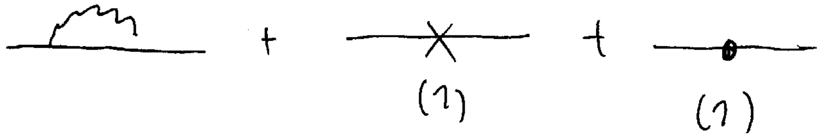
# (6) IR PROBLEM IN QED

## ELECTRON SELF ENERGY

$-i\Sigma(p) =$




$$\hat{G}^{(2)}(\not p) = \frac{i}{\not p - m - \Sigma}$$

$$\Sigma(\not p, m) = \Sigma_V(p^2, m)\,\not p + \Sigma_m(p^2, m)\,\hat{1}$$

ALSO, CONVENIENT TO DEFINE

$$\Sigma(\not p, m) = \Sigma_0(\not p = m_F, m) + (\not p - m_F)\,\Sigma_1(\not p = m_F, m) + $$
$$+ (\not p - m_F)^2\,\Sigma_2(\not p = m_F, m)$$

$m_F^2$ DEFINED AS SOLUTION OF $\not p - m - \Sigma = 0$, ~~TO~~ PERTURBATIVELY TO GIVEN ORDER

$$m_F^2 = \left[\frac{m + \Sigma_m(m_F)}{1 - \Sigma_V(m_F)}\right]^2$$

$$\begin{cases} \not p\,(1 - \Sigma_V) - m - \Sigma_m = 0 \\ p^2\,(1 - \Sigma_V)^2 = (m + \Sigma_m)^2 \end{cases}$$

FEYNMAN DIAGRAM: (IN FEYNMAN GAUGE $a=0$ - RESULT GAUGE-DEP!)

$$-\Sigma(\not p) = (ie)^2 \int \frac{d^n k}{(2\pi)^n}\,\gamma^\mu\,\frac{-i g_{\mu\nu}}{k^2 + i\varepsilon}\,\frac{i}{\not p + \not k - m + i\varepsilon}\,\gamma^\nu$$
$$- i m\,(Z_0 - 1)^{(1)} + i\,(Z_2 - 1)^{(1)}\,\not p$$

$$\mathrm{Tr}\,\Sigma(p) = 4\Sigma_m$$

$$\mathrm{Tr}\left[\not p\,\Sigma(p)\right] = 4p^2\Sigma_v(p^2)$$

DEFINE $N = \gamma^\mu(\not p + \not k + m)\gamma^\nu g_{\mu\nu}$

$$\mathrm{Tr}\,N = 4nm$$

$$\mathrm{Tr}(\not p N) = 4\left[p^2(2-n) + (pk)(2-n)\right] = 4(2-n)(p^2+pk)$$

WITH FEYNMAN PRESCRIPTION

$$\frac{1}{k^2}\,\frac{1}{(p+k)^2-m^2} = \int_0^1 dx\,\frac{1}{\left[k^2 + xp\,2k + x(p^2-m^2)\right]^2}$$

ONE GETS:

$$\Sigma_m = nm\,C\int_0^1 X\,dx - (Z_0-1)^{(1)}\,m$$

$$\Sigma_v = (2-n)\,C\int_0^1(1-x)X\,dx - (Z_2-1)^{(1)}$$

$$C = \frac{\alpha}{4\pi}\left(-\frac{1}{4\pi}\right)^{-\varepsilon/2}\Gamma\left(\frac{\varepsilon}{2}\right)$$

$$\varepsilon = \frac{1}{4-n}$$

IN MS:

$$(Z_0-1)^{(1)} = \frac{2\alpha}{\pi}\,\frac{1}{\varepsilon}$$

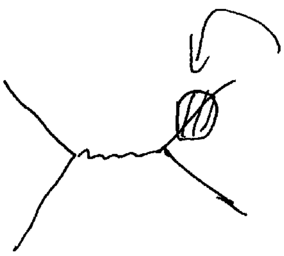
$$(Z_2-1)^{(1)} = -\frac{\alpha}{2\pi}\,\frac{1}{\varepsilon}$$

$$\Sigma_m(p^2) = -\frac{\alpha m}{\pi}\left[\int_0^1 dx\,\overbrace{\ln\frac{xm^2 - x(1-x)p^2}{4\pi\mu^2}}^{X} + \gamma + \frac{1}{2}\right]$$

EXERCISE $\rightarrow$ SHOW $m_F = m(\mu)\left(1 + \frac{\alpha(\mu)}{\pi}\right)$ IN $\overline{MS}$

$p^2 \to m^2 \Rightarrow \Sigma_m, \Sigma_V$ IR FINITE

BUT IN CROSS-SECTION CALCULATIONS $\Sigma_1$ APPEARS (LSZ RULE)



$(1+\frac{1}{2}\Sigma_1)$

$$\Sigma_1 = \frac{\partial \Sigma}{\partial \not p} = \frac{\partial \Sigma_m}{\partial p^2}\Big|_{p^2=m_F} 2m_F + \Sigma_V(m_F^2) + 2m_F^2 \frac{\partial \Sigma_V(p^2)}{\partial p^2}\Big|_{p^2=m_F^2}$$

$$\frac{\partial \Sigma_m}{\partial p^2} = n m \, C \left(-\frac{1}{2}\right)\varepsilon \int_0^1 dx \, \frac{x(1-x)}{M^2} X^{1+\frac{2}{\varepsilon}}$$

$$\frac{\partial \Sigma_m}{\partial p^2} = \frac{n}{m}\frac{\alpha}{4\pi}\left(1-\gamma\frac{\varepsilon}{2}\right)\left(\frac{m^2}{4\pi\mu^2}\right)^{-\varepsilon/2}\frac{\Gamma(-\varepsilon)\Gamma(2)}{\Gamma(2-\varepsilon)} \quad \left(m_F = m \text{ IN THIS ORDER}\right)$$

UV FINITE BUT IR DIVERGENT DUE TO INTEGRATION OVER FEYNMAN PARAMETER!

AGAIN WE CAN REGULARIZE IT BY KEEPING $\varepsilon \neq 0$, BUT NOW WE NEED $\varepsilon < 0$! DEFINE: $n = 4 - \varepsilon_{UV}$ FOR UV REGULARIZATION AND $n = 4 + \varepsilon_{IR}$ FOR IR REG., BOTH $\varepsilon_{UV}, \varepsilon_{IR} > 0$

$$\frac{\partial \Sigma_m}{\partial p^2}\Big|_{p^2=m^2} = \frac{1}{m}\frac{\alpha}{4\pi}\left[4\left(\frac{1}{\varepsilon_{IR}} + \frac{1}{2}\gamma - 1 + \frac{1}{2}\ln\frac{m^2}{4\pi\mu^2}\right) + 2\right]$$

$$\frac{\partial \Sigma_V}{\partial p^2}\Big|_{p^2=m^2} = -\frac{1}{2m^2}\frac{\alpha}{\pi}\left[\frac{1}{\varepsilon_{IR}} - 1 + \frac{1}{2}\gamma + \frac{1}{2}\ln\frac{m^2}{4\pi\mu^2}\right]$$

$$\Sigma_V(p^2=m^2) = \frac{\alpha}{4\pi}\left(\ln\frac{m^2}{4\pi\mu^2} + \gamma - 2\right)$$

SO

$$\Sigma_1 = \left.\frac{\partial\Sigma}{\partial \not p}\right|_{\not p=m} = \frac{\alpha}{\pi}\left(\frac{1}{\varepsilon_{IR}} + \frac{3}{4}\ln\frac{m^2}{4\pi\mu^2} - 1 + \frac{3}{4}\gamma\right)$$

"MORE PHYSICAL" APPROACH — PHOTON MASS REGULARIZATION

$$\frac{1}{k^2} \to \frac{1}{k^2-\lambda^2}$$

$$\Sigma_m(p^2) = -\frac{\alpha m}{\pi}\int dx \ln\frac{x^2p^2 - x(p^2-m^2+\lambda^2) + \lambda^2 - i\epsilon}{4\pi\mu^2} + \text{CONST}$$

$$\Sigma_V(p^2) = \frac{\alpha}{2\pi}\int dx(1-x)\ln(\ldots) + \text{CONST}$$

$$\left.\frac{\partial\Sigma_m}{\partial p^2}\right|_{p^2=m^2} = \frac{\alpha m}{\pi}\frac{1}{2m^2}\ln\frac{m^2}{\lambda^2} + \text{CONST}$$

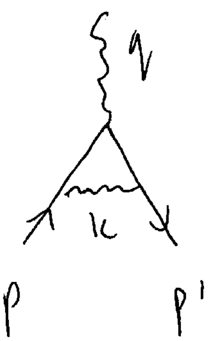
$$\left.\frac{\partial\Sigma_V}{\partial p^2}\right|_{p^2=m^2} = \frac{\alpha}{\pi}\left(-\frac{1}{4m^2}\right)\ln\frac{m^2}{\lambda^2} + \text{const.}$$

$$\Sigma_1 = \frac{\alpha}{\pi}\frac{1}{2}\ln\frac{m^2}{\lambda^2} + \frac{\alpha}{\pi}\frac{1}{4}\ln\frac{m^2}{4\pi\mu^2} + \text{const.}$$

$$\frac{2}{\varepsilon_{IR}} \to \ln\frac{m^2}{\lambda^2}$$

# VERTEX CORRECTION



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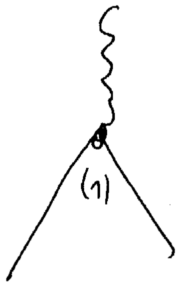


$= ie\Lambda_\mu(p,p')$

$$\Lambda_\mu(p,p') = (ie)^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma_\alpha \frac{i}{\not p' + \not k - m} \gamma_\mu \frac{i}{\not p + \not k - m} \gamma_\beta \left(-i g^{\alpha\beta}\right) \frac{1}{k^2}$$

$$+ (Z_1 - 1)^{(1)} \gamma_\mu$$