(F) SCALE INVARIANCE, RGE AND OPE

$$X \to X' = e^{\varepsilon} X \to \text{DILATATION} \quad (\varepsilon \text{ REAL})$$

FIELD TRANSFORMATION

$$U(\varepsilon)\,\Phi(x)\,U^{-1}(\varepsilon) = T^{-1}(\varepsilon)\,\Phi(e^{\varepsilon}x)$$

$$\Phi'(x') = T(\varepsilon)\,\Phi(x)$$

FULLY REDUCIBLE ? $\to$ DILATATION SHOULD NOT MIX FIELDS

$$T(\varepsilon) = \exp(-d_{\Phi}\,\varepsilon)$$

$d_{\Phi} \to$ "CANONICAL FIELD DIMENSION"

FREE FIELD MASSLESS THEORY

NO DIMENSIONFUL CONSTANT $\to$ LAGR. INV. UNDER DILATATIONS $\to$ KOMMUTATION RELATIONS: ALSO

$$[\varphi(x,t), \dot{\varphi}(y,t)] = i\,\delta^{(3)}(\bar{x}-\bar{y})$$

$$[\psi(x,t), \psi^{+}(y,t)] = \delta^{(3)}(\bar{x}-\bar{y})$$

$$e^{-2d_{\psi}\varepsilon} = e^{-3\varepsilon} \Rightarrow d_{\psi} = 3/2$$

$$e^{-2d_{\varphi}\varepsilon}\,e^{-\varepsilon} = e^{-3\varepsilon} \Rightarrow d_{\varphi} = 1$$

INFINITESIMAL TRANSF.:

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$$U(\varepsilon)\,\Phi(x)\,U^{-1}(\varepsilon) = \Phi(x) + \varepsilon\left(d_\Phi + x_\mu \partial_\mu\right)\Phi(x) + O(\varepsilon^2)$$

LETS CONSIDER (FOR BARE FIELDS $d_\varphi = 1,\ d_\psi = 3/2$ EVEN WITH INTER.)

$$L = i\bar\Psi_B \not\partial \Psi_B - \frac{1}{2}(\partial_\mu \Phi_B)(\partial^\mu \Phi_B) + g\bar\Psi_B \gamma_5 \Psi_B \varphi_B - \frac{\lambda}{4!}\Phi_B^4$$

$$\delta\mathcal{L} = ULU^{-1} - L \simeq \varepsilon\left(4 + x_\mu \partial_\mu\right) L$$

$$= \varepsilon\,\partial_\mu (x^\mu L)$$

$$\delta I = \int d^4x\, L = 0$$

TERM $m^2\Phi_B^2$ SPOILS SCALE INV.!

COMMUTATORS WITH LORENTS GROUP:

$$U(a) = e^{iaP}$$
$$U(\omega) = e^{-\frac{1}{2} i \omega^{\mu\nu} M_{\mu\nu}}$$
$$U(\varepsilon) \equiv e^{i\varepsilon D}$$

$$\Phi(x) \xrightarrow[\varepsilon]{} e^{d_\Phi \varepsilon}\,\Phi(e^\varepsilon x) \xrightarrow[a]{} e^{d_\Phi \varepsilon}\,\Phi(e^\varepsilon x + a)$$

$$\xrightarrow[a]{} \Phi(x+a) \xrightarrow[\varepsilon]{} e^{d_\Phi \varepsilon}\,\Phi(e^\varepsilon(x+a))$$

$$U(\varepsilon)U(a) = U(e^\varepsilon a)\,U(\varepsilon)$$

$$i[D, P_\mu] = P_\mu \quad \Big| \quad \text{ALSO } [D, M_{\mu\nu}] = 0$$

$$i[D, \Phi(x)] = (d_\Phi + x_\mu \partial^\mu)\Phi(x)$$ 3

DILATATION CURRENT.

SCALE INVARIANCE $\Rightarrow$ CONSERVED CURRENT EXISTS

$$D_\mu(x) = \Pi_\mu d_\Phi \Phi + x^\nu \Theta^{CAN}_{\mu\nu}$$

$$\Theta^{CAN}_{\mu\nu} = \Pi_\mu \frac{\partial}{\partial x^\nu}\Phi(x) - g_{\mu\nu}\mathcal{L}$$

MORE CONVINIENT ~ DEFINE $(\lambda\Phi^4)$:

$$\Theta_{\mu\nu} = \Theta^{CAN}_{\mu\nu} - \frac{1}{6}(\partial_\mu\partial_\nu - g_{\mu\nu}\Box)\Phi^2$$

$$\begin{cases} \Theta_{\mu\nu} = \Theta_{\nu\mu} \quad \partial^\mu\Theta_{\mu\nu} = 0 \\ P_\mu = \int d^3\vec{x}\,\Theta_{0\mu} \\ M_{\mu\nu} = \int d^3\vec{x}\,(x_\mu\Theta_{0\nu} - x_\nu\Theta_{0\mu}) \end{cases}$$ $\rightarrow$ GOOD ENERGY-MOMENTUM TENSOR

USING $\Pi_\mu = \partial_\mu\varphi \Rightarrow D_\mu = x^\nu\Theta_{\mu\nu} - \frac{1}{6}\partial_\nu(x_\mu\partial^\nu - x^\nu\partial_\mu)\varphi^2$

$$\partial_\mu(\ldots) = 0$$

$$D_\mu(x) = x^\nu\Theta_{\mu\nu}(x)$$

$\partial^\mu D_\mu = \Theta_\mu{}^\mu$ AND $\Theta^\mu_\mu = m^2\varphi^2$ WITH MASS TERM

COULD BE GENERALIZED TO HIGHER SPINS

SCALE INV. HAS TO BE BROKEN!

$$[D, P^2] = -2iP^2$$

$$e^{i\varepsilon D} P^2 e^{-i\varepsilon D} = e^{2\varepsilon} P^2$$

LETS CONSIDER STATE WITH MOMENTUM p:

$$P^2 |p\rangle = p^2 |p\rangle$$

THEN

$$P^2 \underbrace{e^{-i\varepsilon D}|p\rangle}_{|e^{\varepsilon} p\rangle} = e^{2\varepsilon} p^2 \underbrace{e^{-i\varepsilon D}|p\rangle}_{|e^{\varepsilon} p\rangle}$$

ALL MASS SPECTRA CONTINUOUS!! - IMPOSSIBLE

~~CON~~ $\partial_\mu D^\mu \Rightarrow \theta^\mu{}_\mu$ $\rightarrow$ "WARD IDENTITIES" FOR THE DILAT. CURRENT

$$\frac{\partial}{\partial y_\mu} \langle 0| T\, D_\mu(y)\, \varphi(x_1) \dots \varphi(x_n)|0\rangle = \langle 0| T\, \theta^\mu_\mu(y)\, \varphi(x_1) \dots \varphi(x_n)|0\rangle$$

$$- i \sum_i \delta(x_i - y_i) \langle 0| T\, \varphi(x_1) \dots \delta\varphi(x_k) \dots \varphi(x_n)|0\rangle$$

WARD ID:

$$\left[-\sum_{i=1}^{n-1} p_i \frac{\partial}{\partial p_i} + n(d_\varphi - 4) + 4\right] G^{(n)}(p_1, \dots, p_{n-1})$$

$$= -i\, G_\theta^{(n)}(0, p_1, \dots, p_{n-1})$$

USING UNIFORM p SCALING, $p_i = \rho \hat{p}_i \Rightarrow \sum_i p_i \frac{\partial}{\partial p_i} = \rho \frac{\partial}{\partial \rho}$

AND $t = \ln \rho$

$$\left[-\frac{\partial}{\partial t} + \underbrace{(4 - n d_\varphi)}_{D}\right] \Gamma^{(n)}(e^t p_i) = -i\, \Gamma_\theta^{(n)}(0, e^t p_i)$$

FOR $\theta_\mu^\mu = 0$

$$\Gamma^{(n)}(\rho p_i) = \rho^D \Gamma^{(n)}(p_i)$$ – CANONICAL SCALING AT CLASSIC LEVEL, BROKEN BY DIMENS. CUTOFF μ AT THE QUANTUM LEVEL

BUT, WE KNOW FROM RGE:

$$\Gamma^{(n)}(e^t p_i, \lambda, m, \mu) = e^{Dt - n\int_0^t \gamma(\bar{\lambda}(t'))\,dt'} \Gamma^{(n)}\left(p_i, \bar{\lambda}, \frac{\bar{m}(t)}{e^t}, \mu\right)$$

$$\int_\lambda^{\bar{\lambda}(t)} \frac{dx}{\beta(x)} = t$$

$$\bar{m}(t) = m\, e^{\int_0^t \gamma_m(\bar{\lambda}(t'))\,dt'} = m\, e^{\int_\lambda^{\bar{\lambda}} \frac{\gamma_m}{\beta} dx}$$

ONLY FOR $\beta = \gamma = 0$ WE RECOVER CANONICAL SCALING

IN GENERAL – $\theta^{\mu}_{\ \mu} \neq 0$, OR FIELD TRANSFORMS ANOMALOUSLY, OR BOTH.

CONSIDER UV FIXED POINT λ_f

$$e^{-n\int_0^t \gamma(\bar\lambda(t'))dt'} = \rho^{-n\gamma(\lambda_f) + \varepsilon(t)}$$

$$\varepsilon(t) = -\frac{1}{t}\int_{\lambda}^{\bar\lambda} \frac{n(\gamma(x) - \gamma(\lambda_f))}{\beta(x)}\, dx$$

INTEGRAL FINITE $\Rightarrow$ $\varepsilon(t) = O(\frac{1}{t})$ $\to$ THEORY IS ASYMPTOTICALLY, $t \to \infty$, SCALE INVARIANT, BUT WITH $D \to D - n\gamma(\lambda_f)$!

$$\Gamma^{(n)}(\rho p_i, \lambda, \mu) \sim \rho^{D - n\gamma(\lambda_f)}\, \Gamma^{(n)}(p_i, \lambda_f, \mu)$$

LEADING CORRECTION:

$$\Gamma^{(n)}(\rho p_i, \lambda, \mu) = \rho^{D - n\gamma_f}\, e^{-\int_{\lambda}^{\lambda_f} \frac{\gamma - \gamma_f}{\beta} d\lambda}\, \Gamma^{(n)}(p_i, \lambda_f, \mu) \otimes$$

$$\otimes \left[1 + O(\rho^{-|\beta'(\lambda^f)|})\right]$$

HOW ABOUT QCD

$$\beta(x) \cong -b x^3 + \dots$$

$$\gamma(x) - \gamma(0) = 2 c x^2 + \dots$$

THUS $\varepsilon(t) = -\frac{1}{t} \exp \int_{\lambda}^{\bar{\lambda}} \frac{2c}{b} \frac{dx}{x} \sim -\frac{1}{t} \left(2b \lambda^2 t \right)^{-c/b}$

$$\uparrow$$

$$\bar{\lambda}^2(t) \sim \frac{1}{2bt}$$

LOGARITHMIC DEVIATIONS FROM ASYMPTOTIC SCALING!

$$\varrho^{\varepsilon(t)} = e^{t\varepsilon(t)} = e^{-(2b\lambda^2 t)^{-c/b}}$$

CAN BE STRICTLY PROVEN

$$\left(\Theta_\mu^{\mu} \right)_{QCD} = \frac{\beta(g)}{2g^3} G^a_{\mu\nu} G^{\mu\nu}_a \quad !$$

MASS BEHAVIOUR

$$m\frac{\partial}{\partial m}\frac{i}{p^2-m^2} = \frac{i}{p^2-m^2}\left(-2im^2\right)\frac{i}{p^2-m^2}$$

THUS, $m\frac{\partial}{\partial m}$ EQUIVALENT TO INSERTING VERTEX

$-im^2\Psi^2$ AT $p=0$

$$\left[-\frac{\partial}{\partial t}+D\right]G^{(n)}(e^t p_i) = -i\,G_\theta^{(n)}(0, e^t p_i)$$

IDENTICAL WITH

$$\left[-\frac{\partial}{\partial t} - m\frac{\partial}{\partial m} + D\right]G^{(n)}(e^t p_i) = 0$$

AT THE QUANTUM LEVEL

$$\left[-\frac{\partial}{\partial t}+\beta\frac{\partial}{\partial\lambda}+(\gamma_m-1)m\frac{\partial}{\partial m} - n\gamma + D\right]G^{(n)}(e^t p_i) = 0$$

$t\to\infty$

$$\bar{m}(\rho) = m\,\rho^{\gamma_m(\lambda_+)-1}\, e^{\int_\lambda^{\bar\lambda}\frac{\gamma_m-\gamma_m(\lambda_+)}{\beta}dx}$$

$$\to m\rho^{\gamma_+-1}\times\left[\text{logs of }\rho\right]$$

$$\bar{m}(\rho)\xrightarrow[\rho\to\infty]{} 0 \quad \text{IF} \quad \gamma(\lambda_+)<1$$

SCALE DIM FOR MASS OPERATOR:

$$d_{\Delta m} = \begin{cases} 3+\gamma_m(\lambda_+) \\ 2+2\gamma_m(\lambda_+)\end{cases} \qquad \Delta m = \begin{cases} m\bar\Psi\Psi \\ m^2\varphi^2 \end{cases} \quad d_{\Delta m}<4 \to \bar{m}\to 0$$

$d_{\Delta m} < 4 \to$ "SOFTLY BROKEN" SCALE INV.

MASS INSERTIONS DOES NOT AFFECT HIGH ENERGY BEHAVIOUR OF THE GREEN'S FUNCTIONS.