Cosmological Models

- The de Sitter Model
- The Standard Model of Cosmology
- The Acceleration of the Universe.
- The Age of the Universe
- Future of the Universe
- Cosmological Distances
- Non-homogeneous Universe; the Lamaître-Tolman cosmological model

The de Sitter Model

The Einstein's Field Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G T_{\mu\nu}$$

don't allow for static solutions when applied for homogeneous and isotropic Universe. The idea of a static universe or "Einstein's universe" is one which demands that space is not expanding nor contracting but rather is dynamically stable. Albert Einstein proposed such a model as his preferred cosmology by adding a cosmological constant to his equations of general relativity to counteract the dynamical effects of gravity which in a universe of matter would cause the universe to collapse. This motivation evaporated after the discovery by Edwin Hubble that the universe is not static, but expanding; in particular, Hubble discovered a relationship between redshift and distance, which forms the basis for the modern expansion paradigm. This led Einstein to declare this cosmological model, and especially the introduction of the cosmological constant, his "biggest blunder".

Even after Hubble's observations, Fritz Zwicky proposed that a static universe could still be viable if there was an alternative explanation of redshift due to a mechanism that would cause light to lose energy as it traveled through space, a concept that would come to be known as "tired light". However, subsequent cosmological observations have shown that this model has not been a viable alternative either, leading most astrophysicists to conclude that the static universe is not the correct model of our universe.

The Einstein's equations now read

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

The Λ term could be written as a part of the energy-momentum tensor:

$$T^{\Lambda}_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} = \rho_\Lambda g_{\mu\nu} \quad \text{for} \quad \rho_\Lambda \equiv \frac{\Lambda}{8\pi G}$$

Then the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G \left(T_{\mu\nu} + T^{\Lambda}_{\mu\nu} \right)$$

Recall that in the rest frame of an element of perfect fluid $T_{\mu\nu}$ has the form:

$$T_{\mu\nu} = \begin{pmatrix} \rho & & & \\ & p & & \\ & & p & \\ & & & p \end{pmatrix}$$

while $T^{\Lambda}_{\mu\nu}$ in a freely falling system reads

$$T^{\Lambda}_{\mu\nu} = \begin{pmatrix} \rho_\Lambda & & & \\ & -\rho_\Lambda & & \\ & & -\rho_\Lambda & \\ & & & -\rho_\Lambda \end{pmatrix}$$

So, $\rho = \rho_\Lambda$ and $p = -\rho_\Lambda$ (*negative pressure!*).
We assume an empty space (no matter, so $T_{\mu\nu} = 0$), but $\Lambda \neq 0$ and solve the Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\Lambda g_{\mu\nu}$$

The cosmological principle implies that

$$d\tau^2 \equiv g_{\mu\nu}dx^\mu dx^\nu = dt^2 - R^2(t)\left\{\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2\sin^2\theta d\varphi^2\right\}$$

Using the results we obtained for the FLRW metric we have

$$R_{tt} = 3\frac{\ddot{R}}{R} \qquad R_{it} = R_{ti} = 0 \qquad R_{ij} = -\tilde{g}_{ij}(R\ddot{R} + 2\dot{R}^2 + 2k)$$

The Einstein's equations could be written as

$$R_{\mu\nu} = -8\pi G S_{\mu\nu}$$

where

$$S_{\mu\nu} \equiv T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T^\lambda{}_\lambda = (\rho + p)U_\mu U_\nu - \frac{1}{2}(\rho - p)g_{\mu\nu}$$

for $T^{\mu\nu} = -pg^{\mu\nu} + (p+\rho)U^\mu U^\nu$ ($T^\alpha{}_\alpha = \rho - 3p$). So for $U^t = 1$ and $U^i = 0$ we have

$$S_{tt} = \frac{1}{2}(\rho + 3p) = -\rho_\Lambda \qquad S_{ti} = S_{it} = 0 \qquad S_{ij} = \frac{1}{2}(\rho - p)R^2\tilde{g}_{ij} = R^2\rho_\Lambda\tilde{g}_{ij}$$

One gets the Friedmann's equations for this case

- $(0,0)$ component:

$$3\frac{\ddot{R}}{R} = -8\pi G(-\rho_\Lambda) = \Lambda$$

- (i,i) component:

$$-(R\ddot{R} + 2\dot{R}^2 + 2k)\tilde{g}_{ij} = -8\pi G\rho_\Lambda R^2\tilde{g}_{ij} = -R^2\Lambda\tilde{g}_{ij}$$

In other terms

$$3\frac{\ddot{R}}{R} = \Lambda \quad \text{and} \quad \frac{\ddot{R}}{R} + 2\left(\frac{\dot{R}}{R}\right)^2 + 2\frac{k}{R^2} = \Lambda$$

Eliminating $\ddot{R}$ we get

$$3\left(\frac{\dot{R}}{R}\right)^2 + 3\frac{k}{R^2} = \Lambda$$

For $k = 0$ and $\Lambda > 0$ we get

$$H^2(t) \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{\Lambda}{3} \quad \Longrightarrow \quad R(t) = R(t_0)e^{H\cdot(t-t_0)} \quad \text{for} \quad H^2 = \frac{\Lambda}{3} = \text{const.}$$

This is the exponential *inflation*: exponential growth of the scale factor.
It is easy to show (see class) that the necessary and sufficient conditions for the exponential inflation are:

- $k = 0$,
- $p = -\rho$.

$$3\left(\frac{\dot{R}}{R}\right)^2 + 3\frac{k}{R^2} = \Lambda$$

Comments:

- In principle, there is also a possibility of $H =$ const. if $\rho = \frac{3}{8\pi G}\left(H_0^2 + \frac{k}{R^2}\right)$ ($H_0 =$ const.) that we disregard as unphysical. $p \simeq -\rho$ could be easily arranged within a scalar field theory.
- For small R the term $\propto k/R^2$ dominates. However if R is growing then for R large enough, the curvature term $3\frac{k}{R^2} \sim \Lambda$. Since the Universe is expanding the Λ will dominate and the expansion will be exponential.

The Standard Model of Cosmology

The cosmological Principle implies

$$d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu = dt^2 - R^2(t) \left\{ \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2 \right\}$$

The Friedmann equations read

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho_{\rm tot} \quad \text{for} \quad \rho_{\rm tot} \equiv \sum_i \rho_i = \rho_m + \rho_{\rm rad} + \rho_\Lambda \tag{1}$$

$$2\frac{\ddot{R}}{R} + \left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = -8\pi G \sum_i p_i \quad \text{for} \quad p_{\rm tot} \equiv \sum_i p_i = p_{\rm rad} + p_\Lambda \tag{2}$$

where the sum runs over all contributions to the energy density and pressure. The conservation of the energy-momentum tensor ($T^{\mu\nu}{}_{;\nu} = 0$) implies

$$\dot{p}R^3 = \frac{d}{dt}\left[R^3(\rho + p)\right] \quad \Longrightarrow \quad \frac{d}{dt}\left(\rho R^3\right) = -p\frac{d}{dt}R^3 \tag{3}$$

Hereafter we will assume the following equation of state

$$p = w\rho$$

- For the non-relativistic matter (see class): $\rho \simeq nm + \frac{3}{2}p$, so if $p \ll nm$ then $w = 0$,
- For ultra-relativistic matter (e.g. photons): $p = \frac{1}{3}\rho$, so $w = \frac{1}{3}$.
- For the cosmological constant: $p = -\rho$, so $w = -1$.

Let's solve (3) for $p = w\rho$:

$$\frac{d}{dt}\left(\rho R^3\right) = \dot{\rho} R^3 + \rho 3R^2\dot{R} = -p\frac{d}{dt}R^3 = -w\rho 3R^2\dot{R}$$

$$\Downarrow$$

$$\frac{\dot{\rho}}{\rho} = -3(w+1)\frac{\dot{R}}{R} \quad \Longrightarrow \quad \rho(t) = \rho^0\left(\frac{R(t)}{R_0}\right)^{-3(w+1)} \propto R(t)^{-3(w+1)}$$

- Matter dominated Universe ($w = 0$), so called dust: $\rho \propto R^{-3}$

- Radiation dominated Universe ($w = \frac{1}{3}$): $\rho \propto R^{-4}$
  We have shown that for photons emitted at t_0 and detected at t the following relation holds
  $$\lambda(t) = \lambda(t_0)\frac{R(t)}{R(t_0)} \propto R(t)$$
  Since $\nu\lambda = c = 1$, so we have
  $$\nu(t) = \nu(t_0)\frac{R(t_0)}{R(t)} \propto R^{-1}(t)$$
  Therefor the photon energy $E = h\nu$ suffers from another extra suppression because of the expansion, so
  $$\rho \propto R^{-3}R^{-1} = R^{-4}$$

- Cosmological constant dominated Universe: ($w = -1$): $\rho =$ const.

Comment: If the Universe is composed of several components then the result $\rho_i(t) \propto R(t)^{-3(w_i+1)}$ is valid only if interaction between the components could be neglected, see (3).

Now we can try to solve the Friedmann equation (1) assuming $p = w\rho$

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho_{\rm tot}$$

First let's neglect the curvature k, then we have

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho_{\rm tot} = \frac{8\pi G}{3}\rho^0 R_0^{3(w+1)} R^{-3(w+1)}$$

$$\Downarrow$$

$$\dot{R} \propto R^{-\frac{1}{2}(3w+1)} \tag{4}$$

We can look for a power-like solution $R(t) \propto t^\alpha$, then substituting into (4) we can determine α:

$$t^{\alpha-1} = t^{-\frac{\alpha}{2}(3w+1)}$$

hence $\alpha = \frac{2}{3(w+1)}$. Therefore

$$R(t) \propto t^{\frac{2}{3(w+1)}} = \begin{cases} t^{2/3} & \text{matter} \\ t^{1/2} & \text{radiation} \\ e^{Ht} & \text{cosmological constant} \end{cases}$$

The Acceleration of the Universe

Subtracting (1) and (2) we get

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) \tag{5}$$

Since presently $p \sim 0$ and $\rho > 0$ therefore we would be tempted to conclude that the Universe is decelerating at present. This is not consistent with observations which suggest that $\ddot{R} > 0$. If we add Λ then we get an extra contribution to the rhs: $(\rho + 3p) = \rho_\Lambda + 3(-\rho_\Lambda) = -2\rho_\Lambda < 0$, then we can obtain $\ddot{R} > 0$.
Let's expand $R(t)$ around the present time $t = t_0$:

$$R(t) = R_0 + R_0 \frac{\dot{R}}{R}_{|t=t_0}(t - t_0) - \frac{1}{2}R_0 \underbrace{\left[-\frac{\ddot{R}}{R}_{|t=t_0}\frac{1}{H_0^2}\right]}_{\equiv q_0} H_0^2(t - t_0)^2 + \cdots$$

where q_0 is the *deceleration* parameter. Now let's expand the Hubble parameter $H(t)$

$$H(t) \equiv \frac{\dot{R}(t)}{R(t)} = H_0\left[1 - (q_0 + 1)H_0(t - t_0) + \cdots\right]$$

So, in general $H(t)$ is not constant, but a time dependent function.
Questions to test students alertness :

1. What is the condition to obtain $H(t) = \text{const.} + \mathcal{O}[H_0^2(t-t_0)^2]$?

2. What kind of matter in the Universe leads to $H(t) = \text{const.}$ at any time?

Answers:

1. In the next to the leading order: $q_0 = -1$.

2.
$$H(t) = \frac{\dot{R}}{R} = \text{const.} \qquad \Longrightarrow \qquad R(t) \propto e^{Ht} \qquad \text{the cosmological constant}$$

Define the critical density of the Universe as

$$\rho_{\rm crit} \equiv \frac{3H^2}{8\pi G}$$

Note that $\rho_{\rm crit}$ is a function of time, at present

$$\rho^0_{\rm crit} = 1.9 \times 10^{-32} h^2 \frac{\text{kg}}{\text{cm}^3} \quad \text{for} \quad h = \frac{H_0}{100\ \text{km s}^{-1}\ \text{Mpc}^{-1}} \quad \text{for} \quad 0.6 \lesssim h \lesssim 0.8$$

Then we can rewrite the Friedmann equation

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho$$

as follows

$$\frac{k}{R^2 H^2} = \Omega - 1 \quad \text{for} \quad \Omega \equiv \frac{\rho}{\rho_{\rm crit}}$$

$$\Downarrow$$

The geometry of the Universe is determined by Ω

- $\Omega > 1 \quad \Longrightarrow \quad k = +1$ closed Universe
- $\Omega < 1 \quad \Longrightarrow \quad k = -1$ open Universe
- $\Omega = 1 \quad \Longrightarrow \quad k = 0$ flat Universe

Let's calculate the deceleration parameter q_0 assuming $p_i = w\rho_i$ and using (5)

$$q_0 = \frac{4\pi G}{3H_0^2}\sum_i (\rho_i^0 + 3p_i^0) = \frac{4\pi G}{3H_0^2}\sum_i (1+3w_i)\rho_i^0 = \frac{1}{2}\sum_i \Omega_i^0 (1+3w_i)$$

for

$$\Omega_i^0 \equiv \frac{\rho_i^0}{\rho_{\mathrm{crit}}^0} = \frac{\rho_i^0}{\frac{3H_0^2}{8\pi G}}$$

Since $w = -1$ for Λ, therefore in the presence of the cosmological constant q_0 may be negative.

Consider energy density composed of matter, radiation and the cosmological constant, then we can rewrite the Friedmann equation at the present time as

$$H_0^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3}(\rho_m^0 + \rho_{\mathrm{rad}}^0 + \rho_\Lambda)$$

Dividing by H_0^2 and adopting the definition $\rho_{\mathrm{crit}}^0 \equiv \frac{3H_0^2}{8\pi G}$ we obtain

$$1 = -\frac{k}{H_0^2 R_0^2} + \Omega_m^0 + \Omega_{\mathrm{rad}}^0 + \Omega_\Lambda^0$$

Introducing $\Omega_k^0 \equiv -\frac{k}{H_0^2 R_0^2}$ we have

$$1 = \Omega_k^0 + \Omega_m^0 + \Omega_{\mathrm{rad}}^0 + \Omega_\Lambda^0$$

Let's return to the Friedmann equation

$$H^2 = -\frac{k}{R^2} + \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad} + \rho_\Lambda) \tag{6}$$

The matter and radiation densities scale as

$$\rho_m = \rho_m^0 \left(\frac{R_0}{R}\right)^3 \quad \text{and} \quad \rho_{\rm rad} = \rho_{\rm rad}^0 \left(\frac{R_0}{R}\right)^4$$

while ρ_Λ remains constant. The fractional energy densities are defined as follows

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\rm crit}} \quad \text{for} \quad \rho_{\rm crit} \equiv \frac{3H^2}{8\pi G}$$

while at the present time

$$\Omega_i^0 \equiv \frac{\rho_i^0}{\rho_{\rm crit}^0} \quad \text{for} \quad \rho_{\rm crit}^0 \equiv \frac{3H_0^2}{8\pi G}$$

So the densities could be rewritten as

$$\begin{aligned}\rho_{\rm rad} &= \rho_{\rm rad}^0\left(\frac{R_0}{R}\right)^4 = \frac{3}{8\pi G}H_0^2\Omega_{\rm rad}^0\left(\frac{R_0}{R}\right)^4\\ \rho_m &= \rho_m^0\left(\frac{R_0}{R}\right)^3 = \frac{3}{8\pi G}H_0^2\Omega_m^0\left(\frac{R_0}{R}\right)^3\\ \rho_\Lambda &= \rho_\Lambda^0 = \frac{3}{8\pi G}H_0^2\Omega_\Lambda^0\end{aligned}$$

The curvature terms will be written as

$$-\frac{k}{R^2} = \underbrace{-\frac{k}{R_0^2H_0^2}}_{\Omega_k^0}H_0^2\left(\frac{R_0}{R}\right)^2 = \Omega_k^0H_0^2\left(\frac{R_0}{R}\right)^2$$

Now, using the relation $1+z=\frac{R_0}{R}$, we are ready to express the densities corresponding to a given scale factor as functions of the red-sihft:

$$\rho_{\rm rad} = \frac{3}{8\pi G}H_0^2\Omega_{\rm rad}^0\left(\frac{R_0}{R}\right)^4 = \frac{3}{8\pi G}H_0^2\Omega_{\rm rad}^0(1+z)^4$$

$$
\begin{aligned}
\rho_m &= \frac{3}{8\pi G} H_0^2 \Omega_m^0 \left(\frac{R_0}{R}\right)^3 = \frac{3}{8\pi G} H_0^2 \Omega_m^0 (1+z)^3 \\
-\frac{k}{R^2} &= \Omega_k^0 H_0^2 \left(\frac{R_0}{R}\right)^2 = H_0^2 \Omega_k^0 (1+z)^2 \\
\rho_\Lambda &= \frac{3}{8\pi G} H_0^2 \Omega_\Lambda^0
\end{aligned}
$$

Let's insert the above formulas into the Friedmann equation (6):

$$
H^2 = H_0^2 \left[\Omega_{\mathrm{rad}}^0 (1+z)^4 + \Omega_m^0 (1+z)^3 + \Omega_k^0 (1+z)^2 + \Omega_\Lambda^0\right]
$$

So, we have shown how to determine the expansion rate at a given epoch (z) knowing its present value and present energy densities.

The Age of the Universe

♠ Matter or Radiation dominated Universe

The Friedmann equation could be integrated to give the age of the Universe. Two periods must be separately considered (the possibility of the existence of Λ will not be considered at this moment): radiation domination (early Universe) when $\rho = \rho_{\rm rad} = \rho^0 (R_0/R)^4$ and matter domination (present Universe) when $\rho = \rho_m = \rho^0 (R_0/R)^3$:

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad} + \rho_\Lambda) \leftarrow \rho_m = \rho^0 \left(\frac{R_0}{R}\right)^3, \ \rho_{\rm rad} = \rho_\Lambda = 0 \ \text{(MD)}$$

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad} + \rho_\Lambda) \leftarrow \rho_{\rm rad} = \rho^0 \left(\frac{R_0}{R}\right)^4, \ \rho_m = \rho_\Lambda = 0 \ \text{(RD)}$$

$$\left(\frac{\dot{R}}{R_0}\right)^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3}\rho^0 \frac{R_0}{R} \qquad \text{(MD)} \qquad (7)$$

$$\left(\frac{\dot{R}}{R_0}\right)^2 + \frac{k}{R_0^2} = \frac{8\pi G}{3}\rho^0 \left(\frac{R_0}{R}\right)^2 \qquad \text{(RD)} \qquad (8)$$

The age t of the Universe of a size $R(t)$ is defined as

$$t \equiv \int_{t(0)}^{t(R)} dt'$$

Changing integration variables to $R' = R'(t')$ we can write

$$t = \int_0^{R(t)} \frac{dR'}{\dot{R}'}$$

Using the relation

$$\frac{k}{R_0^2 H_0^2} = \Omega_m^0 + \Omega_{\mathrm{rad}}^0 + \Omega_\Lambda^0 - 1$$

to eliminate $\frac{k}{R_0^2}$ and defining $x \equiv \frac{R}{R_0}$ one can rewrite (7) for $\Omega_{\mathrm{rad}}^0 = \Omega_\Lambda^0 = 0$ and $\Omega_m^0 = \Omega^0$

$$\left(\frac{\dot{R}}{R_0}\right)^2 = -H_0^2(\Omega^0 - 1) + \underbrace{\frac{8\pi G}{3}\rho^0}_{H_0^2\Omega^0} x^{-1} = H_0^2\left(\Omega^0 x^{-1} + 1 - \Omega^0\right)$$

So,

$$\dot{R}'(x) = R_0 H_0 \left(\Omega^0 x^{-1} + 1 - \Omega^0\right)^{1/2} \quad \text{and} \quad dR' = R_0 dx$$

Then expressing the scale factor R in terms of the redshift z $(1 + z = \frac{R_0}{R} = \frac{1}{x})$ one gets the age for MD as

$$t_0^{(MD)} = \int_0^{R(t)} \frac{dR'}{\dot{R}'} = \int_0^{(1+z)^{-1}} \frac{R_0 dx}{\dot{R}'(x)}$$

$$t_0^{(MD)} = H_0^{-1} \int_0^{(1+z)^{-1}} \frac{dx}{\left[1 - \Omega^0 + \Omega^0 x^{-1}\right]^{1/2}} \quad \text{(MD)} \qquad (9)$$

$$t_0^{(RD)} = H_0^{-1} \int_0^{(1+z)^{-1}} \frac{dx}{\left[1 - \Omega^0 + \Omega^0 x^{-2}\right]^{1/2}} \quad \text{(RD)} \qquad (10)$$

Comments:

- For $R \lesssim l_{Pl}$ (the Planck length $l_{Pl} \equiv \left(\frac{\hbar G}{c^3}\right)^{1/2}$) our knowledge of the Universe is uncertain. However if we assume that $R(t) = R_0 \left(\frac{t}{t_0}\right)^n$ $(n < 1)$ then this first period of the expansion (from $R = 0$ till $R = l_{Pl}$) contributes a tiny piece to the total age

$$\int_0^{l_{Pl}} \frac{dR'}{\dot{R}'} = \frac{t_0}{nR_0^{1/n}} \int_0^{l_{Pl}} \frac{dR'}{R'^{\,(n-1)/n}} = t_0 \left(\frac{l_{Pl}}{R_0}\right)^{1/n}$$

Since $1/n > 1$ this contribution could be neglected.

- Note that if I wanted to include more Universe components in the same spirit as in (9-10) then I would be allowed just to add add various Ω's only if interaction between them could be neglected. Otherwise the scaling of $\rho = \rho(R)$ is more complicated. For instance the interaction between matter and radiation, could be neglected after neutral atoms were created (the recombination), so that photons stopped interacting with matter.

First the (MD) Universe. The present age could be obtained substituting $z = 0$ in (9), then integrating for $\Omega^0 > 1$ we obtain

$$t_0^{(MD)} = H_0^{-1} \frac{\Omega^0}{2(\Omega^0 - 1)^{3/2}} \left[\cos^{-1}\left(2\Omega^{0\,-1} - 1\right) - \frac{2}{\Omega^0}\left(\Omega^0 - 1\right)^{1/2}\right] \qquad (11)$$

and for $\Omega^0 < 1$

$$t_0^{(MD)} = H_0^{-1} \frac{\Omega^0}{2(1-\Omega^0)^{3/2}} \left[\frac{2}{\Omega^0} \left(1-\Omega^0\right)^{1/2} - \cosh^{-1}\left(2\Omega^{0\,-1} - 1\right) \right] \qquad (12)$$

For $\Omega^0 = 1$ we have $t_0^{(MD)} = \frac{2}{3} H_0^{-1}$. Note that $t_0 = t_0(\Omega^0)$ is a decreasing function of Ω^0.
Expanding (11-12) around $\Omega^0 = 1$ we obtain

$$t_0^{(MD)} = \frac{2}{3} H_0^{-1} \left[1 - \frac{1}{5}(\Omega^0 - 1) + \cdots \right]$$

The present age of the (MD) Universe could be easily estimated assuming $\Omega^0 \simeq 1$

$$t_0^{(MD)} = \frac{2}{3} H_0^{-1} = 6.5 \times 10^9 h^{-1} \, \text{yr} \quad \text{for} \quad 0.6 \lesssim h \lesssim 0.8$$

for $H_0 = 100$ km s^{-1} Mpc^{-1}h. The above estimates assumes that the Universe was MD from the very beginning till today.

For the (RD) Universe we obtain at $z = 0$

$$t_0^{(RD)} = H_0^{-1} \frac{\Omega^{0\,1/2} - 1}{\Omega^0 - 1} = \frac{1}{2} H_0^{-1} \left[1 - \frac{1}{4} \left(\Omega^0 - 1 \right) + \cdots \right]$$

Matter domination leads to larger age: $t_0^{(MD)}(\Omega^0) > t_0^{(RD)}(\Omega^0)$.

♠ Universe made of Matter and Cosmological Constant

Let's now discuss the age of the Universe for a model that is flat $(k = 0)$ but that contains both matter and $\Lambda > 0$, so

$$\left(\frac{\dot{R}}{R} \right)^2 = \frac{8\pi G}{3} (\rho_m + \rho_\Lambda)$$

Following the same steps as above for the (MD) case and using the fact that $\Omega_m^0 + \Omega_\Lambda^0 = 1$ (it follows from $k = 0$) we find (see class) that

$$t_0^{(\Lambda)} = H_0^{-1} \int_0^1 \frac{dx}{(\Omega_m^0 x^{-1} + \Omega_\Lambda^0 x^2)^{1/2}} = \frac{2}{3} H_0^{-1} \frac{1}{\Omega_\Lambda^{0\,1/2}} \ln \left[\frac{1 + \Omega_\Lambda^{0\,1/2}}{(1 - \Omega_\Lambda^0)^{1/2}} \right]$$

Comments:

- For $\Omega_\Lambda^0 \gtrsim \frac{3}{4}$, $t_0^{(\Lambda)} \gtrsim H_0^{-1}$, unlike $t_0^{(MD)}$ and $t_0^{(RD)}$.

- $t_0^{(\Lambda)} = t_0^{(\Lambda)}(\Omega_\Lambda)$ is an increasing function of Ω_Λ,

$$\lim_{\Omega_\Lambda^0 \to 0} t_0^{(\Lambda)} = \frac{2}{3} H_0^{-1} \quad \text{and} \quad \lim_{\Omega_\Lambda^0 \to 1} t_0^{(\Lambda)} = \infty$$

♠ The General Case

The expansion rate at a given epoch (z) as a function of its present value and present energy densities:

$$H^2 = H_0^2 \left[\Omega_{\text{rad}}^0 (1+z)^4 + \Omega_m^0 (1+z)^3 + \Omega_k^0 (1+z)^2 + \Omega_\Lambda^0 \right] \tag{13}$$

Using the above form of the Friedmann equation we will derive a general formula that allows to determine the age of the Universe at a given redshift (the "lookback time"). The Hubble parameter could be written as

$$H = \frac{d}{dt} \ln\left(\frac{R(t)}{R_0}\right) = \frac{d}{dt} \ln\left(\frac{1}{1+z}\right) = \frac{-1}{1+z}\frac{dz}{dt}$$

Then using (13) we get

$$\frac{dt}{dz} = H_0^{-1} \frac{-1}{1+z} \frac{1}{[\Omega^0_{\mathrm{rad}}(1+z)^4 + \Omega^0_m(1+z)^3 + \Omega^0_k(1+z)^2 + \Omega^0_\Lambda]^{1/2}}$$

Integrating we obtain

$$t_0 - t = H_0^{-1} \int_0^z \frac{dz'}{(1+z')[\Omega^0_{\mathrm{rad}}(1+z')^4 + \Omega^0_m(1+z')^3 + \Omega^0_k(1+z')^2 + \Omega^0_\Lambda]^{1/2}} \quad (14)$$

Note that Ω^0_i are not independent as they satisfy

$$1 = \Omega^0_k + \Omega^0_m + \Omega^0_{\mathrm{rad}} + \Omega^0_\Lambda$$

Choosing $t = 0$ and $z = \infty$ in (14) we have the present age of the Universe. Note that (as we have anticipated) the scale of the lookback time is set by H_0^{-1}, which is called the Hubble time.

Future of the Universe

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}(\rho_m + \rho_{\mathrm{rad}} + \rho_\Lambda) \tag{15}$$

$$\rho_m \propto R^{-3} \qquad \rho_{\mathrm{rad}} \propto R^{-4} \qquad \rho_\Lambda = \text{const.}$$

In general (15) is difficult to solve. However at present $\rho_{\mathrm{rad}} = 2 \times 10^5$ eV m^{-3} for the CMB while $\rho_{\mathrm{baryon}} = 10^9$ eV m^{-3}, so we can assume that $\rho_m \gg \rho_{\mathrm{rad}}$.

♠ No cosmological constant: $\rho_\Lambda = 0$

Therefore we have (neglecting ρ_Λ temporarily)

$$\rho = \rho^0 \frac{R_0^3}{R^3} \quad \Longrightarrow \quad \dot{R}^2 + k = \frac{8\pi G}{3} \rho^0 R_0^3 R^{-1} \tag{16}$$

- Suppose for a moment that $\rho^0 = 0$, then $k < 0$ is required to have real-valued solutions for R (so called Milne model). The solution is (in general $\pm|k|^{1/2}t$+const.)

$$R_{\mathrm{Milne}}(t) = |k|^{1/2} t = t$$

- For $\rho^0 > 0$ and $k = 0$ the solution reads

$$R(t) = R_0 \left(\frac{t}{t_0}\right)^{2/3} \propto t^{2/3}$$

- For $\rho^0 > 0$ and $k = -1$ we observe that

$$\left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{R^2} > 0 \tag{17}$$

Since $\rho = \rho^0 \frac{R_0^3}{R^3}$ we find that

$$\dot{R}^2 + k = \frac{8\pi G}{3}\rho^0 R_0^2 \frac{R_0}{R}$$

Therefore one can see that if R is large enough (and $\dot{R} > 0$ at some moment), then matter term becomes sub-dominant and the Universe turns out to expand forever a'la Milne: $R(t) \propto t$.

- Now let's consider the case $\rho^0 > 0$ and $k = +1$

$$\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi G}{3}\rho_m$$

Then we observe that in this case there is $R = R_{\text{crit}}$ such that $\dot{R}(t) = 0$ for $R = R_{\text{crit}}$:

$$\frac{8\pi G}{3}\rho^0 \left(\frac{R_0}{R_{\text{crit}}}\right)^3 = \frac{k}{R_{\text{crit}}^2} \quad \Longrightarrow \quad R_{\text{crit}} = \frac{8\pi G \rho^0 R_0^3}{3}$$

Since we know that

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}(\rho + 3p) = -\frac{4\pi G}{3}\rho_m$$

therefore in our case (no Λ and $p = 0$) we have $\ddot{R} < 0$. Hence the Universe is decelerating and at $R = R_{\text{crit}}$ the expansion stops and the contraction ends as a "Big Crunch".

♠ Cosmological constant: $\rho_\Lambda \neq 0$

Let's now consider $\Lambda \neq 0$ (still neglecting $\rho_{\rm rad}$), then we should solve

$$\dot{R}^2 = \frac{8\pi G}{3}\rho^0 \frac{R_0^3}{R} - k + \frac{\Lambda R^2}{3} \tag{18}$$

Comments:

- From (18) we can see that even if Λ was negligible for small R (at the beginning of the expansion) it will eventually dominate over all other forms of matter (including curvature).

- If $\Lambda < 0$ then (18) tells us that $R(t)$ cannot be arbitrarily large since $\dot{R}(t)$ must be real. So, the maximal size of the scale factor is determined by the solution of

$$\frac{8\pi G}{3}\rho^0 \frac{R_0^3}{R} = k + \frac{|\Lambda| R^2}{3}$$

$$\frac{\ddot{R}}{R} = -\frac{4\pi G}{3}\sum_i(\rho_i + 3p_i) = -\frac{4\pi G}{3}\left(-2\rho_\Lambda + \rho^0 \frac{R_0^3}{R^3}\right) = -\frac{4\pi G}{3}\left(\frac{|\Lambda|}{4\pi G} + \rho^0 \frac{R_0^3}{R^3}\right) < 0$$

The Universe is decelerating, so again we have an oscillating Universe (regardless of the value of k).

- If $\Lambda > 0$ and $k = 0$ or $k = -1$ we have

$$\dot{R}^2 = \frac{8\pi G}{3}\rho^0 \frac{R_0^3}{R} + |k| + \frac{\Lambda R^2}{3} > 0$$

So, the Universe is expanding forever (as it was expanding at the beginning: $R(t) \propto t^{2/3}$) and after some time the cosmological constant starts to dominate and enter a period of exponential expansion (de Sitter model).

- For $\Lambda > 0$ and $k = +1$ the picture is more complicated. It is possible to find $\Lambda = \Lambda_E$ such that $\dot{R}(t) = \ddot{R}(t) = 0$ for some $R = R_E$ (see class). This is a static Universe, the existence of this solution (not consistent with the present data) motivated Einstein to introduce Λ.

  - For $\Lambda = \Lambda_E$ the Universe is static (Is it stable?).
  - For $\Lambda > \Lambda_E$ the repulsion from Λ (Why is Λ repulsive?) dominates and the Universe expands forever.

– For $\Lambda < \Lambda_E$ there is a range of R: $R_{\min} \le R \le R_{\max}$ such that

$$\dot{R}^2 = \frac{8\pi G}{3}\rho^0 \frac{R_0^3}{R} - k + \frac{\Lambda R^2}{3} \le 0$$

that is forbidden (see class). So, for $0 < \Lambda < \Lambda_E$ and $k = +1$ the Universe is:

* oscillating between $R = 0$ and $R = R_{\min}$, or
* always expands if $\dot{R}$ was positive at some moment, or contracts (if $\dot{R} < 0$ at some initial moment) until $R = R_{\max}$ is reached then it bounces away and expands forever (no Big Bang in this case).

Cosmological Distances

♠ The Luminosity Distance

The total power $\mathcal{F}$ (the energy *per time* per area measured by the detector) of the light received by a telescope on Earth from an object of emitting power $\mathcal{L}$, called luminosity (energy produced *per time* by the source) can be calculated as follows. A "flash" of $N_{\rm emitt}$ photons is emitted isotropically at the time $t = t_{\rm emitt}$ from a source located at the radial coordinate r. If there were no expansion then a telescope located at $r = 0$ would detect the total power

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi[R(t_{\rm emitt})r]^2}$$

Note that $4\pi[R(t_{\rm emitt})r]^2$ is the area of the sphere containing photons emitted at $t = t_{\rm emitt}$.

The two-sphere analogy could be helpful to understand the presence of $[R(t_{\rm emitt})r]$.

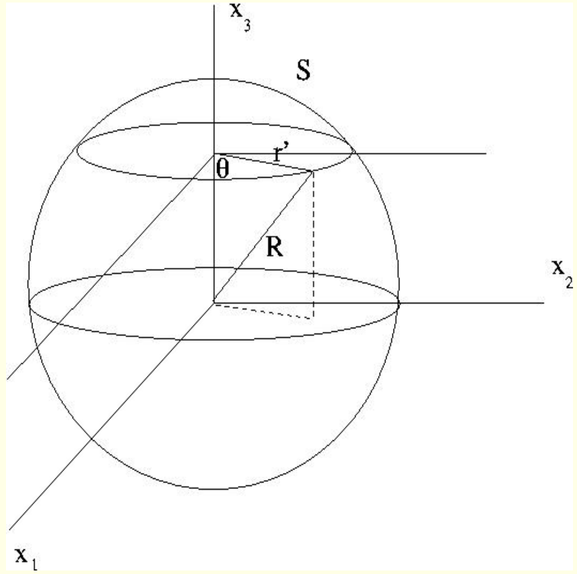

Figure 1: *The two sphere.*

However, because of the expansion of the sphere (the space time is expanding while the photon is traveling), at the detection time $t = t_{\rm obser}$, the area of the spherical shell within which the photons travel has expanded to $4\pi[R(t_{\rm obser})r]^2$, therefore the fraction should be corrected

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi[R(t_{\rm obser})r]^2}$$

To compute properly the total power, two other effects must be taken into account:

- Each emitted photon has its energy redshifted by the factor $\frac{\nu_{\rm emitt}}{\nu_{\rm obser}} = 1 + z$, so the photon energy is rescaled by the factor $\frac{1}{1+z}$.

- The observed power is defined as energy *per time*, so that must be taken into account. If the time distance between photon flashes at the source is $\delta t_{\rm emitt}$, then the time distance between the detection of those flashes, $\delta t_{\rm obser}$, will be increased according to the relation which we have obtained earlier:

$$\frac{\delta t_{\rm emitt}}{\delta t_{\rm obser}} = \frac{R(t_{\rm emitt})}{R(t_{\rm obser})} = \frac{1}{1+z}$$

  So, the detected power is suppressed by the factor $\frac{1}{1+z}$.

$$\Downarrow$$

The total power observed now reads

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi d_L^2} \quad \text{for} \quad d_L \equiv R(t_0) r (1+z)$$

where d_L is called the *luminosity distance* and the detection time is denoted by t_0. From now on the emission time will be denoted by t. Note that r is unknown radial coordinate of the source. However, if the solution of the Friedmann equation is known then r could be related to the redshift z as follows. Let's recall the expansion of the scale factor around the present time:

$$R(t) = R_0 + R_0 \frac{\dot{R}}{R}_{|t=t_0}(t-t_0) - \frac{1}{2} R_0 \underbrace{\left[-\frac{\ddot{R}}{R}_{|t=t_0} \frac{1}{H_0^2} \right]}_{\equiv q_0} H_0^2 (t-t_0)^2 + \cdots$$

where q_0 is the *deceleration* parameter. We can eliminate the ratio $\frac{R(t)}{R(t_0)}$ using the relation $\frac{R(t)}{R(t_0)} = \frac{1}{1+z}$ (t_0 is the detection moment), so that

$$\frac{1}{1+z} = 1 + H_0(t-t_0) - \frac{1}{2} q_0 H_0^2 (t-t_0)^2 + \cdots$$

Inverting we get

$$z = -H_0(t-t_0) + \left(1 + \frac{q_0}{2}\right) H_0^2 (t-t_0)^2 + \cdots = H_0(t-t_0)[-1 + \left(1 + \frac{q_0}{2}\right) H_0(t-t_0) + \cdots]$$

Therefore we can express the time difference $t_0 - t$ as a function of z:

$$t_0 - t = z H_0^{-1} \left[1 - \left(1 + \frac{q_0}{2}\right) z + \cdots \right]$$

Let's now recall the relation we have obtained for a massless wave traveling on a geodesic $d\tau^2 = 0$:

$$\int_t^{t_0} \frac{dt'}{R(t')} = \int_0^r \frac{dr'}{(1 - kr'^{\,2})^{1/2}} = \begin{cases} \sin^{-1} r = r + \frac{r^3}{6} + \cdots & k = +1 \\ r & k = 0 \\ \sinh^{-1} r = r - \frac{r^3}{6} + \cdots & k = -1 \end{cases} \tag{19}$$

Let's use the expansion

$$R(t) = R_0 + R_0 H_0 (t - t_0) - \frac{1}{2} R_0 q_0 H_0^2 (t - t_0)^2 + \cdots$$

on the lhs of (19) and keep only $\propto r$ terms on the rhs, then we get

$$R^{-1}(t_0) \left[(t_0 - t) + H_0 \frac{1}{2} (t_0 - t)^2 + \cdots \right] = r + \cdots$$

Substituting $t_0 - t = z H_0^{-1} \left[1 - \left(1 + \frac{q_0}{2} \right) z + \cdots \right]$ and keeping only terms $\mathcal{O}(z^2)$ we get

$$r = R_0^{-1} H_0^{-1} \left[z - \frac{1}{2} (1 + q_0) z^2 \right]$$

Now we are ready to use the above result in the expression for the luminosity distance $d_L = R(t_0)r(1+z)$

$$d_L = H_0^{-1}\left[z + \frac{1}{2}(1-q_0)z^2\right]$$

where we have kept only terms $\mathcal{O}(z^2)$. The above result yields a version of the Hubble law

$$H_0 d_L = z + \frac{1}{2}(1-q_0)z^2 + \cdots$$

Note that the above formula differs from the linear Hubble law for $q_0 \neq 1$, even though it was obtained for small z. Since q_0 depends on the cosmological model

$$q_0 = \frac{4\pi G}{3H_0^2}\sum_i(\rho_i^0 + 3p_i^0) = \frac{4\pi G}{3H_0^2}\sum_i(1+3w_i)\rho_i^0 = \frac{1}{2}\sum_i \Omega_i^0(1+3w_i)$$

therefore the measurement of $H_0 d_L$ offers the way to determine the fate of the Universe.

♠ The Angular Distances

Assume that there is an object of known diameter D located at the coordinate $r = r$, which emitted light at $t = t$, observed at $t = t_0$ at $r = 0$. From the FLRW metric we know that the angular diameter of the source, δ is given by

$$\delta = \frac{D}{R(t)r}$$

The angular distance d_A is defined as

$$d_A \equiv \frac{D}{\delta} = R(t)r$$

Since the luminosity distance is given by $d_L = R(t_0)r(1+z)$ and we know the relation between the size of the scale factor at the corresponding redshift: $\frac{1}{1+z} = \frac{R(t)}{R(t_0)}$ therefore we can derive the relation between d_L and d_A:

$$d_A = \frac{d_L}{(1+z)^2}$$

♠ Determination of Cosmological Parameters

Here we will discuss the determination of cosmological parameters such as H_0 and Ω_i^0 through a measurement of the luminosity distance d_L.
The luminosity distance d_L is defined through the total observed power

$$\mathcal{F} = \frac{\mathcal{L}}{4\pi d_L^2} \quad \text{for} \quad d_L \equiv R(t_0) r (1+z)$$

where $R(t_0)$, r and t are related by the equation of radial, null (light-like) geodesics for the FLRW metric ($d\theta = d\varphi = 0$):

$$d\tau = 0 \quad \Longrightarrow \quad \frac{dr}{dt} = \frac{(1-kr^2)^{1/2}}{R(t)}$$

Using the relation between the scale factor $R(t)$ and the redshift $1+z = \frac{R_0}{R(t)}$ we get

$$R_0 \frac{dr}{(1-kr^2)^{1/2}} = (1+z)dt$$

The following relation (obtained earlier)

$$\frac{dt}{dz} = H_0^{-1}\frac{-1}{1+z}\,\frac{1}{[\Omega^0_{\rm rad}(1+z)^4 + \Omega^0_m(1+z)^3 + \Omega^0_k(1+z)^2 + \Omega^0_\Lambda]^{1/2}}$$

could be adopted to change dt into dz such that the integration could be performed

$$R_0\int_0^r \frac{dr'}{(1-kr'^{\,2})^{1/2}} = H_0^{-1}\int_0^z \frac{dz'}{[\Omega^0_{\rm rad}(1+z')^4 + \Omega^0_m(1+z')^3 + \Omega^0_k(1+z')^2 + \Omega^0_\Lambda]^{1/2}}$$

The lhs could be easily integrated

$$R_0\int_0^r \frac{dr'}{(1-kr'^{\,2})^{1/2}} = R_0\begin{cases} \sin^{-1} r & k=+1 \\ r & k=0 \\ \sinh^{-1} r & k=-1 \end{cases}$$

Thus we are able to express r as a function of z, this is exactly what is needed to find the luminosity distance as a function of z, that way we get e.g. for $k=+1$

$$r(z) = \sin\left\{(R_0H_0)^{-1}\int_o^z \frac{dz'}{[\Omega^0_{\rm rad}(1+z)^4 + \Omega^0_m(1+z)^3 + \Omega^0_k(1+z)^2 + \Omega^0_\Lambda]^{1/2}}\right\}$$

Using the definition of $\Omega_k^0 = \frac{-k}{(R_0 H_0)^2}$ we will get rid of $R_0 H_0$ obtaining

- $k = +1$

$$\begin{aligned} d_L &= R(t_0)(1+z)r(z) = \frac{R_0 H_0}{H_0}(1+z)r(z) = H_0^{-1}(1+z)\left(|\Omega_k^0|\right)^{-1/2} \times \\ &\quad \sin\left\{\left(|\Omega_k^0|\right)^{1/2}\int_o^z \frac{dz'}{[\Omega_{\rm rad}^0(1+z)^4 + \Omega_m^0(1+z)^3 + \Omega_k^0(1+z)^2 + \Omega_\Lambda^0]^{1/2}}\right\} \\ \Omega_k^0 &= 1 - \Omega_{\rm rad}^0 - \Omega_m^0 - \Omega_\Lambda < 0 \end{aligned}$$

- $k = 0$

$$\begin{aligned} d_L &= R(t_0)(1+z)r(z) = \frac{R_0 H_0}{H_0}(1+z)r(z) = \\ &\quad H_0^{-1}(1+z)\int_o^z \frac{dz'}{[\Omega_{\rm rad}^0(1+z)^4 + \Omega_m^0(1+z)^3 + \Omega_k^0(1+z)^2 + \Omega_\Lambda^0]^{1/2}} \\ \Omega_k^0 &= 0 \end{aligned}$$

- $k = -1$

$$\begin{aligned}
d_L &= R(t_0)(1+z)r(z) = \frac{R_0 H_0}{H_0}(1+z)r(z) = H_0^{-1}(1+z)\left(|\Omega_k^0|\right)^{-1/2} \times \\
&\quad \sinh\left\{\left(|\Omega_k^0|\right)^{1/2} \int_o^z \frac{dz'}{[\Omega_{\rm rad}^0(1+z)^4 + \Omega_m^0(1+z)^3 + \Omega_k^0(1+z)^2 + \Omega_\Lambda^0]^{1/2}}\right\} \\
\Omega_k^0 &= 1 - \Omega_{\rm rad}^0 - \Omega_m^0 - \Omega_\Lambda > 0
\end{aligned}$$

So, a measurement of d_L provides a constraint on H_0 and $\Omega_{\rm rad}^0$, Ω_m^0 and Ω_Λ^0.

♠ The General form of the redshift Dependence of Particle Horizon

As we have shown the distance to the particle horizon is given by

$$d_{ph}(t) = R(t) \int_0^t \frac{dt'}{R(t')}$$

Our goal is to find the distance d_{ph} as a function of z (earlier we obtained $d_L = d_L(z)$ for small z), therefore it is convenient to change variables from t' to z'. For that we can adopt the relation obtained earlier

$$\frac{dt}{dz} = H_0^{-1} \frac{-1}{1+z} \frac{1}{[\Omega_{\rm rad}^0 (1+z)^4 + \Omega_m^0 (1+z)^3 + \Omega_k^0 (1+z)^2 + \Omega_\Lambda^0]^{1/2}}$$

Then

$$d_{ph}(t) = R(t) \int_0^t \frac{dt'}{R(t')} = R(t) \int_\infty^z R_0^{-1} \frac{R_0}{R(t')} \frac{dt'}{dz'} dz'$$

Inserting $\frac{dt}{dz}$ we obtain

$$d_{ph}(z) = \frac{R(t)}{R_0} \int_\infty^z (1+z') H_0^{-1} \frac{-1}{1+z'} \frac{dz'}{[\Omega_{\rm rad}^0 (1+z')^4 + \Omega_m^0 (1+z')^3 + \Omega_k^0 (1+z')^2 + \Omega_\Lambda^0]^{1}$$

Using $1+z=\frac{R_0}{R(t)}$ we have

$$d_{ph}(z)=\frac{1}{H_0(1+z)}\int_z^\infty \frac{dz'}{[\Omega^0_{\rm rad}(1+z')^4+\Omega^0_m(1+z')^3+\Omega^0_k(1+z')^2+\Omega^0_\Lambda]^{1/2}}$$

If we allow for an extra component of the Universe with the equation of state $p=w_x\rho$ then the above result is modified such that the horizon distance reads

$$\begin{aligned} d_{ph}(z) &= \frac{1}{H_0(1+z)}\times \\ & \int_z^\infty \frac{dz'}{[\Omega^0_{\rm rad}(1+z')^4+\Omega^0_m(1+z')^3+\Omega^0_k(1+z')^2+\Omega^0_\Lambda+\Omega_x(1+z')^{3(1+w_x)}]^{1/2}}\end{aligned}$$

Comment:

- It is important to realize that various powers of $(1+z)$ present above (or just on the rhs of the Friedmann equation, where they come from) originate from different dependence of energy densities on R (e.g. $\propto R^{-3}$ for matter, $\propto R^{-4}$ for radiation). The dependence on R was derived from the first law of thermodynamics $separately$ for each kind of Universe constituents while the the first law of thermodynamics applies for the $total$ energy density and pressure. In general (before decoupling) non-relativistic matter interacts with radiation and the precise picture is more involved. So, strictly speaking what we are doing applies for the period when the radiation and the non-relativistic matter do not interact.

♠ Measurements of Distances and Observation of Standard Candles

Parallax-based methods:

The most important direct distance measurements come from the parallax. The Earth's motion around the sun causes small shifts in stellar positions. These shifts are angles in a right triangle, with 1 AU making the short leg of the triangle and the distance to the star being the long leg. One pc is the distance of a star whose parallax is one arc second.

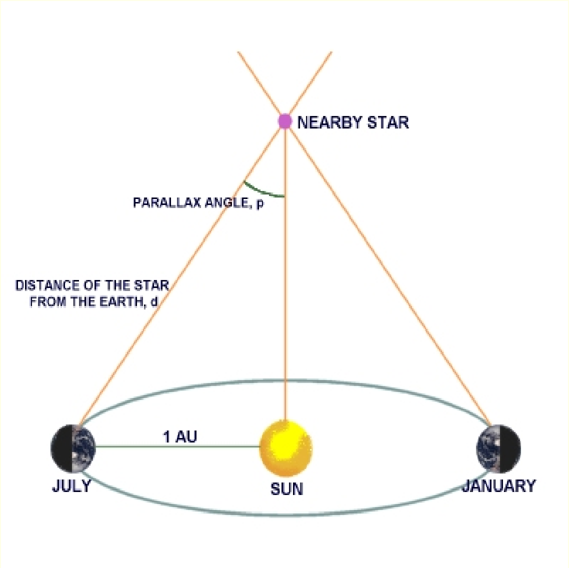

Figure 2: *The parallax.*

A standard candle is a class of astrophysical objects, such as supernovae or variable stars, which have known luminosity due to some characteristic quality possessed by the entire class of objects.

- Cepheids
  Cepheid is a variable star that has a fairly tight correlation between their period of variability and intrinsic brightness. Because of this correlation (discovered and stated by Henrietta Swan Leavitt in 1908 and given precise mathematical form by her in 1912), a Cepheid can be used as a so called "standard candle" to determine the distance to its host cluster or galaxy.

  – The variation in luminosity is caused by a cycle of ionization of helium in the star's atmosphere, followed by expansion and deionization. While ionized, the atmosphere is more opaque to light.
  – The luminosity of cepheid stars range from 10^3 to 10^4 times that of the Sun. A three-day period Cepheid has a luminosity of about 800 times that of the Sun. A thirty-day period Cepheid is 10^4 times as bright as the Sun. The scale has been calibrated using nearby Cepheid stars, for which the distance was already known (a source of some uncertainties). This high luminosity, and the precision with which their distance can be estimated, makes Cepheid stars the ideal standard candle to measure the distance of clusters and external galaxies.
  – First define apparent magnitude m of a celestial body as a measure of its

brightness as seen on Earth:

$$m = -2.5 \log_{10} \mathcal{F} + \text{const.}$$

where $\mathcal{F}$ is the total power (energy/area/time) observed on Earth while const. is a constant to be determined by the requirement that the star Vega has apparent magnitude $m = 0$. Then the period-luminosity relationship could be written as follows:

$$M = -2.81 \log_{10}(P) - (1.43 \pm 0.1)$$

where M is the absolute magnitude (an apparent magnitude of the object if it would be at 10 pc distance from the observer) and P is the period measured in days. The above relation was obtained by Henrietta Leavitt. She was working at the Harvard College Observatory, studying photographic plates of the Large (LMC) and Small (SMC) Magellanic Clouds, compiled a list of 1777 periodic variables. Eventually she classified 47 of these in the two clouds as Cepheid variables and noticed that those with longer periods were brighter than the shorter-period ones. She correctly inferred that as the stars were in the same distant clouds they were all at much the same relative distance from us. Any difference in apparent magnitude was therefore related to a difference in absolute magnitude. When she plotted her results for the two clouds she noted that they formed distinct relationships between brightness and period.

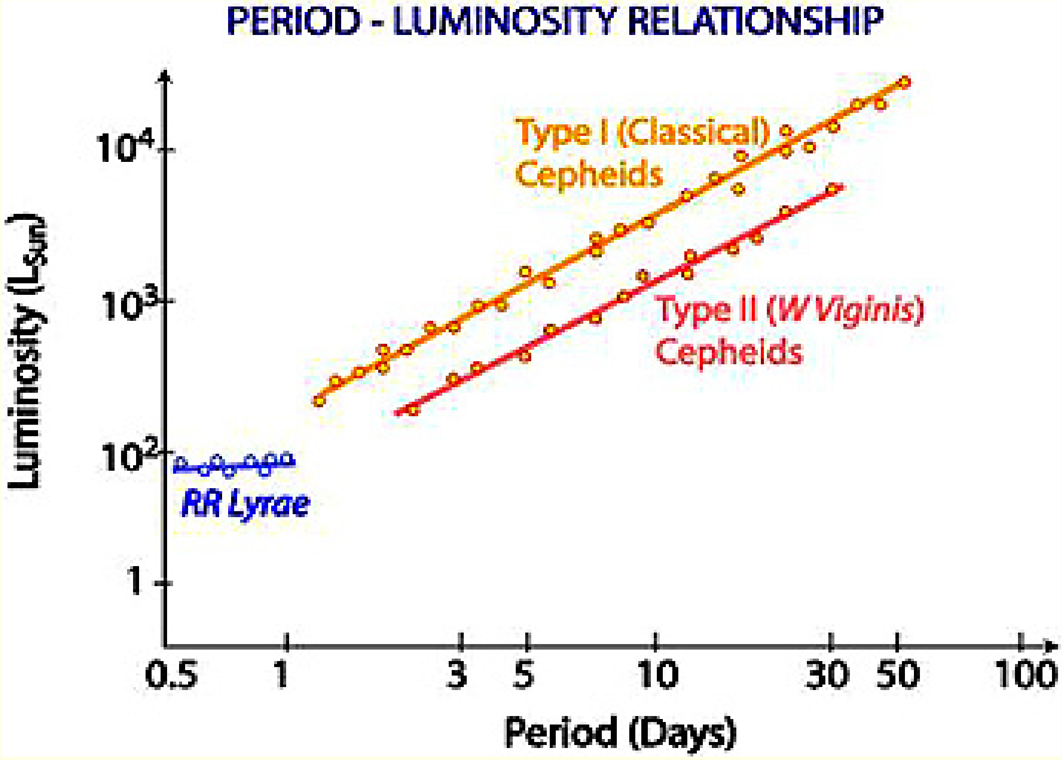

Figure 3: *Period-luminosity relationship for Cepheids and RR Lyrae stars.*

Let us now see how this relationship can be used to determine the distance to a Cepheid. For this procedure we will assume that we are dealing with a Type I, Classical Cepheid but the same method applies for W Virginis and RR Lyrae-type stars.

1. Photometric observations, by the naked-eye estimates, photographic plates, or photoelectric CCD images provide the apparent magnitude values for the Cepheid.

2. Plotting apparent magnitude values from observations at different times results in a light curve such as that below for a Cepheid in the LMC.

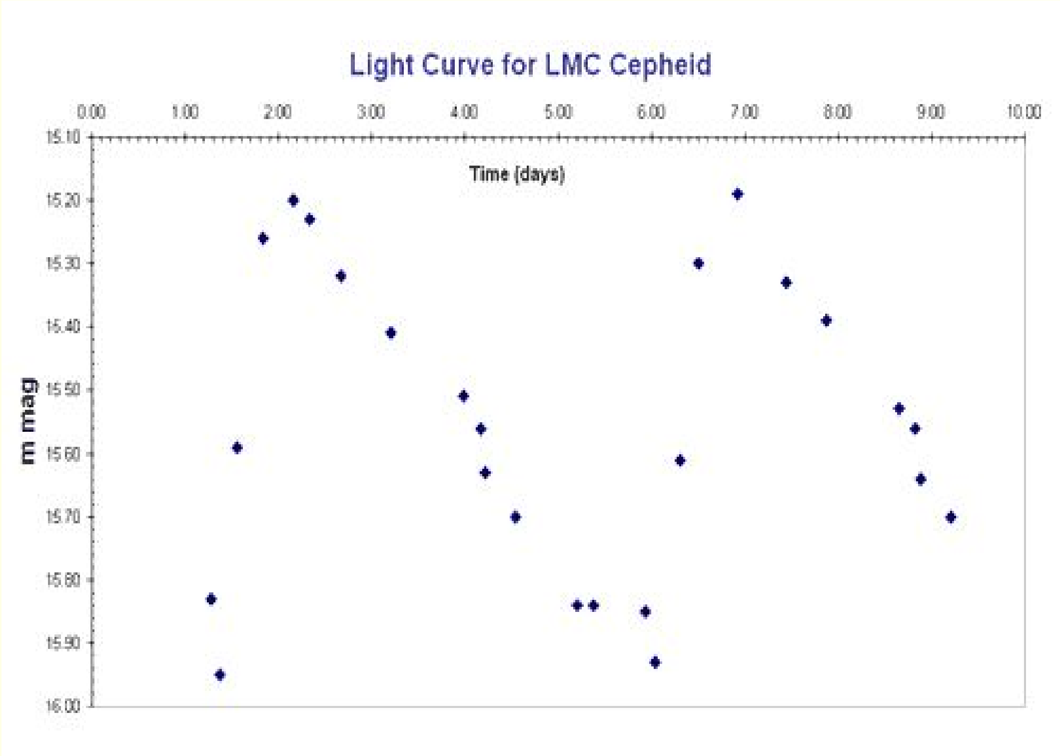

Figure 4: *The light curve for LMC Cepheid.*

3. From the light curve and the photometric data, two values can be determined; the average apparent magnitude, m, of the star and its period in days. In the example above the Cepheid has a mean apparent magnitude of 15.56 and a period of 4.76 days.
4. Knowing the period of the Cepheid we can now determine its mean absolute

magnitude, M, by adopting the relation found by Henrietta Leavitt

$$M = -2.81 \log_{10}(P) - (1.43 \pm 0.1)$$

Alternatively one can put a Cepheid on the period-luminosity plot as shown in (5). The one shown below is based on Cepheids within the Milky Way. The vertical axis shows absolute magnitude whilst period is displayed as a log value on the horizontal axes.

5. Once both apparent magnitude, m, and absolute magnitude, M are known we can simply substitute in to the distance-modulus formula and rewrite it to find a value for d_L the luminosity distance to the Cepheid.

$$M = -\frac{5}{2}\log_{10}\mathcal{F}_{10} + \text{const.} \quad \text{and} \quad m = -\frac{5}{2}\log_{10}\mathcal{F} + \text{const.}$$

where $\mathcal{F}_{10}$ is the total power observed at the distance of 10 pc (according to the definition of M). Since $\mathcal{F} \propto d_L^{-2}$ we obtain

$$5\log_{10}\left(\frac{d_L}{\text{Mpc}}\right) = m - M - 25, \tag{20}$$

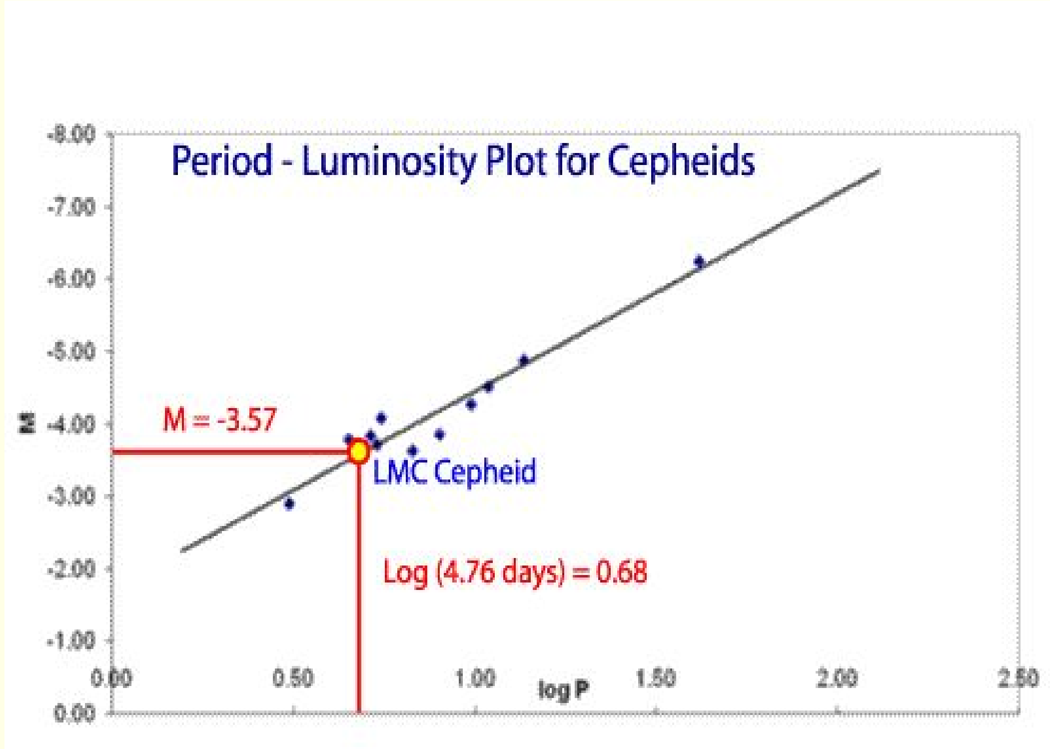

Figure 5: *The log of 4.76 days = 0.68. When this is plotted a value of about -3.6 results for absolute magnitude.*

- Type Ia Supernovas
  A supernova (plural: supernovae or supernovas) is a stellar explosion that creates an extremely luminous object. A supernova causes a burst of radiation that may briefly outshine its entire host galaxy before fading from view over several weeks or months. During this short interval, a supernova can radiate as much energy as

the Sun could emit over its life span. The explosion expels much or all of a star's material at a velocity of up to a tenth the speed of light, driving a shock wave into the surrounding interstellar medium.
Type Ia Supernova could be formed as follows. If a carbon-oxygen white dwarf accreted enough matter to reach the Chandrasekhar limit (the maximum non-rotating mass which can be supported against gravitational collapse) of about 1.38 solar masses, (note that this is for white dwarfs, not for any stars) it would no longer be able to support the bulk of its plasma and would begin to collapse. Increasing temperature and density inside the core triggers carbon fusion. Within a few seconds, a substantial fraction of the matter in the white dwarf undergoes nuclear fusion, releasing enough energy ($1 - 2 \times 10^{44}$ J) to unbind the star in a supernova explosion. An outwardly expanding shock wave is generated, with matter reaching velocities on the order of $5,000 - 20,000$ km/s, or roughly 3% of the speed of light. There is also a significant increase in luminosity, reaching an absolute magnitude of -19.3 (or 5 billion times brighter than the Sun), with little variation.
One model for the formation of a Type Ia explosion involves the merger of two white dwarf stars, with the combined mass momentarily exceeding the Chandrasekhar limit. A white dwarf could also accrete matter from other types of companions (if the orbit is sufficiently close). For the list of supernovae see http://www.cfa.harvard.edu/iau/lists/Supernovae.html. Supernovae are very rare,

one per few hundred years per galaxy, however since there are many galaxies we can observe many supernovae "simultaneously".

Figure 6: *Two closely orbiting violet-hot carbon-oxygen white dwarfs are spiraling into one another. The two stars are destined to merge, which will bring the new star over the Chandrasekhar limit, leading to carbon-oxygen fusion and a Type Ia supernova explosion.*

Figure 7: *Type Ia supernova observed by the Hubble Space Telescope in 1994.*

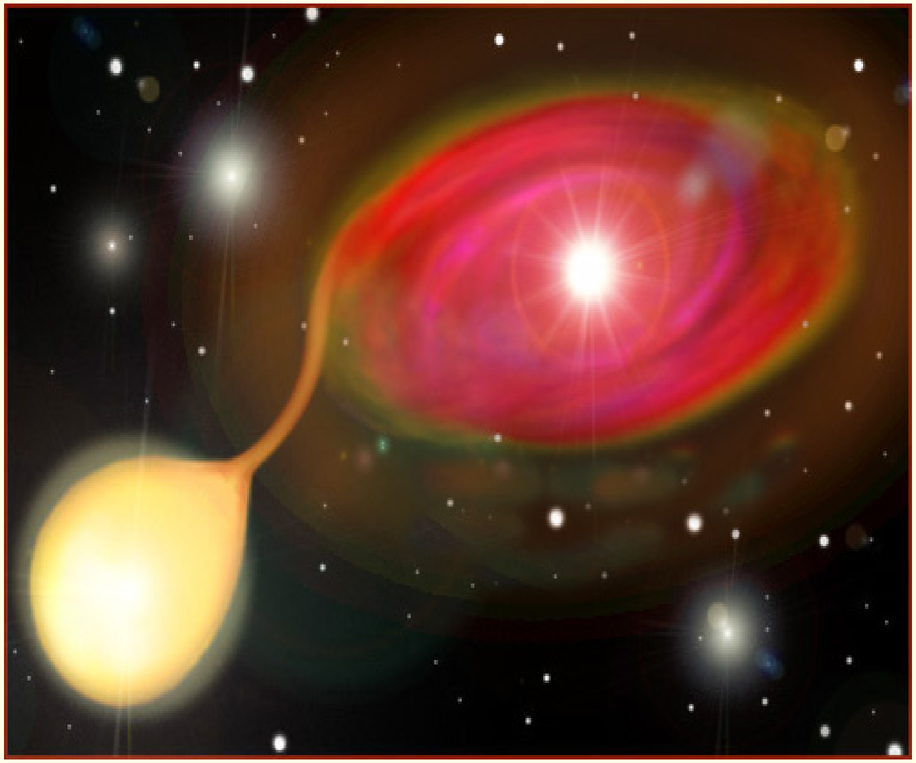

Figure 8: *A binary system before the explosion.*

The supernova explosions always release roughly the same amount of energy, and studies of relatively nearby type Ia supernovae have shown that they reach almost the same peak brightness in every case. Therefore it can be used as standard candle to determine their true distance. Fig. 7 is a Type Ia supernova observed in 1994. It is the bright spot on the lower left at the brink of the galaxy. Fig. 8 shows such binary system before the explosion. The absolute magnitude for the Type Ia supernovae has been calibrated to be $M = -19.33 \pm 0.25$, therefore a measurement of the apparent luminosity m allows us to determine the luminosity distance d_L according to (20).

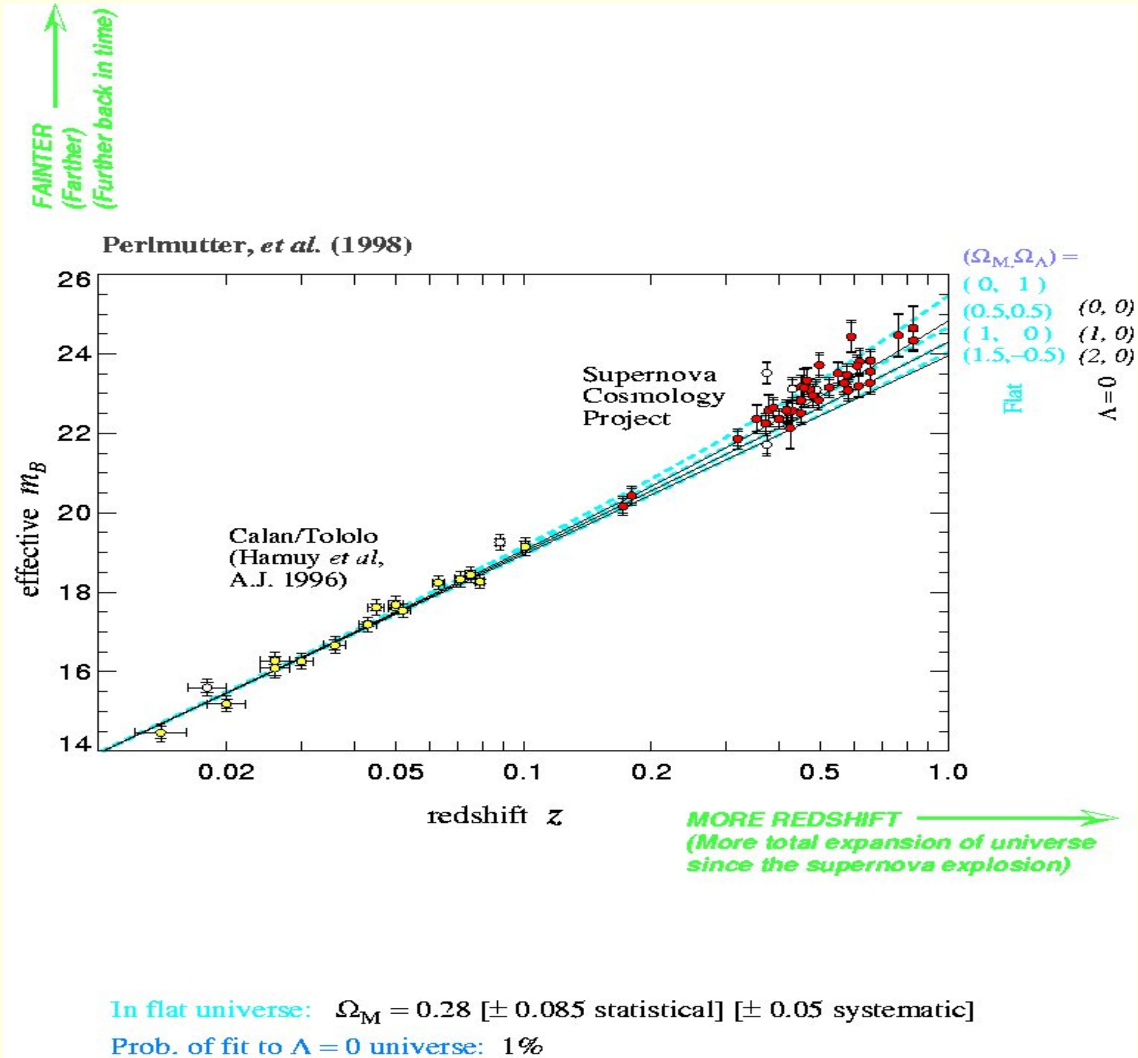

Figure 9: *Hubble diagram with 42 high-redshift supernovae (log redshift scale), from SCP.*

The data (from the *Supernova Cosmology Project* shown in (9) favour a flat ($k = 0$) Universe with a positive cosmological constant, $\Omega_\Lambda = 0.75 \pm 0.08$. The current data set of high-redshift Type Ia supernovas is not sufficient to break the degeneracy of the density terms, see (10). The results can be approximated by the linear combination $0.8\Omega_m - 0.6\Omega_\Lambda \simeq -0.2 \pm 0.1$.

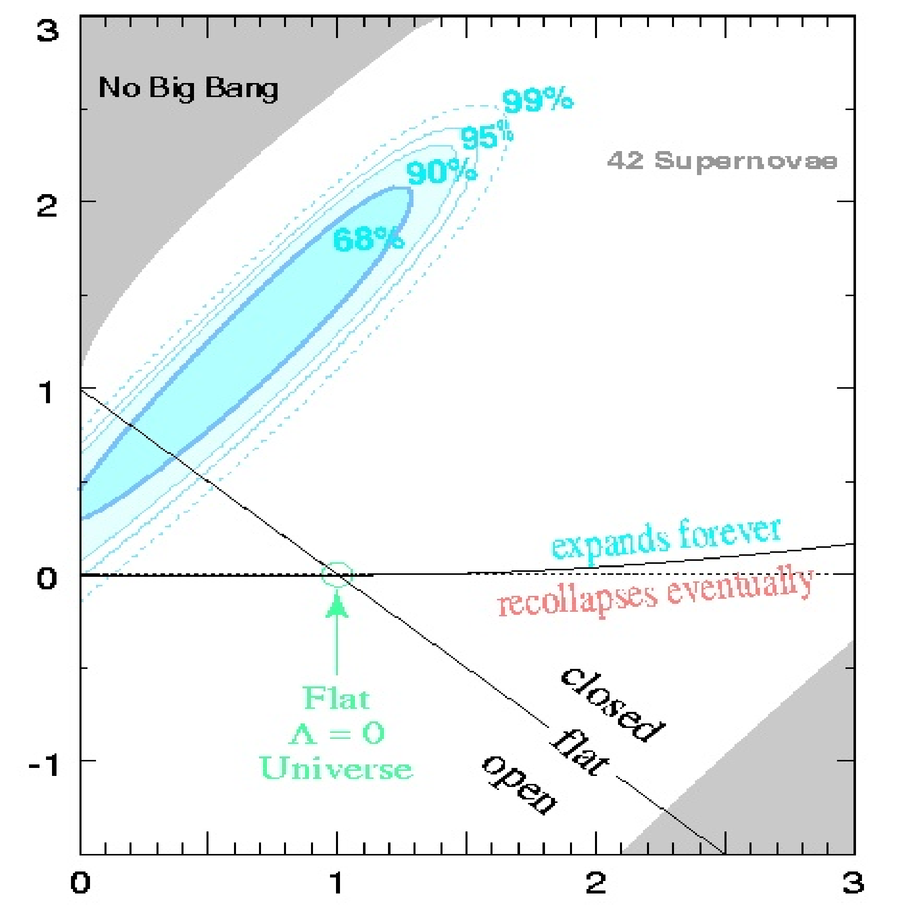

Figure 10: *Confidence region on Ω_m vs. Ω_Λ plane, from SCP.*

The geometry of the Universe is determined by $\Omega = \Omega_m + \Omega_\Lambda$:

– $\Omega > 1 \quad \Longrightarrow \quad k = +1$ closed Universe
– $\Omega < 1 \quad \Longrightarrow \quad k = -1$ open Universe
– $\Omega = 1 \quad \Longrightarrow \quad k = 0$ flat Universe

$\Omega_\Lambda = 1 - \Omega_m$ separates regions of closed $(k = +1)$ and open $(k = -1)$ Universes.

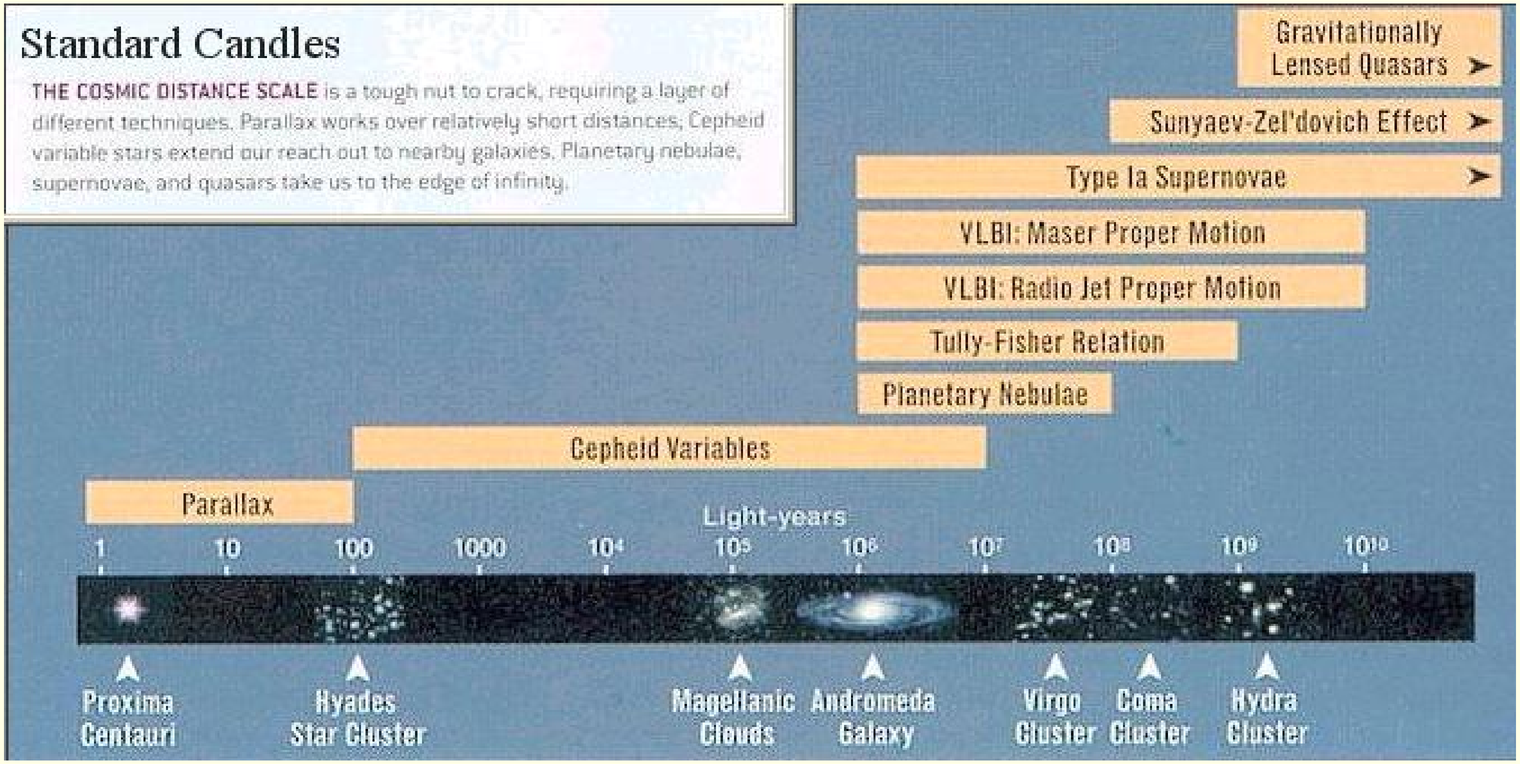

Figure 11: *The cosmic distance ladder*

Comments:

- An explanation of *smallness* of the cosmological constant is one of the most outstanding problems of modern theoretical physics. In units with $\hbar = c = 1$, the energy density for $\Omega_\Lambda \simeq 1$ is $\rho_\Lambda \simeq 10^{-46}$ GeV^4. Since the origin of Λ seems to be gravitational, therefore the natural size of ρ_Λ should be a 4th power of the Planck mass, $\sim \mathcal{O}(M_{Pl}^4)$, $M_{Pl} = 1.2 \cdot 10^{19}$ GeV, that gives $\rho_\Lambda \simeq 10^{76}$ GeV^4, while the observed value is smaller by 122 orders of magnitude! Theoretically, it is much easier to explain that a quantity is zero, then to show that it is so small, unfortunately the data require $\Omega_\Lambda \simeq 1$.

- There are some problems concerning the distance determination using standard candles. The principal one is calibration, determining exactly what the absolute magnitude of the candle is. This includes defining the class well enough that members can be recognized, and finding enough members with well-known distances that their true absolute magnitude can be determined with enough accuracy. The second lies in recognizing members of the class, and not mistakenly using the standard candle calibration upon an object which does not belong to the class. At extreme distances, which is where one most wishes to use a distance indicator, this recognition problem can be quite serious.

♠ Non-homogeneous universe; the Lamaître-Tolman cosmological model

Consider spherically symmetric dust universe with radial inhomogeneities observed from the origin ($x^i = 0$). The line element takes the following form

$$d\tau^2 = dt^2 - X^2(r,t)dr^2 - R^2(r,t)(d\theta^2 + \sin^2\theta d\phi^2)$$

The FLRW metric is a limiting case of the Lamaître-Tolman (LT):

$$X(r,t) \to \frac{R(t)}{(1-kr^2)^{1/2}}, \qquad R(r,t) \to R(t)r$$

The energy-momentum tensor in that case reads

$$T_{\alpha\beta}(r,t) = \rho_m(r,t)U_\alpha U_\beta$$

for U_α being perfect fluid 4-velocity, so $U_0 = 1$ and $U_i = 0$ in the comoving frame. The Einstein equations leads to the following set of differential equations:

$$
\begin{aligned}
-2\frac{R''}{RX^2}+2\frac{R'X'}{RX^3}+2\frac{\dot{X}\dot{R}}{XR}+\frac{1}{R^2}+\left(\frac{\dot{R}}{R}\right)^2-\left(\frac{R'}{RX}\right)^2 &= 8\pi G\rho_m \\
\dot{R}' &= R'\frac{\dot{X}}{X} \qquad (21) \\
2\frac{\ddot{R}}{R}+\frac{1}{R^2}+\left(\frac{\dot{R}}{R}\right)^2-\left(\frac{R'}{RX}\right)^2 &= 0 \\
-\frac{R''}{RX^2}+\frac{\ddot{R}}{R}+\frac{\dot{X}\dot{R}}{XR}+\frac{R'X'}{RX^3}+\frac{\ddot{X}}{X} &= 0
\end{aligned}
$$

where $R' \equiv \partial R/\partial r$ and $\dot{R} \equiv \partial R/\partial t$. Only three of the above four equations are independent. Eq.21 could be easily solved by

$$X(r,t) = C(r)R'(r,t)$$

The function $C(r)$ (to be determined by boundary conditions) could be written as follows:

$$C(r) \equiv \frac{1}{[1-k(r)]^{1/2}}$$

Then the LT metric could rewritten as

$$d\tau^2 = dt^2 - \frac{[R'(r,t)]^2}{1-k(r)}dr^2 - R^2(r,t)(d\theta^2 + \sin^2\theta d\phi^2)$$

(The FLRW case could be obtained for $k(r) \to kr^2$ and $R(r,t) \to R(t)r$.) Then the two independent Einstein equations read

$$\frac{\dot{R}^2 + k(r)}{R^2} + \frac{2\dot{R}\dot{R}' + k'(r)}{RR'} = 8\pi G\rho_m \quad (22)$$

$$\dot{R}^2 + 2R\ddot{R} + k(r) = 0 \quad (23)$$

It is easy to verify (apply $\partial/\partial t$) that the first integral of (23) is

$$R\dot{R}^2 = F(r) - Rk(r)$$

for $F(r)$ to be determined by boundary conditions. Then we get the generalized Friedmann equation for the *local* Hubble parameter $H(r,t) \equiv \dot{R}(r,t)/R(r,t)$:

$$H^2(r,t) + \frac{k(r)}{R^2} = \frac{F(r)}{R^3} \quad (24)$$

Instead of $F(r)$ and $k(r)$ one can define $\Omega_m^0(r)$ and $\Omega_k^0(r)$

$$\begin{aligned} F(r) &= H_0^2(r)\Omega_m^0(r)R_0^2(r) \\ k(r) &= -H_0^2(r)\Omega_k^0(r)R_0^2(r) \end{aligned}$$

where

$$\Omega_m^0(r) \equiv \frac{\rho_m(r,t_0)}{\rho_{\text{crit}}(r,t_0)}, \quad \Omega_k^0(r) \equiv \frac{\rho_k(r,t_0)}{\rho_{\text{crit}}(r,t_0)}, \quad H_0(r) \equiv H(r,t_0) \ \text{ and } \ R_0(r) \equiv R(r,t_0)$$

Then the generalized Friedmann equation (24) reads

$$H^2(r,t) = H_0^2(r)\left[\Omega_k^0(r)\left(\frac{R_0(r)}{R(r,t)}\right)^2 + \Omega_m^0(r)\left(\frac{R_0(r)}{R(r,t)}\right)^3\right]$$

That should be compared with the FLRW Friedmann equation in the presence of the cosmological constant

$$H^2(t) = H_0^2\left[\Omega_k^0\left(\frac{R_0}{R(t)}\right)^2 + \Omega_m^0\left(\frac{R_0}{R(t)}\right)^3 + \Omega_\Lambda\right]$$

Comments

- The "observed" acceleration of the Universe is not a direct measurement, but a consequence of interpretation of the supernova data within the standard (FLRW) cosmology. Within FLRW Ω_Λ is a possible explanation of the observed maximal luminosity of supernovae (the observed luminosity is lower than one expected in FLRW model with $\Omega_\Lambda = 0$). Therefore in the concordance model we found $\Omega_\Lambda \simeq 0.7$ and $\Omega_m \simeq 0.3$. *Non-zero* Ω_Λ *and the standard Friedmann's equations imply* $\ddot{R} > 0$:

$$q_0 = \frac{4\pi G}{3H_0^2} \sum_i (\rho_i^0 + 3p_i^0) = \frac{4\pi G}{3H_0^2} \sum_i (1 + 3w_i)\rho_i^0 = \frac{1}{2} \sum_i \Omega_i^0 (1 + 3w_i)$$

  So, the conclusion that $\ddot{R} > 0$ and $\Omega_\Lambda \neq 0$ are consequences of *the assumed FLRW geometry.*

- When light travels from a supernova toward us it "feels" $H(r,t)$ on its way. That is seen through the expression for luminosity distance d_L. It turns out (see e.g. H. Iguchi, T. Nakamura and K. i. Nakao, "Is dark energy the only solution to the apparent acceleration of the present universe?", Prog. Theor. Phys. **108**, 809 (2002) [arXiv:astro-ph/0112419]) that the extra freedom that appears within the

LT geometry (i.e. $H(r,t)$) allows to fit the supernova data without invoking the cosmological constant (no dark energy).