(A) INTRODUCTION - RENORMALIZATION AND RENORMALIZATION GROUP.

$$L = \frac{1}{2}(\partial_\mu \varphi_B)^2 - \frac{1}{2} m_B^2 \varphi_B^2 - \frac{\lambda_B}{4!} \varphi_B^4$$

n-point GREEN'S FUNCTION $= \sum_{k=0}^{\infty} G_k^{(n)}(p, m_B^2)\, \lambda_B^k$

$G_k^{(n)}$ - DIVERGENT IN PERTURBATION THEORY (DIV. MOMENTUM INTEGRATION)

HOW TO INTERPRET PERT. SERIES IN QFT?

1) REGULARIZATION (INTERMEDIATE STEPS)

2) RENORMALIZATION - "BARE" QUANTITIES ARE NOT OBSERVABLES - EXPRESS OBSERVABLES IN TERMS OF MEASURED PARAMETERS.

PHYSICAL FIELDS : CREATE NORMALIZED PARTICLE STATES

$$\langle k | \varphi_F(x) | 0 \rangle = e^{ikx}$$

"PHYSICAL" TRUNCATED GREEN'S FUNCTIONS $\Leftrightarrow$ S MATRIX ELEMENTS

CONSIDER 2-POINT GREEN'S FUNCTION:

$$G^{(2)}(x,y) = \langle 0 | T\, \Phi(x)\, \Phi(y) | 0 \rangle$$

$$G^{(2)}(x,y) = \int \frac{d^4k}{(2\pi)^4}\, \delta(k^2-m^2)\, \langle 0|\Phi(x)|k\rangle\langle k|\Phi(y)|0\rangle$$

$$= \int \frac{d^4k}{(2\pi)^4}\, \frac{i}{k^2-m_F^2+i\epsilon}\, e^{-ik(x-y)}\, \left|\langle 0|\Phi(0)|k\rangle\right|^2$$

FOR PHYSICAL FIELDS

$$G_F^{(2)}(k) \underset{k^2\to m_F^2}{\approx} \frac{i}{k^2-m_F^2+i\epsilon}$$

IN GENERAL

$$G^{(2)}(k) \underset{k^2\to m_F^2}{\approx} \frac{i\langle 0|\Phi(0)|k\rangle^2}{k^2-m_F^2+i\epsilon}$$

FOR <u>BARE FIELDS</u> WE DEFINE

$$G_B^{(2)}(k) \approx \frac{Z_3}{k^2-m_F^2+i\epsilon} \qquad Z_3 \neq 1 \quad (\text{DIVERGENT!})$$

SO $\Phi_B = Z_3^{1/2}\, \Phi_F$

$$G_F^{(n)}(x_1,\dots,x_n) = Z_3^{-n/2}\, G_B^{(n)}(x_1,\dots,x_n)$$

BARE GREEN'S FUNCTIONS ARE NOT OBSERVABLES!

FURTHER STEPS:

$$\Gamma^{(4)}_F(p)\Big|_\mu = \lambda_F \qquad - \text{"PHYSICAL COUPLING"}$$

DEFINE $\Gamma^{(4)}_B(p)\Big|_\mu = Z_1^{-1}\lambda_B \implies \lambda_B = Z_1 Z_3^{-2}\lambda_F$

$$\Gamma^{(2)}(p^2 = m_F^2) = 0 \implies m_F^2 = (Z_3/Z_0)\, m_B^2$$

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \varphi_B)^2 - \tfrac{1}{2} m_B^2 \varphi_B^2 - \tfrac{1}{4!}\lambda_B \varphi_B^4 =$$

$$= \tfrac{1}{2}(\partial_\mu \varphi_F)^2 - \tfrac{1}{2} m_F^2 \varphi_F^2 - \tfrac{\lambda_F}{4!}\varphi_F^4$$

$$\left.+\tfrac{1}{2}(Z_3 - 1)(\partial_\mu \varphi_F)^2 - \tfrac{1}{2}(Z_0 - 1)\, m_F^2 \varphi_F^2 - (Z_1 - 1)\tfrac{\lambda_F}{4!}\varphi_F^2\right\}$$

COUNTER-TERMS

Z_i FIXED BY REN. CONDITIONS:

FOR "PHYSICAL" FIELDS

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}_F\Big|_{p^2 = m_F^2} = 1 \qquad \left(G_F \sim \frac{i}{k^2 - m_F^2 + i\varepsilon}\right)$$

$$\Sigma_F(p^2)\Big|_{p^2 = m_F^2} = 0 \qquad (m_F^2 - \text{real pole})$$

$$\Gamma^{(4)}(p)\Big|_\mu = \lambda_F \qquad (\text{physical coupling})$$

ACTUALLY – EVEN MORE FREEDOM

$$\Gamma^{(4)}_F(p)\big|_\mu = \lambda_F \longrightarrow \Gamma^{(4)}_F(p)_{\mu'} = \lambda'_F$$

CHOICE OF μ, μ' FREE OF COURSE!

ALSO, DEFINE $\Sigma_R(p)\big|_\mu = 0$, $\mu \neq m_F^2 \Rightarrow m_R^2 = \frac{Z_3}{Z_0'} m_B^2$

WE CAN EXPRESS EVERYTHING IN TERMS OF λ'_F, m_R^2. E.G., PHYSICAL MASS NOW GIVEN BY

$$\left[p^2 - m_R^2 - \Sigma_R(p)\right]\Big|_{p^2 = m_F^2} = 0$$

$$m_F^2 = m_F^2(m_R^2) \neq m_R^2$$

EQUIVALENT SET OF REN. CONDITIONS

$$\frac{\partial}{\partial p^2}\Gamma^{(2)}_R(k)\Big|_\mu = 1$$

$$\Sigma_R(k)\big|_\mu = 0$$

$$\Gamma^{(4)}_R(k)\big|_\mu = \lambda_R$$

OTHER CHOICE – MS, $\overline{\text{MS}}$ SCHEME

$$L = L\left(Z_3 \varphi_R, \frac{Z_0}{Z_3} m_R^2, \frac{Z_1}{Z_3^2}\lambda_R\right) = L(\varphi_R, m_R^2, \lambda_R) + c.t.$$

PHYSICAL OBSERVABLES SHOULD NOT DEPEND ON REN. SCHEME – UNFORTUNATELY ONLY IF WE SUM PERT. SERIES TO ALL ORDERS...

RGE DERIVATION

$$\Gamma_R = Z(R)\,\Gamma_B$$

$$\Gamma_{R'} = Z(R')\,\Gamma_B$$

$\cancel{\Gamma_R}$ $\quad \Gamma_{R'} = \dfrac{Z(R')}{Z(R)}\Gamma_R \equiv Z(R',R)\,\Gamma_R$

$$\begin{cases} Z(R'',R) = Z(R'',R')\,Z(R',R) \\ Z^{-1}(R',R) = Z(R,R') \\ Z(R,R) = 1 \end{cases} \Rightarrow \text{GROUP?}$$

NOT REALLY, SOMETIMES:

$Z(R,R')\,Z(R'',R''') \notin G$

LETS ASSUME WE USE DIMENSIONAL REGULARIZATION, $d = 4-\varepsilon$. FOR $\lambda\varphi^4$:

$$\Gamma_B^{(n)}(p_i, \lambda_B, m_B, \varepsilon) = Z_3^{-n/2}\,\Gamma_R^{(n)}(p_i, \lambda_R, m_R, \mu, \varepsilon)$$

$$\Gamma_R^{(n)} = Z_3^{n/2}\,\Gamma_B^{(n)}$$

[REMARK: DIMENSIONLESS $\lambda_R = Z_3^2 Z_1^{-1} \mu^{-\varepsilon}\lambda_B$]

$\dfrac{d}{d\mu}\Gamma_B = 0$, SO

$$\mu\left(\frac{\partial}{\partial\mu} + \frac{d\lambda_R}{d\mu}\frac{\partial}{\partial\lambda_R} + \frac{dm_R}{d\mu}\frac{\partial}{\partial m_R}\right)\Gamma_R^{(n)} = \left(\frac{1}{2}n\frac{1}{Z_3}\mu\frac{d}{d\mu}Z_3\right)\Gamma_R^{(n)}$$

LETS DEFINE:

$$\beta(\lambda_R, \varepsilon) = \mu \frac{d}{d\mu} \lambda_R = \mu \frac{d}{d\mu} \left(\lambda_B \mu^{-\varepsilon} Z_3^2 Z_1^{-1} \right)$$

$$\cancel{= \lambda_B \left(\mu \frac{d\mu}{d\mu} Z_3^2 Z_1^{-1} \right)}$$

$$= \lambda_B \left[-\varepsilon \mu^{-\varepsilon} Z_3^2 Z_1^{-1} + \mu^{-\varepsilon+1} \frac{d}{d\mu} Z_3^2 Z_1^{-1} \right]$$

$$= -\varepsilon \lambda_R + \lambda_R Z_3^{-2} Z_1 \mu \frac{d}{d\mu} Z_3^2 Z_1^{-1}$$

$$= -\varepsilon \lambda_R + \lambda_R \mu \frac{d}{d\mu} \left(\ln Z_3^2 Z_1^{-1} \right) \xrightarrow[\varepsilon \to 0]{} \beta(\lambda_R)$$

$$\gamma(\lambda_R, \varepsilon) = \frac{1}{2} \mu \frac{d}{d\mu} \ln Z_3 \quad \rightarrow \gamma(\lambda_R)$$

$$\gamma_m(\lambda_R, \varepsilon) = \frac{\mu}{m_R} \frac{d m_R}{d\mu} = \frac{1}{2} \mu \frac{d}{d\mu} \ln \left(Z_3 Z_0^{-1} \right) \rightarrow \gamma_m(\lambda_R)$$

FOR $\varepsilon \to 0$

$$\left[\mu \frac{\partial}{\partial \mu} + \beta(\lambda_R) \frac{\partial}{\partial \lambda_R} + \gamma_m(\lambda_R) m_R \frac{\partial}{\partial m_R} - n \gamma(\lambda_R) \right] \Gamma_R^{(n)} = 0$$

HOW TO CALCULATE β, γ, γ_m? DENOTE $Z_\lambda = Z_3^2 Z_1^{-1}$

$$\beta(\lambda, \varepsilon) = \mu \frac{d\lambda}{d\mu} = -\varepsilon \lambda + \lambda Z_\lambda^{-1} \mu \frac{d}{d\mu} Z_\lambda$$

$$\mu \frac{d}{d\mu} Z_\lambda = \mu \frac{d\lambda}{d\mu} \frac{d Z_\lambda}{d\lambda} = \beta(\lambda, \varepsilon) \frac{d Z_\lambda}{d\lambda}$$. THUS:

$$\beta(\lambda, \varepsilon) \left[Z_\lambda + \lambda \frac{d Z_\lambda}{d\lambda} \right] + \varepsilon \lambda Z_\lambda = 0$$

EXERCISE SHOW THAT IF

$$\lambda_B = \mu^\varepsilon Z_\lambda \lambda = \mu^\varepsilon \left[1 + \sum_\nu \frac{a_\nu(\lambda)}{\varepsilon^\nu}\right]\lambda$$

$$a_\nu(\lambda) = a_{\nu 1}\lambda + \dots + a_{\nu n}\lambda^n \quad \text{IN } n\text{-th ORDER}$$

THEN:

$$\beta(\lambda, \varepsilon) = -\varepsilon\lambda + \beta(\lambda)$$

$$\beta(\lambda) = \lambda^2 \frac{da_1}{d\lambda} \qquad \text{— SIMPLE POLES IMPORTANT!}$$

$$\lambda^2 \frac{da_{\nu+1}}{d\lambda} = \beta \frac{d}{d\lambda}(\lambda a_\nu)$$

$$a_{mn} = 0 \quad \text{FOR} \quad m < n$$

HINT: ASSUME $\beta(\lambda, \varepsilon) = \sum_\nu \beta_\nu \varepsilon^\nu$, COMPARE ε^ν COEFFICIENTS

AT HOME (?)

$$\gamma(\lambda) = -\frac{1}{2}\lambda \frac{d}{d\lambda} Z_3^{(1)}(\lambda), \quad \text{WHERE} \quad Z_3 = 1 + \sum_\nu \frac{Z_3^{(\nu)}}{\varepsilon^\nu}$$

$$\gamma_m(\lambda) = \frac{1}{2}\lambda \frac{dZ_m^{(1)}}{d\lambda}, \quad \text{WHERE} \quad m_B^2 = Z_m m^2, \quad Z_m = 1 + \sum_\nu \frac{Z_m^{(\nu)}}{\varepsilon^\nu}$$

$$Z_m = Z_0 Z_3^{-1} \text{ in } \lambda\phi^4$$