β FUNCTION IN QED ©

STANDALONE EXERCISE ?

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu}^{B} F_{B}^{\mu\nu} + \bar{\Psi}_B (i\not{\partial} - q e_B \not{A}_B)\Psi_B - m_B \bar{\Psi}_B \Psi_B + \mathcal{L}_{GAUGE-FIX}$$

$$\mathcal{L}_{GF} = -\frac{1}{2a_B}(\partial_\mu A^\mu_B)^2 \qquad \begin{cases} q = -1 \\ e > 0 \end{cases}$$

$$A_\mu^B = Z_3^{1/2} A_\mu \qquad e_B = Z_1 Z_2^{-1} Z_3^{-1/2} e$$

$$\psi^B = Z_2^{1/2}\psi \qquad m_B = Z_0 Z_2^{-1} m \equiv (m - \delta m) Z_2^{-1}$$

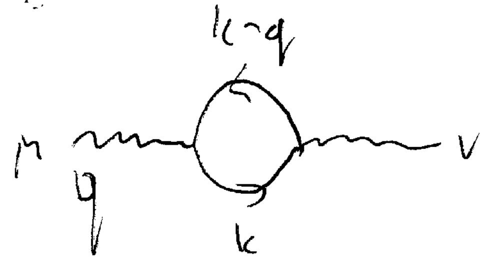
$$a_B = Z_3 a$$

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\not{\partial} + e\not{A})\Psi - m\bar{\Psi}\Psi - \frac{1}{2a}(\partial_\mu A^\mu)^2$$
$$+ (Z_2 - 1)\bar{\Psi} i\not{\partial}\Psi - (Z_0 - 1) m \bar{\Psi}\Psi + (Z_1 - 1) e \bar{\Psi}\not{A}\Psi$$
$$- (Z_3 - 1)\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

WARD-TAKAHASHI IDENTITY $Z_1 = Z_2$ IN QED!

THUS $e_B = Z_3^{-1/2} e$ — CALCULATION OF THE PHOTON WAVE FUNCTION RENORMALIZATION SUFFICIENT FOR $\beta^{QED}(e)$ CALCULATION!

PHOTON SELF-ENERGY AT 1-LOOP



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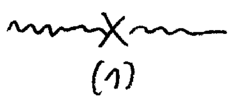

$\equiv i\,\Pi_{\mu\nu}(q^2)$

ONLY TRANSVERSAL PART IS PHYSICAL AND GETS HIGHER ORDER CORRECTIONS

$$i\,\Pi_{\mu\nu}(q) \equiv i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right) q^2\, \Pi(q^2)$$

$$g_{\mu\nu}\,\Pi_{\mu\nu}(q) = (n-1)\, q^2\, \Pi(q^2) \qquad \left[\, g_{\mu\nu} g^{\mu\nu} = n\,! \right]$$

$$\mathrm{Tr}[\gamma_\mu \gamma_\nu] = 4 g_{\mu\nu}$$

$$i\,\Pi_{\mu\nu}(q) = -(ie)^2 \int \frac{d^4k}{(2\pi)^4} \frac{i}{(k^2-m^2)} \frac{i}{(k-q)^2-m^2} \otimes$$

$$\otimes\, \mathrm{Tr}\left[\gamma_\mu(\cancel{k}+m)\gamma_\nu(\cancel{k}-\cancel{q}+m)\right]$$

$$+\, i\,(Z_3-1)^{(1)}\,(q_\mu q_\nu - q^2 g_{\mu\nu})$$

$$\mathrm{Tr}[\gamma_\mu\gamma_\alpha\gamma_\nu\gamma_\beta] = 4\left(g_{\mu\alpha}g_{\nu\beta} + g_{\mu\beta}g_{\nu\alpha} - g_{\mu\nu}g_{\alpha\beta}\right)$$

THUS

$$(n-1)\, q^2\, \Pi(q^2) = ie^2 \int \frac{d^nk}{(2\pi)^n} \int_0^1 dx\, \frac{4(2-n)(k^2-kq) + 4nm^2}{\left[x(k^2-m^2) + (1-x)\left[(k-q)^2-m^2\right]\right]^2}$$

$$+\,\mathrm{c.t.}$$

$$\Pi(q^2) = -\frac{\alpha}{\pi}\,(4\pi)^{\varepsilon/2}\,\Gamma\!\left(\tfrac{1}{2}\varepsilon\right)\int_0^1 dx\,\cancel{(2x-1)}\;2x(1-x)$$

$$\times\left[\frac{m^2 - x(1-x)q^2 - i\varepsilon}{\mu^2}\right]^{-\varepsilon/2} + \text{c.t.}$$

$$= -\frac{\alpha}{\pi}\left[\frac{2}{3}\frac{1}{\varepsilon} + \frac{1}{3}(\ln 4\pi - \gamma) - \int_0^1 dx\, 2x(1-x)\ln\left[\frac{m^2 - x(1-x)q^2 - i\varepsilon}{\mu^2}\right]\right]$$

$$+ O(\varepsilon) - (Z_3 - 1)^{(1)}$$

$$\alpha = \frac{e^2}{4\pi} \to \alpha\,\mu^{\varepsilon}$$

THUS $(Z_3 - 1)^{(1)} = -\frac{\alpha}{\pi}\frac{2}{3\varepsilon}$ + FINITE

$$(Z_3 - 1)^{(1)} = \begin{cases} -\dfrac{2\alpha}{3\pi\varepsilon} & \text{MS} \\[2ex] -\dfrac{\alpha}{3\pi}\left[\dfrac{2}{\varepsilon} - \gamma + \ln 4\pi\right] & \overline{\text{MS}} \\[2ex] -\dfrac{\alpha}{3\pi}\left[\dfrac{2}{\varepsilon} - \gamma + \ln 4\pi - \ln\dfrac{m^2}{\mu^2}\right] & \text{OS} \end{cases}$$

$\Pi^{OS}(q^2 = 0) = 0$ — "PHYSICAL" RENORMALIZATION

USING $\alpha_B = \mu^{\varepsilon} Z_\alpha \alpha$, WE HAVE

$$Z_\alpha = 1 + \frac{\alpha}{\pi}\frac{2}{3\varepsilon} + O(\alpha^2)$$

$$\beta(\alpha) = \alpha^2 \frac{d}{d\alpha} Z_\alpha^{(1)} = \frac{2\alpha^2}{3\pi} + O(\alpha^3)$$

FULL RGE IN QED

$$\Big[\mu \frac{\partial}{\partial\mu} + \beta(\alpha,a)\frac{\partial}{\partial\alpha} + \gamma_\mu(\alpha,a)\, m \frac{\partial}{\partial m} + \delta(a,a)\frac{\partial}{\partial a}$$

$$- n_\gamma \gamma_\gamma(\alpha,a) - n_F \gamma_F(\alpha,a)\Big] \Gamma^{(n_\gamma,n_F)}(p_i;\alpha,m,\mu,a) = 0$$

$$\beta(\alpha,a) = \mu \frac{d\alpha}{d\mu}$$

$$\gamma_m(\alpha,a) = \mu \frac{d Z_m}{d\mu}$$

$$\gamma_\gamma(\alpha,a) = \frac{1}{2}\mu \frac{d}{d\mu} Z_3$$

$$\gamma_F(\alpha,a) = \frac{1}{2}\mu \frac{d}{d\mu} Z_2$$

$$\delta(\alpha,a) = \mu \frac{da}{d\mu} = -a\mu \frac{d}{d\mu} \ln Z_3$$

RUNNING VS. PHYSICAL α

IR FIXED POINT $\Rightarrow$ $\alpha(q^2) \xrightarrow[q^2\to 0]{} 0$?!

$\alpha(q^2)$ NOT A PHYSICAL COUPLING!

LETS DEFINE REN. PHOTON PROPAGATOR (MASS-IND. REN. SCHEME)

$$\cancel{\alpha D^{\mu\nu}(q^2) = -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)}$$

$$\alpha D^{\mu\nu}(q^2) = -i\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)\frac{\alpha \not{q}^2}{1-\Pi(q^2/\mu^2,\alpha)} - \alpha a i \frac{q^\mu q^\nu}{q^2}$$

ALSO DEFINE

$$d(q^2/\mu^2,\alpha) = \frac{1}{1-\Pi(q^2/\mu^2)}$$

$\Pi \to$ PHOTON S.E CALCULATION

FURTHER DEFINE $\alpha_{INV}(q^2) = \alpha\, d(q^2/\mu^2,\alpha)$. WHY INV?

$$\alpha\, d(q^2/\mu^2,\alpha) = \left(\frac{\alpha}{Z_3}\right) Z_3\, d(q^2/\mu^2,\alpha) = \alpha_B\, d_B(q^2)$$

IN QCD $Z_1 \neq Z_2$, SO $\alpha_B^{QCD} \neq \frac{\alpha_1^{QCD}}{Z_3}$! IN QED $\alpha_B d_B$ FINITE, IN QCD IT IS NOT!

IN THE ON-SHELL SCHEME $\Pi^{ON}(q^2=0) = 0$

THUS $\alpha_{INV} = \alpha d = \alpha^{ON}(q^2)\, d^{ON}(q^2/\mu^2,\alpha^{ON}) \xrightarrow[q^2\to 0]{} \alpha^{ON}(0) \equiv \alpha^{ON} = \frac{1}{137..}$

ALSO $\alpha^{ON} = \alpha^{\overline{MS}}(q^2 = m_e^2)$!

REN. INDEPENDENCE $\rightarrow$ TAKE $\mu^2 = q^2$

$$\alpha_{INV}(q^2) = \alpha(q^2)\, d(1, \alpha(q^2))$$

$$\Pi(q^2) = \begin{cases} \frac{\alpha}{3\pi} \ln \frac{m^2}{\mu^2} - \frac{\alpha}{15\pi} \frac{q^2}{m^2} + O\left(\frac{q^4}{m^4}\right) & q^2 \to 0 \\ \frac{\alpha}{3\pi} \ln \frac{|q^2|}{\mu^2} & q^2 \gg m^2 \end{cases}$$

$$q^2 \to \infty \qquad \alpha_{INV}(q^2) = \alpha(q^2) = \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln \frac{|q^2|}{\mu^2}} \longrightarrow \text{RGE SOLUTION!}$$

TO ANY ORDER $\alpha_{INV}(q^2) = \alpha(q^2)$ FOR $q^2 \to \infty$

$q^2 \to 0$

$$\alpha_{INV}(q^2) \underset{q^2\to 0}{=} \frac{1}{137.036} = \frac{\alpha(q^2)}{1 - \frac{\alpha(q^2)}{3\pi} \ln \frac{m^2}{\underset{\substack{\uparrow \\ q^2}}{\mu^2}}} = \frac{0}{0}$$

TO FIRST ORDER

$$\alpha(q^2) \underset{q^2 \to 0}{=} \alpha^{ON}\left(1 + \frac{\alpha^{ON}}{3\pi} \ln \frac{|q^2|}{m^2}\right)$$

$$\alpha^{ON} \ln \frac{q^2}{m^2} \ll 1$$

$|q^2| \to 0 \to$ MORE AND MORE TERMS REQUIRED.