"Home work" in Radiative Corrections

H1. Feynman rules of "scalar electrodynamics"

Consider a hypothetical model, consisting of a complex scalar field $\phi(x)$ and a vector field $A_\mu(x)$. The Lagrangian density of this model reads:

$$\mathcal{L} = (D_\mu\phi)^* D^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\phi) ,$$

where

$$\begin{aligned} D_\mu\phi(x) &= \partial_\mu\phi(x) - ieA_\mu(x)\phi(x) , \\ V(\phi) &= \mu^2 \left|\phi(x)\right|^2 + \frac{\lambda}{4}\left|\phi(x)\right|^4 , \end{aligned}$$

with λ, μ^2 real parameters, $\lambda > 0$.

(a) Determine the Feynman rules for the case $\mu^2 > 0$ ("scalar electrodynamics").

(b) Discuss the ground state of the model for the case $\mu^2 < 0$. Show that the ground state expectation value of the scalar field is non-zero and can be chosen real and positive, since the Lagrangian density is invariant under global phase transformations $\phi \to e^{i\alpha}\phi$, ($\alpha =$ const.):

$$\langle 0 |\, \phi(x)\, | 0 \rangle = \frac{v}{\sqrt{2}} .$$

Let us calculate the particle spectrum of this model, which we shall call "abelian Higgs model". For this purpose, replace the complex scalar field using the (non-linear) field redefinition

$$\phi(x) \to \frac{1}{\sqrt{2}}\left(v + h(x)\right) e^{i\varphi(x)} ,$$

with $h(x), \varphi(x)$ now being real fields. Show that the field $\varphi(x)$ can be completely eliminated from $\mathcal{L}$ by a suitable gauge transformation, and that the vector field acquires the mass $M_A = ev$.

Determine the Feynman rules (propagators and vertices) of this model. Express (if possible) the constants e, μ^2 and λ in terms of the masses M_A and M_h of the vector and scalar field and the vacuum expectation value v.

(Remark: the special gauge that eliminates the unphysical field $\varphi(x)$ from the Lagrangian is usually called *unitary gauge*.)

H2. Dimensional regularization

Show that the surface of a D-dimensional unit sphere is given by

$$\Omega_D = \frac{2\pi^{D/2}}{\Gamma\left(\frac{D}{2}\right)}$$

Hints: calculate the Gaussian integral in D dimensions,

$$\int \mathrm{d}^D\vec{k}\, \exp\left(-\vec{k}^2\right) ,$$

both in cartesian coordinates and in polar coordinates, where Ω_D is just the D-dimensional solid angle. Then use

$$\int_0^\infty x^{\beta-1} e^{-x}\, \mathrm{d}x = \Gamma(\beta) .$$

H3. Generalized Feynman trick I

Show that

$$\frac{1}{\prod_{i=1}^n a_i} = (n-1)! \int_0^1 \prod_{i=1}^n \mathrm{d}x_i\, \frac{\delta\left(1-\sum_{i=1}^n x_i\right)}{\left[\sum_{i=1}^n a_i x_i\right]^n}$$

Hints: use

$$\frac{1}{a_i} = \int_0^\infty \mathrm{d}\lambda_i\, e^{-\lambda_i a_i}$$

and the "rescaling trick"

$$1 = \int_0^\infty \mathrm{d}\lambda\, \delta\left(\lambda - \sum_{i=1}^n \lambda_i\right) .$$

Finally substitute $\lambda_i \to \lambda x_i$.

H4. Generalized Feynman trick II

Having mastered the above steps, now prove ($M = \sum_{i=1}^n m_i$):

$$\frac{1}{\prod_{i=1}^n a_i^{m_i}} = \frac{\Gamma(M)}{\prod_{i=1}^n \Gamma(m_i)} \int_0^1 \prod_{i=1}^n \mathrm{d}x_i\, \delta\left(1-\sum_{i=1}^n x_i\right) \frac{\prod_{i=1}^n x_i^{m_i-1}}{\left[\sum_{i=1}^n a_i x_i\right]^M} .$$

H5. Scalar twopoint function

Calculate $B_0(0; m_1, m_2)$ for arbitrary m_1, m_2. Check your result for $m_1 = m_2 = m$.

Let us determine $B_0(p^2; m, m)$. To achieve this, first assume that $p^2 < 0$, and set $\tau = -4m^2/p^2$. Now try to factorize the argument of the logarithm in the following way:

$$1 + \frac{4x(1-x)}{\tau} = \left(1 - \frac{x}{x_1}\right)\left(1 - \frac{x}{x_2}\right)$$

Can you analytically continue your result to $0 < p^2 < 4m^2$ and to $p^2 > 4m^2$?

H6. A simple scalar threepoint function

Calculate the scalar threepoint function $C_0(0, q^2, 0; m, m, m)$ for $q^2 < 0$.

Hints: use the Feynman parameter representation. With $\tau = -4m^2/q^2$, show that C_0 can be written as

$$C_0(0, q^2, 0; m, m, m) = \frac{1}{q^2} \int\limits_0^1 \mathrm{d}x \, \frac{\log[1 + 4x(1-x)/\tau]}{x} \,.$$

Decompose the argument of the logarithm as in the preceeding problem, and use

$$\mathrm{Li}_2(t) + \mathrm{Li}_2\left(\frac{1}{1-1/t}\right) \;=\; -\frac{1}{2}\log^2(1-t) \qquad \text{for} \quad 0 < t < 1 \,.$$

Find the analytic continuation into the region $q^2 > 0$! (You will need this result if you want to calculate, e.g., the decay of a Higgs-boson into two photons or into two gluons.)

H7. Algebraic reduction

For which choice of momenta p_1, p_2 and $p_{21} = p_2 - p_1$ does our tensor reduction procedure break down for $C_{1,2}$?

For advanced students: how would you modify the tensor reduction?

(see e.g., G. Devaraj, R. Stuart, Nucl. Phys. **B519** (1998) 483;
J.M. Campbell, E.W.N. Glover, D.J. Miller, Nucl. Phys. **B498** (1997) 397)

H8. Photon self energy III

Verify the transversity of the one-loop integral for the photon self energy, $k^\mu \Sigma_{\mu\nu}(k) = 0$, by directly manipulating the integral. Which properties of dimensional regularization are essential for your manipulations?

H9. Photon self energy IV

Calculate the photon self energy in scalar QED.

H10. Self energy of a "scalar electron"

Calculate the self energy of the charged scalar particle in scalar QED.

H11. Renormalization conditions of scalar QED

Formulate the renormalization conditions of scalar QED. For the sake of simplicity, consider only mass and wave-function renormalizations.

H12. Renormalized vacuum polarization in scalar QED

Calculate the renormalized vacuum polarization in scalar QED. Determine the asymptotic behaviour of the contribution of a charged scalar for $|k^2| \gg m^2$, and compare with the contribution of a spin-$\frac{1}{2}$ fermion of the same mass and charge.

H13. Dispersion relations I

Let $f(z)$ be a complex function analytic in the upper half plane ($\mathrm{Im}(z) \geq 0$), and assume that

$$\lim_{|z|\to\infty} |f(z)| = 0 \,, \qquad 0 \leq \arg z \leq \pi \,.$$

(a) Starting from Cauchy's integral formula, show that $f(z_0)$ is given by an integral along the real axis,

$$f(z_0) = \frac{1}{2\pi i} \int\limits_{-\infty}^{\infty} \frac{f(x)}{x - z_0} \, dx \quad \text{for} \quad \mathrm{Im}(z_0) > 0 \,,$$

and that its analytic continuation vanishes for $\mathrm{Im}(z_0) < 0$.

(b) For $z_0 \to x_0$ with x_0 real it follows:

$$f(x_0) = \frac{1}{\pi i} \mathrm{P} \int\limits_{-\infty}^{\infty} \frac{f(x)}{x - x_0} \, dx \,. \qquad \text{(H13.1)}$$

(c) Decompose (H13.1) into real and imaginary parts, $f(x) = u(x) + iv(x)$, and show that

$$\begin{aligned} u(x_0) &= \frac{1}{\pi} \mathrm{P} \int\limits_{-\infty}^{\infty} \frac{v(x)}{x - x_0} \, \mathrm{d}x \, , \\ v(x_0) &= -\frac{1}{\pi} \mathrm{P} \int\limits_{-\infty}^{\infty} \frac{u(x)}{x - x_0} \, \mathrm{d}x \, . \end{aligned} \tag{H13.2}$$

H14. **Dispersion relations II**

For k^2 real, calculate $\mathrm{Im}\,(B_0(k^2; m, m))$ directly from the parameter representation of the scalar twopoint function using

$$\log{(x - i\epsilon)} = -i\pi\theta(-x) \, .$$

Use the above result to obtain the real part of

$$f(z) = \frac{B_0(z; m, m) - B_0(0; m, m)}{z} \, ,$$

where it is assumed that $B_0(z; \ldots)$ is analytic in the half plane $\mathrm{Im}(z) \geq 0$. Why did we need to "subtract once" in the present case? Compare your result with that of problem H5.

Hint: use the substitution $t = \sqrt{1 - 4m^2/x}$.

H15. **Asymptotic behaviour of the electron-photon vertex**

(a) Derive the asymptotic behaviour of $C_0(m^2, k^2, m^2; \lambda, m, m)$ in the region $|k^2| \gg m^2 \gg \lambda^2$.

(b) Using the Gordon identity, the on-shell vertex can be rewritten as

$$\hat{\Gamma}_\mu(k, -p', p) = ie \left[\gamma_\mu F_1(k^2) + \frac{i}{2m} \sigma_{\mu\nu} (p' - p)^\nu F_2(k^2) \right] \, ,$$

with form factors $F_1(k^2)$ und $F_2(k^2)$. Calculate the asymptotic behaviour of the F_i for $|k^2| \gg m^2$!

H16. **Soft-photon approximation**

Discuss the soft-photon approximation for scalar QED.