Inflation

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Problems of the Standard Big-Bang Model

♠ The horizon problem

- 1991 COBE (Cosmic Background Explorer) satellite shows that the Universe is extremely isotropic on large scales ($\sim 10^3$ Mpc), the temperature fluctuations are

$$\frac{\Delta T}{T} \sim 2 \cdot 10^{-5}$$

- However not whole Universe is causally connected, so why is it so isotropic?

As we have shown the distance to the particle horizon is given by

$$d_{ph}(t) = R(t) \int_0^t \frac{dt'}{R(t')}$$

In order to find $d_{ph}(z)$ we can adopt the relation obtained earlier

$$\frac{dt}{dz} = H_0^{-1} \frac{-1}{1+z} \frac{1}{[\Omega^0_{\rm rad}(1+z)^4 + \Omega^0_m(1+z)^3 + \Omega^0_k(1+z)^2 + \Omega^0_\Lambda]^{1/2}}$$

Then

$$d_{ph}(t) = R(t) \int_0^t \frac{dt'}{R(t')} = R(t) \int_\infty^z R_0^{-1} \frac{R_0}{R(t')} \frac{dt'}{dz'} dz'$$

Inserting $\frac{dt}{dz}$ we obtain

$$d_{ph}(z) = \frac{R(t)}{R_0} \int_\infty^z (1+z') H_0^{-1} \frac{-1}{1+z'} \frac{dz'}{[\Omega^0_{\text{rad}}(1+z')^4 + \Omega^0_m (1+z')^3 + \Omega^0_k (1+z')^2 + \Omega^0_\Lambda]^{1}}$$

Using $1 + z = \frac{R_0}{R(t)}$ we have

$$d_{ph}(z) = \frac{1}{H_0(1+z)} \int_z^\infty \frac{dz'}{[\Omega^0_{\text{rad}}(1+z')^4 + \Omega^0_m (1+z')^3 + \Omega^0_k (1+z')^2 + \Omega^0_\Lambda]^{1/2}}$$

For a single component Universe and neglected curvature (even if $k \neq 1$, for early Universe Ω_k could be neglected) we had for the scale factor:

$$R(t) \propto t^{\frac{2}{3(1+w)}} \quad \text{for} \quad p = w\rho$$

Let's consider simple cases of MD and RD Universes:

$$R(t) \propto \begin{cases} t^{2/3} & \text{for} \quad w = 0 \quad \text{(MD)} \\ t^{1/2} & \text{for} \quad w = \frac{1}{3} \quad \text{(RD)} \end{cases}$$

Then integrating we obtain

$$d_{ph}(t) = R(t) \int_0^t \frac{dt'}{R(t')} = \begin{cases} 3t & \text{for} \quad w = 0 \quad \text{(MD)} \\ 2t & \text{for} \quad w = \frac{1}{3} \quad \text{(RD)} \end{cases}$$

Thus we can see that fraction of the visible Universe is varying with time

$$\frac{d_{ph}(t)}{R(t)} = \int_0^t \frac{dt'}{R(t')} \propto \begin{cases} t^{1/3} & \text{for} \quad w = 0 \quad \text{(MD)} \\ t^{1/2} & \text{for} \quad w = \frac{1}{3} \quad \text{(RD)} \end{cases}$$

So, we conclude that in the early times a much smaller fraction of the Universe was causally connected (in other words visible). The CMB photons were emitted at the time $t_{\text{rec}} \sim 1.4 \cdot 10^5 \, h^{-1}$ yr, since that time we can see more and more and still all that is so isotropic? How is that possible?

Let's estimate how many causally connected regions there were at the time of recombination that are now within the observable Universe. Assume MD at the moment of recombination (reasonable as $T_{\text{eq}} \simeq 0.37$ eV while $T_{\text{rec}} \simeq 0.26$ eV). We want to know how many horizon volumes at the time t_{rec} has expanded to fill the presently observed Universe. Let $V_0(t_0)$ be the volume of the presently observed Universe and $V_{\text{rec}}(t_{\text{rec}})$ be the horizon volume at the recombination. Then, since $RT =$ const. (because of the conservation of entropy) therefore

$$\frac{1}{V_{\text{rec}}(t_{\text{rec}})} V_0(t_{\text{rec}}) = \frac{1}{V_{\text{rec}}(t_{\text{rec}})} V_0(t_0) \left[\frac{R(t_{\text{rec}})}{R(t_0)}\right]^3 = \frac{V_0(t_0)}{V_{\text{rec}}(t_{\text{rec}})} \left[\frac{T_0}{T_{\text{rec}}}\right]^3$$

Assume now that at $t = t_{\text{rec}}$ and $t = t_0$ the Universe is MD (so $d_{ph}(t) \propto t$), then

$$\frac{V_0(t_0)}{V_{\text{rec}}(t_{\text{rec}})} = \left(\frac{d_{ph}(t_0)}{d_{ph}(t_{\text{rec}})}\right)^3 = \left(\frac{t_0}{t_{\text{rec}}}\right)^3$$

So

$$\frac{V_0(t_{\text{rec}})}{V_{\text{rec}}(t_{\text{rec}})} = \left(\frac{t_0}{t_{\text{rec}}}\right)^3 \left(\frac{T_0}{T_{\text{rec}}}\right)^3$$

Now let's eliminate time. Since $R(t) \propto t^{2/3}$ for MD (from recombination till now) therefore from the Friedmann equation

$$H = \frac{2}{3}\frac{t^{-1/3}}{t^{2/3}} = \frac{2}{3}\frac{1}{t} \propto \frac{T^{3/2}}{M_{\mathsf{Pl}}} \quad \text{for} \quad \text{(MD)}$$

so $t \propto T^{-3/2}$, thus

$$\frac{V_0(t_{\mathsf{rec}})}{V_{\mathsf{rec}}(t_{\mathsf{rec}})} = \left(\frac{t_0}{t_{\mathsf{rec}}}\right)^3 \left(\frac{T_0}{T_{\mathsf{rec}}}\right)^3 = \left(\frac{T_0}{T_{\mathsf{rec}}}\right)^{-9/2} \left(\frac{T_0}{T_{\mathsf{rec}}}\right)^3 = \left(\frac{T_{\mathsf{rec}}}{T_0}\right)^{3/2} \simeq 3.6 \cdot 10^4$$

for $T_{\mathsf{rec}} \simeq 3.0 \cdot 10^3$ K and $T_0 \simeq 2.73$. This is the number of horizon regions which expanded from the recombination time to presently observable Universe.

Let's now find entropy within causally connected regions of the Universe. First recall the approximate (valid for $(1+z) \gg (\Omega_i^0)^{-1}$) relations between the Universe age and the redshift for the two cases:

$$t \simeq \begin{cases} \frac{2}{3}(1+z)^{-3/2} H_0^{-1} (\Omega_m^0)^{-1/2} & \text{for} \quad w = 0 \quad \text{(MD)} \\ \frac{1}{2}(1+z)^{-2} H_0^{-1} (\Omega_r^0)^{-1/2} & \text{for} \quad w = \frac{1}{3} \quad \text{(RD)} \end{cases}$$

- (MD):

From the entropy conservation we have

$$s = s_0 \left(\frac{R_0}{R}\right)^3 = s_0(1+z)^3$$

where the present entropy density is known (see class): $s_0 = \frac{2\pi^2}{45} g_{\star\,S} T^3|_{\text{today}} \simeq 2970$ cm^{-3} ($g_{\star\,S}|_{\text{today}} = 3.91$). So we obtain for the entropy contained within a causally connected region

$$\begin{aligned} s_{\text{HOR}}^{(\text{MD})} &= \frac{4\pi d_{ph}^3}{3} s = \frac{4\pi}{3}\left[3 \cdot \overbrace{\frac{2}{3}(1+z)^{-3/2} H_0^{-1} (\Omega_m^0)^{-1/2}}^{t}\right]^3 s_0 (1+z)^3 \\ &\simeq 7.9 \cdot 10^{88} (h^2 \Omega_m^0)^{-3/2} (1+z)^{-3/2} \end{aligned}$$

where I have used $H_0^{-1} = 9.2503 \cdot 10^{27} h^{-1}$ cm.

- (RD):
  The entropy density reads

$$s = \frac{2\pi^2}{45} g_{\star\,S} T^3$$

Hence we get for the entropy in the causally connected region

$$
\begin{aligned}
s_{\mathsf{HOR}}^{(\mathsf{RD})} &= \frac{4\pi d_{ph}^3}{3} s = \frac{4\pi}{3}\left(2\cdot\overbrace{0.30\frac{M_{\mathsf{Pl}}}{g_\star^{1/2}T^2}}^{t}\right)^3 \frac{2\pi^2}{45} g_{\star\,S} T^3 \\
&\simeq 0.4\frac{g_{\star\,S}}{g_\star^{3/2}}\left(\frac{M_{\mathsf{Pl}}}{T}\right)^3 = \frac{0.4}{g_\star^{1/2}}\left(\frac{M_{\mathsf{Pl}}}{T}\right)^3
\end{aligned}
$$

Thus at the recombination ($T \simeq 3500$ K ~ 0.3 eV, $z \simeq 1300$) the entropy within the horizon was about $1.7 \cdot 10^{85}$, while the entropy within the presently observable Universe is $\sim 8.2 \cdot 10^{90}$, a factor of $(1+z_{\mathsf{rec}})^{3/2} \sim 10^5$ larger (the (MD) approximation was adopted in both cases, note that $z_{\mathsf{eq}} \sim 3500$). So, roughly there were 10^5 causally disconnected regions at the moment of recombination that are seen now with a very small non-isotropy $\sim 10^{-5}$! This is the horizon problem.

Let's estimate the maximal angle $\Delta\theta$ on the sky today that would correspond to the causally connected region at the moment of recombination:

- Assume (for simplicity, and also since it is experimentally favored) $k = 0$, $\Omega^0_{\text{rad}} = 0$ and $\Omega_\Lambda = 0$ (crude approximation), so $\Omega^0_m = 1$.

- Calculate $d_{ph}(t_{\text{rec}}) \simeq 3t_{\text{rec}}$ (MD was assumed as $z_{\text{rec}} \simeq 1300 \ll z_{\text{eq}} \simeq 3400$). As we have found earlier $t_{\text{rec}} \simeq \frac{2}{3}(1+z_{\text{rec}})^{-3/2} H_0^{-1} \Omega_m^{0\,-1/2}$ (valid for $(1+z) \gg (\Omega^0_m)^{-1}$ - well satisfied for z_{rec}).

- Calculate the distance Δl that light traveled from the last scattering till now assuming MD (as $z_{\text{rec}} \simeq 1300 \ll z_{\text{eq}} \simeq 3400$). From our general formula for MD we have for $\Omega^0 = 1$

$$t_0^{(MD)} = \frac{2}{3} H_0^{-1}$$

  Since $t_{\text{rec}} \simeq \frac{2}{3}(1+z_{\text{rec}})^{-3/2} H_0^{-1} \ll t_0^{(MD)}$, so the time CMB photons traveled is $\Delta t \equiv t_0^{(MD)} - t_{\text{rec}} \simeq t_0^{(MD)}$. Therefore the distance the photons traveled could be approximated by the present distance to the horizon:

$$\Delta l \simeq d_{ph}(0) = 3t_0^{(MD)} = 3\frac{2}{3} H_0^{-1}$$

- Note that while the CMB photons were traveling the space expanded by a factor of $(1+z_{\text{rec}})$ and that must be taken into account while comparing with the distance to the last scattering surface. Therefore (assuming Euclidean geometry) the angle reads

$$
\begin{aligned}
\Delta\theta &= \frac{d_{ph}(t_{\text{rec}})(1+z_{\text{rec}})}{\Delta l} = \frac{3t_{\text{rec}}(1+z_{\text{rec}})}{3t_0^{(MD)}} = \frac{3\frac{2}{3}(1+z_{\text{rec}})^{-\frac{3}{2}+1}H_0^{-1}}{3\frac{2}{3}H_0^{-1}} = (1+z_{\text{rec}})^{-1/2} \simeq \\
&\simeq 2.8\cdot 10^{-2}\text{rad} = 1.6\ ^0
\end{aligned}
$$

♠ The flatness problem

As we have shown the Friedmann equation could be written as follows:

$$\Omega_{\text{rad}} + \Omega_m + \Omega_k + \Omega_\Lambda = 1$$

where

$$\Omega_i \equiv \rho_i/\rho_{\text{crit}} \quad \text{for} \quad \rho_{\text{crit}} \equiv \frac{3H^2}{8\pi G} \quad \text{and} \quad \Omega_k \equiv \frac{-k}{(RH)^2}$$

Note that Ω_i's are functions of time. The above equation could be rewritten as

$$\underbrace{\Omega_{\text{rad}} + \Omega_m + \Omega_\Lambda}_{\Omega} - 1 = -\Omega_k = \frac{k}{(RH)^2} \tag{1}$$

So, at present we have

$$\underbrace{\Omega^0_{\text{rad}} + \Omega^0_m + \Omega^0_\Lambda}_{\Omega^0} - 1 = -\Omega^0_k = \frac{k}{(R_0 H_0)^2}$$

From observations we know that $\Omega^0 \simeq 1$.
Assuming that the Universe is dominated by just one component we have

$$R(t) = R_0 \cdot \begin{cases} \left(\frac{t}{t_0}\right)^{2/3} & \text{for} \quad w = 0 \quad \text{(MD)} \\ \left(\frac{t}{t_0}\right)^{1/2} & \text{for} \quad w = \frac{1}{3} \quad \text{(RD)} \end{cases}$$

Therefore we can estimate the rhs of (1) as a function of time:

$$-\Omega_k = \frac{k}{(RH)^2} = \frac{k}{(\dot{R})^2} \propto k \begin{cases} t^{2/3} \propto R & \text{for} \quad t \gtrsim t_{EQ} \quad \text{(MD)} \\ t \propto R^2 & \text{for} \quad t \lesssim t_{EQ} \quad \text{(RD)} \end{cases}$$

The above equation has dramatic consequences. It shows that $\frac{k}{(RH)^2} \to 0$ as $t \to 0$. Since Ω is close to 1 at present therefore it must be very close to 1 at early times.

For instance

$$\Omega(t) = \begin{cases} 1 \pm 10^{-16} & \text{for} \quad t = 1 \text{ s} \quad \text{(BBN)} \\ 1 \pm 10^{-60} & \text{for} \quad t = 10^{-43} \text{ s} \quad \text{(Planck time)} \end{cases}$$

The flatness problem is to explain why the Universe was so flat at the beginning? In other words one can say that flatness problem is caused by the instability of the initial value $\Omega \simeq 1$. Note what is the variation of $\Omega - 1$ as a function of time

$$\Omega - 1 \propto \begin{cases} t^{2/3} \propto R & \text{for} \quad t \gtrsim t_{EQ} \quad \text{(MD)} \\ t \propto R^2 & \text{for} \quad t \lesssim t_{EQ} \quad \text{(RD)} \end{cases}$$

Therefore $\Omega - 1$ is an increasing function of time, this is why such a high precision for the initial value of Ω is necessary.

♠ The monopole problem

At early times the evolution of the Universe was dominated by the presence of radiation for which the energy density decreases very fast (faster than for other components)

$$\rho_{\text{rad}} \propto \frac{1}{R^4}$$

Therefore if in the (RD) Universe there was some amount of non-relativistic matter ($\rho_m \propto R^{-3}$) it will soon dominate. If matter particles are not very heavy (as it happens in the SM) then they thermalize easily and contribute as a radiation. However GUT predicts at $T \sim 10^{15}$ GeV production of some very heavy particles, monopoles. They would be non-relativistic for the most of the expansion time and should dominate also today, when we don't observe them. So their density must be somehow diluted. This is the monopole problem. That reasoning applies also to other possible heavy particles that could be produced in the early epochs, like heavy gravitinos or modulus fields.

♠ The small-scale inhomogeneities problem

Even though the Universe is very homogeneous at large scales, there is a lot of structures at scales ranging from 1 to 100 Mpc. The problem is that we don't know where do the inhomogeneities necessary for the structure formation come from.

♠ The cosmological constant problem

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}$$

The Λ term could be written as a part of the energy-momentum tensor:

$$T^{\Lambda}_{\mu\nu} = \frac{\Lambda}{8\pi G} g_{\mu\nu} = \rho_\Lambda g_{\mu\nu} \quad \text{for} \quad \rho_\Lambda \equiv \frac{\Lambda}{8\pi G}$$

Then the Einstein's Field Equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G \left(T_{\mu\nu} + T_{\mu\nu}^{\Lambda} \right)$$

Let's return to the Friedmann equation

$$H^2 = -\frac{k}{R^2} + \frac{8\pi G}{3}(\rho_m + \rho_{\mathrm{rad}} + \rho_\Lambda) = -\frac{k}{R^2} + \frac{8\pi G}{3}(\rho_m + \rho_{\mathrm{rad}}) + \frac{\Lambda}{3}$$

The ratio of the two last terms is know from observations to be at present

$$r_\Lambda \equiv \frac{\Lambda}{8\pi G(\rho_m + \rho_{\mathrm{rad}})} \lesssim 1$$

That implies that at the Planck time $r_\Lambda \lesssim 10^{-122}$, impressively small number! The difficulty to explain that constitutes the cosmological constant problem.

The Basic Mechanism of Inflation

There was an epoch (the de Sitter phase) when the vacuum energy (the cosmological constant) was the dominant component of the energy density of the Universe, then the expansion was exponential.
The Einstein's Field Equations with the cosmological constant are as follows

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi G \left(T_{\mu\nu} + T^{\Lambda}_{\mu\nu} \right)$$

Then the Friedmann equation reads

$$H^2 = -\frac{k}{R^2} + \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad} + \rho_\Lambda) = -\frac{k}{R^2} + \frac{8\pi G}{3}(\rho_m + \rho_{\rm rad}) + \frac{\Lambda}{3} \simeq \frac{\Lambda}{3}$$

So that

$$H^2(t) \equiv \left(\frac{\dot{R}}{R}\right)^2 = \frac{\Lambda}{3} \quad \Longrightarrow \quad R(t) = R(t_0) e^{H\cdot(t-t_0)} \quad \text{for} \quad H^2 = \frac{\Lambda}{3} = \text{const.}$$

This is the exponential *inflation*: exponential growth of the scale factor.
During this epoch a small, smooth, and causally connected patch of the size less then

H^{-1} grows to such a size that at present it contains the whole observable Universe. It is usually assumed that a scalar field is responsible for inflation: it provides the necessary equation of state $p = -\rho$.
Basic steps in the evolution of the scalar field responsible for inflation:

- Transition from $\phi = 0$ to $\phi = \phi_{\text{in}}$ (spatially uniform) through a possible barrier (denoted as (a) in the figure 3).

- Classical evolution (the "slow-roll") toward the minimum (b) of the potential according to (see class)

$$\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0$$

  It is just the equation for sliding down hill with a friction ($3H$). If the potential is flat enough one can have the time of "slow-rolling" Δt large comparing to the Hubble time: $H\Delta t \gg 1$.

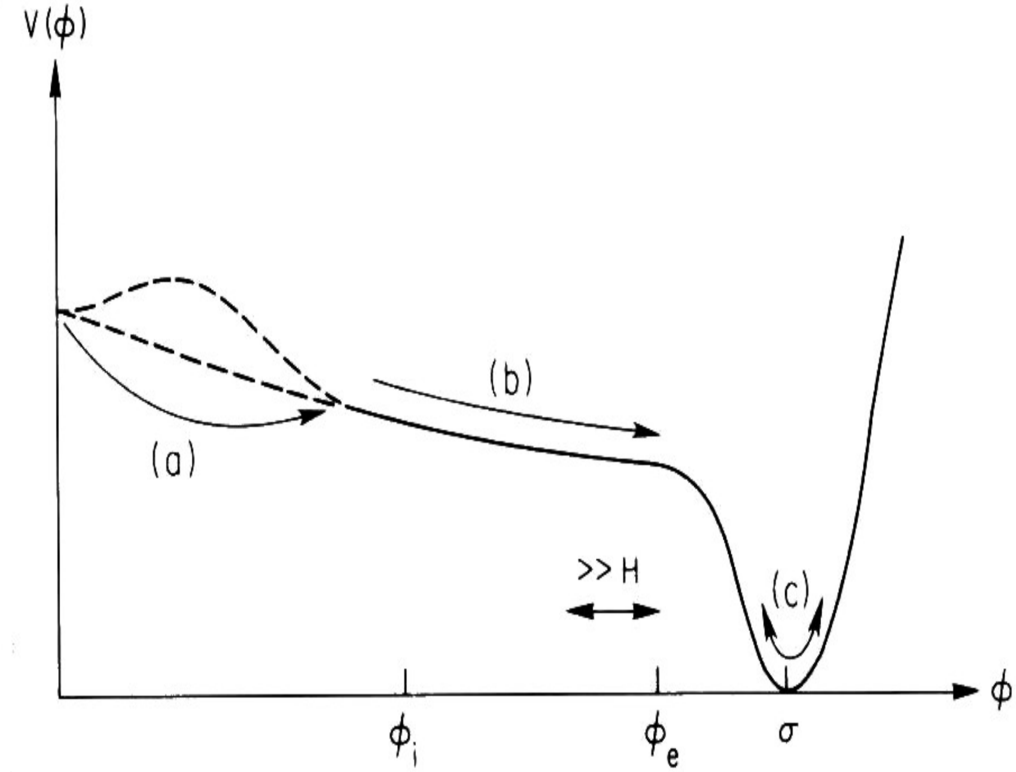

Fig. 8.1: Schematic illustration of an inflationary potential. Notice the "flatness" of the potential—a feature common to all inflationary models. The three qualitative phases of inflation are depicted: (a) barrier penetration (if necessary); (b) slow roll; and (c) coherent oscillations about the minimum of the potential.

Figure 1: *The slow-roll potential (from Kolb & Turner).*

Is that possible ? Note that often the universe age $t_U \sim H^{-1}$.

For the de Sitter phase $H =$ const., so

$$t - t_{\text{Pl}} = \int_{R_{\text{Pl}}}^{R(t)} \frac{dR'}{\dot{R}'} = H^{-1} \int_{R_{\text{Pl}}}^{R(t)} \frac{dR'}{R'} = H^{-1} \ln \left(\frac{R(t)}{R_{\text{Pl}}} \right)$$

Therefore as long as $R(t) \gg R_{\text{Pl}}$ we obtain $t - t_{\text{Pl}} \gg H^{-1}$. In other words it is possible that $H\Delta t \gg 1$. Note that if $V(\phi = 0) \equiv M^4$ then during inflation period we have roughly

$$R(t) \propto e^{Ht} \quad \text{for} \quad H^2 \simeq \frac{8\pi G}{3} V(\phi = 0) \simeq \frac{M^4}{M_{\text{Pl}}^2}$$

So for $H\Delta t \gg 1$ one needs M large enough.

Assume:

- ϕ is spatially uniform
- The size of the initial patch $R_{\text{in}} = 10^{-23}$ cm (exact number is not very relevant)
- $M = 10^{14}$ GeV

- $\Delta t = 100 H^{-1} \quad \Longrightarrow \quad R(t_{\text{fin}}) = R(t_{\text{in}}) \times e^{100} \simeq R(t_{\text{in}}) \times 3 \cdot 10^{43}$

- During inflation

$$H^{-1} \simeq \frac{M_{\text{Pl}}}{M^2} = \frac{1.22 \cdot 10^{19}\ \text{GeV}}{10^{28}\ \text{GeV}^2} \simeq 8.02 \cdot 10^{-34}\ \text{s} \quad \text{and} \quad \Delta t = 8.02 \cdot 10^{-32}\ \text{s}$$

for $\text{GeV}^{-1} = 6.5822 \cdot 10^{-25}$ s.

- When ϕ reaches σ (the minimum) it starts oscillate (c) coherently as a uniform field around $\phi = \sigma$. The oscillations are dumped by interactions between ϕ and the SM: the energy of oscillations is transfered (through production of SM particles) from ϕ to the SM. That results in an increase of the SM temperature (the reheating):

$$\rho_{SM} \sim V(\phi = 0) = M^4 \quad \Longrightarrow \quad T_{RH} \sim M$$

Note that the energy density remains the same during inflation (since $V(\phi = 0) \sim M^4$). The expanding Universe is being filled with the constant field - the cosmological constant. During the reheating the whole energy is transfered quickly (with a small change of the scale factor) to SM particles, therefore $T_{RH} \sim M$.

– The entropy at the beginning of inflation (with temperature T_{in}) reads:

$$\begin{aligned} S_{\text{in}} &\sim (T_{\text{in}}R_{\text{in}})^3 = (10^{14}\ \text{GeV} \times 10^{-23}\ \text{cm})^3 \simeq \left(5.07 \times 10^{14-23+14}\right)^3 \\ &\simeq 1.25 \times 10^{14} \ll S_0 \simeq 10^{88} \end{aligned}$$

where S_0 is the entropy in the presently observed Universe and $1\ \text{GeV} \times 1\ \text{cm} = 5.07 \cdot 10^{13}$ was used. During inflation the entropy is conserved. Since $R(t) \propto e^{Ht}$ therefore $T \propto e^{-Ht}$, the patch "supercools".

– Massive entropy production happens during the reheating (after inflation ended). The temperature returns to it's value at the beginning of inflation i.e. $T \sim 10^{14}$ GeV and the final entropy is

$$S_{\text{fin}} \sim \left(T_{\text{RH}} e^{H\Delta t} R_{\text{in}}\right)^3 \sim 2.53 \cdot 10^{144} \gg S_0 \simeq 10^{88}$$

So the reheating process increases the entropy by a factor 10^{130}.

– The final size of the Universe after inflation is $e^{H\Delta t} R_{\text{in}} \sim 2.7 \cdot 10^{20}$ cm

How does the inflation cures the problems of the standard cosmology?

- The horizon problem

  The presently observable Universe contains (10^{88}) much less entropy than the entropy contained in initial patch after inflation (10^{144}), so the whole content of the presently observable Universe could have easily been in a causal contact at the moment of the recombination. So, no wonder it is so smooth.

The presently observed Universe contains (without inflation) about 10^5 regions which were causally disconnected at the moment of recombination. The distance to the horizon is given by

$$d_{ph}(t) = R(t) \int_0^t \frac{dt'}{R(t')}$$

Assuming $R(t) \propto e^{Ht}$ and neglecting the period preceding inflation we obtain

$$d_{ph}(t) = H^{-1}(e^{Ht} - 1)$$

The meaning of that is that when the Universe reheated after inflation to the temperature $T \sim M$ typical for the period prior to inflation, the distance to the horizon was by the factor $e^{H\Delta t} \simeq 3 \cdot 10^{43}$ larger than at the moment before inflation (when the temperature was the same as after inflation). Inflation breaks the standard relation between temperature and distance to the horizon. The reason is the massive entropy non-conservation. Therefore it is easy for the horizon at the recombination to contain many times over the volume which will expand to the presently seeable Universe.

- The flatness problem

$$\underbrace{\Omega_{\text{rad}} + \Omega_m + \Omega_\Lambda}_{\Omega} - 1 = -\Omega_k = \frac{k}{(RH)^2} \tag{2}$$

During inflation $H^2 \sim \frac{M^4}{M_{\text{Pl}}^2} = \text{const.}$ while $R^2 \to e^{200} R^2$, therefore

$$\Omega - 1 \sim \frac{k}{R^2} \frac{M_{\text{Pl}}^2}{M^4} \to \frac{1}{e^{200}} \times \frac{k}{R^2} \frac{M_{\text{Pl}}^2}{M^4}$$

In other words whatever was the initial value of $\Omega - 1$ it is very close to 0 after inflation.

- The monopole problem
The concentration of any relic produced before inflation is reduced by the factor e^{-300}, so the concentration of monopoles would be negligible after inflation.

- <u>The small-scale inhomogeneity problem</u>
  The post inflationary patch is exactly homogeneous as a consequence of homogeneity of the scalar field ϕ, however some inhomogeneities are needed a seeds to build large scale structures. Those will be provided by the "quantum" fluctuations of the ϕ.

- <u>The cosmological constant problem</u>
  No solution is offered here by inflation.

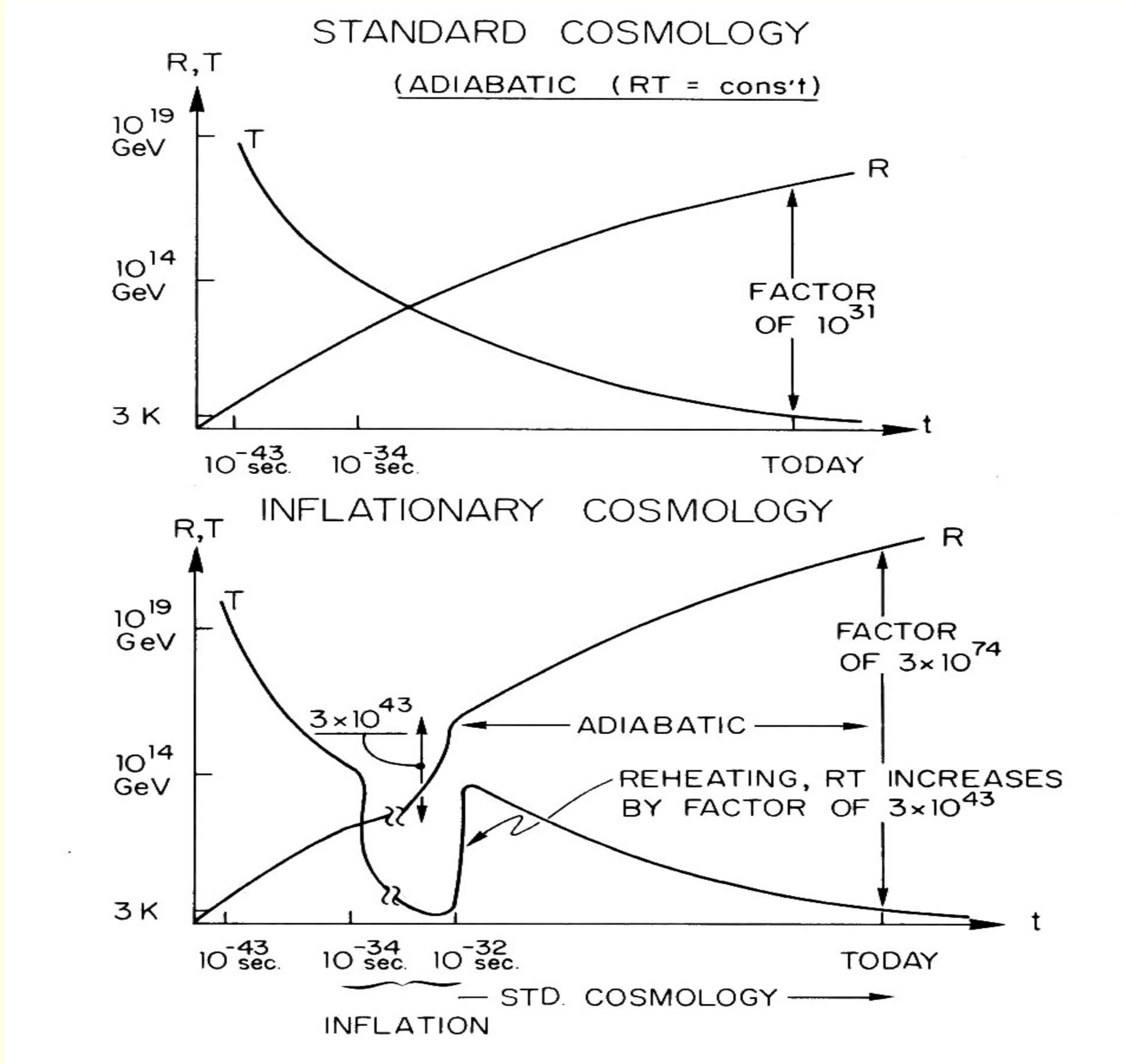

Fig. 8.2: Comparison of the evolution of R and T in the standard and inflationary cosmologies. Note the enormous jump in entropy ($S \propto R^3T^3$) at the end of inflation.

Figure 2: *Evolution of temperature and scale factor in the standard and inflationary cosmologies (from Kolb & Turner).*

Models of Inflation

Assumptions:

- A single scalar real field ϕ is responsible for inflation.

- $$H^2 = \frac{8\pi G}{3}\rho_\phi - \frac{k}{R^2}$$

  where ρ_ϕ is the energy density of the scalar filed, so ϕ dominates the energy density.

- The scalar field is homogeneous with the initial value $\phi_{\text{ini}} \neq \sigma$ for σ being the global minimum of the potential, i.e. $V(\sigma) = V'(\sigma) = 0$.

- "Quantum" fluctuations of the scalar field are "small" compared to the classical solution:

  $$\phi(t) = \phi_{\text{cl}} + \delta\phi \quad \text{with} \quad \delta\phi \ll \phi_{\text{cl}}$$

- The scalar field is described by

  $$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - V(\phi)$$

The energy-momentum tensor reads (see class)

$$T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \mathcal{L} g^{\mu\nu}$$

For the perfect fluid we had

$$T^{\mu\nu} = (p + \rho) U^\mu U^\nu - p g^{\mu\nu}$$

where U^μ is the 4-velocity. To match the above forms one requires (assuming spatial homogeneity)

$$\begin{array}{ccc} (0,0) & \Longrightarrow & p + \rho - p g^{00} = \dot{\phi}^2 - \mathcal{L} g^{00} \\ (i,i) & \Longrightarrow & -p g^{ii} = -\mathcal{L} g^{ii} \end{array} \tag{3}$$

that implies $p = \mathcal{L} = \frac{1}{2}(\dot{\phi})^2 - V(\phi)$ and $\rho = \frac{1}{2}(\dot{\phi})^2 + V(\phi)$.
So for spatially homogeneous field we obtain:

$$\begin{array}{llll} \rho_\phi & \equiv T^{00} & = (\dot{\phi})^2 - [\frac{1}{2}(\dot{\phi})^2 - V(\phi)] g^{00} & = \frac{1}{2}(\dot{\phi})^2 + V(\phi) \\ p_\phi & \equiv T^{ii} & = -[\frac{1}{2}(\dot{\phi})^2 - V(\phi)] g^{ii} & = \frac{1}{2}(\dot{\phi})^2 - V(\phi) \end{array}$$

- The classical equation of motion for ϕ could be obtained from variation of the action $S = \int d^4x\sqrt{-g}\mathcal{L}$ (see class) or just from the appropriate replacement in the Klein-Gordon equation:

$$\phi^{,\mu}{}_{;\mu} = -V'(\phi)$$

From that we get (see class)

$$\ddot{\phi} + 3H\dot{\phi} + V(\phi)' = 0 \tag{4}$$

♠ The "slow-roll"

We will look for the "slow-roll" solutions of (4), i.e. such that $\ddot{\phi}$ could be neglected, so

$$\dot{\phi} = -\frac{V'(\phi)}{3H} \quad \text{for} \quad \left|\frac{\ddot{\phi}}{3H\dot{\phi}}\right| \ll 1 \quad \text{and} \quad \left|\frac{\ddot{\phi}}{V'(\phi)}\right| \ll 1 \tag{5}$$

$\ddot{\phi}$ could be estimated from (5):

$$\ddot{\phi} = -\frac{1}{3H}V''(\phi)\dot{\phi} + \frac{1}{3}\frac{\dot{H}}{H^2}V'(\phi)$$

We can also estimate H^2:

$$H^2 = \frac{8\pi G}{3}\rho_\phi \sim \frac{8\pi G}{3}V(\phi) \quad \Longrightarrow \quad 2H\dot{H} \sim \frac{8\pi G}{3}V'(\phi)\dot{\phi} \tag{6}$$

where I neglected the kinetic term in

$$\rho_\phi = \frac{1}{2}(\dot{\phi})^2 + V(\phi) \sim V(\phi)$$

As we will see in the slow-roll region the above approximation is justified. Now, using (6) we can rewrite the first condition that allows us to neglect $\ddot{\phi}$:

$$\left|\frac{\ddot{\phi}}{3H\dot{\phi}}\right| = \left|-\frac{1}{9H^2}V''(\phi) + \frac{8\pi G}{54H^4}[V'(\phi)]^2\right| \ll 1$$

To ensure the above inequality we must require

$$\frac{|V''(\phi)|}{9H^2} \ll 1 \quad \text{and} \quad \frac{8\pi}{54M_{\mathsf{Pl}}^2H^4}[V'(\phi)]^2 \ll 1$$

Using (6) we can obtain the slow-roll conditions (should be satisfied during inflation)

$$\eta(\phi) \equiv \frac{M_{\mathsf{Pl}}^2}{8\pi}\frac{|V''(\phi)|}{V(\phi)} \ll 3 \quad \text{and} \quad \epsilon(\phi) \equiv \frac{M_{\mathsf{Pl}}^2}{16\pi}\left(\frac{V'(\phi)}{V(\phi)}\right)^2 \ll 3$$

The second condition that allows us to neglect $\ddot{\phi}$ reads

$$\left|\frac{\ddot{\phi}}{V'(\phi)}\right| \ll 1$$

Using the same arguments as for the first condition one can show that

$$\left|\frac{\ddot{\phi}}{V'(\phi)}\right| = \left|\frac{\eta(\phi)}{3} - \frac{2\epsilon(\phi)}{3}\right| \ll 1$$

Therefore it is also satisfied when the slow-roll conditions $\eta(\phi) \ll 3$ and $\epsilon(\phi) \ll 3$ are imposed.

Now we can show that indeed if the slow-roll conditions are satisfied then the energy density is dominated by the potential energy. Form (5) we obtain

$$\frac{1}{2}(\dot{\phi})^2 = \frac{[V'(\phi)]^2}{18H^2}$$

Let's denote

$$\kappa \equiv \frac{\frac{1}{2}(\dot{\phi})^2}{V(\phi)}$$

Then we can write the Fiedmann equation as

$$H^2 = \frac{8\pi G}{3} V(\phi)(\kappa + 1)$$

So, we obtain

$$\frac{\frac{1}{2}(\dot{\phi})^2}{V(\phi)} \equiv \kappa = \frac{[V'(\phi)]^2}{18H^2} \frac{1}{V(\phi)} = \underbrace{\frac{M_{\mathsf{Pl}}^2}{16\pi} \left(\frac{V'(\phi)}{V(\phi)}\right)^2}_{\epsilon} \frac{1}{3(\kappa+1)} = \frac{\epsilon(\phi)}{3(\kappa+1)} \ll \frac{1}{(\kappa+1)}$$

which could be written as

$$\kappa(\kappa+1) \ll 1$$

So indeed, the energy density is dominated by the potential energy.
Now we can calculate how many e-folds of inflation happened when the scalar field slow-rolled from ϕ_{in} to ϕ_{fin}:

$$N_e \equiv \ln \frac{R_{\text{fin}}}{R_{\text{in}}}$$

Since

$$\int_{t_{\text{in}}}^{t_{\text{fin}}} H dt = \int_{t_{\text{in}}}^{t_{\text{fin}}} \frac{\dot{R}(t)}{R(t)} dt = \int_{R_{\text{in}}}^{R_{\text{fin}}} \frac{dR'}{R'} = \ln \frac{R_{\text{fin}}}{R_{\text{in}}} \quad \Longrightarrow \quad N_e = \int_{t_{\text{in}}}^{t_{\text{fin}}} H dt$$

Thus since $\dot{\phi} = -\frac{V'(\phi)}{3H}$ we have

$$N_e = \ln \frac{R_{\text{fin}}}{R_{\text{in}}} = \int_{\phi_{\text{in}}}^{\phi_{\text{fin}}} H \frac{d\phi}{\dot{\phi}} = -3 \int_{\phi_{\text{in}}}^{\phi_{\text{fin}}} H^2 \frac{d\phi}{V'(\phi)} = -\frac{8\pi}{M_{\text{Pl}}^2} \int_{\phi_{\text{in}}}^{\phi_{\text{fin}}} \frac{V(\phi)}{V'(\phi)} d\phi \qquad (7)$$

Now we approximate the derivative of the potential

$$V'(\phi) \simeq V'(\phi_{\text{in}}) + V''(\phi_{\text{in}})(\phi - \phi_{\text{in}}) \quad \text{with} \quad V'(\phi_{\text{in}}) \simeq 0$$

and use it to estimate variation of $\phi = \phi(t)$

$$\dot{\phi} = -\frac{V'(\phi)}{3H} = -\frac{V''(\phi_{\text{in}})}{3H}(\phi - \phi_{\text{in}}) \qquad \Longrightarrow \qquad \frac{\dot{\phi}}{\phi - \phi_{\text{in}}} = -\frac{V''(\phi_{\text{in}})}{3H}$$

Neglecting small variation of H we obtain

$$\phi - \phi_{\text{in}} \sim \exp\left[-\frac{V''(\phi_{\text{in}})}{3H}(t - t_{\text{in}})\right]$$

To retain the slow motion of the field we must limit the rolling to the period that is not larger than

$$\Delta t \sim \frac{3H}{|V''(\phi_{\text{in}})|}$$

Then

$$N_e = \ln\frac{R_{\text{fin}}}{R_{\text{in}}} = \int_{t_{\text{in}}}^{t_{\text{fin}}} H dt \sim H\Delta t \sim \frac{3H^2}{|V''(\phi_{\text{in}})|}$$

Since $H^2 \sim \frac{8\pi G}{3} V(\phi)$ we have

$$N_e \sim \frac{8\pi V(\phi)}{M_{\text{Pl}}^2 |V''(\phi_{\text{in}})|} \gg \frac{1}{3}$$

so it is possible to obtain many e-folds of inflation. The above relation is a crude approximation, to get more precise estimate one would need to adopt (7).

♠ The reheating - coherent oscillations

The slow-roll ends when ϕ reaches the region of steeper potential that is closed to its absolute minimum. Then the period of coherent (spatially uniform field) oscillations starts. We assume that ϕ couples to the SM, so the oscillations are dumped through production of SM particles: inflaton energy is converted into energy of SM particles. Hereafter we assume that the decay rate Γ_ϕ of inflaton satisfies: $\Gamma_\phi \lesssim H_{\text{osc}}$ where H_{osc} is the Hubble parameter when the oscillations start.

In order to take into account the damping we modify the equation of motion for ϕ

$$\ddot{\phi} + 3H\dot{\phi} + \Gamma_\phi \dot{\phi} + V'(\phi) = 0$$

For small oscillations around $\phi = \sigma$, when the friction terms ($\propto \dot{\phi}$) are neglected we obtain just harmonic oscillations with frequency $\omega^2 = V''(\sigma)$.

Since $\rho_\phi = \frac{1}{2}(\dot{\phi})^2 + V(\phi)$ we can rewrite the above equation (first multiplying by $\dot{\phi}$)

$$\dot{\rho}_\phi + (3H + \Gamma_\phi)\dot{\phi}^2 = 0 \tag{8}$$

For simple harmonic oscillations, the average of the kinetic energy over an oscillation period equals the average of the potential energy over a period

$$\frac{1}{2}\langle\dot{\phi}^2\rangle = \langle V(\phi)\rangle = \frac{1}{2}\langle\rho_\phi\rangle$$

Therefore we replace in (8): $\dot{\phi}^2 \to \langle\dot{\phi}^2\rangle = \langle\rho_\phi\rangle$, i.e.

$$\dot{\rho}_\phi + (3H + \Gamma_\phi)\rho_\phi = 0 \tag{9}$$

where from now on ρ_ϕ denotes the averaged energy density. The equation we have obtained is a Boltzmann-like equation which describes the evolution of energy density of massive particles that can decay (and therefore disappear). Its solution (see class) reads

$$\rho_\phi = M^4 \left(\frac{R_{\text{osc}}}{R}\right)^3 e^{-\Gamma_\phi(t - t_{\text{osc}})}$$

where $_{\text{osc}}$ refers to the moment when the oscillations start while M^4 denotes the energy density at that time.
We assume that ϕ is so heavy that its decay products are highly relativistic, then we

have in addition the following relevant equations:

$$\dot{\rho}_{\mathsf{rel}} + 4H\rho_{\mathsf{rel}} = \Gamma_\phi \rho_\phi \tag{10}$$

$$H^2 = \frac{8\pi}{3M_{\mathsf{Pl}}^2}(\rho_\phi + \rho_{\mathsf{rel}}) \tag{11}$$

where ρ_{rel} is the energy density of the relativistic decay products of ϕ. The equations (9-11) describe the reheating. Let's summarize the important aspects of the reheating

- From $t \simeq t_{\mathsf{osc}}$ till $t \simeq t_{\mathsf{osc}} + \Gamma_\phi^{-1}$, inflatons (NR by assumption) dominate the energy density (the coherent ϕ oscillations), so the Universe behaves like in the MD phase: $R \propto t^{2/3}$.

- During the de Sitter phase the Universe goes through supercooling, so at the beginning of oscillations we have $\rho_{\mathsf{rad}} \simeq 0$

- During the ϕ-dominated epoch one can find (see class) the following approximate solution of (10)

$$\rho_{\mathsf{rad}} \simeq \frac{M_{\mathsf{Pl}}^2 \Gamma_\phi}{10\pi} \frac{1 - \left(\frac{t_{\mathsf{osc}}}{t}\right)^{5/3}}{t} \simeq$$

$$\simeq \quad \frac{6^{1/2}}{\pi^{1/2}10} M_{\text{Pl}} \Gamma_\phi M^2 \left(\frac{R_{\text{osc}}}{R}\right)^{3/2} \left[1 - \left(\frac{R_{\text{osc}}}{R}\right)^{5/2}\right] \qquad (12)$$

where $\propto t^{-1}$ is a special solution of (10) with non-zero rhs, while $\propto t^{-8/3}$ is the general solution of the homogeneous equation (as $\rho_{\text{rad}} \propto R^{-4}$). So ρ_{rad} first grows rapidly from zero to $M_{\text{Pl}}\Gamma_\phi M^2$ and then it decreases as $R^{-3/2}$, so the temperature increases only at the very beginning of oscillation period, then it decreases, see figure. The maximal temperature achieved is

$$T_{\text{max}} \simeq 0.8 g_\star^{-1/4} M^{1/2} \left(\Gamma_\phi M_{\text{Pl}}\right)^{1/4}$$

- For $t \simeq \Gamma_\phi^{-1}$, inflatons start to decay efficiently, so that the Universe becomes radiation dominated, the temperature at the beginning of the standard RD phase is

$$T_{RH} \equiv T(t = \Gamma_\phi^{-1}) \simeq 0.55 g_\star^{-1/4} \left(M_{\text{Pl}}\Gamma_\phi\right)^{1/2}$$

Note that the reheat temperature is determined by Γ_ϕ, not by the initial vacuum energy M.

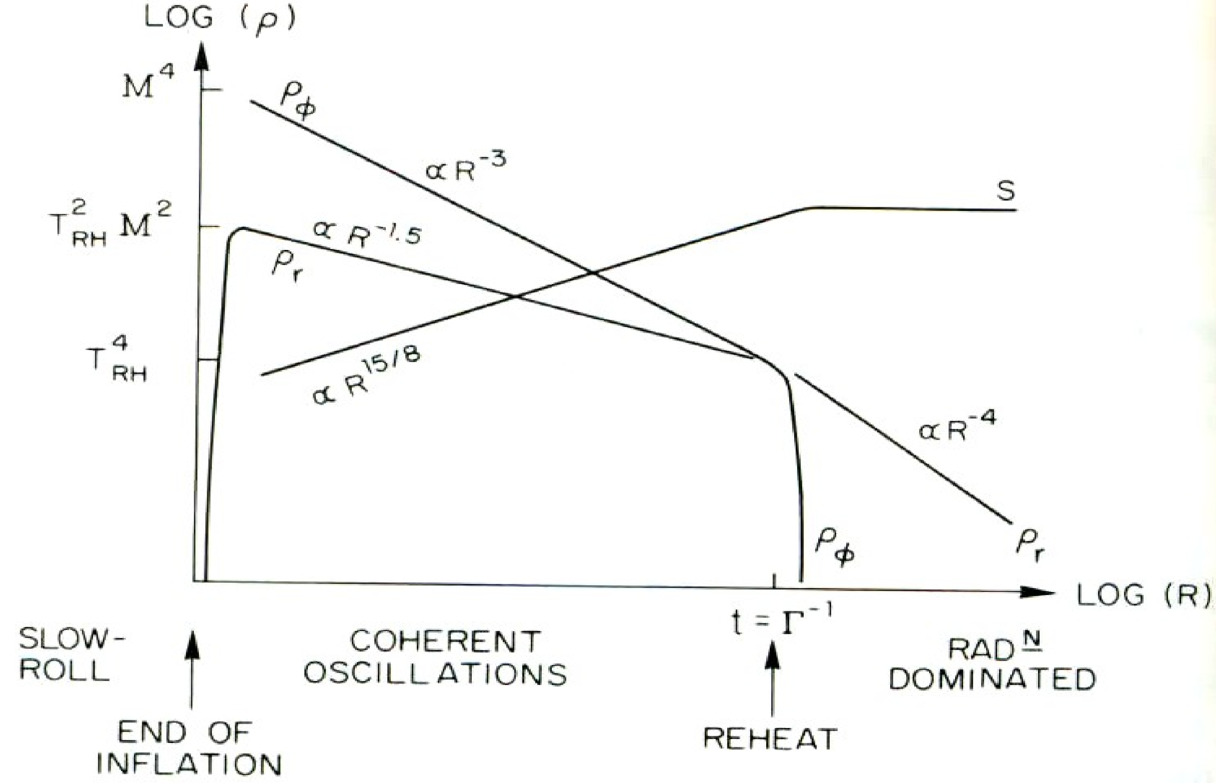

Fig. 8.3: Summary of the evolution of ρ_ϕ, ρ_R, and S during reheating.

Figure 3: *Evolution during reheating (from Kolb & Turner).*

- Soon after or even during the reheating the baryon asymmetry should be generated. If it happened before inflation it would be exponentially diluted. The basic mechanisms are
  - Through CP-violating decays of X bosons within a GUT.

– Through the reheating itself, so in the process of CP-violating inflaton decays that lead to production of SM particles.

♠ How many e-folds is needed to solve the horizon problem?
After inflation the Universe must contain at least 10^{88} of entropy. Suppose that the initial size of the region that would grow to our observable Universe is of the size of $H^{-1} \sim \frac{M_{\rm Pl}}{M^2}$. During inflation its size grows by the factor of e^{N_e} while during reheating by

$$\frac{R_{RH}}{R_{\rm osc}} = \left(\frac{t_{RH}}{t_{\rm osc}}\right)^{2/3}$$

Since during the reheating we assumed MD, therefore

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \left(\frac{2}{3}\right)^2 \frac{1}{t^2} = \frac{8\pi}{3M_{\rm Pl}^2}\rho$$

At the beginning of reheating $\rho \propto M^4$ while at the end (pure radiation) $\rho \propto T_{RH}^4$, so we obtain

$$\frac{R_{RH}}{R_{\rm osc}} = \left(\frac{t_{RH}^2}{t_{\rm osc}^2}\right)^{1/3} \simeq \left(\frac{\rho_{\rm osc}}{\rho_{RH}}\right)^{1/3} \simeq \left(\frac{M^4}{T_{RH}^4}\right)^{1/3}$$

So the entropy at the end of reheating is

$$S_{\mathsf{fin}} \simeq e^{3N_e}\left(\frac{M}{T_{RH}}\right)^4 \left(H^{-1}T_{RH}\right)^3 \simeq e^{3N_e}\frac{M^4}{T_{RH}^4}\frac{T_{RH}^3}{H^3} \simeq e^{3N_e}\frac{M^4}{T_{RH}^4}\frac{T_{RH}^3}{1}\frac{M_{\mathsf{Pl}}^3}{M^6} = e^{3N_e}\frac{M_{\mathsf{Pl}}^3}{T_{RH}M^2}$$

where I used the fact that $H^{-1} \simeq \frac{M_{\mathsf{Pl}}}{M^2}$. From the condition $S \geq 10^{88}$ we obtain

$$N_e \geq 53 + \frac{2}{3}\ln\left(\frac{M}{10^{14}\ \mathsf{GeV}}\right) + \frac{1}{3}\ln\left(\frac{T_{RH}}{10^{10}\ \mathsf{GeV}}\right)$$

Varying M and T_{RH} between 1 GeV and M_{Pl} the rhs ranges from 24 to 68.

♠ How many e-folds is needed to solve the flatness problem?

Roughly the same amount of inflation is needed for the flatness problem as for the horizon problem.

Dark Energy

Current data shows that the Universe is presently accelerating. For $\Omega_\Lambda \simeq 0.7$ and $\Omega_m^0 \simeq 0.3$ the deceleration parameter is negative

$$q_0 = \frac{1}{2}\sum_i \Omega_i^0 \left(1 + 3\frac{p_i}{\rho_i}\right) \simeq \frac{1}{2}[0.3 \times 1 + 0.7 \times (1-3)] = -0.45$$

The contribution to the energy density in the form of Ω_Λ is called "dark energy". However it could also have its roots in some kind of unknown matter, so called "quintessence". The simplest model of the quintessence is a scalar field Q with slowly rolling potential (as it was for the inflaton). The quintessence is supposed to describe dynamically variation of Λ which is needed in various epochs:

$$\left.\frac{\Omega_m}{\Omega_\Lambda}\right|_{\text{now}} = \mathcal{O}(1)$$

As we know for the scalar field Q the parameter of the equation of state ($p = w\rho$) is the following

$$w_Q = \frac{\frac{1}{2}\dot{Q}^2 - V(Q)}{\frac{1}{2}\dot{Q}^2 + V(Q)}$$

so for $\dot{Q}^2 \ll V(Q)$, $w \to -1$ as for Λ, in general $-1 \leq w \leq 1$. For the accelerating universe (so for the one we observe now) one needs $w < -\frac{1}{3}$ if only one component dominates.
When the potential is specified (e.g. $V(Q) \propto e^{-Q}$ or $V(Q) \propto Q^{-1}$) the Friedmann equation, the equation of motion for Q and the energy "conservation" determine the dynamics:

$$\begin{aligned} & H^2 = \frac{8\pi G}{3}\left(\rho + \frac{1}{2}\dot{Q}^2 + V(Q)\right) \\ & \ddot{Q} + 3H\dot{Q} + V'(Q) = 0 \\ & \dot{\rho} + 3(1+w)H\rho = 0 \end{aligned} \tag{13}$$

where ρ and w refer to the dominant component of the Universe in a given epoch (so not the quintessence).
Example:
I assume no interaction between "the single component" and Q. Let's consider $V(Q) = V_0 e^{-Q/\mu}$ adopting the *ansatz* $R(t) \propto t^\beta$ and $Q = Q_0 \ln t$ and find the time evolution of the dark energy density ρ_{de}.

$$R \propto t^\beta \quad \Longrightarrow \quad H = \frac{\dot{R}}{R} = \frac{\beta}{t}$$

From $Q = Q_0 \ln t$ we obtain

$$\dot{Q} = \frac{Q_0}{t} \quad \text{and} \quad \ddot{Q} = -\frac{Q_0}{t^2}$$

Inserting into the equation of motion for Q we get

$$-\frac{Q_0}{t^2} + 3\frac{\beta}{t}\frac{Q_0}{t} - \frac{1}{t^{Q_0}} = 0$$

where I have used the fact that $V'(Q) = -e^{-Q} = -t^{-Q_0}$. Therefore we get $Q_0 = 2$ and $\beta = \frac{1}{2}$. Then

$$\rho_Q = \frac{1}{2}\dot{Q}^2 + V(Q) = \frac{1}{2}\left(\frac{Q_0}{t}\right)^2 + \frac{1}{t^2} = \frac{3}{t^2}$$

Since both for RD ($\rho \propto R^{-4}$ and $R \propto t^{1/2}$) and MD ($\rho \propto R^{-3}$ and $R \propto t^{2/3}$) we have $\rho(t) \propto t^{-2}$, we conclude that the exponential potentials are not good as for they

$$\frac{\rho_m}{\rho_Q} = \text{const.} \neq \frac{1}{R^3}$$

Other potential may provide different time dependence of ρ_Q and ρ_m. However the existing models must cope with the problem of very small mass for the quintessence particles $m_Q \sim H_0 \simeq 10^{-33}$ eV.

Summary of the Universe evolution

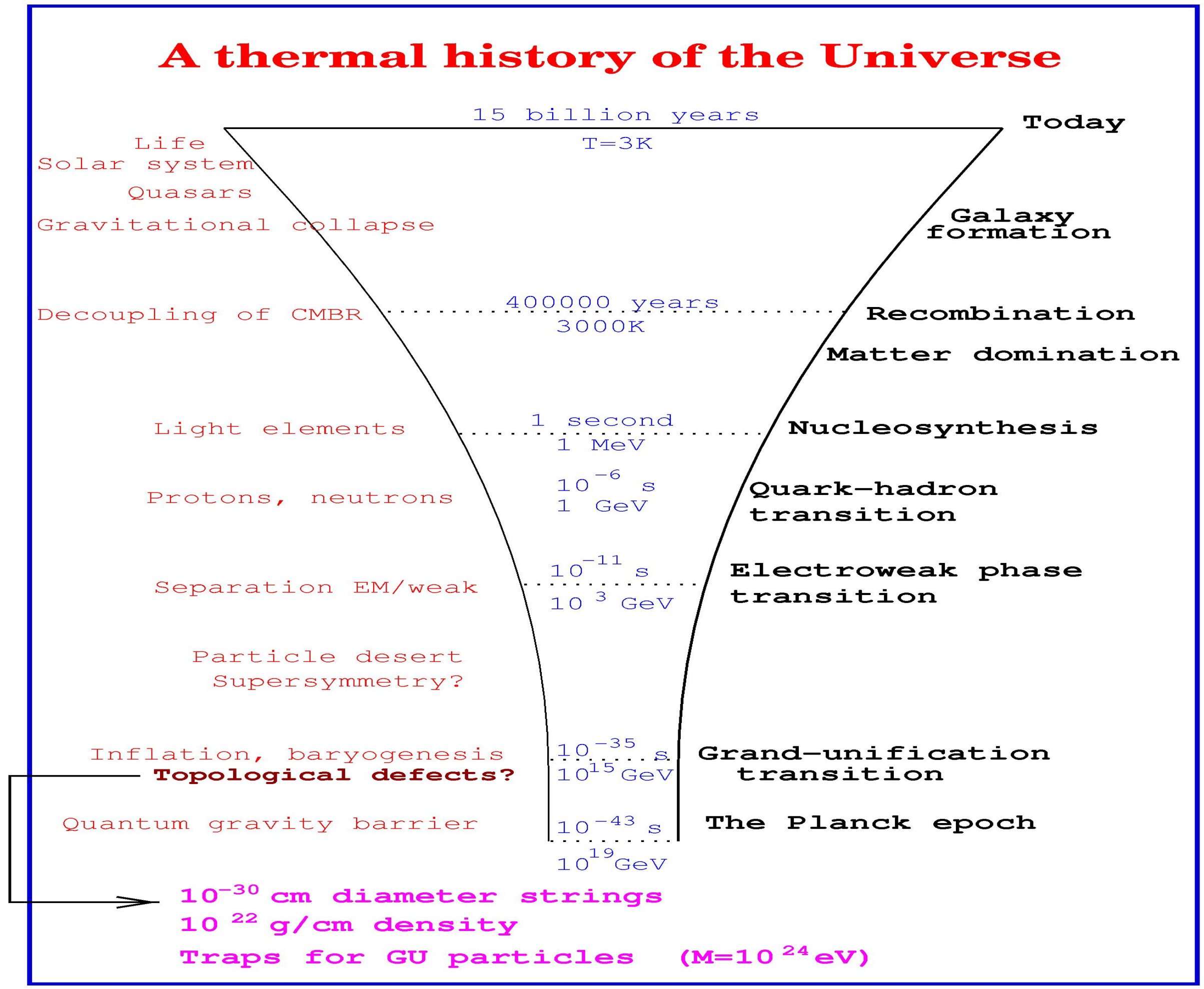

Figure 4: *History of the Universe. Form lpnhe-auger.in2p3.fr/slides/vulg/.*