Renormalization and Renormalization Group

Prof. A. Buras

Exercises 1 and 2 (27.10.00 and 3.11.00)

Lets define 1- and 2-point scalar, vector and tensor integrals as:

$$\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2} = \frac{-i}{(4\pi)^2}A_0(m^2) \tag{1}$$

$$\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-m_1^2][(p+k)^2-m_2^2]} = \frac{i}{(4\pi)^2}B_0(p^2,m_1^2,m_2^2) \tag{2}$$

$$\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{k^\alpha}{[k^2-m_1^2][(p+k)^2-m_2^2]} = \frac{i}{(4\pi)^2}p^\alpha B_1(p^2,m_1^2,m_2^2) \tag{3}$$

$$\begin{aligned}\mu^{4-d}\int\frac{d^dk}{(2\pi)^d}\frac{k^\alpha k^\beta}{[k^2-m_1^2][(p+k)^2-m_2^2]} &= \frac{i}{(4\pi)^2}\Big(p^\alpha p^\beta B_{21}(p^2,m_1^2,m_2^2)\\ &+ g^{\alpha\beta}B_{22}(p^2,m_1^2,m_2^2)\Big)\end{aligned} \tag{4}$$

Problem 1. Show that

$$A_0(m^2) = -m^2\left(\Delta - \log\frac{m^2}{\mu^2} + \mathcal{O}(\epsilon)\right) \tag{5}$$

where $\Delta = \frac{1}{\epsilon} - \gamma + \log 4\pi$, $\epsilon = \frac{4-d}{2}$ and $\gamma = 0.5772\ldots$ is the Euler constant.

Problem 2. Show that

$$B_0(p^2,m_1^2,m_2^2) = \Delta - \int_0^1 \log f(x)dx + \mathcal{O}(\epsilon) \tag{6}$$

where $f(x) = m_1^2x + m_2^2(1-x) - p^2x(1-x)$

Problem 3. Show that

$$B_1(p^2,m_1^2,m_2^2) = \frac{A_0(m_2^2) - A_0(m_1^2) + (m_2^2 - m_1^2 - p^2)B_0(p^2,m_1^2,m_2^2)}{2p^2} \tag{7}$$

Problem 4. Find Feynman representation (i.e. in the form of 1-dimensional integral) for B_{21}, B_{22}.

Problem 5. Find divergent parts (i.e. coefficients of Δ term) of A, B integrals.

Problem 6. Prove the "generalized Feynman integral" formulae:

$$\begin{aligned}F_n &= \frac{1}{A_1^{\alpha_1}A_2^{\alpha_2}\ldots A_n^{\alpha_n}}\\ &= \frac{\Gamma(\alpha_1+\ldots+\alpha_n)}{\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}\int_0^1 dx_1\ldots\int_0^1 dx_n\frac{x_1^{\alpha_1-1}\ldots x_n^{\alpha_n-1}\delta(1-x_1-\ldots-x_n)}{(x_1A_1+\ldots+x_nA_n)^{\alpha_1+\ldots+\alpha_n}}\end{aligned} \tag{8}$$

Hint:

i) Calculate directly F_2. Show that

$$F_2 = \frac{\Gamma(\alpha_1+\alpha_2)}{\Gamma(\alpha_1)\Gamma(\alpha_2)}\int_0^1 dx\frac{x^{\alpha_1-1}(1-x)^{\alpha_2-1}}{(xA_1+(1-x)A_2)^{\alpha_1+\alpha_2}} \tag{9}$$

and use the Möbius transform $x = y/(1+y)$ and definition of the Euler Beta function to get the explicit expression for F_2.
ii) Prove eq. (8) by complete induction (use $\delta(ax) = \delta(x)/|a|$).

J. Rosiek

Renormalization and Renormalization Group

Prof. A. Buras

Exercise 3 (10.11.00)

Lets define 3-point scalar, vector and tensor integrals as ($C_{ij} \equiv C_{ij}(p, q, m_1^2, m_2^2, m_3^2)$):

$$
\begin{aligned}
\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{1}{[k^2-m_1^2][(p+k)^2-m_2^2][(p+q+k)^2-m_3^2]} &= \frac{-i}{(4\pi)^2}C_0 \\
\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{k^\alpha}{[k^2-m_1^2][(p+k)^2-m_2^2][(p+q+k)^2-m_3^2]} &= \frac{-i}{(4\pi)^2}(p^\alpha C_{11}+q^\alpha C_{12}) \\
\mu^{4-d}\int \frac{d^dk}{(2\pi)^d}\frac{k^\alpha k^\beta}{[k^2-m_1^2][(p+k)^2-m_2^2][(p+q+k)^2-m_3^2]} &= \frac{-i}{(4\pi)^2}(p^\alpha p^\beta C_{21}+q^\alpha q^\beta C_{22} \\
&+ (p^\alpha q^\beta + q^\alpha p^\beta)C_{21} + g^{\alpha\beta}C_{24})
\end{aligned}
$$

Problem 1. Find Feynman representation of C functions (i.e. perform the momentum integration leaving the result in the form of 2-dimensional integral over the auxiliary parameters).

Problem 2. Find the divergent parts of C functions.

Problem 3. Calculate explicit form of the C_0 function for the special case $m_1 = m_3 = 0$, $m_2 = M$, $p^2 = q^2 = 0$, $(p+q)^2 = s$ (such an integral appears e.g. in calculations of the 1-loop corrections to $Z^0\bar{f}f$ vertex). Express the result in terms of *dilogarithm* function:

$$\mathrm{Li}_2(z) \equiv \int_0^1 \frac{\log(1+zx)}{x}dx \tag{1}$$

Some useful identities:

$$
\begin{aligned}
\mathrm{Li}_2(-1) &= \frac{\pi^2}{6} \\
\mathrm{Li}_2(-1-z) + \mathrm{Li}_2(z) &= -\frac{\pi^2}{6} + \log(-z)\log(1+z)
\end{aligned} \tag{2}
$$

Problem 4 Prove the Gordon identity:

$$\bar{u}(p_2)\gamma_\mu u(p_1) = \frac{1}{2m}\bar{u}(p_2)[(p_1+p_2)_\mu + i\sigma_{\mu\nu}(p_2-p_1)^\nu]u(p_1) \tag{3}$$

where $\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_{nu}]$.

Hint: start from the identity, given simply by the Dirac equation and valid for any vector a:

$$\bar{u}(p_2)[\not{a}(\not{p}_1 - m) + (\not{p}_2 - m)\not{a}]u(p_1) = 0 \tag{4}$$

Find Gordon identity for $\bar{u}(p_2)\gamma_\mu\gamma_5 u(p_1)$

J. Rosiek

Renormalization and Renormalization Group

Prof. A. Buras

Exercise 4 (16.11.00)

Problem 1. Consider Lie algebra of $SU(N)$ group with N^2-1 generators T^a fulfilling the relation $[T^a, T^b] = if^{abc}T^c$. The generators are normalized by the condition $\mathrm{Tr}(T^aT^b) = \delta^{ab}$.
i) Prove that in the fundamental (N-dimensional) representation matrices T^a fulfill the relation:

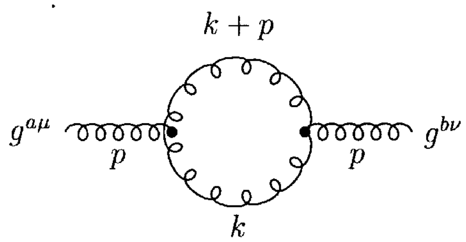
$$T^a_{ij}T^a_{kl} = \frac{1}{2}\left[\delta_{il}\delta_{jk} - \frac{1}{N}\delta_{ij}\delta_{kl}\right] \tag{1}$$

(hint: every $N \times N$ matrix can be written in the form $A = c_0\hat{1} + c_aT^a$).
ii) Prove that (the Jacobi identity):

$$f^{abe}f^{cde} + f^{cbe}f^{dae} + f^{dbe}f^{ace} = 0 \tag{2}$$

iii) Use i) and ii) to calculate $Tr(T^aT^bT^c)$, $Tr(T^aT^bT^cT^d)$, T^aT^a, $T^aT^bT^a$, $T^aT^bT^cT^a$, $f^{abc}f^{bcd}$.

Problem 2. Calculate the 1-loop correction to the gluon propagator given by the gluon self-interaction diagram:



J. Rosiek

Renormalization and Renormalization Group

Prof. A. Buras

Exercise 5 (23.11.00)

Please consider in details the renormalization of $\lambda\phi^4$ theory with spontaneously broken symmetry:

$$L = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda\phi^4 \tag{1}$$

where $m^2 < 0$.

1. Calculate the vacuum expectation value $v \equiv <\phi>$ of the ϕ field and express the Lagrangian in terms of the shifted field $\phi' = \phi - <\phi>$ (keep v explicitly in the Lagrangian, do not express it by m^2 and λ).

2. Define the renormalized quantities:

$$\begin{aligned}
\phi_R &= Z_\phi^{-1/2}\phi_0 \\
\lambda_R &= Z_\lambda^{-1}Z_\phi^2\lambda_0 \\
v_R &= Z_\phi^{-1/2}v_0 - \delta v_0 \\
m_R^2 &= Z_\phi m_0^2 - \delta m_0^2
\end{aligned}$$

   Express Lagrangian in terms of renormalized quantities. Expand renormalization constants and find all counterterms.

3. Fix δv_0 imposing the condition that terms linear in ϕ must disappear from the Lagrangian in all orders of perturbation expansion (find explicit expression for δv_0 at 1-loop level).

4. Find 1-loop expressions for other renormalization constants in $\overline{MS}$ and on-shell renormalization schemes (the latter is defined by the conditions the physical renormalized m_R^2 and λ_R are equal to the tree level ones and by the additional requirement that the residuum of the scalar propagator is equal to 1).

J. Rosiek

Adriatic School on Particle Physics and Physics Informatics

Split, September 2001

Exercises on "Renormalization Group Methods" (advanced group)

Janusz Rosiek

1. General introduction on regularization and renormalization procedures.
   - Derivation of Renormalization Group Equations.
   - Expressing β functions and anomalous dimensions in terms of the first order poles of renormalization constants.
2. Detailed derivation of the RG equations for the scalar $\lambda\phi^4$ theory.
   - Renormalized Lagrangian.
   - Infinite parts of the 2- and 4-point Green's functions at 1-loop.
   - Renormalization constants.
   - β function and anomalous dimensions.
   - Some discussion of higher orders.
3. Derivation of β function in QED.
4. Solving the RGE:
   - General expressions for the Green's functions.
   - Momentum scaling.
   - IR/UV fixed points, asymptotic freedom, Landau pole, dimensional transmutation.
   - Stability of the fixed points - dependence on the choice of the renormalization scheme and/or choice of the gauge. Proof of the gauge independence of β function in MS scheme.
   - Perturbative solution for the running coupling.
   - $\alpha_s(M_Z)$ as a function of $\alpha_{em}(M_Z)$ and $\sin^2\theta_W$ in SM and MSSM.
5. Introduction to Operator Product Expansion and RG for composite operators. Practical application: derive and solve RG equations for Wilson coefficients in the effective Hamiltonian describing $\Delta F = 2$ transitions ($\bar{K}^0 K^0$, $\bar{B}_d B_d$, $\bar{B}_s B_s$ mixing) in the Standard Model.
6. (optional, if time permits) Introduction to breaking of scale invariance and RG applications in this case (Bjorken scaling, deep inelastic scattering etc.)