Gravitational Lensing

- Freely falling particle in a static isotropic gravitational field
- Deflection of light by the Sun
- Observation of gravitational lensing

Freely falling particle in a static isotropic gravitational field

For static and isotropic gravitational field the metric in the standard form reads

$$d\tau^2 = B(r)dt^2 - A(r)dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

where $B(r) = \frac{1}{A(r)} = 1 - \frac{r_s}{r}$ for $r_s = 2GM$.
The equations of motion for a free fall read

$$0 = \frac{d^2x^\lambda}{dp^2} + \Gamma^\lambda_{\mu\nu}\frac{dx^\mu}{dp}\frac{dx^\nu}{dp}$$

where p is a parameter describing the trajectory. Since we are going to consider massless particles as well therefore we don't use the proper time τ, as $d\tau = 0$ in that case. The explicit meaning of p will be clear later when we determine relation between p and the coordinates t and r. To write down the explicit form of the equations of motion we need non-zero elements of the connection (found earlier):

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \tfrac{1}{2}B'B^{-1}$$
$$\Gamma^r_{tt} = \tfrac{1}{2}B'A^{-1} \qquad \Gamma^r_{rr} = \tfrac{1}{2}A'A^{-1} \qquad \Gamma^r_{\theta\theta} = -rA^{-1} \quad \Gamma^r_{\varphi\varphi} = -rA^{-1}\sin^2\theta$$
$$\Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = -r^{-1} \qquad \Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta$$
$$\Gamma^\varphi_{r\varphi} = \Gamma^\varphi_{\varphi r} = r^{-1} \qquad \Gamma^\varphi_{\varphi\theta} = \cot\theta$$

Then we get

$$0 = \frac{d^2t}{dp^2} + \frac{B'}{B}\frac{dt}{dp}\frac{dr}{dp} \tag{1}$$

$$0 = \frac{d^2r}{dp^2} + \frac{1}{2}\frac{B'}{A}\left(\frac{dt}{dp}\right)^2 + \frac{1}{2}\frac{A'}{A}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\theta}{dp}\right)^2 - \frac{r}{A}\sin^2\theta\left(\frac{d\varphi}{dp}\right)^2 \tag{2}$$

$$0 = \frac{d^2\theta}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\theta}{dp} - \sin\theta\cos\theta\left(\frac{d\varphi}{dp}\right)^2 \tag{3}$$

$$0 = \frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\varphi}{dp} + 2\frac{\cos\theta}{\sin\theta}\frac{d\varphi}{dp}\frac{d\theta}{dp} \tag{4}$$

The field is isotropic, so we can consider the orbit in the equator plane, so $\theta = \frac{\pi}{2}$.

The (3) is trivially satisfied, while using

$$\frac{d}{dp}\ln B = \frac{B'}{B}\frac{dr}{dp} \quad \text{and} \quad \frac{dt}{dp}\,\frac{d}{dp}\ln\left(\frac{dt}{dp}\right) = \frac{d^2t}{dp^2}$$

from (1), dividing by $\frac{dt}{dp}$, we get

$$\frac{d^2t}{dp^2} + \frac{B'}{B}\frac{dt}{dp}\frac{dr}{dp} = 0 \quad \Longrightarrow \quad \frac{d}{dp}\left[\ln\left(\frac{dt}{dp}\right) + \ln B\right] = 0$$

From (4), dividing by $\frac{d\varphi}{dp}$, we get another constant of motion

$$\frac{d^2\varphi}{dp^2} + \frac{2}{r}\frac{dr}{dp}\frac{d\varphi}{dp} + 2\frac{\cos\theta}{\sin\theta}\frac{d\varphi}{dp}\frac{d\theta}{dp} = 0 \quad \Longrightarrow \quad \frac{d}{dp}\left[\ln\left(\frac{d\varphi}{dp}\right) + \ln r^2\right] = 0$$

So we have

$$\frac{dt}{dp}B = \text{const.}$$

We can choose the normalization of p such that the const. $= 1$, so

$$\frac{dt}{dp}B = 1 \tag{5}$$

Since $B(r) = 1 - \frac{r_s}{r}$ is close to 1, therefore p is almost the time t.
The second constant of motion (corresponding to angular momentum) gives

$$\frac{d\varphi}{dp} r^2 = J = \text{const.} \tag{6}$$

Now, using $\theta = \frac{\pi}{2}$, $\frac{dt}{dp} = B^{-1}$ and $\frac{d\varphi}{dp} r^2 = J$ in (2) we obtain

$$\begin{aligned} \frac{d^2 r}{dp^2} + \frac{1}{2}\frac{B'}{A}\left(\frac{dt}{dp}\right)^2 + \frac{1}{2}\frac{A'}{A}\left(\frac{dr}{dp}\right)^2 - \frac{r}{A}\left(\frac{d\theta}{dp}\right)^2 - \frac{r}{A}\sin^2\theta\left(\frac{d\varphi}{dp}\right)^2 &= 0 \\ \Longrightarrow \quad \frac{d}{dp}\left[A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B}\right] &= 0 \end{aligned}$$

So, we have the next constant of motion (related to energy)

$$A\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B} = -E = \text{const.} \tag{7}$$

The proper time τ could be now determined from the length element

$$d\tau^2 = B(r)\left(\frac{dt}{dp}\right)^2 dp^2 - A(r)\left(\frac{dr}{dp}\right)^2 dp^2 - r^2\left(\frac{d\varphi}{dp}\right)^2 dp^2 = E dp^2$$

Therefore

$$\frac{d\tau}{dp} = E^{1/2} = \text{const.} \quad \text{for} \quad E \geq 0 \tag{8}$$

for $\begin{cases} E = 0 & \text{for photons} \\ E > 0 & \text{for material particles} \end{cases}$.

From (7) we have

$$A\left(\frac{dr}{dp}\right)^2 + \left\{\frac{J^2}{r^2} + E\right\} = \frac{1}{B}$$

and since $A(r) = \left(1 - \frac{r_s}{r}\right)^{-1}$, so for $r > r_s$, we have $A(r) > 0$, therefore the range of r is determined by

$$\frac{J^2}{r^2} + E \leq \frac{1}{B} = \frac{1}{1 + \frac{r_s}{r}}$$

The parameter p could be eliminated from equations of motion using (5-8):

$$\frac{d\varphi}{dp}r^2 = \frac{d\varphi}{dt}\underbrace{\frac{dt}{dp}}_{B^{-1}}r^2 = J \quad \Rightarrow \quad \frac{d\varphi}{dt}r^2 = BJ \tag{9}$$

$$A\left(\frac{dr}{dt}\right)^2\left(\frac{dt}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B} = -E \quad \Rightarrow \quad \frac{A}{B^2}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B} = -E \tag{10}$$

and

$$d\tau^2 = E dp^2 = E\left(\frac{dp}{dt}\right)^2 dt^2 = EB^2\, dt^2$$

To find the trajectory of a particle we may eliminate p from (6-7) using

$$\frac{dr}{dp} = \frac{dr}{d\varphi}\frac{d\varphi}{dp} = \frac{dr}{d\varphi}\frac{J}{r^2}$$

We can get a single equation for $r = r(\varphi)$:

$$\frac{A(r)}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2 B(r)} = -\frac{E}{J^2} \tag{11}$$

with the solution determined by a quadrature

$$\varphi - \varphi_0 = \pm \int_{r(\varphi_0)}^{r(\varphi)} \frac{A^{1/2}(r')\,dr'}{r'^{\,2}\left[J^{-2}B^{-1}(r') - EJ^{-2} - r'^{\,-2}\right]^{1/2}} \tag{12}$$

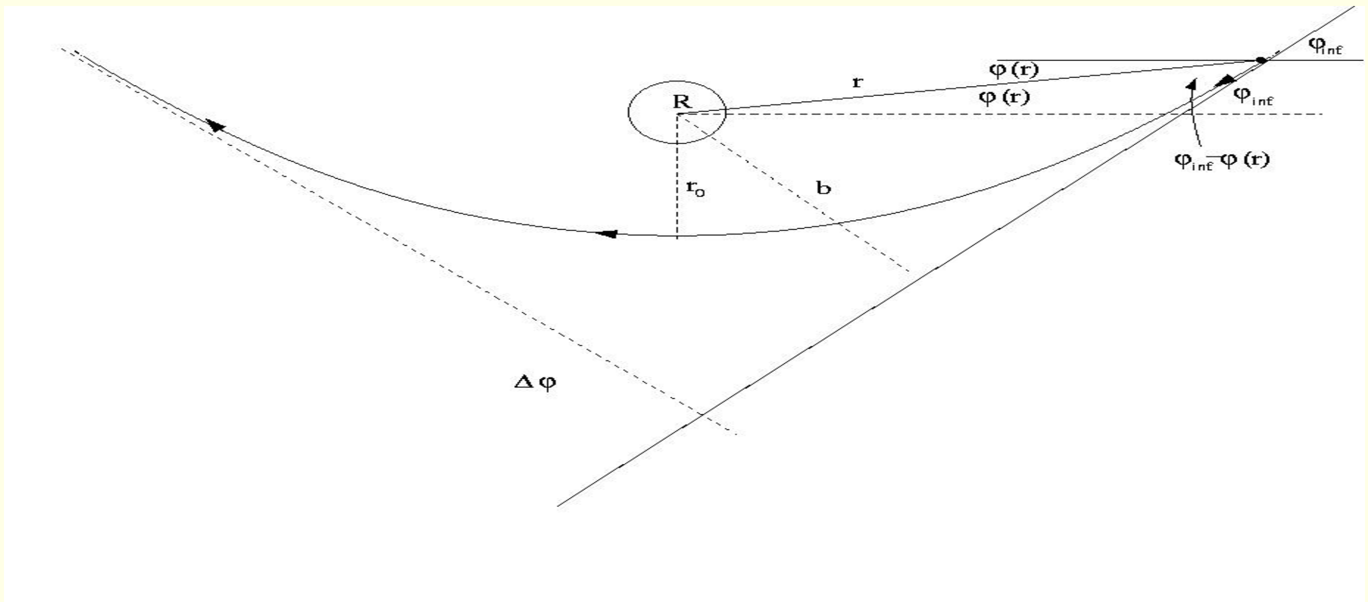

Figure 1: *Bending light around a massive object.*

Now we will try to solve the equation in the case of deflection by Sun taking into account that the bending is very small.
From the Fig.1 we can see that for small deflection (φ is measured with respect to

any fixed direction)

$$\sin(\varphi_\infty - \varphi) \simeq \frac{b}{r} \qquad \Longrightarrow \qquad \frac{dr}{d\varphi} = \frac{b}{(\varphi_\infty - \varphi)^2}$$

Note that $dr/d\varphi$ should be positive. Denoting the size of the velocity by v

$$-v \simeq \frac{d}{dt}\left[r\cos(\varphi_\infty - \varphi)\right] \simeq \frac{dr}{dt}$$

Then using the fact that at infinity the metric becomes Minkowskian ($A(\infty) = B(\infty) \simeq 1$), we can determine the constants J (< 0) and E:

$$\frac{d\varphi}{dt}r^2 = \frac{d\varphi}{dr}\frac{dr}{dt}r^2 = \frac{(\varphi_\infty - \varphi)^2}{b}(-v)r^2 = -bv(\varphi_\infty - \varphi)^2\left(\frac{r}{b}\right)^2 = B(\infty)J \;\Rightarrow\; J = -bv$$

While from (10)

$$\frac{A}{B^2}\left(\frac{dr}{dt}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B} = -E \qquad \Longrightarrow \qquad v^2 + \frac{(bv)^2}{r^2} - 1 = -E \qquad \Longrightarrow \qquad E = 1 - v^2$$

For photons $v = 1$, so $E = 0$.
We will express J in terms of r_0, the minimal distance to the sun. At $r = r_0$ we have $\frac{dr}{d\varphi} = 0$, so from (11)

$$\frac{A(r)}{r^4}\left(\frac{dr}{d\varphi}\right)^2 + \frac{1}{r^2} - \frac{1}{J^2 B(r)} = -\frac{E}{J^2} \quad \Longrightarrow \quad \frac{1}{r_0^2} - \frac{1}{J^2 B(r_0)} = -\frac{E}{J^2}$$
$$\Longrightarrow \quad J^2 = r_0^2 \left(\frac{1}{B(r_0)} - 1 + v^2\right)$$

Using $E = 1 - v^2$ and the above in (12) we get

$$[J^{-2}B^{-1}(r') - EJ^{-2} - r'^{\,-2}] = \{J^{-2}[B^{-1}(r') - 1 + v^2] - r'^{\,-2}\} =$$
$$\{r_0^{-2}\left[B^{-1}(r') - 1 + v^2\right]\left[B^{-1}(r_0) - 1 + v^2\right] - r'^{\,-2}\}$$

So the trajectory reads

$$\varphi(r) = \varphi_\infty - \int_\infty^r \frac{A^{1/2}(r')\,dr'}{r'^{\,2}\left\{r_0^{-2}\left[B^{-1}(r') - 1 + v^2\right]\left[B^{-1}(r_0) - 1 + v^2\right] - r'^{\,-2}\right\}^{1/2}}$$

The deflection angle $\Delta\varphi$ is defined as follows

$$\Delta\varphi = 2[\varphi(r_0) - \varphi_\infty] - \pi$$

For a photon, $v = 1$ so we have

$$\varphi(r) = \varphi_\infty + \int_r^\infty A^{1/2}(r') \left[\left(\frac{r'}{r_0} \right)^2 \frac{B(r_0)}{B(r')} - 1 \right]^{-1/2} \frac{dr'}{r'} \tag{13}$$

In general we have got an elliptic integral, so we will expand in small quantities $\frac{GM}{r}$ and $\frac{GM}{r_0}$ in order to simplify the task:

$$A(r) = \left(1 - \frac{2MG}{r}\right)^{-1} = 1 + \frac{2MG}{r} + \cdots \quad \text{while} \quad B(r) = 1 - \frac{2MG}{r}$$

Then the factor in the integrand could be written as

$$\left(\frac{r'}{r_0}\right)^2 \frac{B(r_0)}{B(r')} - 1 = \left(\frac{r'}{r_0}\right)^2 \frac{1 - \frac{2MG}{r_0}}{1 - \frac{2MG}{r'}} - 1 = \left[\left(\frac{r'}{r_0}\right)^2 - 1\right] \left[1 - \frac{2MGr'}{r_0(r' + r_0)} + \cdots\right]$$

Then from (14) we obtain

$$\varphi(r) = \varphi_\infty + \int_r^\infty \frac{dr'}{r'}\left(1 + \frac{MG}{r} + \cdots\right)\left[\left(\frac{r'}{r_0}\right)^2 - 1\right]^{-1/2}\left[1 + \frac{MGr'}{r_0(r'+r_0)} + \cdots\right]$$

Then introducing $x \equiv \frac{r'}{r_0}$ we get

$$\varphi(r) = \varphi_\infty + \int_{r/r_0}^\infty \frac{dx}{x}\frac{1}{(x^2-1)^{1/2}}\left\{1 + \frac{MG}{r_0}\frac{1}{x} + \frac{MG}{r_0}\frac{x}{1+x} + \cdots\right\}$$

For $r = r_0$ we have

$$\begin{aligned}
\varphi(r_0) - \varphi_\infty &= \frac{\pi}{2} + \frac{MG}{r_0}\left[\underbrace{\int_1^\infty \frac{dx}{x^2}\frac{1}{(x^2-1)^{1/2}}}_{1} + \underbrace{\int_1^\infty \frac{dx}{(1+x)(x^2-1)^{1/2}}}_{1} + \cdots\right] \\
&= \frac{\pi}{2} + \frac{2MG}{r_0}(1 + \cdots)
\end{aligned}$$

Hence

$$\Delta\varphi = 2[\varphi(r_0) - \varphi_\infty] - \pi = \frac{4MG}{r_0}(1 + \cdots) = \frac{4MG}{b}(1 + \cdots)$$

For the Sun $M = M_\odot = 1.97 \times 10^{30}$ kg and $MG = M_\odot G = 1475$ km. The minimal r is $r_0 = R_\odot = 6.96 \times 10^8 m$, so

$$\Delta\varphi = \frac{R_\odot}{r_0}\theta_\odot \quad \text{for} \quad \theta_\odot = \frac{4M_\odot G}{R_\odot} = 1.75''$$

Observation of gravitational lensing

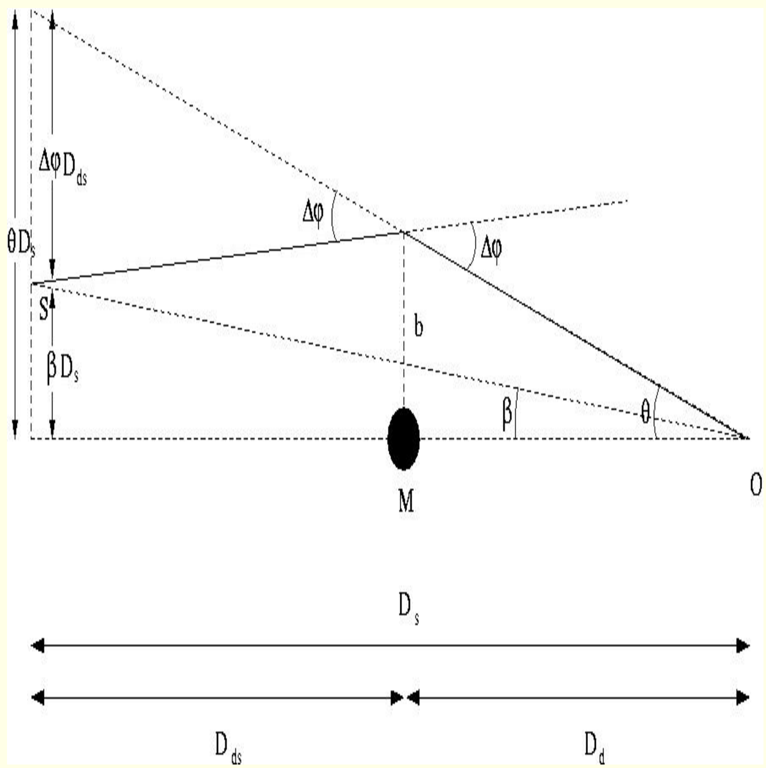

Figure 2: *Gravitational lensing: S denotes the source, M is the lens and O is the observer.*

From the Fig.2 it is easy to derive the lens equation noting that

$$\beta D_s = \theta D_s - \Delta\varphi D_{ds}$$

Then we get the lens equation

$$\beta(\theta) = \theta - \frac{D_{ds}}{D_s}\Delta\varphi = \theta - \frac{D_{ds}}{D_s}\frac{4MG}{b} = \theta - \frac{D_{ds}}{D_s D_d}\frac{4MG}{\theta}$$

If $\beta = 0$ (perfect alignment) we get the Einstein ring

$$\theta_E^2 = \frac{D_{ds}}{D_s D_d} 4MG$$

while for $\beta \neq 0$ there are two solutions

$$\theta_{1/2} = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

A gravitational lens is formed when the light from a very distant, bright source (such as a quasar) is "bent" around a massive object (such as a cluster of galaxies) between the source object and the observer.
There are three classes of gravitational lensing:

1. Strong lensing:
   The most extreme bending of light is when the lens is very massive such that

multiple images may appear. The first example of a double image was found in 1979, of a quasar. If the source varies with time, the multiple images will vary with time as well. However, the light doesn't travel the same distance to each image, due to the bending of space. So there will be time delays for the changes in the images. These time delays can be used to calculate the Hubble constant H_0. A few systems with these time delays have been found and are under study. Much of the subtlety in this work lies with constructing the model of the mass distribution forming the lens. In some special cases the alignment of the source and the lens will be such that light will be deflected to the observer in an "Einstein ring." One of these was observed in 1998. More often than a ring, the source may get stretched out and curved, and form a tangential or radial arc. A lot of mass is needed to cause an arc to appear, so that properties of arcs (numbers, size, geometry) can often be used to study massive objects like clusters. One can also, given a set of images, try to reconstruct the lens mass distribution.

2. Weak lensing:
   In many cases the lens is not strong enough to form multiple images or arcs. However, the source can still be distorted: both stretched and magnified. If all sources were well known in size and shape, one could just use properties of images to deduce the properties of the lens. However, usually one does not know the intrinsic properties of the sources, but has information about the average

properties. The statistics of the sources can then be used to get information about the lens. For instance, galaxies in general aren't perfectly spherical, but if one has a collection of galaxies one doesn't expect them all to be lined up. Thus, if this set of galaxies is lensed, on average, or statistically, there will be some overall shear and/or convergence imposed on the distribution, which will give information about the intervening lens(es). There is a distribution of galaxies far enough away that can be treated as sources, and thus clusters nearby can be "weighed" (i.e. have their mass measured) using their lensing. Superclusters have been considered as well. In addition, theories of cosmology predict the distribution of large scale structure, the distribution of matter in the universe. The statistical properties of the large scale structure (e.g. the probability of finding a galaxy at one place when there is another a certain distance away) can also be measured by weak lensing, because the matter will produce shear and convergence in distant sources (which can be galaxies, or the cosmic microwave background, for example). Weak lensing is a useful complement to measures of the distribution of luminous mass such as galaxy surveys. Lensing measures all the mass, in particular the dark matter as well as the luminous matter.

3. Microlensing:
   For M equal to the mass of the Sun, $D_d = 4$ Mpc, and $D_s = 8$ Mpc (typical for a Bulge microlensing event), the Einstein angular radius $\theta_E = 0.001$ arcseconds

(1 milliarcsecond). By comparison, ideal Earth-based observations have angular resolution around 0.4 arcseconds, 400 times greater. So, in those cases the lensing is so weak that one doesn't see the multiple images – the additional light bent towards the observer just means that the source appears brighter. (The surface brightness remains unchanged but as more images of the object appear the object appears bigger and hence brighter.) This lensing can have effects in many measurements, as sources which would have otherwise been too dim become visible. This can be helpful, as when one wants to view objects that would otherwise be too far away. It can also be a problem, for example when one is trying to measure all objects brighter than a certain amount in a certain region and lensing introduces objects by magnifying objects enough to bring them into the sample.

There are ongoing searches to use lensing to find a type of dark matter called MACHOs (massive compact halo objects). Although MACHOs, as dark matter, cannot be seen themselves, if they pass in front of a source (e.g. a star nearby), they can cause the star to become brighter for a while, e.g. days or weeks. This effect has been observed but determinations of the dark matter are not yet conclusive. Observations are underway by many groups.

Gravitational lenses act equally on all kinds of electromagnetic radiation, not just visible light. Weak lensing effects are being studied for the cosmic microwave background as well. Strong lenses have been observed in radio and x-ray regimes as well.

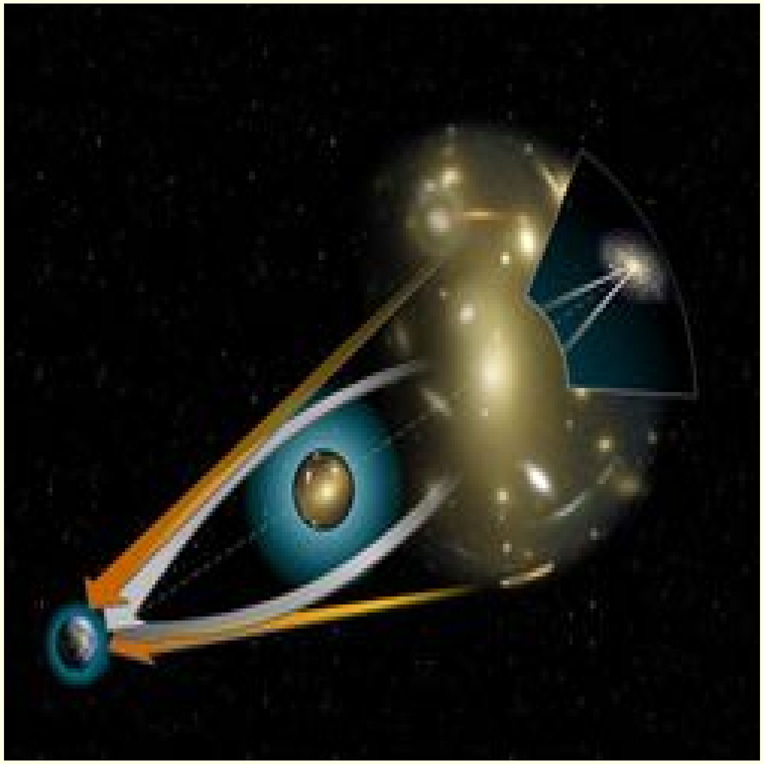

Figure 3: *Bending light around a massive object from a distant source. The orange arrows show the apparent position of the background source. The white arrows show the path of the light from the true position of the source.*

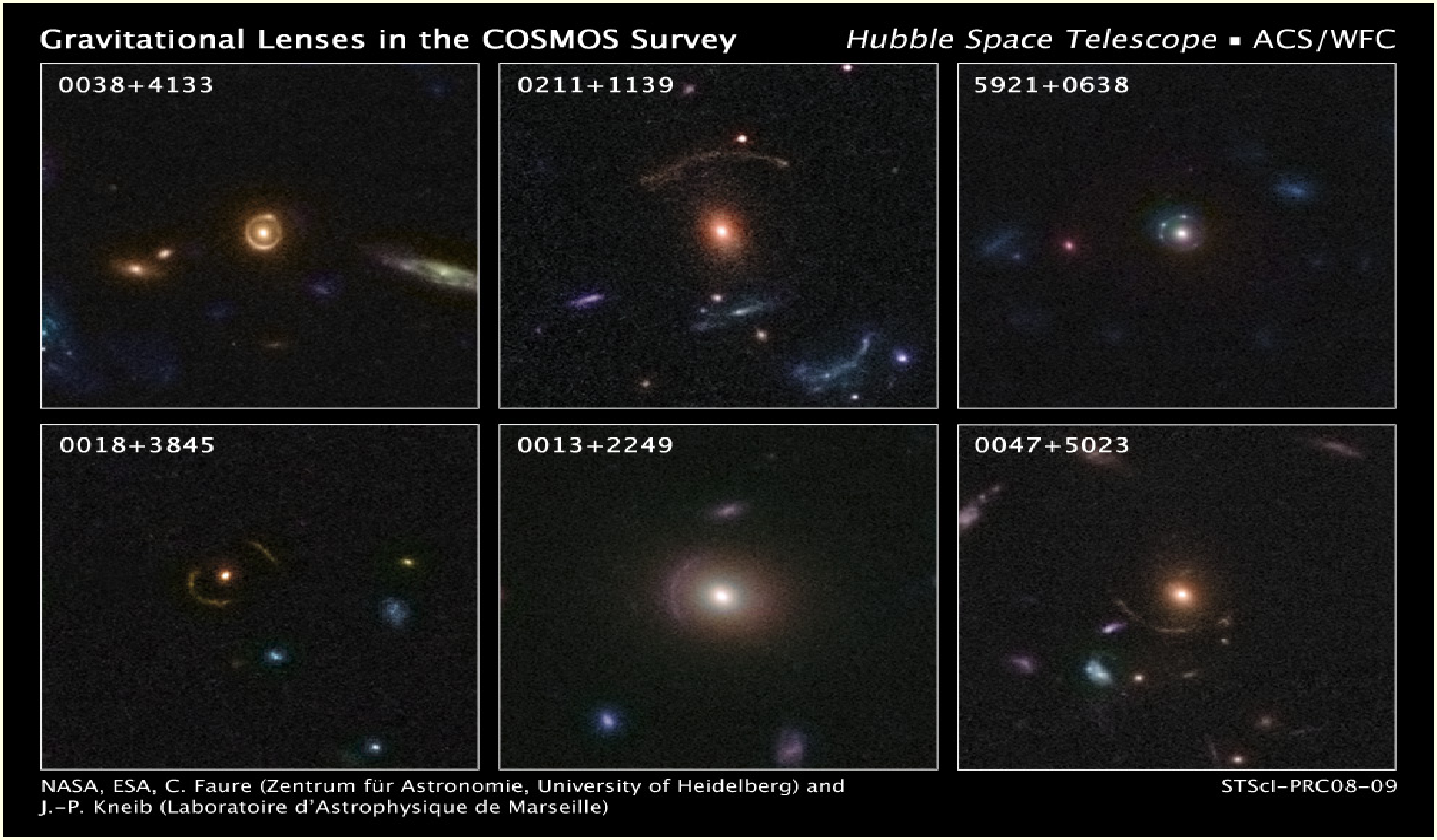

Figure 4: *Astronomers using NASA's Hubble Space Telescope have compiled a large catalog of gravitational lenses in the distant universe. The catalog contains 67 new gravitationally lensed galaxy images found around massive elliptical and lenticular-shaped galaxies.*

Figure 5: *In the formation known as Einstein's Cross four images of the same distant quasar appears around a foreground galaxy due to strong gravitational lensing*

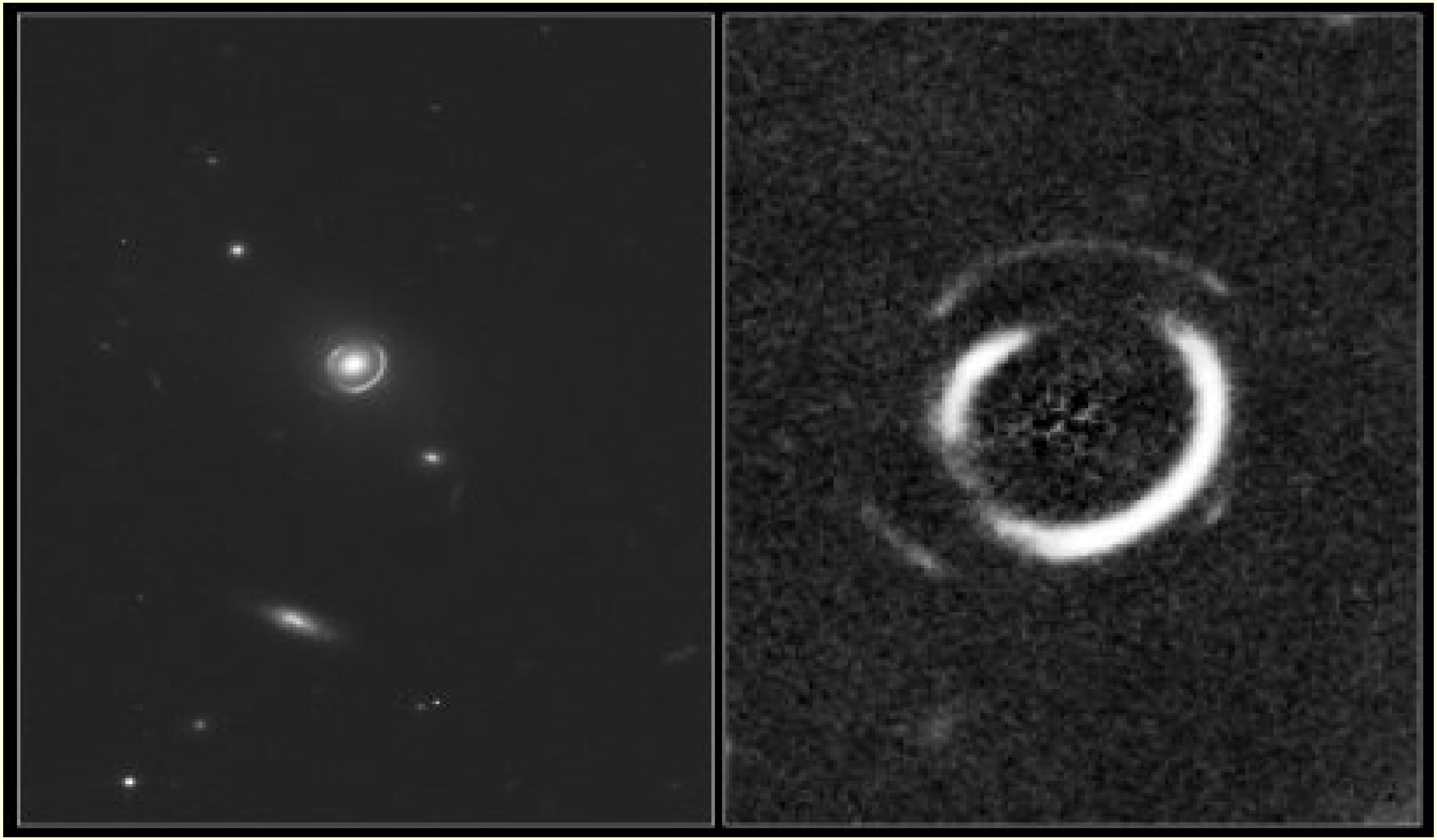

Figure 6: *The NASA/ESA Hubble Space Telescope has revealed (2008) a never-before-seen optical alignment in space: a pair of glowing rings, one nestled inside the other like a bull's-eye pattern (SDSSJ0946+1006). The double-ring pattern is caused by the bending of light from two distant galaxies that both lie behind a foreground massive galaxy. The left picture shows the double Einstein ring and the right one is a close-up with the central foreground galaxy image subtracted.*