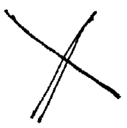
13 DERIVATION OF RGE IN $\lambda\varphi^4$

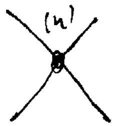
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2 - \frac{\lambda\mu^\varepsilon}{4!}\varphi^4 + \frac{1}{2}(Z_3-1)(\partial_\mu\varphi)^2 - \frac{1}{2}(Z_0-1)m^2\varphi^2 - (Z_1-1)\frac{\lambda\mu^\varepsilon}{4!}\varphi^4$$

$$Z_i = 1 + \sum_n (Z_i-1)^{(n)} \equiv 1 + t_i^{(n)}\lambda^n$$

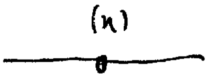
FEYNMAN RULES:

 $\frac{i}{p^2-m^2+i\varepsilon}$

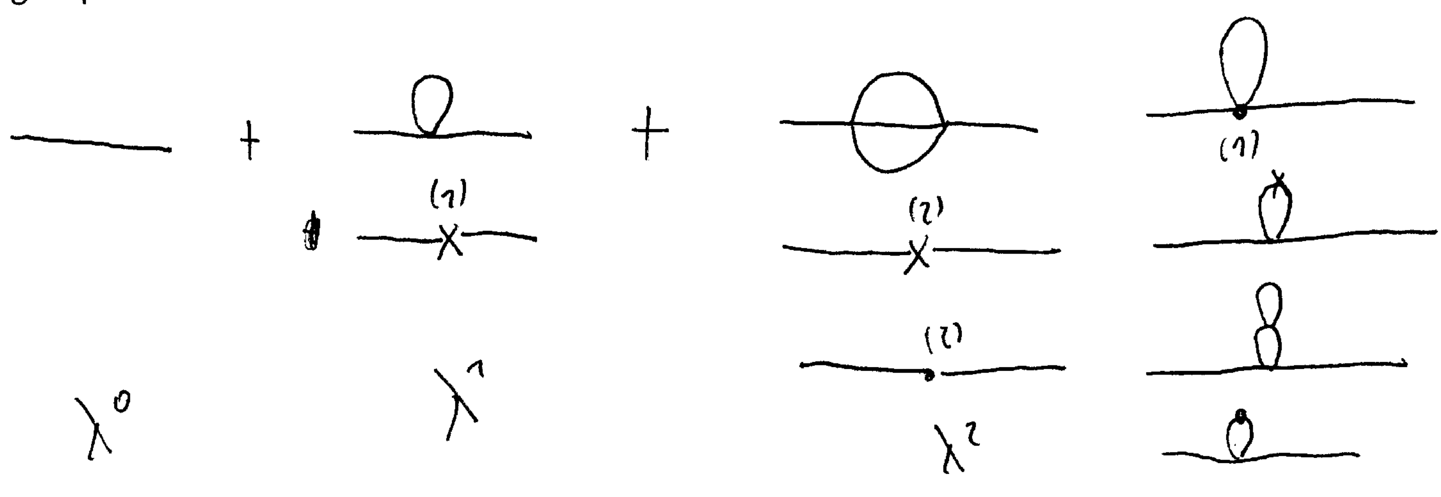
 $-i\lambda\mu^\varepsilon$

 $-i\lambda(Z_1-1)^{(n)}\mu^\varepsilon$

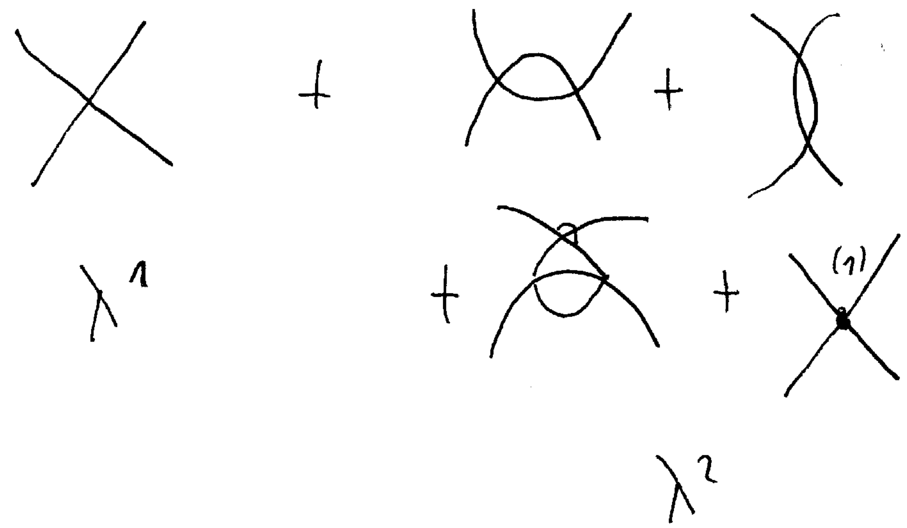
 $-i(Z_0-1)^{(n)}m^2$

 $i(Z_3-1)^{(n)}p^2$

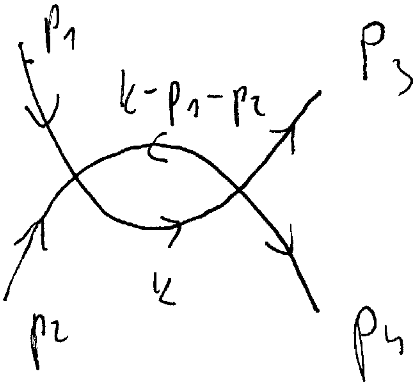
2-POINT GREEN'S FUNCTION



4-POINT GREEN'S FUNCTION



START FROM 4-POINT



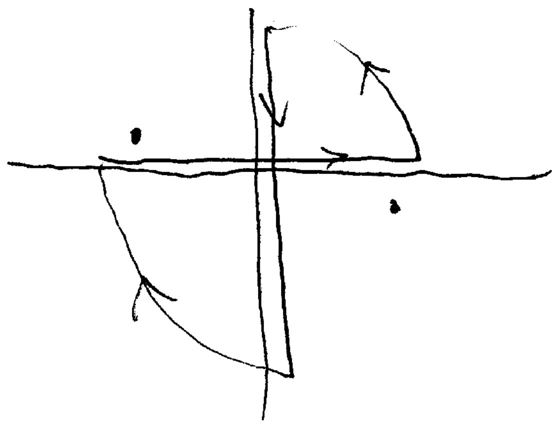
$$\Gamma_{4a} = N_c \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\varepsilon} \frac{i}{(k-p)^2-m^2+i\varepsilon} \left(-i\lambda \mu^{\varepsilon}\right)^2$$

$$N_c = \frac{1}{2}\left(\frac{1}{4!}\frac{1}{4!}\, 4!\,4!\,2!\right)$$

USING $\frac{1}{ab} = \int_0^1 \frac{dx}{[ax+b(1-x)]^2}$ ~~for~~ AND $k-Px \to k$ ONE GET'S

$$A_1 = \frac{1}{2}\lambda^2 \mu^{2\varepsilon} \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{1}{\left(k^2+p^2x(1-x)-m^2+i\varepsilon\right)^2}$$

WICK ROTATION



$$k_4 = i k_0$$

$$d k_0 = -i\, d k_4$$

$$k_0^2 - \bar{k}^2 = -\bar{k}_E^2$$

$$\int_{i\infty}^{-i\infty} d k_0 = i \int_{-\infty}^{\infty} d k_4$$

$$A_1 = \frac{1}{2} \lambda^2 \mu^{2\varepsilon} \int_0^1 dx \int \frac{d^n k_E}{(2\pi)^n} \frac{i}{\left[k_E^2 - p^2 x(1-x) + m^2 - i\varepsilon\right]^2}$$

$$\int d^n k = \frac{2\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)} \int k^{n-1} dk$$

$$A_1 = \frac{1}{2} \lambda^2 \mu^{2\varepsilon} \int_0^1 dx \frac{2\pi^{n/2}}{(2\pi)^n} \int_0^\infty \frac{i\, k^{n-1} dk}{\left[k^2 - p^2 x(1-x) + m^2\right]^2} \quad (*)$$

29.10.01

MORE GENERAL

$$\int \frac{d^n k}{(2\pi)^n} \frac{1}{(k^2+b)^\alpha} = \frac{1}{(4\pi)^{n/2}} \frac{b^{n/2-\alpha}\, \Gamma\left(\alpha - \frac{1}{2} n\right)}{\Gamma(\alpha)}$$

$$A_1 = \frac{1}{2} i (\lambda \mu^\varepsilon) \frac{\lambda}{(4\pi)^{n/2}} \frac{\Gamma(\frac{\varepsilon}{2})}{\Gamma(2)} \int_0^1 dx \left[\frac{m^2 - p^2 x(1-x) - i\varepsilon}{\mu^2} \right]^{-\varepsilon/2}$$

$$\Gamma(x) \xrightarrow[x \to 0]{} \frac{1}{x} - \gamma + O(x)$$

$$x^y \doteq 1 + y \ln x + O(y^2)$$

$$A_1 = i\lambda\mu^\varepsilon \frac{\lambda}{2(4\pi)^2} \left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi \right]$$

$$- i\lambda\mu^\varepsilon \frac{\lambda}{2(4\pi)^2} \int_0^1 dx \ln \frac{m^2 - p^2 x(1-x) - i\varepsilon}{\mu^2}$$

$$\cancel{= -i \frac{\lambda(\lambda\mu^\varepsilon)}{2(4\pi)^2} \left[-\frac{2}{\varepsilon} + \gamma - \ln 4\pi + \ln \frac{m^2}{\mu^2} \right.}$$

$$= i\lambda\mu^\varepsilon \frac{\lambda}{2(4\pi)^2} \left[\frac{2}{\varepsilon} - \gamma + \ln 4\pi \right] - i A_1(s)$$

$$s = p^2, \quad A_1(s) = \frac{\lambda(\lambda\mu^\varepsilon)}{2(4\pi)^2} \left[\gamma - \ln 4\pi - 2 + \ln \frac{m^2}{\mu^2} \right.$$

$$\left. + 2 \underbrace{\left(\frac{1}{4} - \frac{m^2}{s} \right)^{1/2}}_{\sqrt{}} \ln \frac{\frac{1}{2} + \sqrt{}}{\frac{1}{2} - \sqrt{}} \right]$$

WITH CROSSED DIAGRAMS

$$\Gamma^{(4)}_{1\text{-loop}} = \lambda - \frac{3\lambda}{(4\pi)^2} \frac{1}{\varepsilon} \lambda \mu^\varepsilon + A_1(s) + A_1(t) + A_1(u)$$

$$+ \lambda (Z_1 - 1)^{(1)} \mu^\varepsilon$$

THUS:

$$(Z_1-1)^{(1)} = \underbrace{\frac{1}{\epsilon}\frac{3\lambda}{(4\pi)^2}}_{\text{MS SCHEME}} + \text{FINITE TERMS}$$

$$(Z_1-1)^{(1)}_{\overline{MS}} = \frac{1}{2}\frac{3\lambda}{(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma+\ln 4\pi\right]$$

2-POINT FUNCTION

$$= \frac{1}{2}(-i\lambda)\mu^{\epsilon}\int\frac{d^nk}{(2\pi)^n}\frac{i}{k^2-m^2+i\epsilon}$$

$$= \frac{1}{2}(-i\lambda)\mu^{\epsilon}\frac{1}{(4\pi)^2}(4\pi)^{\epsilon/2}\left(m^2\right)^{1-\frac{\epsilon}{2}}\Gamma\left(\frac{1}{2}\epsilon-1\right)$$

$$= i\frac{\lambda m^2}{2(4\pi)^2}\left[\frac{2}{\epsilon}-\gamma+\ln 4\pi\right] - i\frac{\lambda m^2}{2(4\pi)^2}\left[-1+\ln\frac{m^2}{\mu^2}\right]$$

$$= -i(Z_0-1)^{(1)}m^2$$

THUS $(Z_0-1)^{(1)} = \frac{1}{\epsilon}\frac{\lambda}{(4\pi)^2}$ + FINITE TERMS.

λ^2 ORDER – MORE COMPLICATED → AT HOME.

ALL CALCULATIONS A BIT TEDIOUS BUT EASY,

ONE EXCEPTION:  $\equiv I$

$$I = \frac{1}{6}\lambda^2 \mu^{2\epsilon}\, i \int \frac{d^n k}{(2\pi)^n}\frac{d^n l}{(2\pi)^n}\,\frac{1}{k^2-m^2+i\epsilon}\,\frac{1}{l^2-m^2+i\epsilon}\,\otimes$$

$$\otimes\ \frac{1}{(p+l-k)^2-m^2+i\epsilon}$$

RESULT: COLLINS PRD10 (1974) 1213

$$I^{INF} = \frac{i\lambda^2}{[(4\pi)^2]^2}\left[-\frac{m^2}{\epsilon^2}+\frac{1}{\epsilon}\left[m^2\ln\frac{m^2}{4\pi\mu^2}+\frac{p^2}{12}+\left(\gamma-\frac{3}{2}\right)m^2\right]\right]\Big\}$$

ALL TOGETHER

$$\begin{cases} Z_0 = 1+\frac{\lambda}{(4\pi)^2}\frac{1}{\epsilon}+\left[\frac{\lambda}{(4\pi)^2}\right]^2\left[\frac{2}{\epsilon^2}-\frac{1}{2\epsilon}\right]+O(\lambda^3) \\ Z_1 = 1+\frac{\lambda}{(4\pi)^2}\frac{3}{\epsilon}+O(\lambda^2) \\ Z_3 = 1-\left[\frac{\lambda}{(4\pi)^2}\right]^2\frac{1}{12\epsilon}+O(\lambda^3) \end{cases}$$

$$Z_\lambda = Z_1 Z_3^{-2} = 1+\frac{\lambda}{(4\pi)^2}\frac{3}{\epsilon}+O(\lambda^2)$$

$$Z_m = Z_0 Z_3^{-1} = 1+\frac{1}{\epsilon}\left[\frac{\lambda}{(4\pi)^2}-\frac{5}{12}\left(\frac{\lambda}{(4\pi)^2}\right)^2\right]+\frac{2}{\epsilon^2}\left(\frac{\lambda}{(4\pi)^2}\right)^2+O(\lambda^3)$$

$$m_B^2 = Z_m m_R^2$$

$$\lambda_B = Z_\lambda \lambda_R^2$$

COMING BACK TO RGE:

$$\beta(\lambda) = \lambda^2 \frac{d}{d\lambda} Z_\lambda^{(1)} = \lambda^2 \frac{d}{d\lambda} \frac{3\lambda}{(4\pi)^2} = \frac{3\lambda^2}{(4\pi)^2} + O(\lambda^3)$$

$$\gamma_m(\lambda) = \frac{1}{2}\lambda \frac{d Z_m^{(1)}}{d\lambda} = \frac{1}{2}\lambda \frac{d}{d\lambda}\left[\frac{\lambda}{(4\pi)^2} - \frac{5}{12}\left(\frac{\lambda}{(4\pi)^2}\right)^2\right]$$

$$= \frac{1}{2}\frac{\lambda}{(4\pi)^2} - \frac{5}{12}\left(\frac{\lambda}{(4\pi)^2}\right)^2 + O(\lambda^3)$$

$$\gamma(\lambda) = -\frac{1}{2}\lambda \frac{d}{d\lambda} Z_3^{(1)}(\lambda) = \frac{1}{12}\left(\frac{\lambda}{(4\pi)^2}\right)^2 + O(\lambda^3)$$

SOME BRAVE RUSSIANS CALCULATED FIRST 4 (5?) TERMS OF $\beta(\lambda)$ IN $\lambda\varphi^4$!