{u} = \frac{v}{\sqrt{2}}\tilde{\Gamma}_{ij} \quad \mathcal{M}_{ij}^{d} = \frac{v}{\sqrt{2}}\Gamma_{ij} \quad \Rightarrow \quad \text{no FCNC for one Higgs boson doublet}$$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{L,R} = U_{L,R} \begin{pmatrix} u \\ c \\ t \end{pmatrix}_{L,R} \qquad \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_{L,R} = D_{L,R} \begin{pmatrix} d \\ s \\ b \end{pmatrix}_{L,R}$$

$$U_R^\dagger \mathcal{M}^u U_L = \mathrm{diag}(m_u, m_c, m_t) \qquad D_R^\dagger \mathcal{M}^d D_L = \mathrm{diag}(m_d, m_s, m_b)$$

$$\Downarrow$$

$$\tilde{\Gamma}, \Gamma \quad \text{diagonal} \quad (g_f = \sqrt{2}\frac{m_f}{v}) \quad \Rightarrow \quad \text{no FCNC}$$

- charged currents: $\sum \bar{u}_{i\,L} \gamma^\mu d_{i\,L} = (\bar{u}, \bar{c}, \bar{t})_L \underbrace{U_L^\dagger D_L}_{U_{CKM}} \gamma^\mu \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L$

- neutral currents: $\sum \bar{u}_{i\,L} \gamma^\mu u_{i\,L}$, $\sum \bar{d}_{i\,L} \gamma^\mu d_{i\,L}$ remain unchanged upon $U_{L,R}$, $D_{L,R}$ transformations

U_{CKM}:

- unitary complex $N \times N$ matrix, $q_{i\,L} \to e^{i\alpha_i} q_{i\,L} \;\Rightarrow\; \frac{1}{2}(N-1)(N-2)$ phases in U_{CKM}

- $N \geq 3 \quad \Rightarrow \quad$ CP violation in charged currents

♠ Masses in the SM: $\quad m_V \propto gv \quad m_h \propto \lambda^{1/2} v \quad m_f \propto g_f v$

Leptons:

$$\begin{array}{lll} m_{\nu_e} \lesssim 3 \text{ eV} & m_{\nu_\mu} \lesssim 0.2 \text{ MeV} & m_{\nu_\tau} \lesssim 18 \text{ MeV} \\ m_e = 0.5 \text{ MeV} & m_\mu = 105.5 \text{ MeV} & m_\tau = 1.78 \text{ GeV} \end{array}$$

Quarks:

$$\begin{array}{lll} m_u \simeq 2 \text{ MeV} & m_c \simeq 1.2 \text{ GeV} & m_t \simeq 174 \text{ GeV} \\ m_d = 5 \text{ MeV} & m_s = 0.1 \text{ GeV} & m_b = 4.3 \text{ GeV} \end{array}$$

Bosons:

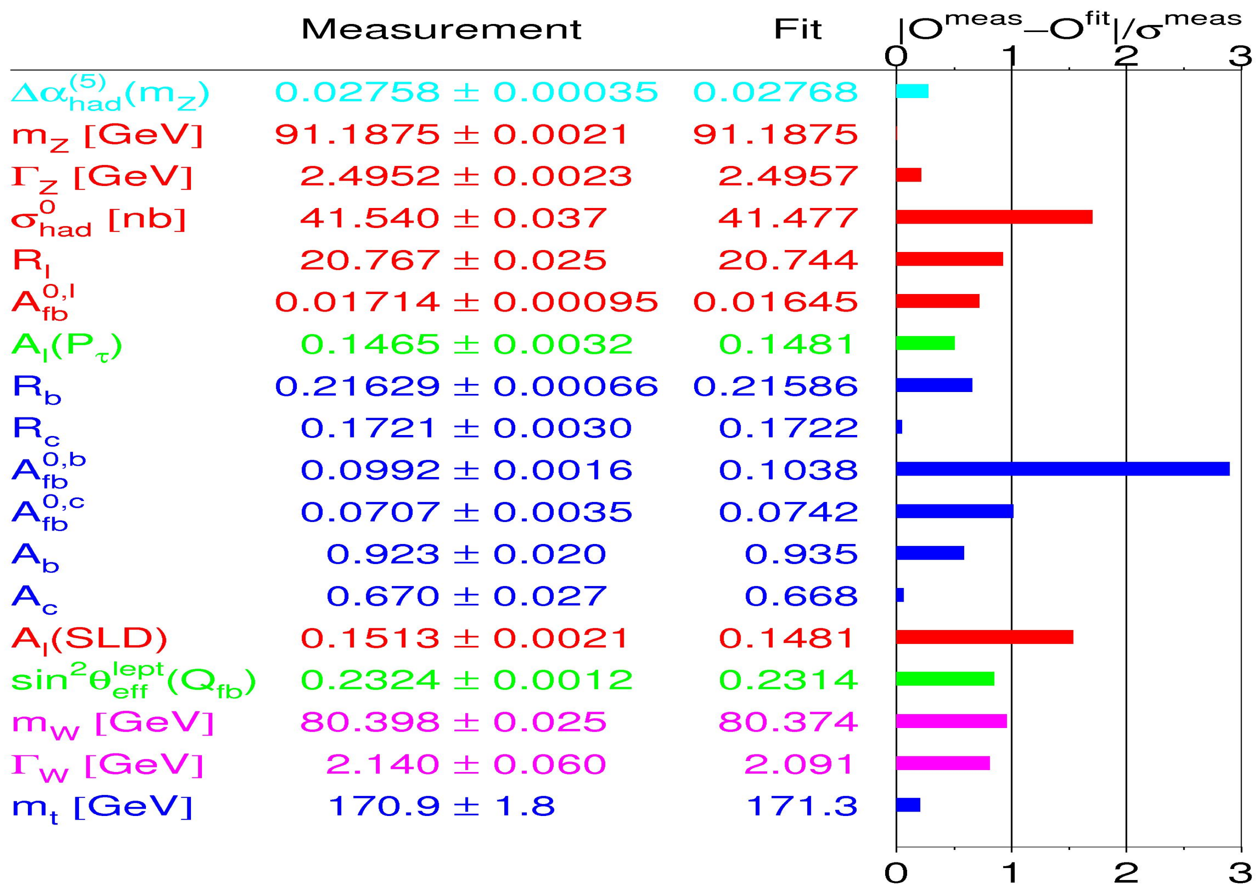
$$m_{W^\pm} = 80.4 \text{ GeV} \quad m_Z = 91.2 \text{ GeV} \quad m_h \geq 115 \text{ GeV}$$

$$\Downarrow$$

Fine tuning:

$$\frac{m_{\nu_e}}{m_t} \lesssim 0.5 \cdot 10^{-11} \quad \Rightarrow \quad \frac{g_{\nu_e}}{g_t} \lesssim 0.5 \cdot 10^{-11}$$

- Perfect agreement with the existing data



- The scalar sector weakly constrained
  - Higgs-boson representation:

$$\rho \equiv \frac{m_W^2}{m_Z^2 \cos^2\theta_W}, \qquad \text{SM} \qquad \Rightarrow \qquad \rho = 1 + \mathcal{O}(\alpha)$$

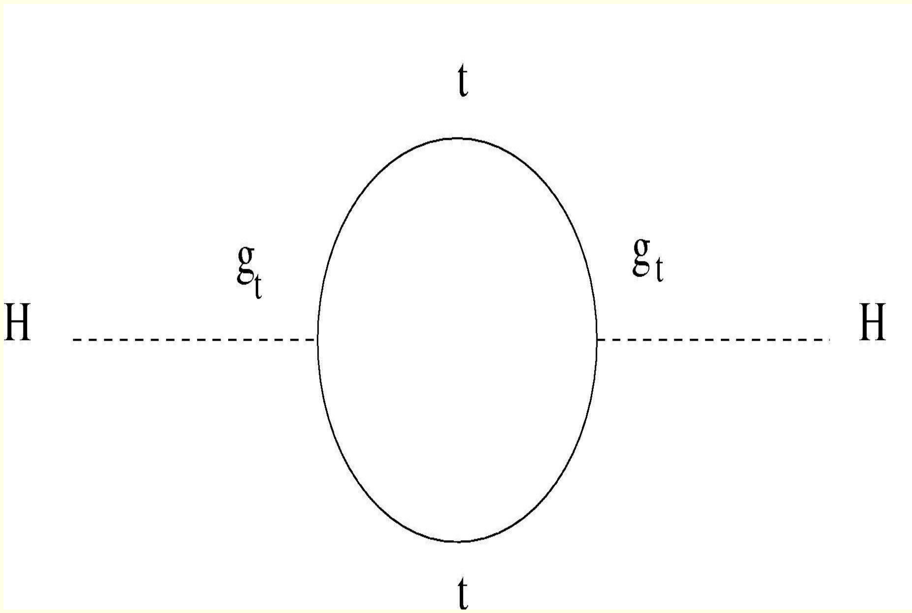
    for general Higgs multiplets: $\rho = \frac{\sum_i \left[T_i(T_i+1) - T_{i\,3}^2\right] v_i^2}{\sum_i 2T_{i\,3}^2 v_i^2}$

    data: $\rho = 1.0002 \begin{cases} +0.0024 \\ -0.0009 \end{cases} \qquad \Rightarrow \qquad T = \frac{1}{2} \qquad\qquad (\text{doublets are favored})$

  - Higgs-boson interactions: no direct tests of the scalar potential

Outstanding problems of the SM

♠ Gauge-Higgs sector:



$$\Rightarrow \qquad m_h^2 = m_h^{(\mathrm{tree})\,2} - c \cdot \Lambda^2$$

- The hierarchy problem

if $m_h \simeq 100$ GeV and $c \simeq \frac{g_t^2}{4\pi^2} \simeq \frac{1}{40}$ then

$$1 = \left(\frac{10 m_h^{(\mathrm{tree})}}{1\ \mathrm{TeV}}\right)^2 - \left(\frac{1.6\Lambda}{1\ \mathrm{TeV}}\right)^2$$

If $\Lambda \gg 1$ TeV then a fine tuned cancellation is needed to obtain $m_h \simeq 100$ GeV, e.g. for $\Lambda = M_{Pl} = 10^{19}$ GeV one has

$$1 = \left(\frac{10 m_h^{(\mathrm{tree})}}{1\ \mathrm{TeV}}\right)^2 - 2.5 \cdot 10^{30} \quad \Rightarrow \quad \text{to avoid fine tuning} \quad \Lambda \lesssim 5 m_h \lesssim 1\ \mathrm{TeV}$$

$$\Downarrow$$

The New Physics is expected at $E \simeq 1$ TeV

- Why is there only one Higgs boson?
  - The Higgs field was introduced just to make the model renormalizable (unitary)
  - There exist many fermions and vector bosons, so why only one scalar? Why, for instance, not a dedicated scalar for each fermion?

- The strong CP problem:
  - symmetries of the SM allow for

$$\mathrm{tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right) \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}\mathrm{tr}\left(F_{\mu\nu}F_{\alpha\beta}\right) \xrightarrow{P} -\mathrm{tr}\left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)$$

  - odd under CP

$$\mathcal{L}_\theta = \theta\frac{g_s^2}{32\pi^2}F^{a\,\mu\nu}\tilde{F}^a_{\mu\nu} \quad \Rightarrow \quad \mathrm{neutron-EDM} \quad D_n \simeq 2.7\cdot 10^{-16}\theta \text{ e cm}$$

$$\Downarrow$$

$$\text{data: } D_n \lesssim 1.1\cdot 10^{-25} \text{ e cm} \quad \Rightarrow \quad \theta \lesssim 3\cdot 10^{-10}$$

The strong CP problem: why is θ so small?

♠ The flavor sector:

- parity violation:

$$W^{+\,\mu}\,\bar{u}_i\gamma_\mu(1-\gamma_5)d_j \quad \xrightarrow{P} \quad W^{+\,\mu}\,\bar{u}_i\gamma_\mu(1+\gamma_5)d_j$$

Maximal parity violation, why?

- Charge quantization, why $q_u = \frac{2}{3}$, $q_d = -\frac{1}{3}$ and $q_l = -1$?

- Number of generations, why $N = 3$?

- Why is the top quark so heavy ($m_t \simeq 174$ GeV while $m_b \simeq 4.3$ GeV) ?

$$m_t \simeq v = \langle 0|H|0\rangle \simeq 246 \text{ GeV}$$
$$\Downarrow$$

top quark is very different (possibly sensitive to the spontaneous symmetry breaking)

- Mixing angles and fermion masses:

$$\mathcal{L} \supset -\sum_{i,j=1}^{3} \left(\tilde{\Gamma}_{ij}\bar{u}_{i\,R}\tilde{H}^{\dagger}Q_{j\,L} + \Gamma_{ij}\bar{d}_{i\,R}H^{\dagger}Q_{j\,L} + \text{H.c.}\right)$$

$$\Downarrow$$

$$\mathcal{L}_{\rm q\ mass} = -\sum_{i,j=1}^{3} \left(\bar{u}_{i\,R}\mathcal{M}^{u}_{ij}u_{j\,L} + \bar{d}_{i\,R}\mathcal{M}^{d}_{ij}d_{j\,L} + \text{H.c.}\right) \quad \text{for} \quad \mathcal{M}^{u}_{ij} = \frac{v}{\sqrt{2}}\tilde{\Gamma}_{ij},\ \mathcal{M}^{d}_{ij} = $$

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}_{L,R} = U_{L,R}\begin{pmatrix} u \\ c \\ t \end{pmatrix}_{L,R} \qquad \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}_{L,R} = D_{L,R}\begin{pmatrix} d \\ s \\ b \end{pmatrix}_{L,R}$$

$$U_R^{\dagger}\mathcal{M}^{u}U_L = \mathrm{diag}(m_u, m_c, m_t) \qquad D_R^{\dagger}\mathcal{M}^{d}D_L = \mathrm{diag}(m_d, m_s, m_b)$$

$$\Downarrow$$

$$\sum \bar{u}_{i\,L}\gamma^{\mu}d_{i\,L} = (\bar{u}, \bar{c}, \bar{t})_L \underbrace{U_L^{\dagger}D_L}_{U_{CKM}}\gamma^{\mu}\begin{pmatrix} d \\ s \\ b \end{pmatrix}_L$$

It is natural to expect that $U_{CKM} = U_{CKM}(m_q/m'_q)$.

♠ Parameters of the SM:

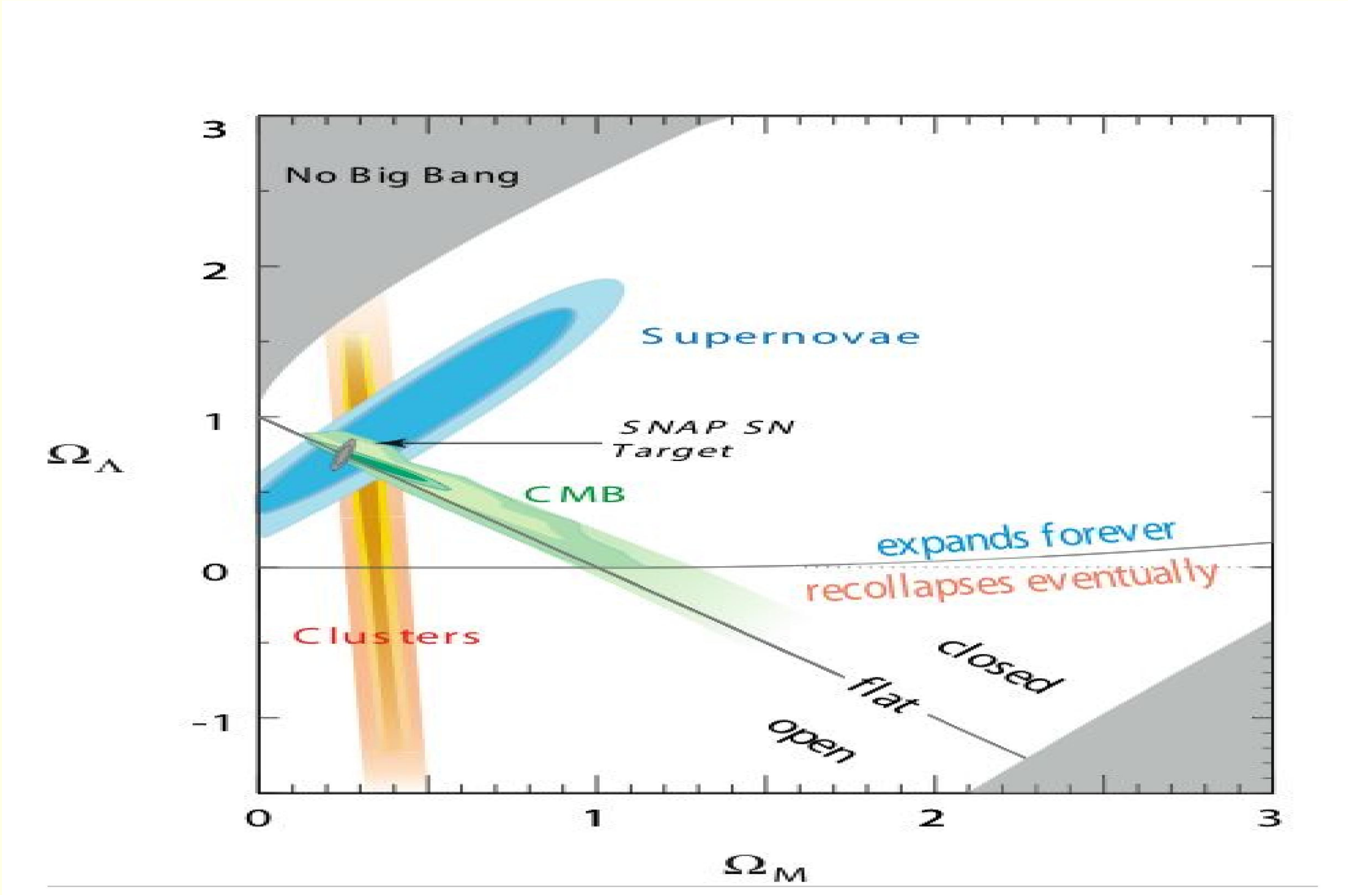
$$\begin{array}{cccccc} m_e & m_\mu & m_\tau & m_u & m_c & m_t \\ m_{\nu_e} & m_{\nu_\mu} & m_{\nu_\tau} & m_d & m_s & m_b \end{array}$$

$$\underbrace{g \quad , \quad g'}_{(\alpha_{QED}, \sin\theta_W)} \;,\; \underbrace{g_s}_{(\alpha_{QCD})} \;,\; \underbrace{m_h, \;\lambda}_{(\mu, \lambda)} \;,\; \underbrace{U_{CKM}}_{\theta_{1,2,3}, \delta_{CP}}$$

$$21 \quad \text{parameters !}$$

♠ Cosmology:

- Dark matter and dark energy



$$\Omega_i \equiv \frac{\rho_i}{\rho_c} \quad \text{for} \quad \rho_c = \frac{3H_0^2}{8\pi G_N} = 11 h^2 m_p/m^3 \quad \text{for} \quad h \simeq 0.7$$

data $\Rightarrow$ $\Omega_\Lambda = \frac{\Lambda}{3H_0^2} \simeq 70\%$, $\Omega_{DM} \simeq 27\%$ and $\Omega_B \simeq 3\%$

- SM has no candidate for dark matter

- $\Omega_\Lambda = \frac{\rho_\Lambda}{\rho_c} \simeq 0.7 \quad \Rightarrow \quad \rho_\Lambda \simeq 10^{-120} M_{Pl}^4 = (10^{-3}\text{ eV})^4$ while typical scale of the SM is $\mathcal{O}(100\text{ GeV})$! Fine tuning again!

- Inflation: period of fast expansion of the very early Universe, $R(t) \simeq \exp\left(\sqrt{\frac{\Lambda}{3}}\, t\right)$
  Again the SM has no means to explain the inflation (no inflaton in the SM). For a typical inflaton $m_\varphi \sim 10^{13}$ GeV and $\lambda \sim 10^{-13}$, so the SM Higgs boson is not an inflaton.

- Baryogenesis and SM CP violation $\eta \equiv \frac{n_b - n_{\bar{b}}}{n_\gamma} \simeq \frac{n_b}{n_\gamma} \simeq 6 \cdot 10^{-10}$
  The Sakharov's necessary conditions for baryogenesis:
  - B number violation
  - C and CP violation
  - Departure from thermal equilibrium

  SM:

  - B number violation: **OK**
  - C and CP violation: too weak CP violation $\propto \mathbf{Im}Q$, for $Q \equiv U_{ud} U_{cb} U_{ub}^\star U_{cd}^\star$ (re-phasing invariant)

– Departure from thermal equilibrium: no electroweak phase transition for $m_h \gtrsim 73$ GeV

Conclusion: the SM doesn't explain the baryogenesis

- Why is gravity so weak? Or, why $M_{Pl} = 10^{19}$ GeV $\gg v = 246$ GeV?

♠ Extra Higgs bosons

- SM single Higgs doublets quite unnatural, why only one?
- extra sources of CP violation from the scalar sector (needed for baryogenesis)
- hope for an explanation of weak mixing angles through horizontal symmetries

$$\mathcal{L} \supset -\sum_{\alpha=1}^{N_H} \sum_{i,j=1}^{3} \left(\tilde{\Gamma}^{\alpha}_{ij} \bar{u}_{i\,R} \tilde{H}^{\alpha\,\dagger} Q_{j\,L} + \Gamma^{\alpha}_{ij} \bar{d}_{i\,R} H^{\alpha\,\dagger} Q_{j\,L} + \text{H.c.} \right)$$

$$H^{\alpha} \to \mathcal{H}^{\alpha}_{\beta} H^{\beta}\,, \quad u_{i\,R} \to \mathcal{U}^{j}_{i} u_{j\,R}\,, \quad d_{i\,R} \to \mathcal{D}^{j}_{i} u_{j\,R}\,, \quad Q_{i\,L} \to \mathcal{Q}^{j}_{i} Q_{j\,L}$$

$$\Downarrow$$

constraints on fermion mass-matrices:

$$\mathcal{M}^{u}_{ij} = \sum_{\alpha=1}^{N_H} \tilde{\Gamma}^{\alpha}_{ij} \frac{v_\alpha}{\sqrt{2}}, \quad \mathcal{M}^{d}_{ij} = \sum_{\alpha=1}^{N_H} \Gamma^{\alpha}_{ij} \frac{v_\alpha}{\sqrt{2}}$$

$$U^{\dagger}_{R} \mathcal{M}^{u} U_{L} = \mathrm{diag}(m_u, m_c, m_t) \quad D^{\dagger}_{R} \mathcal{M}^{d} D_{L} = \mathrm{diag}(m_d, m_s, m_b)$$

If $\mathcal{M}^{u,d}$ sufficiently constraints, then $U_{CKM} \equiv U_L^\dagger D_L = U_{CKM}(m_q/m_{q'})$

- Multi-doublet models favored by the ρ measurement

- An example of extra Higgs boson scenario: the 2 Higgs Doublet Model:

$$\begin{aligned}
V(\phi_1,\phi_2) \;\; = \;\; & m_1^2|\phi_1|^2 + m_2^2|\phi_2|^2 + m_3^2(e^{i\delta_3}\phi_1^\dagger\phi_2 + e^{-i\delta_3}\phi_2^\dagger\phi_1) + \\
& +\lambda_1(\phi_1^\dagger\phi_1)^2 + \lambda_2(\phi_2^\dagger\phi_2)^2 + \lambda_3(\phi_1^\dagger\phi_1)(\phi_2^\dagger\phi_2) + \\
& +\lambda_4(\phi_1^\dagger\phi_2)(\phi_2^\dagger\phi_1) + \lambda_5\left[e^{i\delta_5}(\phi_1^\dagger\phi_2)^2 + \text{H.c.}\right] + \\
& +\lambda_6(\phi_1^\dagger\phi_1)\left[e^{i\delta_6}\phi_1^\dagger\phi_2 + \text{H.c.}\right] + \lambda_7(\phi_2^\dagger\phi_2)\left[e^{i\delta_7}\phi_1^\dagger\phi_2 + \text{H.c.}\right]
\end{aligned}$$

where m_i^2 and δ_i real

$$\text{under CP: } \phi_i(t,\vec{x}) \quad \xrightarrow{CP} \quad e^{i\alpha_i}\phi_i^\star(t,-\vec{x}) \quad \text{for} \quad i=1,2$$

- explicit CP violation: $\delta_i \neq 0$

$$\phi_1^\dagger\phi_2 \quad \xrightarrow{CP} \quad e^{i(\alpha_2-\alpha_1)}\phi_2^\dagger\phi_1$$

- spontaneous CP violation ($\delta_i = 0$)

$$\langle \phi_1 \rangle = \begin{pmatrix} 0 \\ \frac{v_1}{\sqrt{2}} \end{pmatrix} \qquad \langle \phi_2 \rangle = \begin{pmatrix} 0 \\ \frac{v_1 e^{i\theta}}{\sqrt{2}} \end{pmatrix}$$

$\cos\theta = \frac{2m_3^2 - \lambda_6 v_1^2 - \lambda_7 v_2^2}{4\lambda_5 v_1 v_2}$ $\quad\Rightarrow\quad$ SCPV if $\theta \neq 0, \pi$

- Difficulties of extra-Higgs-boson scenarios:
  - many new parameters $(m_i^2, \lambda_i, \delta_i)$
  - tree-level FCNC to be suppressed

$$\mathcal{M}_{ij}^u = \sum_{\alpha=1}^{N_H} \tilde{\Gamma}_{ij}^\alpha \frac{v_\alpha}{\sqrt{2}}, \qquad \mathcal{M}_{ij}^d = \sum_{\alpha=1}^{N_H} \Gamma_{ij}^\alpha \frac{v_\alpha}{\sqrt{2}}$$

♠ Extra gauge symmetries

- GUTs, e.g. $SU(5)$: unification of gauge couplings, . . .
- $L-R$ symmetry, $SU(2)_L \times SU(2)_R \times U(1)$: spontaneous parity violation
- $SU(2)_L \times U(1) \times U(1)'$: just extra Z'

♠ Extra dimensions (more special dimensions)

Motivations:

- Unification of gravity and gauge interactions in g_{AB} (Kaluza & Klein)
- Quantization of gravity (strings)
- Solution (amelioration) of the hierarchy problem

The interaction rates Γ_i

♠ Definition of the Cross-Section:
The transition matrix element $w_{i\to f}$ gives the probability for the transition to occur:

$$P_{i\to f} = |w_{i\to f}|^2 = |\langle f|i\rangle|^2$$

The translational invariance allows to write the matrix element as

$$w_{i\to f} = \delta_{if} + i(2\pi)^4\delta^4(p_f - p_i)T_{i\to f}$$

The above formula defines the transition matrix T.
Let's consider the following scattering process

$$a + b \to c_1 + c_2 + \cdots + c_n$$

We assume that b is at rest, and the velocity of a is $v = |\vec{p}_a|/E_a$. The number of particles b per target volume is (that defines the normalization of plane waves): $2E_b = 2m_b$ as b is at rest. The incident flux is the velocity of a times their number density $2E_a$, so $2|\vec{p}_a|$. If the reaction volume is V and the reaction takes place during the time T, then *the cross-section σ is defined* such that the transition probability per unit time and unit volume equals the target density $\times$ the incident flux $\times$ the cross-section σ, that is, $2m_b \times 2|\vec{p}_a| \times \sigma$. On the other hand it is equal to $|w_{i\to f}|^2/(VT)$. Hence summing over all available momenta for the final state we get

$$\begin{aligned}\sigma(a+b &\rightarrow c_1 + c_2 + \cdots + c_n) = \\ &= \frac{1}{4m_b|\vec{p}_a|}\int \prod_{j=1}^{n} \frac{d^3p_j}{2E_j(2\pi)^3}(2\pi)^4\delta^4(p_a+p_b-p_1-\cdots-p_n)|\tilde{T}|^2\end{aligned}$$

where for unpolarized initial state we have

$$|\tilde{T}|^2 = \frac{1}{S}\frac{1}{(2s_a+1)(2s_b+1)}\sum_{\text{final spins}}|T_{i\to f}|^2$$

The spins of initial states are denoted by s_a and s_b. The symmetry factor S appears because in quantum mechanics we can't distinguish between two final states which differ only by the exchange of identical particles, in general, if there are k groups of n_i $(i=1,2,\ldots,k)$ identical particles in the final state, one has $S = n_1!n_2!\ldots n_k!$. In order to have the cross-section in a Lorentz invariant form one has to replace

$$m_b|\vec{p}_a| \rightarrow \left[(p_a\cdot p_b)^2 - m_a^2m_b^2\right]^{1/2}$$

For decays

$$a \to c_1 + c_2 + \cdots + c_n$$

we get instead of the cross-section the decay width

$$\begin{aligned} \Gamma(a &\to c_1 + c_2 + \cdots + c_n) = \\ &= \frac{1}{4m_a} \int \prod_{j=1}^{n} \frac{d^3 p_j}{2E_j (2\pi)^3} (2\pi)^4 \delta^4 (p_a - p_1 - \cdots - p_n) |\tilde{T}|^2 \end{aligned}$$

for

$$|\tilde{T}|^2 = \frac{1}{S} \frac{1}{(2s_a + 1)} \sum_{\text{final spins}} |T_{i \to f}|^2$$

Summing over all final states we get the total width

$$\Gamma_{\text{tot}} = \sum_{\text{final states f}} \Gamma(a \to f)$$

Then the life time is given by

$$\tau = \frac{1}{\Gamma_{\text{tot}}}$$

while the branching ratio reads

$$BR(a \to f) = \frac{\Gamma(a \to f)}{\Gamma_{\text{tot}}(a)}$$

♠ Strong and Electroweak Transitions:
Estimates of cross-sections:

- $$\sigma_{\text{em}}(e^+e^- \to \mu^+\mu^-) \sim \left(\frac{e^2}{4\pi}\right)^2 \frac{1}{s} \quad \text{for} \quad s \equiv (p_{e^+} + p_{e^-})^2 \gg m_e^2$$

  where $\frac{e^2}{4\pi} \equiv \alpha_{\text{QED}} \simeq \frac{1}{128}$, for $\sqrt{s} \simeq 100$ GeV.

- $$\sigma_{\text{strong}}(q\bar{q} \to q\bar{q}) \sim \left(\frac{g_{\text{QCD}}^2}{4\pi}\right)^2 \frac{1}{s} \quad \text{for} \quad s \gg m_q^2$$

  where $\frac{g_{\text{QCD}}^2}{4\pi} \equiv \alpha_{\text{QCD}} \simeq 10^{-1}$,

- $$\sigma_{\text{weak}}(\nu_e + e^+ \to \nu_\mu + \mu^+) \sim \left(\frac{g_{\text{weak}}^2}{4\pi}\right)^2 \frac{s}{(s - m_W^2)^2}$$

where $\frac{g_{\rm weak}^2}{4\pi} = \frac{e^2}{4\pi \sin^2\theta_W} = \frac{\alpha_{\rm QED}}{0.23}$

♠ The Interaction Rate:

If interactions between species are fast enough they could be in local equilibrium (state of maximal entropy). The reaction rate responsible for establishing equilibrium can be characterized by the *collision time*:

$$t_c \equiv \frac{1}{\sigma n v}$$

where σ is the cross-section, n is the number density of the target particles and v is their relative velocity. In order to maintain the equilibrium this time must be much shorter than the Universe age $t_H \sim H^{-1}$:

$$t_c \ll t_H \tag{1}$$

Then the local equilibrium is reached before the expansion becomes relevant.
Let's consider $T \gtrsim 500$ GeV, then the cross-section for strong and electroweak interactions could be estimated applying just dimensional analysis for typical energy-

momentum $p \sim T$ (masses are irrelevant at that energy)

$$\sigma \sim \frac{\alpha^2}{T^2}$$

where α is the fine structure constant for strong or electroweak interactions $\alpha \simeq 10^{-1} - 10^{-2}$. Taking into account that the number density of relativistic species behaves (see next section for details) as $n \sim R^{-3} \sim T^3$ we obtain

$$t_c \sim \frac{1}{\alpha^2 T}$$

If the Universe is dominated by relativistic species then we have (see next section for details)

$$t_H \sim \frac{1}{H} \sim \frac{1}{(\rho_{\rm rad}/M_{Pl}^2)^{1/2}} \sim \frac{M_{Pl}}{T^2}$$

Hence we can see that the collision (reaction) time t_c decreases slower than the Hubble time t_H, so if T is too large then (1) can not be satisfied. Note that since $\rho_{\rm rad} \sim T^4$ during the radiation dominated epoch we have $H \sim T^2/M_{Pl}$ (see next section for details). Therefore at temperatures $T \sim \alpha^2 M_{Pl} \simeq 10^{15} - 10^{17}$ GeV, we obtain $t_c \simeq t_H$. So for $T \lesssim 10^{15} - 10^{17}$ GeV but above few hundred GeV (where

$\sigma \sim \frac{\alpha^2}{T^2}$) the inequality (1) is satisfied and the Universe made of quarks, leptons, gauge bosons and Higgses remain in equilibrium. Above 10^{17} GeV the interaction that we know are too slow too keep the universe in equilibrium. Below 100 GeV, the masses of gauge bosons $W^{\pm}$ ($m_W = 80.403 \pm 0.029$ GeV) and Z ($m_Z = 91.1876 \pm 0.0021$ GeV) become relevant and the cross-sections scale as $\alpha_{\rm weak}^2 T^2/m_W^4$, so

$$t_c \sim \frac{1}{\alpha_{\rm weak}^2}\left(\frac{m_W}{T}\right)^4 \frac{1}{T}$$

for $T \lesssim 100$ GeV. Therefore the interactions become too slow to maintain the equilibrium, as a consequence, e.g. neutrinos decouple at $T \simeq 1$ MeV (more on that later).

Rudiments of Equilibrium Thermodynamics

Assumptions

- The Universe is a dilute and weakly interacting gas.
- If rates of interactions between constituents of the Universe are large enough, then we assume the Universe is in *local equilibrium* (so the state of maximal entropy, see Mukhanov for detailed discussion).

Then the number density n_i, the energy density ρ_i, and the pressure for particles with g_i internal degrees of freedom (massless gauge boson has g=2, massive gauge boson has g=3, massless fermion has $g = 1$, massive fermion has $g = 2$, the same for anti-fermions) is given by the following integrals of the expected number density of particles in states with energy E_i (phase space distribution or occupancy functions) $f_i(\vec{p}, T)$:

$$n_i(T) = g_i \int f_i(\vec{p}, T) \frac{d^3p}{(2\pi)^3} \tag{2}$$

$$\rho_i(T) = g_i \int E_i(\vec{p}) f_i(\vec{p}, T) \frac{d^3p}{(2\pi)^3} \quad \text{for} \quad E_i(\vec{p}) = (|\vec{p}|^2 + m_i^2)^{1/2} \tag{3}$$

$$p_i(T) = g_i \int \frac{|\vec{p}|^2}{3E_i(\vec{p})} f_i(\vec{p}, T) \frac{d^3p}{(2\pi)^3} \tag{4}$$

The phase space distribution (the expected number of particles in an energy state) is given by the Fermi-Dirac (for fermions, $+$ sign below) or Bose-Einstein (for bosons, $-$ sign below) distributions

$$f_i(\vec{p}, T) = \frac{1}{e^{[E_i(\vec{p}) - \mu_i]/T} \pm 1}$$

where μ_i is the chemical potential of the species, for our unit choice $k_B = 1$. It will be usually assumed that μ can be neglected in the early Universe. Performing the angular integrations and changing variables from $|\vec{p}|$ to $E = (|\vec{p}|^2 + m^2)^{1/2}$, so $|\vec{p}| d|\vec{p}| = EdE$, so that $d^3p \to 4\pi(E^2 - m^2)^{1/2} EdE$ and we obtain

$$n(T) = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp\left[E(\vec{p}) - \mu\right]/T \pm 1} EdE$$

$$\rho(T) = \frac{g}{2\pi^2} \int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp\left[E(\vec{p}) - \mu\right]/T \pm 1} E^2 dE$$

$$p(T) = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2-m^2)^{3/2}}{\exp\left[E(\vec{p})-\mu\right]/T \pm 1} dE$$

In the relativistic limit ($T \gg m$) we get

$$n(T) = \begin{cases} \frac{\zeta(3)}{\pi^2} g T^3 & \text{for bosons} \\ \frac{3}{4}\frac{\zeta(3)}{\pi^2} g T^3 & \text{for fermions} \end{cases} \qquad \rho(T) = \begin{cases} \frac{\pi^2}{30} g T^4 & \text{for bosons} \\ \frac{7}{8}\frac{\pi^2}{30} g T^4 & \text{for fermions} \end{cases} \qquad p(T) = \frac{\rho(T)}{3} \tag{5}$$

where $\zeta(3) = 1.202\ldots$ is the Riemann zeta function of 3.
In the non-relativistic limit ($T \ll m$) there is no difference between fermions and bosons

$$n(T) = g\left(\frac{mT}{2\pi}\right)^{3/2} \exp(-m/T) \qquad \rho(T) = m\, n(T) \qquad p(T) = n(T)T \ll \rho(T) \tag{6}$$

For non-relativistic species the average energy per particle reads

$$\langle E \rangle \equiv \frac{\rho}{n} = \begin{cases} \frac{\pi^4}{30\zeta(3)} T \simeq 2.701\, T & \text{for bosons} \\ \frac{7\pi^4}{180\zeta(3)} T \simeq 3.151\, T & \text{for fermions} \end{cases} \tag{7}$$

For the rhs of Friedmann equations we need the total contribution to the energy density and the pressure, that is

$$\rho_{\rm tot} = T^4 \sum_i \left(\frac{T_i}{T}\right)^4 \frac{g_i}{2\pi^2} \int_{x_i}^{\infty} \frac{(y^2 - x_i^2)^{1/2} y^2 dy}{\exp(y) \pm 1} \tag{8}$$

$$p_{\rm tot} = T^4 \sum_i \left(\frac{T_i}{T}\right)^4 \frac{g_i}{2\pi^2} \int_{x_i}^{\infty} \frac{(y^2 - x_i^2)^{3/2} y^2 dy}{\exp(y) \pm 1} \tag{9}$$

where $x_i \equiv m_i/T$ and $y = E/T$, and it has been taken into account that some species may have decoupled (remaining in equilibrium) so that they may have different temperature T_i.
Note that at a given temperature the ratio of the energy density for non-relativistic species to relativistic one reads

$$\frac{\rho_{\rm nrel}}{\rho_{\rm rel}} \propto \left(\frac{m}{T}\right)^{5/2} e^{-m/T}$$

For the species to be non-relativistic one needs $m \gg T$ so the $e^{-m/T}$ is a strong suppression factor, so that we will neglect contributions from non-relativistic species while calculating total energy density. In that case we get

$$\rho_{\rm tot}(T) = \frac{\pi^2}{30} g_\star T^4 \quad \text{and} \quad p(T) = \frac{\rho(T)}{3} = \frac{\pi^2}{90} g_\star T^4 \tag{10}$$

where $g_\star$ counts only massless ($m_i \ll T$) degrees of freedom:

$$g_\star = \sum_{\rm bosons} g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8} \sum_{\rm fermions} g_i \left(\frac{T_i}{T}\right)^4 \tag{11}$$

Note that $g_\star = g_\star(T)$ is a function of temperature. An exact form of $g_\star(T)$ could be easily obtained from (8) and (9). For $T \gg 100$ MeV $g_\star = 106\frac{3}{4}$, for $T \ll 1$ MeV $g_\star = 3.36$, while for 100 MeV $\gtrsim T \gtrsim 1$ MeV one gets $g_\star = 10\frac{3}{4}$ (see class).
During the radiation dominated epoch ($t \lesssim 4 \times 10^{10}$ s see class for this number), $\rho_{\rm tot} = \rho_{\rm rad}$ hence inserting (10) into the Friedmann equation one gets the very important formula for the physics of early Universe:

$$H = \left[\frac{8\pi G}{3} \rho_{\rm tot}(T)\right]^{1/2} = \left[\frac{8\pi G}{3} \frac{\pi^2}{30} g_\star T^4\right]^{1/2} = 1.66 \frac{g_\star^{1/2} T^2}{M_{Pl}}$$

For the radiation dominated Universe we have obtained earlier the following time dependence of the scale factor

$$R(t) \propto t^{1/2}$$

So, for the radiation domination one has

$$H \equiv \frac{\dot{R}}{R} = \frac{1}{2t}$$

Hence the time – temperature relation could be obtained

$$t = 0.30 \frac{M_{Pl}}{g_\star^{1/2} T^2} \sim \left(\frac{1 \text{ MeV}}{T}\right)^2 \text{ s}$$

The above is a useful formula to memorize as $T \simeq 1$ MeV is a very important temperature in the evolution of the early Universe.

Entropy

Let's define the entropy through its differential

$$T dS(V,T) \equiv d[\rho(T)V] + p(T)dV = d\left[(\rho + p)V\right] - V dp \tag{12}$$

In general we have

$$dS(V,T) = \frac{\partial S(V,T)}{\partial V} dV + \frac{\partial S(V,T)}{\partial T} dT$$

So, we get

$$\frac{\partial S(V,T)}{\partial V} = \frac{1}{T}[\rho(T) + p(T)] \quad \text{and} \quad \frac{\partial S(V,T)}{\partial T} = \frac{V}{T}\frac{d\rho(T)}{dT}$$

The integrability condition tells us that

$$\frac{\partial^2 S(V,T)}{\partial T \partial V} = \frac{\partial^2 S(V,T)}{\partial V \partial T} \quad \Longrightarrow \quad \frac{\partial}{\partial T}\left[\frac{1}{T}[\rho(T) + p(T)]\right] = \frac{\partial}{\partial V}\left[\frac{V}{T}\frac{d\rho(T)}{dT}\right]$$

$$\Downarrow$$

$$\frac{dp(T)}{dT} = \frac{1}{T}\left[\rho(T) + p(T)\right] \quad \Longleftrightarrow \quad dp(T) = \frac{\rho(T) + p(T)}{T} dT \qquad (13)$$

The above equation could be derived (see class) from

$$\rho(T) = \frac{g_i}{2\pi^2}\int_m^\infty \frac{(E^2 - m^2)^{1/2}}{\exp\left[E(\vec{p}) - \mu\right]/T \pm 1} E^2 dE, \; p(T) = \frac{g}{6\pi^2}\int_m^\infty \frac{(E^2 - m^2)^{3/2}}{\exp\left[E(\vec{p}) - \mu\right]/T \pm 1} dE$$

Inserting (13) into (12) we get

$$dS = \frac{1}{T} d\left[(\rho + p)V\right] - V\left[\rho(T) + p(T)\right]\frac{dT}{T^2} = d\left\{\frac{V}{T}[\rho(T) + p(T)] + \text{const.}\right\}$$

So the entropy, up to an integration constant is given by

$$S(V, T) = \frac{V}{T}[\rho(T) + p(T)]$$

Recall now the "first law of thermodynamics" (equivalently ${T^{\mu\nu}}_{;\nu} = 0$)

$$R^3 \frac{dp(T)}{dt} = \frac{d}{dt}\left\{R^3[\rho(T) + p(T)]\right\}$$

Combining with (13) we get

$$R^3 \underbrace{\frac{1}{T}\frac{dT}{dt}}_{-T\frac{d}{dt}\left(\frac{1}{T}\right)} [\rho(T) + p(T)] = \frac{d}{dt}\left\{R^3[\rho(T) + p(T)]\right\}$$

Hence

$$\frac{d}{dt}\left\{\frac{R^3}{T}[\rho(T) + p(T)]\right\} = 0$$

Therefore, identifying volume with R^3 we can conclude that the entropy in the volume V is conserved. It proves useful to define the entropy density

$$s(T) \equiv \frac{S(T)}{V} = \frac{\rho(T) + p(T)}{T}$$

Since for the relativistic particles both $\rho(T)$ and $p(T)$ are dominated by relativistic species, the same happens for the entropy density. Using (5) one gets:

$$s = \frac{2\pi^2}{45} g_{\star\,S} T^3$$

where

$$g_{\star\,S} = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3 \tag{14}$$

Since $n_\gamma \propto T^3$:

$$n_\gamma = \frac{2\zeta(3)}{\pi^2} T^3$$

therefore one can derive the following relation

$$s = \frac{\pi^4}{45\zeta(3)} g_{\star\,S} n_\gamma \simeq 1.8 g_{\star\,S} n_\gamma$$

Note that the entropy conservation implies that $g_{\star\,S} T^3 R^3 = \text{const.}$, therefore in the early Universe ($R \sim 0$) the temperature was maximal (roughly $T \propto R^{-1}$), consequently all species can be treated as highly relativistic.

Let's now illustrate the possibility of some species having different temperatures by the decoupling of neutrinos at about $T \sim 1$ MeV. For weak interactions we had

$$\sigma_{\rm weak}(e^+ + e^- \to \nu_i + \bar{\nu}_i) \sim \left(\frac{g_{\rm weak}^2}{4\pi}\right)^2 \frac{s}{(s - m_W^2)^2} \overset{s \ll m_W^2}{\simeq} \left(\frac{g_{\rm weak}^2}{4\pi}\right)^2 \frac{s}{m_W^4}$$

So, since $\langle E \rangle \sim 3T$ therefore at $T \ll m_W$ we get

$$\sigma_{\rm weak}(e^+ + e^- \to \nu_i + \bar{\nu}_i) \simeq \left(\frac{g_{\rm weak}^2}{4\pi}\right)^2 \frac{T^2}{m_W^4}$$

Since the interaction rate $\Gamma_{\rm int} \equiv t_c^{-1} = n\sigma v$ therefore we get for $n \sim T^3$ and $v = 1$

$$\Gamma_{\rm int} \simeq \frac{\alpha_{\rm weak}^2 T^5}{m_W^4} \simeq G_F^2 T^5$$

where $G_F = 1.1664 \times 10^{-5}$ GeV^{-2} is the Fermi constant. Let's compare the interaction rate with the expansion rate $H \sim g_\star^{1/2} T^2 / M_{Pl}$

$$\frac{\Gamma_{\rm int}}{H} \simeq \frac{G_F^2 T^5}{g_\star^{1/2} T^2 / M_{Pl}} \simeq \frac{G_F^2 T^5}{T^2 / M_{Pl}} \simeq \left(\frac{T}{0.7\ \text{MeV}}\right)^3$$

So, at $T \lesssim 1$ MeV the interactions are too slow to provide an equilibrium between leptons and neutrinos. Neutrinos decouple ("the freeze-out") from the SM and evolve separately, so the possibility for neutrinos to have different temperature appears. So, their energy (temperature) is being redshifted the same way as for photons

$$T_\nu = T_{\rm dec} \frac{R_{\rm dec}}{R} \sim \frac{1}{R}$$

The entropy is separately conserved for each decoupled species, so

$$g_{\star\, S} (RT)^3 = \text{const.} \qquad \Longrightarrow \qquad T \sim (g_{\star\, S})^{-1/3} \frac{1}{R}$$

Hence we can see that neutrino distribution will be *the same as if it was still in thermal equilibrium with photons as long as* $(g_{\star\, S})$ *does not change*. However around the same temperature electrons become non-relativistic $m_e \simeq 0.5$ MeV so they annihilate $e^+ e^- \to \gamma\gamma$ (the inverse process is being suppressed as the averaged energy decreases roughly below $2m_e$). So, the number of relativistic degrees of freedom (rdf) drops down:

- for $T \gtrsim 2m_e \simeq 1$ MeV:

$$g_{\star\, S} = \sum_{\text{bosons}} g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8} \sum_{\text{fermions}} g_i \left(\frac{T_i}{T}\right)^3 = 2 + \frac{7}{8} \times 4 = \frac{11}{2}$$

- for $T \ll 2m_e$:

$$g_{\star\, S} = 2$$

From continuity of the entropy we get the following condition

$$\left[g_{\star\, S}(RT)^3\right]_{\text{before}} = \left[g_{\star\, S}(RT)^3\right]_{\text{after}}$$

which implies

$$\frac{11}{2}(RT)^3_{\text{before}} = 2(RT)^3_{\text{after}} \quad \Longrightarrow \quad T_{\text{before}} = \left(\frac{4}{11}\right)^{1/3} T_{\text{after}}$$

For the temperature "before", the neutrinos even though they decoupled a bit earlier, have the same temperature at photons, however at $T \sim 2m_e$ photons are heated up by $e^+e^- \to \gamma\gamma$ as the entropy is transfered (since it is continues function of T) from e^+e^- to photons. The already decoupled neutrinos do not benefit from that reheating, since for them $(RT_\nu)^3_{\rm before} = (RT_\nu)^3_{\rm after}$ (in other words the entropy of neutrinos is conserved separately after the decoupling). Consequently there is a difference in temperatures for neutrinos and photons after e^+e^- freeze-out:

$$T_\nu = \left(\frac{4}{11}\right)^{1/3} T_\gamma$$

Strictly speaking the photon temperature does not jump at $T = 2m_e$, but rather start to decrease slower already at temperatures slightly above $T = 2m_e$ (in reality the freeze-out process is smooth and starts already before $T = 2m_e$).

For CMB photons $T_\gamma = 2.73$ K, so there should be also *cosmic neutrino background* with the temperature $T_\nu = 1.95$ K.

Let's now determine the present energy density, number density and entropy density for CMB photons and neutrinos assuming $T_0 = 2.75$ K.

	γ	ν
$g_\star = \sum_b g_i \left(\frac{T_i}{T}\right)^4 + \frac{7}{8}\sum_f g_i \left(\frac{T_i}{T}\right)^4$	2	$\frac{7}{8}\cdot 2\cdot 3\cdot \left(\frac{4}{11}\right)^{4/3} = 1.36$
$g_{\star\,S} = \sum_b g_i \left(\frac{T_i}{T}\right)^3 + \frac{7}{8}\sum_f g_i \left(\frac{T_i}{T}\right)^3$	2	$\frac{7}{8}\cdot 2\cdot 3\cdot \frac{4}{11} = 1.91$
$\rho = \frac{\pi^2}{30} g_\star T^4$	$4.64\cdot 10^{-34}$ g cm^{-3}	$3.16\cdot 10^{-34}$ g cm^{-3}
$n = \frac{2\zeta(3)}{\pi^2} T^3$	410 cm^{-3}	149 cm^{-3}
$s = \frac{2\pi^2}{45} g_{\star\,S} T^3$	1478 cm^{-3}	1412 cm^{-3}
$\Omega h^2 = \rho \frac{8\pi G}{3(H_0/h)^2}$	$2.47\cdot 10^{-5}$	$1.68\cdot 10^{-5}$

Where I used the following conversion factors: 1 K $= 4.3668$ cm^{-1} $= 8.6170\cdot 10^{-14}$ GeV $= 1.5361\cdot 10^{-37}$ g, 1 Mpc $= 1.5637\cdot 10^{38}$ GeV^{-1}, G $= 6.7065\cdot 10^{-39}$ GeV^{-2} and $H_0 = h\,2.1317\cdot 10^{-42}$ GeV.

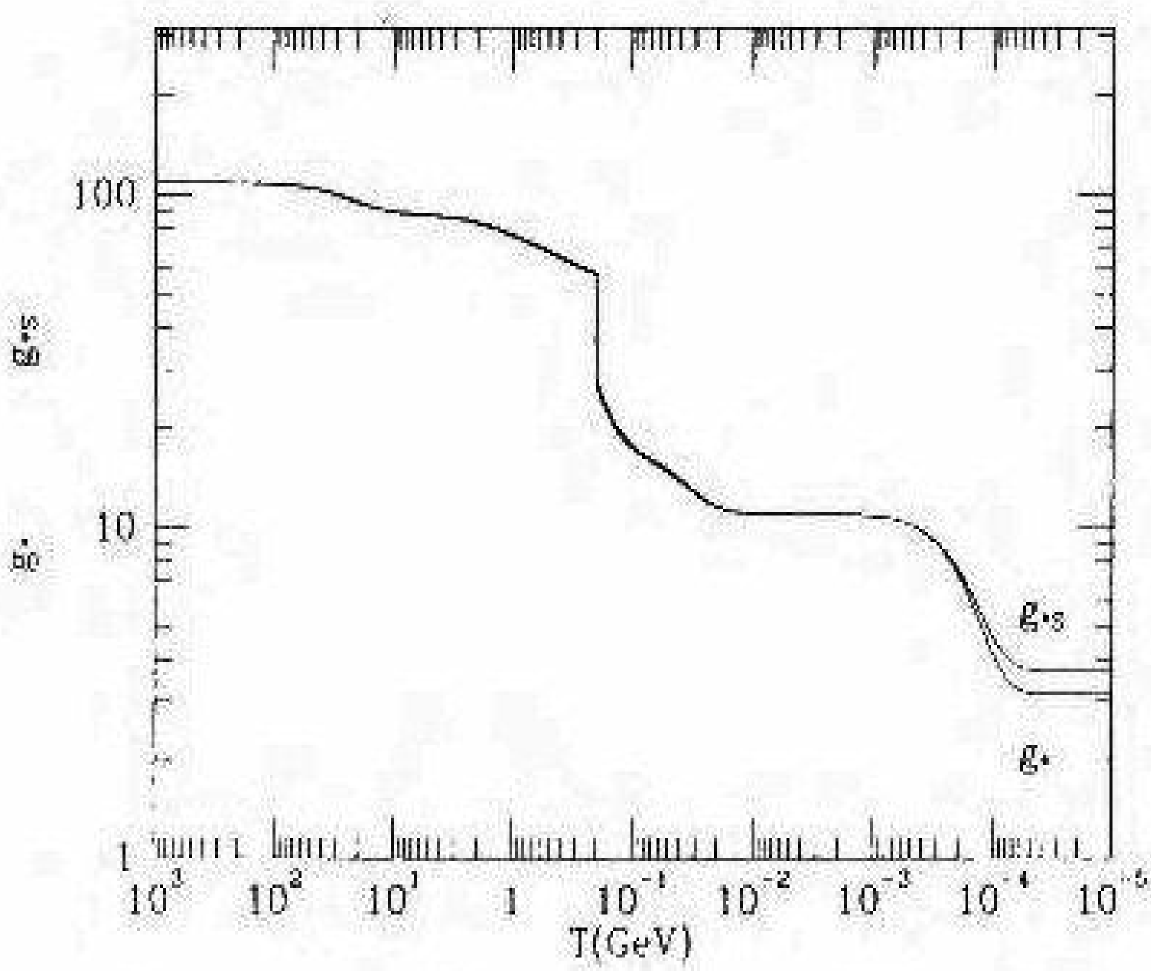


Fig. 3.5: The evolution of $g_*(T)$ as a function of temperature in the $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$ theory.

(plot from Kolb&Turner)

There exists also a possibility for another kind of radiation present as a relic of the early Universe, this is the graviton, the massless quantum excitation of a fluctuation of the gravitational field. The reaction responsible for maintaining the equilibrium would be e.g. $\bar{\psi}\psi \leftrightarrow hh$, where h is the graviton. Gravitons $h_{\mu\nu}$ interact with ordinary matter through the standard Lagrangian $\propto 1/M_{Pl} h_{\mu\nu} T^{\mu\nu}$, where $T^{\mu\nu}$ is the energy momentum tensor, therefore the reaction width (the inverse of the reaction rate) is

$$\Gamma_{\text{grav}} \sim \frac{T^5}{M_{Pl}^4}$$

Since at the early Universe $H \sim g_\star^{1/2} T^2/M_{Pl}$ therefore we get ($g_\star^{1/2} \sim 10$ for the SM at $T \gtrsim 100$ GeV)

$$\frac{\Gamma_{\text{grav}}}{H} \sim \frac{1}{10} \left(\frac{T}{M_{Pl}} \right)^3$$

So the gravitons freeze-out roughly at the Planck temperature $T \sim 2M_{Pl} \sim 10^{19}$ GeV. Using the continuity of entropy at the moment of graviton freeze-out and all the SM thresholds we get the relation between graviton temperature and the CMB photon temperature at the present moment (see class for the discussion):

$$T_{\text{grav}} = \left(\frac{g_{\star\,S}^{\text{now}}}{g_{\star\,S}^{\text{Planck}}}\right)^{1/3} \cdot T_0 \simeq 1 \text{ K}$$

where we have approximated $g_{\star\,S}^{\text{Planck}}$ by its SM value for $T \gtrsim 100$ GeV, i.e. ~ 100. Their contribution to the present energy density is $\rho_{\text{grav}} \sim T^4 \sim 0.018\rho_\gamma$.