(E) OPERATOR PRODUCT EXPANSION
RGE FOR COMPOSITE OPERATORS

WILSON HYPOTHESIS

$$A(x)\,B(y) \underset{x\sim y}{=} \sum_n C^n_{AB}(x-y)\, O_n\left(\tfrac{1}{2}(x+y)\right)$$

$\{O_n(x)\}$ local hermitian operators

C^n_{AB} - c-functions (WILSON COEFFICIENTS)

PROOF - VALID IN FREE-FIELD THEORY

INTERACTIVE THEORY TO ANY ORDER

APPLICATIONS TO RGE - E.G. COMPOSITE OPERATORS

$$L^B_{\varphi^4} \to L^B_{\varphi^4} + \sum_n C^B_n O^B_n$$

$$G^{(n)}_O(x_1,\dots,x_n,x) = \langle 0 | T\,\varphi(x_1)\dots\varphi(x_n)\,O(x)|0\rangle$$

NEW REN. CONSTANTS NECESSARY

$$O^B = Z_O O^R$$

E.G. $(\varphi^2_B)_B = Z_{\varphi^2}(\varphi^2_B)_R = Z_{\varphi^2} Z_3 (\varphi^2_R)_R$

NEW ~~RENO~~ COUNTERTERMS, NEW REN. CONDITIONS ETC.

$$\Gamma^{(n)}_{O,R} = Z_O^{-1} Z_3^{n/2} \Gamma^{(n)}_{O,B}$$

POSSIBLE COMPLICATION — OPERATOR MIXING

$$O_n^B = (Z_O)_{nk} O_k^R$$

$$\Gamma^n_{O_i,R} = (Z_O^{-1})_{ik} Z_3^{n/2} \Gamma^n_{O_k B}$$

RGE — ANALOGICALLY AS BEFORE

$$\left\{\left[-\frac{\partial}{\partial t} + \beta \frac{\partial}{\partial \lambda} + \gamma_m \frac{\partial}{\partial m} - n\gamma + D\right]\delta_{ij} + \gamma_O^{ij}\right]\Gamma^{(n)}_{O_j}(e^t p_i, \lambda(\mu), m, \mu) = 0$$

$$\gamma_O^{ij} = -\left(\mu \frac{d}{d\mu}(Z_O)^{ik}\right)(Z_O^{-1})^{kj} \qquad (*)$$

RGE FOR WILSON COEFFICIENTS

$$A(\tfrac{1}{2}x) B(-\tfrac{1}{2}x) \underset{x^2 \to 0}{\simeq} \sum_i C_i^{AB}(x, g(\mu), \mu)\, O_i(0)$$

HENCE

$$\Gamma^{(n)}_{AB} = \sum_i C_i^{AB}(q, g(\mu), \mu)\, \Gamma^{(n)}_{O_i} \qquad (**)$$

APPLYING $\mathcal{D} = -\frac{\partial}{\partial t} + \beta(g)\frac{\partial}{\partial g}$ TO $(**)$ AND USING $(*)$:

$$\left\{\left[-\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial g} + D_A + D_B - D_O + \gamma_A + \gamma_B\right]\delta_{ij} - \gamma_O^{ij}\right\} C_j^{AB}(e^t q, g(\mu), \mu) = 0$$

NO MIXING OF A, B ASSUMED — SOMETIMES γ_A, γ_B ARE MATRICES
D_A, D_B, D_O — CANONICAL DIM'S OF A, B, O

SOLUTION FOR NO-MIXING CASE $\gamma_0^{ij} = \gamma_i \delta^{ij}$ (3)

$$C_i(e^t q, g(\mu), \mu) = \cancel{e^{-\gamma_i(g_+)}} \left[e^{-\int_g^{\bar g(t)} \frac{\gamma_i(x)}{\beta(x)} dx} \right] C_i(q, \bar g(t), \mu)$$

APPROXIMATE SOLUTION IN ASYMPTOTICALLY-FREE THEORIES – EXERCISE

$$C_i(e^t q, g(\mu)) \equiv e^{-\gamma_i(g_+)t} \, e^{-\int_g^{\bar g(t)} \frac{\gamma_i(x) - \gamma_i(g_+)}{\beta(x)} dx} \; C_i(q, \bar g(t))$$

$$\beta(x) = \frac{2}{16\pi^2} b_1 x^3 + \dots$$

$$\gamma_i(x) = \frac{4}{16\pi^2} \gamma_i^0 x^2 + \dots$$

$$g_+ = 0$$

$$C_i(e^t q, g(\mu)) \cong e^{-\int_{g(\mu)}^{\bar g(t)} \frac{4\gamma_i^0}{2 b_1 x} dx} \; C_i(q, \bar g(t))$$

$$\cong e^{-\frac{2\gamma_i^0}{b_1} \ln \frac{\bar g(t)}{g(\mu)}} \; C_i(q, \bar g(t))$$

$$C_i(e^t q, g(\mu), \mu) \cong \left[\frac{\bar g^2(t)}{g^2(\mu)} \right]^{-\frac{\gamma_i^0}{b_1}} C_i(q, \bar g(t), \mu)$$

PUT $\cancel{t = \frac{1}{2} \ln \frac{Q^2}{\mu^2}}$

$$\cancel{C_i(\mu, g(\mu), \mu) = \left[\frac{\bar g^2(t)}{g^2(\mu)} \right]}^{-\frac{\gamma_i^0}{b_1}} C_i(q, \bar g(t), \mu)$$

PUT $q^2 = \mu^2$

$$C_i(e^t \mu, g(\mu), \mu) = \left[\frac{g^2(t)}{g^2(\mu)}\right]^{-\frac{\gamma_i^0}{b_1}} C_i(\mu, g(t), \mu)$$

IN ASYMPT. FREE THEORIES $g(t) \xrightarrow[t\to\infty]{} 0$

$$C_i(\mu, g(t), \mu) \doteq C_i(\mu, 0, \mu)\left[1 + C_i^1 \frac{g^2(t)}{4\pi} + \dots\right]$$

IN THE LOWEST ORDER

$$C_i(e^t \mu, g(\mu), \mu) \doteq \left[\frac{g^2(t)}{g^2(\mu)}\right]^{-\frac{\gamma_i^0}{b_1}} C_i(\mu, 0, \mu)$$

FINALLY PUT $t = \frac{1}{2} \ln \frac{q^2}{\mu^2}$

$$C_i(q, g(\mu), \mu) = C_i(\mu, 0, \mu)\left[\frac{g^2(t)}{g^2(\mu)}\right]^{-\frac{\gamma_i^0}{b_1}}$$

UFF! PRACTICAL APPLICATION: KK / BB MIXING.

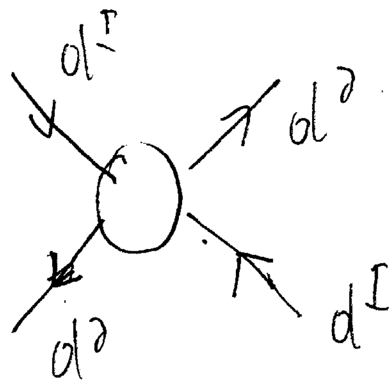
1) CALCULATE WILSON COEFFICIENTS AT HIGH ($\mu \sim M_W, m_t$) SCALE – E.G. BOX DIAGRAMS.

2) EVOLVE THEM TO THE LOW SCALE ($\mu = m_K, m_b$)

3) ADD HADRONIC MATRIX ELEMENTS (E.G., FROM LATTICE).

RESULT: $\varepsilon_K, \Delta m_K, \Delta m_B, \dots$

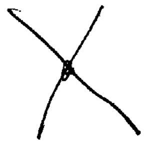
PRACTICAL EXERCISE: $\Delta F=2$ MIXING IN THE SM



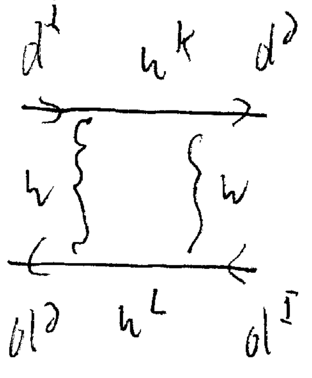
$$\Delta F = 2 \quad (\bar{K}K,\ \bar{B}_d B_d,\ \bar{B}_s B_s)$$

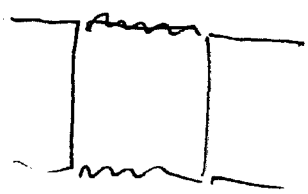
FERMI THEORY - CONTACT 4-FERMION INTERACTION

$$L = \text{const} \cdot (\bar{\Psi}_I \gamma_\mu P_L \Psi_J)(\bar{\Psi}_I \gamma^\mu P_L \Psi_J)$$



STANDARD MODEL:





+ GOLDSTONE

$$\mathcal{L}_{eff}^{SM} = \frac{1}{2}\left(\frac{e}{\sqrt{2}}\right) \sum_{K,L} \overbrace{V^*_{KI} V_{KJ} V^*_{LI} V_{LJ}}^{\text{CKM MATRIX}}$$

$$\propto \int \frac{d^4k}{(2\pi)^4} \frac{[\bar{\Psi}_I \gamma_\mu P_L (\not{k} + m_{u_K}) \gamma_\nu P_L \Psi_J][\bar{\Psi}_I \gamma^\nu P_L (\not{k} + m_{u_L}) \gamma^\mu P_L \Psi_J]}{(k^2 - m_{u_K}^2)(k^2 - m_{u_L}^2)(k^2 - M_W^2)^2}$$

$\longrightarrow$ "const" IN L_{FERMI}

BOX DIAGRAM — WILSON COEFFICIENT AT ELECTROWEAK SCALE

WE NEED IT AT m_K (m_B) SCALE! → RGE

ACTUALLY, FULL EFFECTIVE THEORY: ($K\bar{K}$ ONLY)

$$L_{eff} = L_{QCD} \text{ (gluons, light quarks)}$$
$$- \frac{G_F^2 m_W^2}{4\pi^2} \left(\lambda_c^2 C_c(\mu) + 2\lambda_c \lambda_t C_{ct}(\mu) + \lambda_t^2 C_t(\mu) \right)$$
$$\otimes \left[\bar{\Psi}_s \gamma_\mu P_L \Psi_d \right] \left[\bar{\Psi}_s \gamma^\mu P_L \Psi_d \right]$$

CONCENTRATE ON TOP DIAGRAM ONLY:

$$C_t(\mu) = \tilde{S}(x_t, x_t) + O(\alpha_s) \equiv C(\mu)$$

$\tilde{S}$ — INAMI-LIM FUNCTION — BOX CALCULATION.

$O(\alpha_s)$ ~~$= O(\alpha_s$~~ SMALL FOR $q^2 \sim m_t^2$, BUT WE
$= O(\alpha_s (q^2))$

NEED $\mu \sim m_K$ - RENORMALIZATION

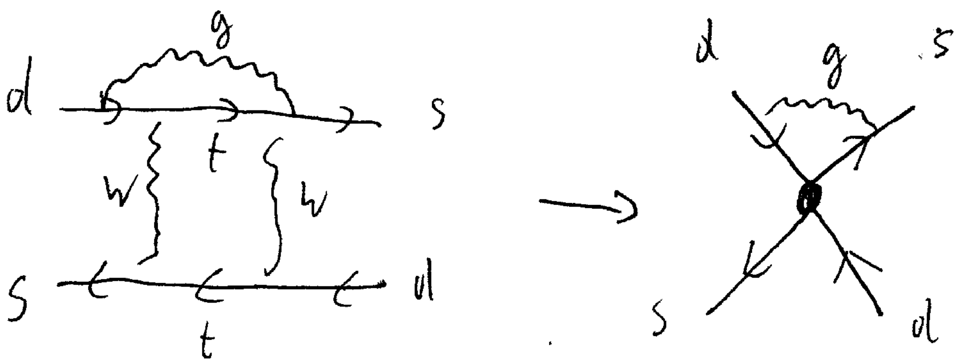
$$C(\mu) \to Z_c C(\mu)$$

$$\left[\bar{\Psi}_s \gamma_\mu P_L \Psi_d \right] \left[\bar{\Psi}_s \gamma^\mu P_L \Psi_d \right] \to Z_2^2 \left[\bar{\Psi}_s \gamma_\mu P_L \Psi_d \right] \left[\bar{\Psi}_s \gamma^\mu P_L \Psi_d \right]$$

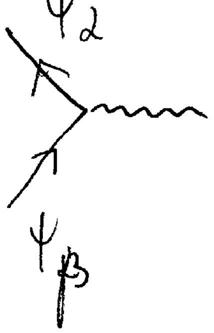
$$\begin{cases} \left(\mu \dfrac{d}{d\mu} - \gamma_c\right) C(\mu) = 0 \\ \gamma_c = -\dfrac{\mu}{Z_c}\dfrac{dZ_c}{d\mu} \end{cases}$$

$$L_{eff} = -\frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} + \sum_f \bar{\Psi}_f (i\not{D} - m_f)\Psi_f + G.F. + F.P.$$
$$- \frac{G_F^2 M_W^2}{4\pi^2} \lambda_t^2 \, C(\mu) \left(\bar{\Psi}_s \gamma_\mu P_L \Psi_d\right)\left(\bar{\Psi}_s \gamma^\mu P_L \Psi_d\right)$$

WE NEED DIAGRAMS LIKE:



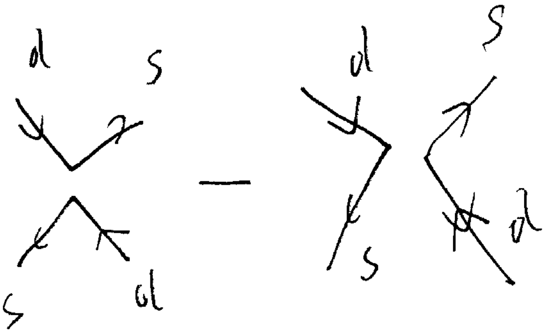
ONLY gluon-fermion COUPLING NECESSARY



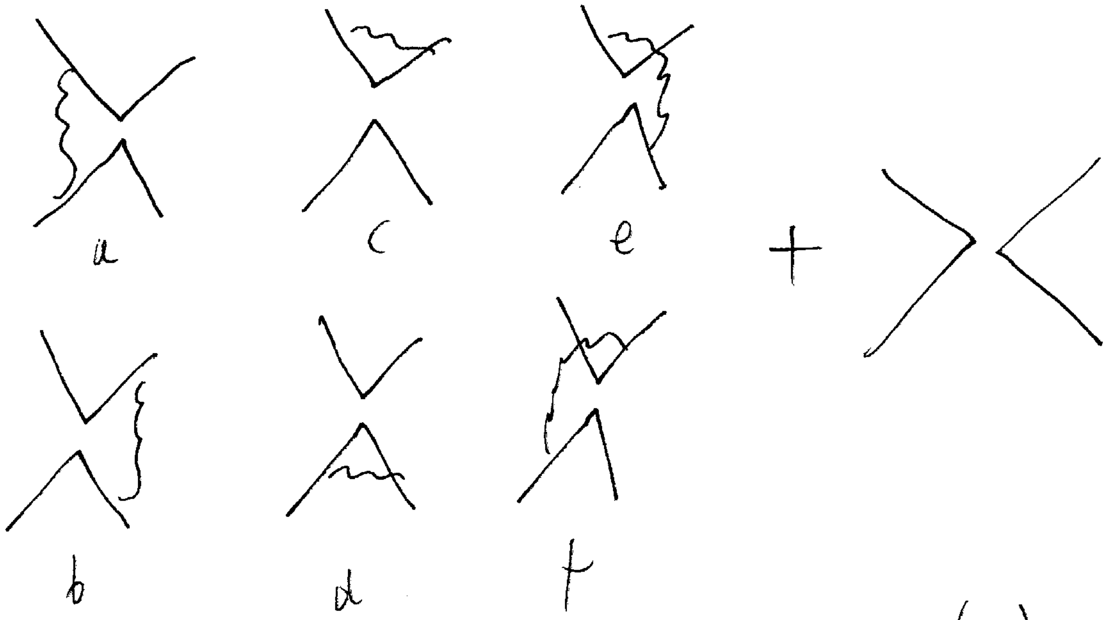
$$-i g_s \gamma_\mu T^a_{\alpha\beta}$$

IMPORTANT

$$\left(\bar{\Psi}_s \gamma_\mu P_L \Psi_d\right)\left(\bar{\Psi}_s \gamma^\mu P_L \Psi_d\right) \rightarrow$$



TO CALCULATE: INFINITE PARTS OF:



(+) DIAGRAMS

(−) DIAGRAMS

$a^+ + b^+ + c^- + d^-$: $\left(X = \frac{G_F^2 M_W^2}{4\pi^2} \lambda_t^2 C(\mu) \right)$

$$L_1 = 2(-X)(-i g_s^2)(-i) i^2 \mu^\varepsilon \int \frac{d^n k}{(2\pi)^n} \frac{\left[\bar\psi_s \gamma_\mu P_L (\not k + m_d) \gamma_\nu T^a \psi_d\right]}{k^2(k^2 - m_d^2)(k^2 - m_s^2)}$$

$$\otimes \left[\bar\psi_s \gamma^\nu T^a (\not k + m_s) \gamma^\mu P_L \psi_d\right]$$

$$\text{INF. PART} \sim \int \frac{d^n k}{(2\pi)^n} \frac{k^\lambda k^\rho}{k^2(k^2 - m_d^2)(k^2 - m_s^2)} = \frac{i g^{\lambda\rho}}{2\varepsilon (4\pi)^2} + \text{FINITE}$$

$$L_1^{POLE} = \frac{-X g_s^2}{(4\pi)^2 \varepsilon} (\bar\psi_s \gamma_\mu P_L \gamma_\rho \gamma_\nu T^a \psi_d)(\bar\psi_s \gamma^\nu \gamma^\rho \gamma^\mu P_L T^a \psi_D)$$

$a^- + b^- + c^+ + d^+$:

$$L_2^{POLE} = \frac{-X g_s^2}{(4\pi)^2 \varepsilon} (\bar\psi_s \gamma_\nu T^a \gamma_\rho \gamma_\mu P_L \gamma^\rho \gamma^\nu T^a \psi_d)(\bar\psi_s \gamma^\mu P_L \psi_d)$$

$\cancel{\mathcal{L}_5^{POLE}} =$

$e^+ + f^+ + e^- + f^-$: (COMB. FACTOR $1/2!$)

$$L_3^{POLE} = + \frac{X g_s^2}{2(4\pi)^2 \epsilon} \left(\bar{\psi}_s \gamma_\nu T^a \gamma_\rho \gamma_\mu P_L \psi_d \right) \left(\bar{\psi}_s \gamma^\nu T^a \gamma^\rho \gamma^\mu P_L \psi_d \right)$$

$$L_4^{POLE} = \frac{+ X g_s^2}{2(4\pi)^2 \epsilon} \left(\bar{\psi}_s \gamma_\mu P_L \gamma_\rho \gamma_\nu T^a \psi_d \right) \left(\bar{\psi}_s \gamma^\mu P_L \gamma^\rho \gamma^\nu T^a \psi_d \right)$$

IDENTITIES ($d \simeq 4!$)

$$(\bar{\psi}_1 \gamma^\lambda \gamma^\epsilon \gamma^\rho P_L \psi_2)(\bar{\psi}_3 \gamma^\rho \gamma^\epsilon \gamma^\lambda \psi_4) = 4 (\bar{\psi}_1 \gamma^\mu P_L \psi_2)(\bar{\psi}_3 \gamma_\mu P_L \psi_4) + O(\epsilon)$$

ALSO:

$$\gamma_\nu \gamma_\rho \gamma_\mu \gamma^\rho \gamma^\nu = 4 \gamma_\mu + O(\epsilon)$$

$$(T^a T^a)_{\alpha\beta} = \frac{4}{3} \delta_{\alpha\beta}$$

$$(T^a)_{\alpha\beta} (T^a_{\gamma\delta}) = \frac{1}{2} \left(\delta_{\alpha\delta} \delta_{\gamma\beta} - \frac{1}{3} \delta_{\alpha\beta} \delta_{\gamma\delta} \right)$$

FIERZ: $(\gamma_\mu P_L)_{ij} (\gamma_\mu P_L)_{kl} = (\gamma_\mu P_L)_{il} (\gamma_\mu P_L)_{kj}$

+ ANTICOMMUTE.

FULL RESULT:

$$\sum_{k=1}^{4} L_k^{POLE} = -\frac{4 X g_s^2}{3(4\pi)^2 \epsilon} \left(\bar{\Psi}_s \gamma^\mu P_L \Psi_d\right)\left(\bar{\Psi}_s \gamma_\mu P_L \Psi_d\right)$$

BUT $\Delta L_{eff} = -X\left(Z_c Z_2^2 - 1\right)\left(\bar{\Psi}_s \gamma^\mu P_L \Psi_d\right)\left(\bar{\Psi}_s \gamma_\mu P_L \Psi_d\right)$

$$Z_c Z_2^2 - 1 = -\frac{4}{3}\frac{g_s^2}{(4\pi)^2 \epsilon} + O(g_s^4)$$

TO BELIEVE:

$$Z_2 = 1 - \frac{8}{3}\frac{g_s^2}{(4\pi)^2 \epsilon} + O(g_s^2)$$ → QUARK WAVE FUNCTION → AT HOME

THUS:

$$Z_c = 1 + \frac{4 g_s^2}{(4\pi)^2 \epsilon} \Rightarrow \gamma_c(\mu) = \frac{4 g_s^2(\mu)}{(4\pi)^2} \Rightarrow \gamma_0 = 1$$

~~WE HAD (PAGE 4)~~

$$\cancel{C_i(q) = C_i(\mu)\left(\frac{g^2(t)}{g^2(\mu)}\right)^{-\frac{\gamma_0}{b_1}}}$$

$$\cancel{C_i(\mu) = C_i(\mu_t)\ \alpha_s}$$

$$\gamma_c = \frac{\alpha_s(\mu)}{\pi} \qquad \left(\mu \frac{d}{d\mu} - \gamma_c\right) C(\mu) = 0$$

$$C(\mu) = e^{\int_{\ln\mu_0}^{\ln\mu} \gamma_C(\mu')\, d\ln\mu'}\, C(\mu_0)$$

BUT: $$\mu \frac{d\alpha_s}{d\mu} = \frac{d\alpha_s}{d\ln\mu} = \alpha_s\left[b_1 \frac{\alpha_s}{\pi} + O(\alpha_s^2)\right]$$

$$C(\mu) = e^{\frac{1}{\pi}\int_{\ln\mu_0}^{\ln\mu} \alpha_s(\mu')\, d\ln\mu'}\, C(\mu_0) =$$

$$= e^{\frac{1}{b_1}\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{d\alpha_s}{\alpha_s}}\, C(\mu_0) = \left[\frac{\alpha_s(\mu_0)}{\alpha_s(\mu)}\right]^{-1/b_1} C(\mu_0)$$

$$b_1^{QCD} = -\frac{1}{2}\left(11 - \frac{2}{3}N_f\right)$$ N_f - no. of <u>active</u> flavours.

FINALLY:

$$C(\mu_{LOW}) = \left[\frac{\alpha_s(\mu_c)}{\alpha_s(\mu_{LOW})}\right]^{6/27}\left[\frac{\alpha_s(\mu_b)}{\alpha_s(\mu_c)}\right]^{6/25}\left[\frac{\alpha_s(\mu_{HIGH})}{\alpha_s(\mu_b)}\right]^{6/23} C(\mu_{HIGH})$$