Exercises to Radiative Corrections

1. **Lorentz algebra**

   Verify (assuming $c = 1; x^0 = t; x^i = (\vec{x})_i, i = 1,2,3$) that:

$$
\begin{aligned}
\partial^\mu &= \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial x^1}, -\frac{\partial}{\partial x^2}, -\frac{\partial}{\partial x^3}\right) = \left(\frac{\partial}{\partial t}, -\nabla\right) , \\
\not{p} &= p^\mu \gamma_\mu = p_\mu \gamma^\mu , \\
\not{p}\not{p} &= p^\mu p_\mu \cdot \mathbf{1} = p^2 \cdot \mathbf{1} , \\
\frac{1}{\not{p} - m} &= \frac{\not{p} + m}{p^2 - m^2} , \\
\{\gamma_\mu, \gamma_5\} &= 0 , \\
\gamma^\rho \gamma_\rho &= 4 \cdot \mathbf{1} , \\
\gamma^\rho \gamma_\mu \gamma_\rho &= -2\gamma_\mu , \\
\mathrm{Tr}\,(\gamma_\mu \gamma_\nu) &= 4 g_{\mu\nu} ,
\end{aligned}
$$

   and make sure that

$$
(i\not{\partial} - m) \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, e^{-ipx} \tilde{S}(p) = \int \frac{\mathrm{d}^4 p}{(2\pi)^4}\, e^{-ipx} (\not{p} - m) \tilde{S}(p) .
$$

2. **Photon propagator I**

   (a) Prove that the Lorentz tensors

$$
\Pi_{\parallel}^{\mu\nu}(k) := \frac{k^\mu k^\nu}{k^2} , \quad \Pi_{\perp}^{\mu\nu}(k) := g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} = g^{\mu\nu} - \Pi_{\parallel}^{\mu\nu}(k) ,
$$

   are projectors, i. e.,

$$
\begin{aligned}
(\Pi_i)^\mu{}_\lambda (\Pi_i)^{\lambda\nu} &= (\Pi_i)^{\mu\nu} , \quad i = \parallel, \perp ; \\
(\Pi_\parallel)^\mu{}_\lambda (\Pi_\perp^{\lambda\nu}) &= 0 = (\Pi_\parallel)^\mu{}_\lambda (\Pi_\perp)^{\lambda\nu} .
\end{aligned}
$$

   (b) Let $D^{\mu\nu}(q) = A(q^2)\Pi_\perp^{\mu\nu}(q) + B(q^2)\Pi_\parallel^{\mu\nu}(q)$. Obtain $(D^{-1}(q))^{\mu\nu}$ from the relation

$$
D^\mu{}_\lambda(q) \left(D^{-1}(q)\right)^{\lambda\nu} = g^{\mu\nu} .
$$

   Now you should be able to verify the expression for the photon propagator $P_{AA}^{\lambda\nu}$ given in the lecture.

3. **Feynman rule for scalar-vector vertex**

   Let $A_\mu(x)$ be a vector field, $B(x)$ and $C(x)$ scalar fields. Determine the Feynman rule for the vertex

$$\mathcal{L}_g = gA^\mu \left(B\partial_\mu C - C\partial_\mu B\right) \ .$$

4. **Photon propagator II**

   The decomposition of the general expression for the photon self energy into Lorentz covariants reads

$$\Sigma^{AA}_{\mu\nu}(k) = \left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)\Sigma^{AA}_T(k^2) + \frac{k_\mu k_\nu}{k^2}\Sigma^{AA}_L(k^2) \ .$$

   Determine the corrected photon propagator.

5. **Naive power counting**

   Try to find a rule-of-thumb for the 'superficial degree of divergence' of an arbitrary one-loop diagram in QED.

   Hint: confine your discussion to the Feynman gauge for the photon propagator.

   (Result: $d = 4 - \frac{3}{2}\cdot \#$(ext. fermion lines) $- \#$(ext. photon lines))

6. **Dimensional regularization**

   Starting from $g^\mu_\mu = D$ and $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\cdot\mathbf{1}$ in D dimensions, prove:

$$\begin{aligned} \gamma^\rho\gamma_\rho &= D\cdot\mathbf{1}\ , \\ \gamma^\rho\gamma_\mu\gamma_\rho &= (2-D)\gamma_\mu\ , \\ \gamma^\rho\gamma_\mu\gamma_\nu\gamma_\rho &= 4g^{\mu\nu}\cdot\mathbf{1} + (D-4)\gamma_\mu\gamma_\nu\ . \end{aligned}$$

7. **Scalar integrals I**

   Calculate the Taylor series of x^y for $y \to 0$.

   Compute the Taylor series of $\Gamma(z-1)$ for $z \to 0$.

   Determine the Taylor series of $1/(D-2)$ for $D \to 4$.

   Derive:

$$\begin{aligned} \left(\frac{m^2}{4\pi\mu^2}\right)^{\frac{D-4}{2}} &= 1 + \frac{D-4}{2}\log\left(\frac{m^2}{4\pi\mu^2}\right) + \mathcal{O}\left((D-4)^2\right)\ , \\ \Gamma\left(\frac{2-D}{2}\right) &= -\left(\frac{2}{4-D} - \gamma_E + 1\right) + \mathcal{O}(D-4)\ , \end{aligned}$$

   and verify the expression for $A_0(m)$.

8. **Feynman trick**

   Prove

$$\frac{1}{ab} = \int_0^1 \mathrm{d}x \, [a(1-x) + bx]^{-2} .$$

9. **Scalar integrals II**

   Calculate $B_0(0,0,m)$, $B_0(m^2,0,m)$ and $B_0(p^2,0,0)$!

   Reduce $A_0(m)$, $B_0(0,0,m)$ and $B_0(m^2,0,m)$ using $B_0(0,m,m)$!

10. **Scalar integrals III**

    Show that for $\lambda \ll m$:

$$B_0'(m^2,\lambda,m) \equiv \left.\frac{\partial B_0(p^2,\lambda,m)}{\partial p^2}\right|_{p^2=m^2} = -\frac{1}{m^2}\left[\log\frac{\lambda}{m} + 1\right] .$$

11. **Algebraic reduction I**

    Calculate

$$A_{\mu\nu}(m_0) = \left\langle \frac{q_\mu q_\nu}{q^2 - m_0^2} \right\rangle_q$$

    and $A_{\mu\nu\rho\sigma}(m_0)$.

    Hint: derive $\langle 1 \rangle_q = 0$ in dimensional regularization.

12. **Algebraic reduction II**

    Reduce B_{00} and B_{11} to A_0, B_0 and B_1. Now take the limit $D \to 4$ and determine the divergent parts.

    Convince yourself by explicit calculation that $B_{11}(p^2;m,m)$ is regular for $p^2 \to 0$, because the apparent $1/p^2$ singularity gets cancelled by a vanishing numerator.

13. **Photon self energy I**

    Calculate the D-dimensional trace that appears in the calculation of the one-loop contribution to the photon self energy.

14. **Photon self energy II**

    Perform the tensor reduction for the transverse and longitudinal parts of the photon self energy.

    Verify explicitly that $\Sigma_L^{AA}(k^2) \equiv 0$.

    Check the expression for $\Sigma_T^{AA}(k^2)$, and calculate $\Sigma_T^{AA}(0)$.

    Determine the asymptotic behaviour of $\Pi^{AA}(k^2)$ for $|k^2| \gg m^2$.

15. **Electron self energy**

    Calculate the one-loop electron self energy! Apply the Dirac algebra in dimensional regularization and perform the tensor reduction. Decompose the result into the covariants $\Sigma_V^{\bar{e}e}$ and $\Sigma_S^{\bar{e}e}$.

16. **One-loop counterterm vertices**

    Specify the Feynman rules for the one-loop counterterm vertices.

17. **Wave function renormalization of the electron**

    Express δZ_ψ in terms of the scalar two-point functions!

18. **Anomalous magnetic moment of the electron**

    Perform the Gordon decomposition for the renormalized (on-shell) one-loop vertex function $\Gamma_{\mu,1}^{A\bar{e}e} \equiv ie\hat{\Lambda}_\mu(p', p)$.

    Now calculate the one-loop contribution to the anomalous magnetic moment of the electron, $g^{(1)}$!

    Hint: derive

    $$B_0'(m^2, m, \lambda) + C_0(m^2, 0, m^2, \lambda, m, m) = -\frac{1}{m^2}$$

    for $\lambda \ll m$, using the Feynman parameter representation of these quantities.