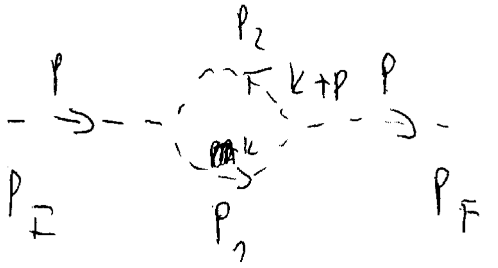
EXERCISES "RENORMALIZATION" OCT. 25, 2000

LOOP INTEGRALS

TYPICAL PROBLEM:



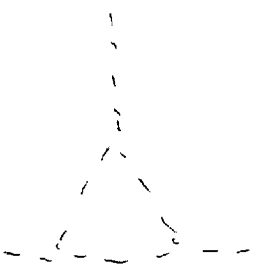
$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m_1^2)((k+p)^2-m_2^2)}$$

ACTUALLY $k^2 - m^2 \equiv k^2 - m^2 + i\varepsilon$

(OR $m^2 \longrightarrow m^2 - i\varepsilon \longrightarrow$ pole prescription treatment)

ANOTHER EXAMPLES

$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2}$$



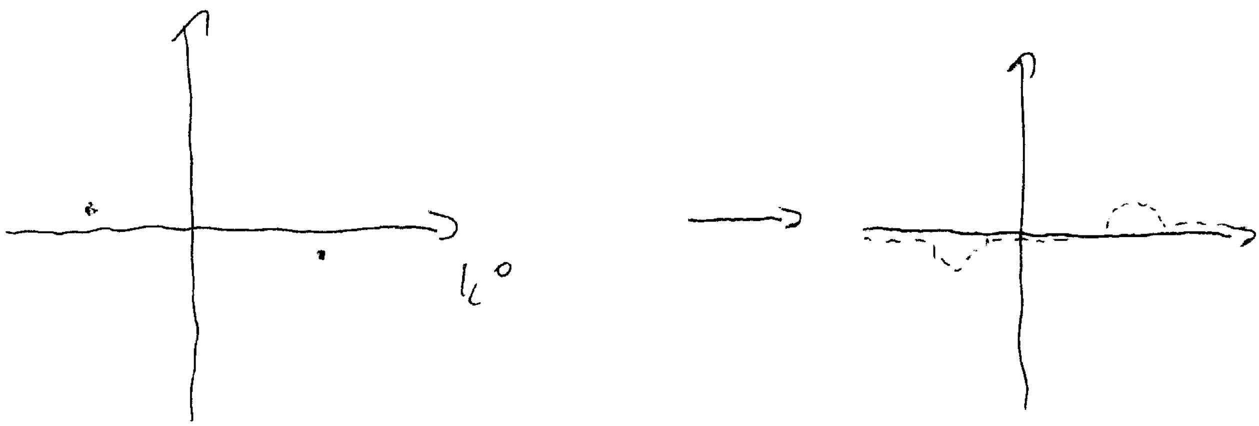
$$\sim \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m_1^2)((p+k)^2-m_2^2)((p+q+k)^2-m_3^2)}$$

$\longrightarrow$ A, B, C, D, ... functions.

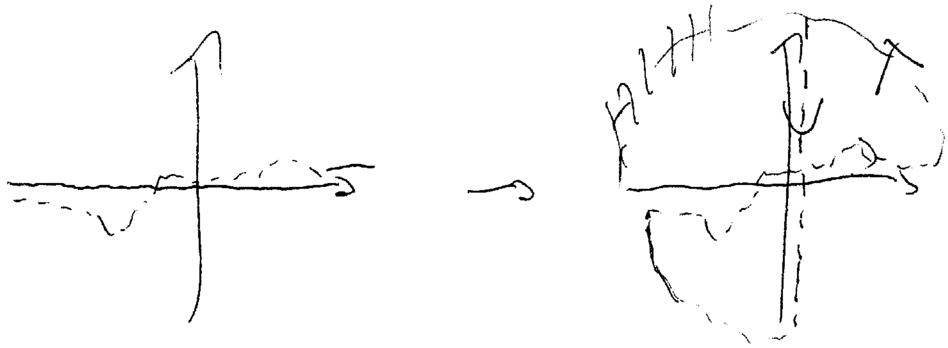
$$k^2 - m^2 + i\epsilon = (k^0)^2 - (\vec{k})^2 - m^2 + i\epsilon$$

$$\int d^4k \to \int dk^0\, d^3k$$

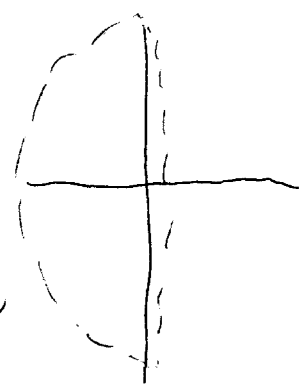
INTEGRATION OVER dk^0 (DETAILS: LECTURE OR LATER)



POLES $k_0 = \pm\sqrt{\vec{k}^2 + m^2 - i\epsilon} = \pm\sqrt{\vec{k}^2 + m^2} \mp i\epsilon$



WICK ROTATION



MORE SIMPLY: $k_0 = i k_4 \quad k_4 = -i k_0$

$$dk^0\, d^3k \to -i\, d^4k_E \,, \quad \int_{+i\infty}^{-i\infty} dk_0 \to -i \int_{+\infty}^{-\infty} dk_4 = i \int_{-\infty}^{\infty} dk_4$$

$$k^2 - m^2 + i\epsilon \to -k_E^2 - m^2 + i\epsilon = -\left(k_E^2 + m^2 - i\epsilon\right)$$

OUR EXAMPLE

$$B_0(p^2) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2-m_1^2+i\varepsilon)((p+k)^2-m_2^2+i\varepsilon)}$$

$$\equiv \frac{i}{(2\pi)^4} \int d^4k_E \frac{1}{(k_E^2+m_1^2)((p_E+k_E)^2+m_2^2)}$$

UGLY INTEGRAL ...

FEYNMAN TRICK

$$\frac{1}{ab} = \int_0^1 \frac{dx}{[ax+b(1-x)]^2}$$ → CHECK?

$$B_0(p^2) = \frac{i}{(2\pi)^4} \int_0^1 dx \int d^4k_E \frac{1}{\left[(k_E^2+m_1^2)\cancel{x}(1-x) + [(p_E+k_E)^2+m_2^2]x\right]^2}$$

$$\text{DEN} = (k_E^2+m_1^2)(1-x) + (p_E^2+2p_Ek_E+k_E^2+m_2^2)x$$

$$= k_E^2 + 2p_Ek_Ex + m_1^2(1-x) + m_2^2x + p_E^2x$$

$$= (k_E+p_E\cancel{x})^2 - p_E^2x^2 + p_E^2x + m_1^2(1-x) + m_2^2x$$

NEW VARIABLE $l = k_E + p_E$ $\quad d^4l = d^4k_E$

ALSO: $p_E^2 = -p^2$

$$B_0(p^2) = \frac{i}{(2\pi)^4} \int_0^1 dx \int d^4 l \; \frac{1}{\left(l^2 + m_1^2(1-x) + m_2^2 x - p^2 x(1-x)\right)^2}$$

DEFINE $q^2 \equiv m_1^2(1-x) + m_2^2 x - p^2 x (1-x)$

$$I(q^2) = \int \frac{d^4 l}{(l^2+q^2)^2}$$

DIVERGENT!

$$I_\Lambda(q^2) = \int^{|l|<\Lambda} \frac{d^4 l}{(l^2+q^2)^2} \sim \int^{|l|<\Lambda} \frac{d^4 l}{l^4} \sim \int_0^{l<\Lambda} \frac{l^3 dl}{l^4} \sim \underset{\to\infty}{\ln\Lambda}$$

"HELP" $\to$ DIMENSIONAL REGULARIZATION

DEFINE

$$I_d(q^2) = \int \frac{d^d p}{(p^2+q^2)^2}$$

POLAR COORDINATES

$$d^d p = r^{d-1} dr \, d\varphi \prod_{k=1}^{d-2} \sin^k \theta_k \, d\theta_k$$

$$0 < r < \infty \qquad 0 < \varphi < 2\pi \qquad 0 < \theta_k < \pi$$

(5)

SO

$$I_d(q^2) = 2\pi \int_0^\infty \frac{r^{d-1}\,dr}{(r^2+q^2)^2} \prod_{k=1}^{d-2} \int_0^\pi \sin^k\theta_k\,d\theta_k$$

$$S_{d-1} = 2\pi \prod_{k=1}^{d-2} \int_0^\pi \sin^k\theta_k\,d\theta_k \longrightarrow$$ SURFACE OF $d-1$ DIMENSIONAL UNIT SPHERE

FORMULAE:

$$\int_0^{\pi/2} (\sin\theta)^{2n-1} (\cos\theta)^{2m-1}\,d\theta = \frac{1}{2}\frac{\Gamma(n)\Gamma(m)}{\Gamma(n+m)}$$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ $\quad m = \frac{1}{2}$ $\qquad 2n-1 = k$

$$\int_0^\pi (\sin\theta)^k\,d\theta = 2\int_0^{\pi/2} (\sin\theta)^k\,d\theta = 2\cdot\frac{1}{2}\frac{\Gamma\left(\frac{k+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)} = \sqrt{\pi}\,\frac{\Gamma\left(\frac{k+1}{2}\right)}{\Gamma\left(\frac{k+2}{2}\right)}$$

$$S_{d-1} = 2\pi\left(\sqrt{\pi}\right)^{d-2} \frac{\Gamma\left(\frac{2}{2}\right)}{\cancel{\Gamma\left(\frac{3}{2}\right)}} \frac{\cancel{\Gamma\left(\frac{2+1}{2}\right)}}{\cancel{\Gamma\left(\frac{3+1}{2}\right)}} \cdots \frac{\cancel{\Gamma\left(\frac{d-2+1}{2}\right)}}{\Gamma\left(\frac{d-2+2}{2}\right)}$$

$$= 2\pi\,\pi^{\frac{d}{2}-1}\,\Gamma(1)\,\frac{1}{\Gamma\left(\frac{d}{2}\right)}$$

$$= \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$$

$$\begin{cases} \Gamma(1) = 0! \equiv 1 \\ \Gamma(2) = 1! = 1\,\Gamma(1) \end{cases}$$

$d=3$ $\quad S_2 = \frac{2\pi^{3/2}}{\Gamma\left(\frac{3}{2}\right)} = \frac{2\pi^{3/2}}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)} = \frac{4\pi^{3/2}}{\pi^{1/2}} = 4\pi$ ✓

$d=4$ $\quad S_3 = \frac{2\pi^2}{\Gamma(2)} = 2\pi^2$

$$I_d(q^2) = S_{d-1}\int_0^\infty \frac{r^{d-1}\,dr}{(r^2+q^2)^\alpha} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty \frac{r^{d-1}\,dr}{(r^2+q^2)^\alpha}$$

EULER BETA FUNCTION

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} = 2\int_0^\infty t^{2x-1}(1+t^2)^{-x-y} \quad \begin{pmatrix} \operatorname{Re} x > 0 \\ \operatorname{Re} y > 0 \end{pmatrix}$$

PUT $r = sq$

$$I_d(q^2) = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\int_0^\infty \frac{q^d s^{d-1}\,ds}{q^{2\alpha}(1+s^2)^\alpha} = \frac{2\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\, q^{d-2\alpha}\int_0^\infty s^{d-1}(1+s^2)^{-\alpha}\,ds$$

$$\begin{cases} 2x-1 = d-1 \\ -x-y = -\alpha \end{cases} \implies \begin{cases} x = d/2 \\ y = -x+\alpha = -\frac{d}{2}-\alpha \end{cases}$$

$$I_d(q^2) = \frac{\pi^{d/2}}{\Gamma\left(\frac{d}{2}\right)}\, q^{d-2\alpha}\, B\left(\frac{d}{2}, +\alpha-\frac{d}{2}\right) = \frac{\pi^{d/2}}{\cancel{\Gamma\left(\frac{d}{2}\right)}}\,\frac{\cancel{\Gamma\left(\frac{d}{2}\right)}\Gamma\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)}\left(q^{d-2\alpha}\right.$$

$$= \pi^{\frac{d}{2}}\,\frac{\Gamma\left(\alpha-\frac{d}{2}\right)}{\Gamma(\alpha)}\, q^{d-2\alpha} = \int \frac{d^d p}{(p^2+q^2)^\alpha}$$

COMING BACK:

$$B_0(p^2) = \frac{i}{(2\pi)^4}\int_0^1 dx \int \frac{d^4 l}{(l^2 + m_1^2(1-x) + m_2^2 x - p^2 x(1-x))^2}$$

$$q^2 = m_1^2(1-x) + m_2^2 x - p^2 x(1-x) \qquad \alpha = 2$$

$$B_0^{REG}(p^2) = \frac{i\mu^{4-d}}{(2\pi)^d}\int_0^1 dx \; \pi^{d/2}\,\frac{\Gamma(2-\frac{d}{2})}{\Gamma(2)}\,(q^2)^{(\frac{d}{2}-2)}$$

$$= \overset{\frac{\mu^{4-d}}{(4\pi)^{d/2}}}{\cancel{\frac{i\,2\,\pi\,\pi^{\frac{d-4}{2}}}{16\,\pi^4}}}\;\Gamma(2-\frac{d}{2})\int_0^1 \frac{dx}{[m_1^2(1-x) + m_2^2 x - p^2 x(1-x)]^{\frac{d}{2}-2}}$$

$$= \frac{i}{(4\pi)^{d/2}}\,\mu^{4-d}\,\Gamma(2-\frac{d}{2})\int_0^1 \frac{dx}{[m_1^2(1-x) + m_2^2 x - p^2 x(1-x)]^{\frac{d}{2}-2}}$$

FULL EVALUATION $\rightarrow$ LATER.

$$d=4 \qquad \Gamma(2-\frac{d}{2}) = \Gamma(0) = \infty \;!$$

$$q^\alpha = e^{\alpha \ln q} \cong 1 + \alpha \ln q \qquad \alpha \ll 1$$

$$\int_0^1 \frac{dx}{[\;]^{\frac{d}{2}-2}} \cong \int_0^1 dx\,\left(1 + \left(\frac{d}{2}-2\right)\ln[\;]\right) = 1 + \left(\frac{d}{2}-2\right)\int_0^1 \ln[\;]\,dx$$

Γ FUNCTION FEATURES.

$$\Gamma(x) \equiv \int_0^\infty t^{x-1} e^{-t}\, dt$$

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = 1$$

$$\Gamma(x+1) = \int_0^\infty t^{x} e^{-t}\, dt = \int_0^\infty t^{x} \left(-\frac{1}{x}\right) d e^{-xt}$$

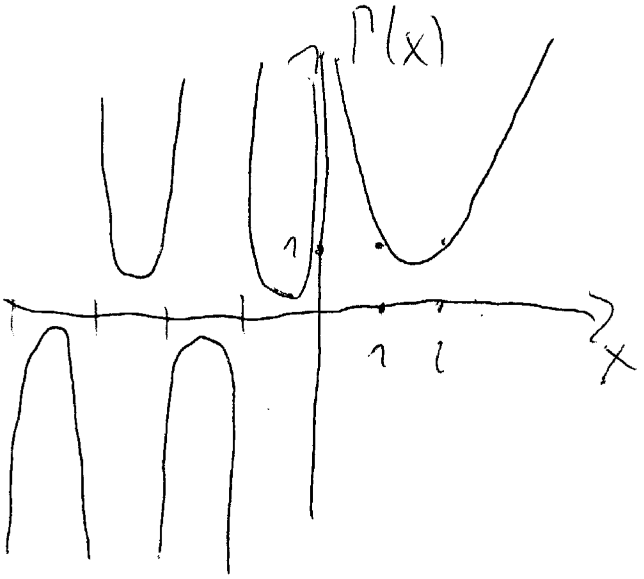
$$= -\left(-\frac{1}{x}\right) \int_0^\infty e^{-xt}\, x\, t^{x-1}\, dt = x\,\Gamma(x)$$

$\Gamma(2) = 1\,\Gamma(1) = 1$

$\Gamma(3) = 2\,\Gamma(2) = 2$

$\Gamma(4) = 3\,\Gamma(3) = 6 = 3!$

$\Gamma(n) = (n-1)!$ $\qquad 0! \equiv 1$



IMPORTANT: $\qquad \gamma = 0.5772157$

$$\Gamma(-n+\varepsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\varepsilon} + 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma + O(\varepsilon)\right]$$

$$\Gamma(\epsilon) = \frac{(-1)^0}{0!}\left[\frac{1}{\epsilon} - \gamma + O(\epsilon)\right] = \frac{1}{\epsilon} - \gamma$$

DEFINE $-\epsilon = \frac{d}{2} - 2 = \frac{d-4}{2}$ $\qquad \epsilon = 2 - \frac{d}{2} = \frac{4-d}{2}$

$$\xcancel{B_0^{REG}(p^2) = \frac{i}{(4\pi)^2}\left(1 + \epsilon \ln \pi\right) 2 \left(\frac{1}{\epsilon} - \gamma\right)\left[1 - \epsilon \int_0^1 \ln[\;]\, dx\right]}$$

$$\xcancel{= \frac{i}{(4\pi)^2} 2 \left[\frac{1}{\epsilon} - \ln \pi - \gamma - \int_0^1 \ln[\;]\, dx\right]}$$

$$B_0^{REG}(p^2) = \frac{i \mu^{2\epsilon}}{(4\pi)^2 (4\pi)^{-\epsilon}} \Gamma(\epsilon) \left(1 - \epsilon \int_0^1 \ln[\;]\, dx\right)$$

$$= \frac{i}{(4\pi)^2} \left(1 + \epsilon \ln 4\pi\mu^2\right)\left(\frac{1}{\epsilon} - \gamma\right)\left(1 + \epsilon F(p^2, m_1^2, m_2^2)\right)$$

$$= \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} + \ln 4\pi\mu^2 - \gamma\right)\left(1 + \epsilon F\right)$$

$$= \frac{i}{(4\pi)^2} \left(\frac{1}{\epsilon} - \gamma + \ln 4\pi\mu^2 + F(p^2, m_1^2, m_2^2)\right)$$

$$\Delta \equiv \frac{1}{\epsilon} - \gamma + \ln 4\pi\mu^2$$

$$B_0(p^2, m_1^2, m_2^2) = \frac{i}{(4\pi)^2}\left(\Delta + B_0^{FIN}\right)$$

$$B_0^{FIN} = -\int_0^1 \ln\left[m_1^2 x + m_2^2 (1-x) - p^2 x (1-x)\right] dx$$

EXERCISE:

$$A_0(m^2) = \left[\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2-m^2}\right] \left[\frac{-i}{(4\pi)^2}\right]^{-1}$$

SHOW
$$A_0(m^2) = \cancel{\frac{i}{(4\pi)^2}} -m^2(\Delta + 1 - \ln m^2)$$
$$= \cancel{\frac{i}{(4\pi)^2}} -m^2\left(\frac{1}{\varepsilon} - \gamma + \ln 4\pi - \ln\frac{m^2}{\mu^2}\right)$$

$$\varepsilon = \frac{4-D}{2} \qquad \Delta = \frac{1}{\varepsilon} - \gamma + \ln 4\pi$$

FURTHER OPTIONS:

$$\int \frac{k^\mu}{(\,)(\,)} = \frac{i}{(4\pi)^2}\, p^\mu B_1$$

$$\int \frac{k^\mu k^\nu}{(\,)(\,)} = \frac{i}{(4\pi)^2}\left(p^\mu p^\nu B_{21} + g^{\mu\nu} B_{22}\right)$$

EXPRESS B_1, B_{21}, B_{22} IN TERMS OF A_0, B_0

FIND DIVERGENT PARTS.