Geometry of null-like surfaces in General Relativity and its application to dynamics of gravitating matter

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Abstract

Geometric tools describing the structure of a null-like surface S (*wave front*) are constructed. They are applied to analysis of interaction between a light-like matter shell and the surrounding gravitational field. It is proved that the Einstein tensor $\mathbf{G}^a{}_b$ describing such a situation may be written in terms of external curvature of S. Conservation laws (Bianchi identities) for $\mathbf{G}$ are proved. Also geometry of non-expanding horizons (surfaces surrounding black holes) is analyzed in terms of the constructed tools. Possibility of application of these results to the problem of motion of isolated objects in General Relativity is discussed.

Key Words: general relativity, differential geometry, black holes

Dedicated to Professor Roman S. Ingarden in honour of his 80-th birthday

1 Introduction

Professor Roman S. Ingarden always taught us how to adapt modern differential geometry to specific needs of theoretical physics. In the present paper we show to what extend the notion of an extrinsic curvature may be generalized to the case of a null-like hypersurface. Such surfaces arise in two important physical situations: 1) they correspond to world-sheets of radiation-like matter-shells and 2) they describe isolated event-horizons in the theory of black holes.

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Geometry of a hypersurface $S \subset M$ in a Riemannian manifold (M, g) may be described by two objects: the restriction g_{ab} of the metric tensor to S (called its "first fundamental form") and the external curvature (called its "second fundamental form"). For many purposes it is useful to represent the latter by the so called Arnowitt-Deser-Misner momentum-density $\mathbf{P}^a{}_b$ (for purposes which will become clear in the sequel we use the mixed contravariant-covariant representation). In a pseudo-Riemannian (Lorentzian) manifold M, the analogous quantity may be easily defined for any submanifold S whose first fundamental form is non-degenerate – see [1] and [2]. This is not true if S is a wave front manifold. In this case the induced metric is degenerate, and the standard construction of external curvature does not make any sense because the inverse metric g^{ab} used in the construction does not exist.

The object $\mathbf{P}^a{}_b$ plays an important role in the theory of a self-gravitating matter-shell – see [4], [5], [6]. Space-time M describing the shell is a union of two pieces stitched along S in such a way that the metric $g_{\mu\nu}$ is continuous and the connection coefficients $\Gamma^\lambda_{\mu\nu}$ admit step discontinuities on S. Einstein tensor of such a space-time contains derivatives of these discontinuities and, therefore, may be defined only in the sense of distribution as $\mathcal{G}^a{}_b = \mathbf{G}^a{}_b \delta_S$, where δ_S is the Dirac's delta distribution concentrated on S and $\mathbf{G}^a{}_b$ is a three-dimensional tensor-density living on S. Actually the following may be proved (see [7], [8] and [9]):

$$\mathbf{G}^a{}_b := [\mathbf{P}^a{}_b] \ , \tag{1.1}$$

where the bracket denotes the jump of the value of the A. D. M. momentum between the two sides of S. The above singular Einstein tensor must be matched by the singular (living on S) energy-momentum tensor of the matter shell. Due to Gauss-Codazzi constraint, these objects must be conserved and the conservation law $\overline{\nabla}_a \mathbf{G}^a{}_b = 0$ may be written in terms of the three-dimensional covariant derivative $\overline{\nabla}$ on S.

In the present paper we show that the above construction may be generalized to the case of a null-like world-surface S (a world-sheet of a light-like matter shell – e. g. a short flash of radiation). Singular Einstein tensor is constructed and its divergence with respect to the degenerate metric of S is uniquely defined. The divergence is shown to vanish as a consequence of Gauss-Codazzi equations. "Hydrodynamics" of radiation-like matter, providing energy-momentum tensors whose divergence vanish automatically, will be analyzed in the next paper.

Unlike in the case of a massive matter shell, the lower index of $\mathbf{G}^a{}_b$ *cannot* be raised because of the degeneracy of the metric tensor g_{ab} and, consequently, the corresponding covariant tensor density $\mathbf{G}^{ab}$ cannot be uniquely defined. This corresponds to the fact that the "symmetric energy-momentum tensor" $\mathbf{T}^{ab}$ of the radiation-like matter, defined as a derivative of the matter Lagrangian with respect to the metric, is not given uniquely, because the latter is subject to a constraint: $\det g_{ab} = 0$. Hence, the derivative is defined only up to an additive term: derivative of the constraint multiplied by an arbitrary Lagrange multiplier (in Section 5 we analyze this non-uniqueness in detail). Nevertheless, as will be shown in the next paper, the canonical energy-momentum tensor $\mathbf{T}^a{}_b$ of such a matter *is* well defined and remains conserved as a consequence of Noether identities. Dynamics of the shell is implied by the singular part of Einstein equations on S: $\mathbf{G}^a{}_b = 8\pi \mathbf{T}^a{}_b$.

In the last Section we apply our construction to the theory of non-expanding horizons. Geometry of a horizon is described by two, mutually conjugate objects: a divergence-free vector-density Λ^a ("first fundamental object") and a gauge potential w_a ("second fundamental object") which is subject to the gradient gauge transformations. Application of these objects to the dynamics of black holes is shortly discussed.

2 Intrinsic geometry of a null hypersurface

A null hypersurface in a Lorentzian space-time M is a three-dimensional submanifold $S \subset M$ such that the restriction g_{ab} of the space-time metrics $g_{\mu\nu}$ to S is degenerate.

We use here adapted coordinates: Cauchy surfaces V_t corresponding to constant value of the "time-like" coordinate $x^0 = t$ are space-like and the x^3 coordinate is constant on S. Space coordinates will be labelled by $k, l = 1, 2, 3$; coordinates on S will be labelled by $a, b = 0, 1, 2$; finally, coordinates on $S_t := V_t \cap S$ will be labelled by $A, B = 1, 2$. Space-time coordinates will be labelled by Greek characters α, β, μ, ν.

The non-degeneracy of the space-time metric implies that the metric g_{ab} on S has signature $(0, +, +)$. This means that there is a non-vanishing null-like vector field X^a on S, such that its four-dimensional embedding X^μ to M is orthogonal to S. Consider integral curves of X^a. We are going to prove that these curves, after a suitable reparameterization, are geodesic curves of the space-time metric $g_{\mu\nu}$. For this purpose consider any smooth function φ with non-vanishing gradient, which is constant on S and take any null-like vector field X^μ in a neighbourhood of S, which coincides with X^a on S. Because X^μ is orthogonal to S, we conclude that X_μ is proportional to the gradient of φ:

$$X_\mu = f \varphi_{,\mu} , \tag{2.1}$$

where f does not vanish on S. Using the symmetry of the second covariant derivatives: $\nabla_\lambda \nabla_\mu \varphi = \nabla_\mu \nabla_\lambda \varphi$, we obtain on S the following identity:

$$\begin{aligned} X^\lambda \nabla_\lambda X_\mu &= \varphi_{,\mu}(X^\lambda \partial_\lambda f) + f X^\lambda \nabla_\lambda \nabla_\mu \varphi = \varphi_{,\mu}(X^\lambda \partial_\lambda f) + f X^\lambda \nabla_\mu \nabla_\lambda \varphi \\ &= (X^\lambda \partial_\lambda f)\varphi_{,\mu} + f X^\lambda \nabla_\mu (\frac{1}{f} X_\lambda) = (X^\lambda \partial_\lambda f)\varphi_{,\mu} + X^\lambda \nabla_\mu X_\lambda \\ &= (X^\lambda \partial_\lambda f)\varphi_{,\mu} + \frac{1}{2}\nabla_\mu (X^\lambda X_\lambda) = (X^\lambda \partial_\lambda \log f) X_\mu \ . \end{aligned} \tag{2.2}$$

This implies that the field $\tilde{X}^\mu := \frac{1}{f} X^\mu$ is geodesic:

$$\tilde{X}^\lambda \nabla_\lambda \tilde{X}^\mu = 0 \ . \tag{2.3}$$

We conclude that the null hypersurface is always a collection of null-like geodesics.

On the other hand, the hypersurface S may be constructed if we only know initial values for these geodesics: a space-like two-surface S_t and a null-like vector field $X^\mu(x)$ defined for $x \in S_t$. More precisely: there are exactly two null hypersurfaces containing given S_t. Indeed, chose any space-like Cauchy surface V_t containing S_t and a three-coordinate system (x^k) on it such that the coordinate x^3 is constant on S_t. At each point $x \in V_t$ there are exactly two null-like directions orthogonal to the two-surface $\{x^3 = \mathrm{const}\}$. Choose a non-vanishing vector $X^\mu(x)$ in one of these directions and suppose that the dependence on x is smooth. There is a unique four-coordinate system (x^μ) in a neighbourhood of V_t satisfying the following conditions:

- coordinates x^k, $k = 1, 2, 3$ are constant along geodesic lines starting from every point $x \in V_t$ in the direction $X^\mu(x)$,
- x^0 is a geodesic parameter along these lines and equals t on V_t.

In this coordinate system we have:

$$X = \frac{\partial}{\partial x^0} \ . \tag{2.4}$$

Because it is null-like, we have: $0 = g(X, X) = g_{00}$. Moreover, the geodesic condition (2.3) reads:

$$0 = \nabla_0 \frac{\partial}{\partial x^0} = \Gamma^\mu_{00} = \frac{1}{2} g^{\mu\nu}(2g_{\nu 0,0} - g_{00,\nu}) = 0 \ . \tag{2.5}$$

These identities imply that

$$\frac{d}{dx^0} g_{\nu 0} = 0 \tag{2.6}$$

along each geodesics. But we had initially:

$$g_{a0}(x^0 = t) = g\left(X(x), \frac{\partial}{\partial x^a}\right) = 0 \ ,$$

because X was orthogonal to surfaces $\{x^3 = \mathrm{const}\}$ $(a = 1, 2)$ and to itself $(a = 0)$. We conclude that $g_{a0} \equiv 0$. Hence $\{x^3 = \mathrm{const}\}$ are null hypersurfaces.

Our construction shows that any null hypersurface S may always be embedded in a 1-parameter congruence of null hypersurfaces $\{x^3 = \mathrm{const}\}$. Moreover, coordinates (x^a) have been constructed in such a way, that the field (2.4) is a null geodesic field in a neighbourhood of S.

For our purposes we will relax the latter condition and use an arbitrary coordinate system, such that coordinate x^3 is constant along null hypersurfaces belonging to the congruence. A general four-metric tensor fulfilling this requirement takes the following form:

$$g_{\mu\nu} = \left[\begin{array}{c|c|c} n^A n_A & n_A & sM + m^A n_A \\ \hline n_A & g_{AB} & m_A \\ \hline sM + m^A n_A & m_A & \left(\frac{M}{N}\right)^2 + m^A m_A \end{array}\right] , \tag{2.7}$$

and

$$
g^{\mu\nu} = \left[\begin{array}{c|c|c}
-\left(\frac{1}{N}\right)^2 & \frac{n^A}{N^2} - s\frac{m^A}{M} & \frac{s}{M} \\
\hline
\frac{n^A}{N^2} - s\frac{m^A}{M} & \tilde{\tilde{g}}^{AB} - \frac{n^A n^B}{N^2} + s\frac{n^A m^B + m^A n^B}{M} & -s\frac{n^A}{M} \\
\hline
\frac{s}{M} & -s\frac{n^A}{M} & 0
\end{array}\right] , \tag{2.8}
$$

where $M > 0$, $s = \pm 1$, g_{AB} is the induced two-metric on surfaces $\{x^0 = \text{const}, x^3 = \text{const}\}$ and $\tilde{\tilde{g}}^{AB}$ is its inverse (contravariant) metric. Both $\tilde{\tilde{g}}^{AB}$ and g_{AB} are used to rise and lower indices $A, B = 1, 2$ of the two-vectors n^A and m^A. Denoting

$$
\lambda := \sqrt{\det g_{AB}} \ , \tag{2.9}
$$

we have that $\sqrt{|\det g_{\mu\nu}|} = \lambda M$.

In this coordinate system the null direction on S may be spanned by the following vector field X:

$$
X = \partial_0 - n^A \partial_A \ , \tag{2.10}
$$

or

$$
X^\mu = \frac{g^{3\mu}}{g^{30}} = M s g^{3\mu} \ . \tag{2.11}
$$

We have:

$$
g(X, X) = g(X, \partial_A) = 0 \ . \tag{2.12}
$$

The triad (X, ∂_A) on S will be used in the sequel for various geometric constructions. It depends upon a particular $(2+1)$-decomposition of S, given by the choice of the time coordinate x^0 on S. As we shall see, several objects constructed by means of the triad will not depend upon this choice and will describe the geometry of S. To prove this independence, observe that we have the following transformation law:

$$
\begin{aligned}
\tilde{X} &= cX \ , &(2.13)\\
\tilde{\partial}_{\tilde{B}} &= C^A_{\tilde{B}} \partial_A + f_{\tilde{B}} X \ , &(2.14)
\end{aligned}
$$

where $(\tilde{X}, \tilde{\partial}_{\tilde{B}})$ is the new triad, corresponding to the new coordinate system $(\tilde{x}^{\tilde{a}})$ on S. The coefficient c may be obtained from the following equation:

$$
\begin{aligned}
1 = \langle d\tilde{x}^0, \tilde{X} \rangle &= \langle \frac{\partial \tilde{x}^0}{\partial x^A} dx^A + \frac{\partial \tilde{x}^0}{\partial x^0} dx^0, cX \rangle \\
&= c\left(-\frac{\partial \tilde{x}^0}{\partial x^A} n^A + \frac{\partial \tilde{x}^0}{\partial x^0} \right) \ , \qquad (2.15)
\end{aligned}
$$

hence,

$$c = \left(\frac{\partial \tilde{x}^0}{\partial x^0} - \frac{\partial \tilde{x}^0}{\partial x^A} n^A \right)^{-1} . \tag{2.16}$$

On the other hand, equation (2.10) implies:

$$\begin{aligned} \partial_{\tilde{B}} &= \frac{\partial x^A}{\partial \tilde{x}^{\tilde{B}}} \partial_A + \frac{\partial x^0}{\partial \tilde{x}^{\tilde{B}}} \left(X + n^A \partial_A \right) \\ &= \left(\frac{\partial x^A}{\partial \tilde{x}^{\tilde{B}}} + \frac{\partial x^0}{\partial \tilde{x}^{\tilde{B}}} n^A \right) \partial_A + \frac{\partial x^0}{\partial \tilde{x}^{\tilde{B}}} X , \end{aligned} \tag{2.17}$$

hence,

$$C_{\tilde{B}}^{\ A} = \frac{\partial x^A}{\partial \tilde{x}^{\tilde{B}}} + \frac{\partial x^0}{\partial \tilde{x}^{\tilde{B}}} n^A , \tag{2.18}$$

$$f_{\tilde{B}} = \frac{\partial x^0}{\partial \tilde{x}^{\tilde{B}}} . \tag{2.19}$$

Now, we are ready to prove that the following quantity:

$$\Lambda^a := \lambda X^a . \tag{2.20}$$

is an invariant vector density on S, given uniquely by the structure of S and independent upon any choice of coordinates, even if the vector field X itself is *not*. Indeed, the transformation law for g_{AB}:

$$g_{\tilde{A}\tilde{B}} = C_{\tilde{A}}^{\ A} C_{\tilde{B}}^{\ B} g \left(\partial_A + f_A X, \partial_B + f_B X \right) = C_{\tilde{A}}^{\ A} C_{\tilde{B}}^{\ B} g_{AB} , \tag{2.21}$$

implies:

$$\tilde{\lambda} = \lambda \det C_{\tilde{A}}^{\ B} . \tag{2.22}$$

Hence, the transformation law for Λ reads:

$$\begin{aligned} \Lambda^{\tilde{a}} = \tilde{\lambda} \tilde{X}^{\tilde{a}} &= \left(\det C_{\tilde{A}}^{\ B} \right) \lambda c X^{\tilde{a}} \\ &= \det \left(\frac{\partial x^c}{\partial \tilde{x}^{\tilde{d}}} \right) \lambda X^a \frac{\partial \tilde{x}^{\tilde{a}}}{\partial x^a} = \det \left(\frac{\partial x^c}{\partial \tilde{x}^{\tilde{d}}} \right) \Lambda^a \frac{\partial \tilde{x}^{\tilde{a}}}{\partial x^a} , \end{aligned}$$

which is precisely the transformation law for vector densities. In the above formula we have used the following algebraic identity, which we prove in the Appendix:

$$c \det C_{\tilde{B}}^{\ A} = \left(\frac{\partial \tilde{x}^0}{\partial x^0} - n^A \frac{\partial \tilde{x}^0}{\partial x^A} \right)^{-1} \det \left(\frac{\partial x^A}{\partial \tilde{x}^{\tilde{B}}} + \frac{\partial x^0}{\partial \tilde{x}^{\tilde{B}}} n^A \right) \equiv \det \left(\frac{\partial x^a}{\partial \tilde{x}^{\tilde{b}}} \right) . \tag{2.23}$$

In order to complete the triad (X, ∂_A) on S to a tetrad in M it is useful to choose a transverse field Y fulfilling the following "normalization conditions":

$$g(Y, X) = 1 , \tag{2.24}$$

$$g(Y, \partial_A) = 0 . \tag{2.25}$$

These equations *do not* determine Y uniquely, but *modulo* an additive term proportional to X: a "gauge transformation"

$$Y \to Y + hX \ , \tag{2.26}$$

with an arbitrary scalar field h is always possible. Extending coordinate x^0 from S to a neighbourhood of S, we may choose the following, transverse field:

$$Y = \frac{s}{M}\left(\partial_3 - m^A \partial_A\right) \ . \tag{2.27}$$

We stress, however, that this particular choice of Y, which we shall always use in the sequel, depends not only upon a $(2+1)$-decomposition of S, but also on a $(3+1)$-decomposition of M in a neighbourhood of S. Because of (2.12) and (2.25), the vectors X and Y span the bundle of vectors normal to S.

The reader may easily check that the transformation law for Y, when passing from one to another $(2+1)$-decomposition of S, reads:

$$\tilde{Y} \ = \ c^{-1}\left(Y - k^A \partial_A\right) + hX \ , \tag{2.28}$$

where the scalar field h is arbitrary (it is determined by the extension of the $(2+1)$-decomposition of S to a $(3+1)$-decomposition of M), and the coefficients k^A are uniquely determined by equation

$$f_{\tilde{B}} = C_{\tilde{B}}^{\ A} g_{AC} k^C \ , \tag{2.29}$$

with $f_{\tilde{B}}$ given by (2.19). Despite of the freedom in choice of Y, some geometric objects constructed with help of the tetrad (X, ∂_A, Y) do not depend upon this choice and characterize only the geometry of $S \subset M$.

3 Extrinsic geometry of a null hypersurface

The covariant derivative of the field X along vectors tangent to S is orthogonal to X and, therefore, tangent to S. Indeed, we have:

$$g(X, \nabla_a X) = \frac{1}{2}\partial_a g(X, X) \equiv 0 \ ,$$

and, consequently:

$$\nabla_a X =: -t^b{}_a \frac{\partial}{\partial x^b} \ . \tag{3.1}$$

Equation (2.2) implies that the field X^a is an eigenvector of $t^b{}_a$. The corresponding eigenvalue vanishes if and only if X is geodesic.

A (2+1)-decomposition of S allows us to split $t^b{}_a$ into two parts:

$$l_{ab} := t^c{}_a g_{cb} = -g(\partial_b, \nabla_a X) = g(\nabla_a \partial_b, X) = X_\mu \Gamma^\mu_{ab} \ , \tag{3.2}$$

and

$$w_a := t^0{}_a = -g(Y, \nabla_a X) = -X^\mu \Gamma^0_{\mu a} \; . \tag{3.3}$$

We note that both l_{ab} and w_a are invariant with respect to any transformation (2.26) because of the following formula:

$$g(Y + hX, \nabla_a X) = g(Y, \nabla_a X) \; .$$

Hence, they depend only upon a (2+1)-decomposition of S.

The matrix l_{ab} is symmetric, because of (3.2). It satisfies three identities

$$0 = l_{ab} X^b = l_{a0} - n^B l_{aB} \; ,$$

implied by (2.2). Hence, we have: $l_{0A} = l_{AB} n^B$ and, consequently, $l_{00} = l_{AB} n^A n^B$. The whole matrix $t^b{}_a$ may be fully reconstructed from the three components of l_{AB} and three of w_a. Namely, we have: $t^0{}_0 = w_0$, $t^0{}_A = w_A$. Moreover,

$$\begin{aligned} t^A{}_0 &= \left(l_{0B} - t^0{}_0 g_{0B} \right) \tilde{\tilde{g}}^{BA} = \left(n^C l_{CB} - w_0 n_B \right) \tilde{\tilde{g}}^{BA} \; , \\ t^C{}_A &= \left(l_{AB} - t^0{}_A g_{0B} \right) \tilde{\tilde{g}}^{BC} = \left(l_{AB} - w_A n_B \right) \tilde{\tilde{g}}^{BC} \; . \end{aligned}$$

The matrix l_{ab} may be expressed in terms of the Lie derivative along X of the metric tensor g_{ab} on S. Because such a three-dimensional Lie derivative is equal to the restriction of the four-dimensional Lie derivative of $g_{\mu\nu}$ to S, we have:

$$\mathcal{L}_X (g_{ab}) = (\mathcal{L}_X g)_{ab} = g_{ac} \nabla_b X^c + g_{cb} \nabla_a X^c = 2 g_{c(a} \nabla_{b)} X^c = 2 g(\partial_a, \nabla_b X) \; ,$$

or

$$l_{ab} = -\frac{1}{2} \mathcal{L}_X g_{ab} \; . \tag{3.4}$$

This implies the following transformation law for the object l_{ab}:

$$\begin{aligned} \tilde{l}_{\tilde{a}\tilde{b}} &= -\frac{1}{2} \mathcal{L}_{\tilde{X}} g_{\tilde{a}\tilde{b}} = -\frac{1}{2} c \left(\mathcal{L}_X g \right)_{\tilde{a}\tilde{b}} \\ &= c l_{\tilde{a}\tilde{b}} = c l_{ab} \frac{\partial x^a}{\partial \tilde{x}^{\tilde{a}}} \frac{\partial x^b}{\partial \tilde{x}^{\tilde{b}}} \; . \end{aligned}$$

It is not a tensorial transformation law, but combining it with the transformation law for λ and using identity (2.23) we obtain the following transformation law for the object $Q_{ab} := \lambda l_{ab}$:

$$\begin{aligned} Q_{\tilde{a}\tilde{b}} = \tilde{\lambda} \tilde{l}_{\tilde{a}\tilde{b}} &= \det C^{\;B}_{\tilde{A}} \lambda c l_{ab} \frac{\partial x^a}{\partial \tilde{x}^{\tilde{a}}} \frac{\partial x^b}{\partial \tilde{x}^{\tilde{b}}} \\ &= \det \left(\frac{\partial x^c}{\partial \tilde{x}^d} \right) Q_{ab} \frac{\partial x^a}{\partial \tilde{x}^{\tilde{a}}} \frac{\partial x^b}{\partial \tilde{x}^{\tilde{b}}} \; . \end{aligned}$$

We conclude that Q_{ab} is an intrinsic three-dimensional tensor density on S, independent upon a particular $(2 + 1)$-decomposition used. Because of (2.2), it is "orthogonal" to X:

$$Q_{ab} X^b = 0 \; .$$

Let us denote by "||" a two-dimensional covariant derivative on each surface $\{x^0 = \text{const}\}$, calculated with respect to the Levi-Civita connection of g_{AB}. Formula (3.4) implies:

$$\begin{aligned} Q_{AB} &= -\frac{1}{2}\lambda \left(g_{AB,0} - n^C g_{AB,C} - n^C{}_{,A} g_{CB} - n^C{}_{,B} g_{AC} \right) \\ &= \frac{1}{2}\lambda \left(n_{A\|B} + n_{B\|A} - \dot{g}_{AB} \right) \ , \end{aligned} \tag{3.5}$$

and, consequently,

$$Q_{AB}\tilde{\tilde{g}}^{AB} = \lambda l_{AB}\tilde{\tilde{g}}^{AB} = \lambda l = -\partial_a \Lambda^a = -\dot{\lambda} + \partial_A(\lambda n^A) \ , \tag{3.6}$$

where $l := l_{AB}\tilde{\tilde{g}}^{AB}$.

Unfortunately, transformation laws for the object w_a, implied by (2.13) and (2.14) are not of tensorial character:

$$\begin{aligned} \tilde{w}_a &= g\left(\frac{1}{c}\left(Y - k^A\partial_A\right) + hX, \nabla_a(cX)\right) \\ &= g\left(\frac{1}{c}Y, c\nabla_a X\right) + g\left(\frac{1}{c}Y, X\right)\nabla_a c + g\left(hX, c\nabla_a X\right) + g\left(hX, X\right)\nabla_a c \\ &\quad -g\left(k^A\partial_A, X\right)\frac{1}{c}\nabla_a c - g\left(k^A\partial_A, \nabla_a X\right) \\ &= w_a + \partial_a\varphi \ - k^B l_{aB}, \end{aligned} \tag{3.7}$$

where $\varphi = \log c$ and k^A are given by (2.29). Although this is not a tensorial transformation law, we shall be able to use this object in a construction of further tensorial objects on S.

4 Gauss-Codazzi Constraints

Similarly as for non-degenerate (e. g. space-like) hypersurfaces S, Einstein equations imply constraints which must be fulfilled by the extrinsic curvature objects. In case of a non-degenerate hypersurface there are four such constraints, corresponding to components $G^3{}_a = g^{3\nu}G_{\nu a}$ and $G^{33} = g^{3\mu}g^{3\nu}G_{\mu\nu}$ of Einstein equations. In the light-like case there are only three independent constraints. Indeed, due to (2.11), the first quantity is proportional to $X^b G_{ba}$, whereas the latter is proportional to $X^a X^b G_{ab}$ and, therefore, equal to linear combination of the first one. This corresponds to the fact, that the vector orthogonal to S coincides with one of the tangent vectors. We conclude that Gauss-Codazzi constraints are equivalent to the following three equations:

$$G_{ab}X^a X^b = 8\pi T_{ab}X^a X^b, \tag{4.1}$$
$$G_{aB}X^a = 8\pi T_{aB}X^a. \tag{4.2}$$

As $g_{ab}X^a = 0$, the above contractions with the Einstein tensor reduce to ones with the Ricci tensor

$$g^{\kappa\lambda}R_{\kappa a\lambda b}X^a = R_{ab}X^a = G_{ab}X^a = 8\pi T_{ab}X^a \ .$$

We prove in the Appendix that equation (4.1) may be rewritten in terms of the quantities l_{ab} and w_a in the following way:

$$\dot{l} - n^A\partial_A l + (w_a X^a)\, l - \frac{1}{2}l^2 - \bar{l}^{AB}\bar{l}_{AB} = 8\pi T_{ab}X^a X^b \ , \tag{4.3}$$

where we have decomposed l_{AB} into its trace l and its traceless part:

$$\bar{l}_{AB} \ := \ l_{AB} - \frac{1}{2}g_{AB}l. \tag{4.4}$$

Moreover, we prove in the Appendix the following form of the second ("vector") constraint (4.2):

$$\dot{w}_B - w_{B\|A}n^A - w_A n^A{}_{\|B} - (w_a X^a)_{\|B} - w_B l + \bar{l}^A_{B\|A} - \frac{1}{2}l_{\|B} = -8\pi T_{aB}X^a. \tag{4.5}$$

In case of vacuum space-times the right-hand sides of both (4.3) and (4.5) vanish.

5 Energy-momentum tensor carried by a connection discontinuity and Bianchi identities

Consider a space-time which is composed of two pieces stitched together along a hypersurface S in such a way that the metric is continuous. We admit, however, step discontinuities of its first derivatives across S.

Consequently, the Riemann tensor contains terms proportional to the Dirac distribution $\delta(x^3)$, where (as usual) we denote by x^3 any coordinate which is constant on S. According to Einstein equations, we interpret these singular terms as the singular energy-momentum tensor carried by a matter which lives on $S = \{x^3 = \text{const}\}$. We assume that the topology of S is equal to that of a world tube $S^2 \times R^1$. The world tube of this "radiation-like" matter shell is a null hypersurface.

Theory of a self-gravitating matter shell was considered by many authors (see e.g. [4] – [7]), but mainly in the context of a massive matter. For a light-like matter (e.g. a short but strong flash of radiation) there are specific problems which we want to discuss here.

The matter shell splits space-time into two parts: the interior and the exterior of the tube S. Both parts fulfill separately vacuum Einstein equations in their interiors. Hence, Einstein tensor vanishes outside of S. There remains, however, its singular part, concentrated on S. It may be calculated from the Ricci tensor:

$$R_{\mu\nu} = \partial_\lambda \Gamma^\lambda_{\mu\nu} - \partial_{(\mu}\Gamma^\lambda_{\nu)\lambda} + \Gamma^\lambda_{\sigma\lambda}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\mu\sigma}\Gamma^\sigma_{\nu\lambda} \ . \tag{5.1}$$

An important simplification is obtained, when we rewrite it in terms of the following combinations of the Christoffel symbols (see [3]):

$$A^{\lambda}_{\mu\nu} := \Gamma^{\lambda}_{\mu\nu} - \delta^{\lambda}_{(\mu}\Gamma^{\kappa}_{\nu)\kappa} \ . \tag{5.2}$$

We have:

$$R_{\mu\nu} = \partial_\lambda A^{\lambda}_{\mu\nu} - A^{\lambda}_{\mu\sigma}A^{\sigma}_{\nu\lambda} + \frac{1}{3}A^{\lambda}_{\mu\lambda}A^{\sigma}_{\nu\sigma}. \tag{5.3}$$

Terms quadratic in A's may have only step-like discontinuities. The derivatives along S are thus bounded and belong to the regular part of the Ricci tensor, which vanishes on both sides of S. The singular part of the Ricci tensor is obtained from the transversal derivative ($\lambda = 3$) only:

$$\mathrm{sing}(R_{\mu\nu}) = \partial_3 A^3_{\mu\nu} = \delta(x^3)[A^3_{\mu\nu}] \ , \tag{5.4}$$

where by δ we denote the Dirac delta-distribution and by square brackets we denote the jump of the value of the corresponding expression between the two sides of S.

Hence, Einstein tensor density reads:

$$\mathcal{G}^{\mu}{}_{\nu} := \sqrt{|g|}\ \mathrm{sing}\left(R^{\mu}{}_{\nu} - \frac{1}{2}R\right) = \delta(x^3)\mathbf{G}^{\mu}{}_{\nu} \ , \tag{5.5}$$

where

$$\mathbf{G}^{\mu}{}_{\nu} := \lambda M \left(\delta^{\beta}_{\nu}g^{\mu\alpha} - \frac{1}{2}\delta^{\mu}_{\nu}g^{\alpha\beta}\right)[A^3_{\alpha\beta}] \tag{5.6}$$

is the three-dimensional quantity living on S, whose geometric character will be discussed later. Now, we are going to prove the following identity:

$$\mathbf{G}^{3}{}_{\nu} \equiv 0 \ . \tag{5.7}$$

For this purpose, on both sides of $S \subset M$ we consider the following combination of the connection coefficients taken in any coordinate system, such that $S = \{x^3 = \mathrm{const}\}$:

$$\mathbf{P}^{\mu}{}_{\nu} := \sqrt{|g|}\left(g^{\mu\alpha}A^3_{\alpha\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}g^{\alpha\beta}A^3_{\alpha\beta}\right) = \pi^{\mu\alpha}A^3_{\alpha\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu}\pi^{\alpha\beta}A^3_{\alpha\beta} \ , \tag{5.8}$$

where we have defined the tensor density:

$$\pi^{\mu\nu} := \sqrt{|g|}\ g^{\mu\nu} \ , \tag{5.9}$$

and A's are given by (5.2). We have:

$$\mathbf{G}^{\mu}{}_{\nu} := [\mathbf{P}^{\mu}{}_{\nu}] \ . \tag{5.10}$$

Due to metricity of the connection, the following identities are satisfied on both sides of S:

$$0 \equiv \nabla_a \pi^{33} = \partial_a\pi^{33} + 2\pi^{3\mu}\Gamma^3_{\mu a} - \pi^{33}\Gamma^{\mu}_{a\mu} = \partial_a\pi^{33} + 2\pi^{3\mu}A^3_{\mu a} \ , \tag{5.11}$$

$$\begin{aligned} 0 &\equiv \nabla_a \pi^{3a} = \partial_a\pi^{3a} + \pi^{\mu a}\Gamma^3_{\mu a} + \pi^{3\mu}\Gamma^a_{\mu a} - \pi^{3a}\Gamma^{\mu}_{a\mu} \\ &= \partial_a\pi^{3a} + \pi^{ab}A^3_{ab} - \pi^{33}A^3_{33} \ , \end{aligned} \tag{5.12}$$

where $a, b = 0, 1, 2$ are coordinates on S. Because $\pi^{\mu\nu}$ are continuous across S, the jump of these expressions between the two sides of S must vanish:

$$\mathbf{G}^3{}_a = \pi^{3\mu}[A^3_{\mu a}] = -\frac{1}{2}[\partial_a \pi^{33}] = 0 \ , \tag{5.13}$$

$$\mathbf{G}^3{}_3 = -\frac{1}{2}\left(\pi^{ab}[A^3_{ab}] - \pi^{33}[A^3_{33}]\right) = \frac{1}{2}[\partial_a \pi^{3a}] = 0 \ , \tag{5.14}$$

which proves (5.7).

To encode the information contained in $\mathbf{G}^\mu{}_\nu$, we could also use the symmetric object $\mathbf{G}^{\mu\lambda} = \mathbf{G}^\mu{}_\nu g^{\nu\lambda}$. Due to (5.7), the entire information about $\mathbf{G}^\mu{}_\nu$ is carried by the three-dimensional, symmetric object $\mathbf{G}^{ab}$.

Unfortunately, this *is not* a tensor density, as will be seen in the sequel. Consequently, also $\mathbf{G}^\mu{}_\nu$ is not. It depends upon the choice of coordinates used in the calculations. Nevertheless, the information contained in $\mathbf{G}$ may be nicely divided into an invariant, coordinate-independent part and the "gauge-dependent" part. The first one is carried by the three-dimensional contravariant-covariant version of $\mathbf{G}$:

$$\mathbf{G}^a{}_b := \mathbf{G}^{a\mu} g_{\mu b} = \mathbf{G}^{ac} g_{cb} \ . \tag{5.15}$$

It will be shown in the sequel that this is a genuine three-dimensional tensor density on S. Due to identity $X^a g_{ab} = 0$, it is orthogonal to X and symmetric after further lowering of indices:

$$\mathbf{G}^a{}_b X^b = 0 \ , \tag{5.16}$$

$$\mathbf{G}_{ab} = \mathbf{G}_{ba} \ . \tag{5.17}$$

It is easy to check that $\mathbf{G}^a{}_b$ contains 5 independent entries. Actually, it may be uniquely reconstructed from $\mathbf{G}^0{}_A$ (2 independent components) and the symmetric two-dimensional matrix $\mathbf{G}_{AB}$ (3 independent components):

$$\mathbf{G}^A{}_B = \tilde{\tilde{g}}^{AC} \mathbf{G}_{CB} - n^A \mathbf{G}^0{}_B \ , \tag{5.18}$$

$$\mathbf{G}^0{}_0 = \mathbf{G}^0{}_A n^A \ , \tag{5.19}$$

$$\mathbf{G}^B{}_0 = \left(\tilde{\tilde{g}}^{BC} \mathbf{G}_{CA} - n^B \mathbf{G}^0{}_A\right) n^A \ . \tag{5.20}$$

The correspondence between $\mathbf{G}^a{}_b$ and $(\mathbf{G}^0{}_A, \mathbf{G}_{AB})$ is one-to-one.

Reconstruction of $\mathbf{G}^{ab}$ from $\mathbf{G}^a{}_b$ is impossible, or *possible modulo* an arbitrary additive term $f X^a X^b$. Actually, let us assign to $\mathbf{G}^a{}_b$ the following, symmetric quantity:

$$\mathbf{F}^{AB} := \tilde{\tilde{g}}^{AC} \mathbf{G}_{CD} \tilde{\tilde{g}}^{DB} - n^A \mathbf{G}^0{}_C \tilde{\tilde{g}}^{CB} - n^B \mathbf{G}^0{}_C \tilde{\tilde{g}}^{CA} \ , \tag{5.21}$$

$$\mathbf{F}^{0A} := \mathbf{G}^0{}_C \tilde{\tilde{g}}^{CA} \ , \tag{5.22}$$

$$\mathbf{F}^{00} := 0 \ . \tag{5.23}$$

One may easily check the following formula:

$$\mathbf{G}^{ab} = \mathbf{F}^{ab} + \mathbf{G}^{00} X^a X^b \ . \tag{5.24}$$

The missing (i. e. not contained in $\mathbf{G}^a{}_b$) information about $\mathbf{G}^{ab}$ is, therefore, encoded in $\mathbf{G}^{00}$. Unlike $\mathbf{G}^a{}_b$, it is not invariant and depends upon the choice of coordinates.

We are going to prove in the Appendix that Gauss-Codazzi constraints obtained in the previous section imply that the "covariant divergence" of $\mathbf{G}^a{}_b$ with respect to the degenerate metric g_{ab}:

$$0 \quad = \quad \partial_a \mathbf{G}^a{}_b - \frac{1}{2}\mathbf{G}^{ac} g_{ac,b} \ , \tag{5.25}$$

must vanish. The formula above mimics the standard formula for the covariant derivative $\overline{\nabla}$ of a symmetric tensor density, taken with respect to the Levi-Civita connection of a non-degenerate metric: $\overline{\nabla}_a \mathbf{G}^a{}_b = \partial_a \mathbf{G}^a{}_b - \mathbf{G}^a{}_c \overline{\Gamma}^c_{ab} = \partial_a \mathbf{G}^a{}_b - \frac{1}{2}\mathbf{G}^{ac} g_{ac,b}$. For our degenerate metric there is no Levi-Civita connection $\overline{\Gamma}$ and the rising of indices of $\mathbf{G}$ makes no sense. Nevertheless, the final formula is well defined because the right-hand side of (5.25) *does not* depend upon any specific reconstruction of $\mathbf{G}^{ac}$ from $\mathbf{G}^a{}_b$. Indeed, adding the term fX^aX^c to $\mathbf{G}^{ac}$ does not change anything, because of the following identity:

$$0 \equiv (X^a X^c g_{ac})_{,\mu} = X^a X^c g_{ac,\mu} + 2X^a g_{ac} X^c{}_{,\mu} = X^a X^c g_{ac,\mu} \ . \tag{5.26}$$

The operator on the right-hand side of (5.25) may thus be called the (three-dimensional) covariant derivative of $\mathbf{G}^a{}_b$ on S with respect to its degenerate metric g_{ab}:

$$0 = \overline{\nabla}_a \mathbf{G}^a{}_b \quad := \quad \partial_a \mathbf{G}^a{}_b - \frac{1}{2}\mathbf{G}^{ac} g_{ac,b} = \partial_a \mathbf{G}^a{}_b - \frac{1}{2}\mathbf{F}^{ac} g_{ac,b} \tag{5.27}$$

$$\quad = \quad \partial_a \mathbf{G}^a{}_b - \frac{1}{2}\left(2\mathbf{G}^0{}_A \, n^A{}_{,b} - \mathbf{G}_{AC} \tilde{\tilde{g}}^{AC}{}_{,b}\right) \ . \tag{5.28}$$

Actually, we have proved that the above operation is well defined for tensor densities $\mathbf{G}^a{}_b$ on S fulfilling conditions (5.16) and (5.17).

The (2+1)-decomposition of (5.25) into the time and the space component gives us its equivalent version:

$$0 = \overline{\nabla}_a \mathbf{G}^a{}_B = \partial_0 \mathbf{G}^0{}_B - \left(n^A \mathbf{G}^0{}_B\right)_{||A} + \left(\tilde{\tilde{g}}^{AC} \mathbf{G}_{CB}\right)_{||A} - \mathbf{G}^0{}_A n^A{}_{||B} \ , \tag{5.29}$$

and

$$0 = \overline{\nabla}_a \mathbf{G}^a{}_0 = \tilde{\tilde{g}}^{AC} \mathbf{G}_{CB} \tilde{\tilde{g}}^{BD} l_{AD} + n^B \overline{\nabla}_a \mathbf{G}^a{}_B \ , \tag{5.30}$$

which, if (5.29) is fulfilled, reduces to a purely algebraic equation:

$$0 = \tilde{\tilde{g}}^{AC} \mathbf{G}_{CB} \tilde{\tilde{g}}^{BD} l_{AD} \ . \tag{5.31}$$

The tensorial character of $\mathbf{G}^a{}_b$ will be obvious if we calculate it in terms of the jump of the "transverse curvature" w_a. Due to formula (3.7), the object $[w_a]$ transforms like an intrinsic three-dimensional covector living on the surface S: the "gauge-like" terms in the transformation law are equal on both sides of S and, therefore, cancel when we calculate the jump $[w_a]$. We are going to prove the following formula for $\mathbf{G}$:

$$\mathbf{G}^a{}_b = s\Lambda^a[w_b] \ . \tag{5.32}$$

For this purpose we first observe that the only non-vanishing discontinuities of $A^3_{\mu\nu}$ are A^3_{33}, A^3_{30} and A^3_{3A}. Indeed, using $g^{33} = 0$ we may prove that $[A^3_{ab}] \equiv 0$ because we have: $A^3_{ab} = \Gamma^3_{ab} = g^{3\mu}\Gamma_{\mu ab} = g^{3c}\Gamma_{cab}$. The latter object contains only the metric components on S together with their derivatives *along* S, i. e. is equal on both sides of S.

Hence, using (2.11) and (2.20), we have:

$$\mathbf{G}^a{}_b \;\; = \;\; \lambda M g^{a\beta}[A^3_{b\beta}] = \lambda M g^{a3}[A^3_{b3}] = s\Lambda^a[A^3_{b3}] \ . \tag{5.33}$$

But

$$[A^3_{b3}] = \frac{1}{2}[\Gamma^3_{3b} - \Gamma^a_{ba}] = \frac{1}{2}g^{3a}[\Gamma_{a3b}] - \frac{1}{2}g^{a\mu}[\Gamma_{\mu ba}] \ . \tag{5.34}$$

Because derivatives of g along S are continuous across S, we have $[g_{\mu\nu,a}] = 0$, and the above expression reduces to

$$[A^3_{b3}] \;\; = \;\; \frac{1}{2}g^{3a}\left([\Gamma_{a3b}] - [\Gamma_{3ba}]\right) = \frac{1}{2}g^{3a}[g_{ab,3}] = \frac{1}{2}g^{03}X^a\,[g_{ab,3}] \tag{5.35}$$

$$\;\; = \;\; -X^a g^{03}\,[\Gamma_{3ab}] = -X^a\left[\Gamma^0_{ab}\right] = -X^\mu\left[\Gamma^0_{\mu b}\right] = [w_b] \ , \tag{5.36}$$

which ends the proof of (5.32) because of (3.3).

The trace of $\mathbf{G}$ vanishes because of the following identity:

$$X^a[w_a] = 0 \ , \tag{5.37}$$

which is an easy consequence of (5.35):

$$X^a[w_a] = \frac{1}{2}g^{03}X^aX^b\,[g_{ab,3}] \equiv 0 \ , \tag{5.38}$$

the last equation implied by (5.26), which is fulfilled on both sides of S.

Similar calculations lead to the following result:

$$\mathbf{G}^{00} = -\frac{1}{M}[\lambda_{,3}] \ , \tag{5.39}$$

which proves that the object $\mathbf{G}^{ab}$ given by (5.24) does not transform like a tensor density on S. In other words, definition of $\mathbf{G}^{ab}$ is non-unique and depends upon an arrangement of coordinates. When considering the complete theory of light-like matter interacting with gravity, the above fact matches the non-uniqueness of the definition of the symmetric energy-momentum tensor of the matter:

$$\mathbf{T}^{ab} := \frac{\partial \mathbf{L}}{\partial g_{ab}} \ , \tag{5.40}$$

where by $\mathbf{L}$ we denote matter Lagrangian-density. Indeed, a light-like matter imposes the constraint $\det g_{ab} = 0$ on the geometry of its world surface. Consequently, the derivative on the right-hand side of (5.40) is non-uniquely defined. Actually, we may always add derivative of the constraint, multiplied by an arbitrary Lagrange multiplier. This is precisely the term fX^aX^b, where X is a null vector of g_{ab}. On the other hand, the canonical energy momentum tensor $\mathbf{G}^a{}_b$ is uniquely defined and Einstein equations may be consistently written as $\mathbf{G}^a{}_b = 8\pi\mathbf{T}^a{}_b$. These issues will be discussed in the next paper.

We stress that identities (5.25) are more fundamental than vacuum Einstein equations used in our paper to prove them. Actually, they follow from the singular version of the Bianchi identities, fulfilled for any space-time with continuous metric and a step discontinuity of the connection. Written for the mixed (contravariant-covariant) Einstein tensor-density, Bianchi identities read:

$$\begin{aligned} 0 = \nabla_\mu \mathcal{G}^\mu{}_c &= \partial_\mu \mathcal{G}^\mu{}_c - \mathcal{G}^\mu{}_\alpha \Gamma^\alpha_{\mu c} = \partial_\mu \mathcal{G}^\mu{}_c - \frac{1}{2}\mathcal{G}^{\mu\lambda} g_{\mu\lambda,c} \\ &= \partial_a \mathcal{G}^a{}_c - \frac{1}{2}\mathcal{G}^{ab} g_{ab,c} \ . \end{aligned} \tag{5.41}$$

The regular part of this expression vanishes on both sides of S, as implied by standard Bianchi identities. Let us calculate its singular part, proportional to δ_S. First, we observe that applying the operator on the right-hand side of (5.41) to the singular (5.5) part of the Einstein tensor, the Dirac delta is not differentiated, because $\mathcal{G}^3{}_c = 0$. This way we obtain the right-hand side of (5.25) multiplied by δ_S. Another δ-like term is obtained from $\partial_\mu \mathcal{G}^\mu{}_c$, when applied to the jump of the regular part $\mathrm{reg}(\mathcal{G})$ of $\mathcal{G}$. This way we obtain the term $[\mathrm{reg}(\mathcal{G})^3{}_c]\delta_S$. Finally, the total singular part of the Bianchi identities reads:

$$[\mathrm{reg}(\mathcal{G})^3{}_c] + \partial_a \mathbf{G}^a{}_c - \frac{1}{2}\mathbf{G}^{ab} g_{ab,c} \equiv 0 \ . \tag{5.42}$$

We are going to prove in the next paper that this is, indeed, a universal identity, fulfilled for any space-time with continuous metric and a step discontinuity of the connection across a hypersurface S. It is interesting that the proof is universal and does not depend upon a specific (degenerate or non-degenerate) character of the induced metric on S. The identity reduces to (5.25) for Einstein space-times, for which the regular part of the Einstein tensor vanishes on both sides of S.

6 Non-expanding horizons

By an non-expanding horizon in a vacuum Einstein space-time M we mean a null hypersurface S for which the density Λ is divergence-free:

$$\partial_a \Lambda^a \equiv 0 \ . \tag{6.1}$$

Due to (3.6), this is equivalent to $l \equiv 0$ (cf. [10]). Vacuum Einstein equations (4.3) imply, that also $\bar{l}_{AB}$ must vanish. We have, therefore, $l_{ab} \equiv 0$ or, consequently,

$$\mathcal{L}_X g_{ab} \equiv 0 \ . \tag{6.2}$$

This means that the geometry of S is "static". More precisely, S is an affine line bundle $\pi : S \to B$ over a base manifold B. Usually, it is assumed that B is topologically isomorphic to a sphere S^2. Fibers of S are integral lines of X and their affine structure is implied by the fact that they are null-geodesic lines in M. The base manifold B is equipped with a Riemannian two-metric tensor γ_{AB}, and the degenerate metric g_{ab} on the fiber manifold is simply the pull back of γ_{AB} from B to S:

$$g = \pi^* \gamma \ .$$

The external curvature object w_a also acquires a nice interpretation, because transformation law (3.7) reduces to a pure "gauge transformation":

$$\tilde{w}_a \ = \ w_a + \partial_a \varphi \ . \tag{6.3}$$

The divergence-free vector density Λ and the gauge field w are mutually conjugate objects, describing boundary data for the gravitational field on S. Both carry two degrees of freedom. These are degrees of freedom of a black hole (or a "white hole" – depending upon a sign of s), interacting with an external gravitational field. In our opinion, analysis of the mixed "boundary value + initial value" problem for the field outside of S might give a deep insight into the problem of motion in General Relativity, where the elementary objects ("particles") are black holes. They play the same role as point particles in electrodynamics, and may be used to model the behaviour of heavy objects (like stars or galaxies) in situations, where the internal structure of the object seems to be irrelevant and only "external properties" (e.g. the total mass, the total angular momentum or the total electric charge of a particle in electrodynamics) are taken into account. In fact, information contained in the boundary data Λ and w on S plays role of these "external properties" of a black hole. Consequently, its equations of motion should be obtained by solving the Cauchy problem for the gravitational field. In particular, static situations of that type describe what is usually called "thermodynamics of black holes". These issues will be discussed in the next paper.

Let us notice, that the field w_a is also "purely static" (of course, only *modulo* gauge transformations (6.3)). Indeed, in a coordinate system compatible with the bundle structure of $\pi : S \to B$, (i. e. such that $n^A = 0$) equation (4.5) reduces to:

$$\dot{w}_B = \partial_B (w_a X^a) \ ,$$

which is a pure gauge term. Using affine coordinates in each fiber, the term $w_a X^a$ is gauged out and we obtain $w_0 = 0$, together with $\dot{w}_B = 0$.

Since the conformal structure contained in g_{AB} is always isomorphic to that of S^2, we may further "standardize" the information about geometry of the horizon S, choosing such coordinates for which $g_{AB} = f h_{AB}$ (by h_{AB} we denote the standard metric tensor on a unit sphere). There remains a three-parameter family of conformal transformation of the unit sphere (boost transformations of the black hole), preserving the above gauge condition. This freedom may be used to further specification of the gauge condition: vanishing of the dipole moment of f. Moreover, using a two-dimensional (time independent) gauge $\varphi = \varphi(x^A)$ in (6.3) (i. e. changing the affine scale in each fiber over B independently), we may kill the longitudinal degree of freedom in w_A. This means that the condition $\partial_A(\sqrt{\det h}\ h^{AB} w_B) = 0$ or, equivalently,

$$w_A = \epsilon_{AB}\ h^{BC} \partial_C \psi \ , \tag{6.4}$$

may always be imposed. Subtracting an irrelevant constant from ψ, we may annihilate its monopole moment. Hence, the entire information about the pair (Λ^a, w_a) (i. e. about "external" properties of the hole) is carried by two functions defined on the unit sphere S^2: the dipole-free function$f$ and the monopole-free function ψ, where $\Lambda^0 = f\sqrt{\det h}$, $\Lambda^A = 0$, $w_0 = 0$ and w_A is given by (6.4). The monopole part of f corresponds to the total mass of the hole and the dipole part of ψ describes its total angular momentum.

In dynamical situations, it is often necessary to relax the above gauge conditions and to allow Λ^a and w_a to be time-dependent, to contain non-vanishing longitudinal part etc. But the gauge-invariant information about the black hole may always be retrieved from two functions: $\det g_{AB}$ and $\epsilon^{AB} \partial_A w_B$.

Appendix

Proof of identity (2.23)

To prove identity (2.23) consider matrix $A^a{}_{\tilde{b}} := \frac{\partial x^a}{\partial \tilde{x}^{\tilde{b}}}$ and its inverse $B^{\tilde{b}}{}_c := \frac{\partial \tilde{x}^{\tilde{b}}}{\partial x^a}$. Put $N^0 = 0$ and $N^A = n^A$. Define the following matrices:

$$C^a{}_{\tilde{b}} := A^a{}_{\tilde{b}} + N^a A^0{}_{\tilde{b}} \ ,$$

and

$$D^{\tilde{b}}{}_c := B^{\tilde{b}}{}_c - B^{\tilde{b}}{}_a N^a \delta^0_c \ .$$

It is easy to check that C and D are mutually inverse. On the other hand, determinants of A and C are equal. Hence:

$$\det A = \det C = \det(C^A{}_{\tilde{B}})(D^{\tilde{0}}{}_0)^{-1} \ ,$$

which ends the proof.

Proof of Gauss-Codazzi constraint

For any tensorial expression $A_{\mu\nu}$, formulae (2.7) and (2.8) imply the following decomposition:

$$g^{\mu\nu} A_{\mu\nu} = \tilde{\tilde{g}}^{AB} A_{AB} + X^c A_c{}^0 + X^c A^0{}_c + \frac{1}{N^2} X^c X^d A_{cd} \ . \tag{6.1}$$

Hence,

$$g^{\kappa\lambda} R_{\kappa a \lambda b} = \tilde{\tilde{g}}^{AB} R_{AaBb} + X^c R^0{}_{acb} + X^c R_{ca}{}^0{}_b + \frac{1}{N^2} X^c X^d R_{cadb} \, , \tag{6.2}$$

and due to symmetries of the Riemann tensor:

$$R_{ab}X^a = \tilde{\tilde{g}}^{AB} R_{AaBb} X^a + X^c R^0{}_{acb} X^a \ . \tag{6.3}$$

Scalar constraint (4.1) is obtained by further contraction of (6.3) with X:

$$R_{ab}X^a X^b = \tilde{\tilde{g}}^{AB} R_{AaBb} X^a X^b \ . \tag{6.4}$$

Riemann tensor may be written in the following way:

$$\begin{aligned} R_{AaBb} &= \frac{1}{2}\left(g_{bA,aB} + g_{aB,bA} - g_{AB,ab} - g_{ab,AB}\right) \\ &\quad + g_{\mu\nu}\left(\Gamma^\mu_{aB}\Gamma^\nu_{Ab} - \Gamma^\mu_{ab}\Gamma^\nu_{AB}\right) \ . \end{aligned} \tag{6.5}$$

Using identity (6.1) we obtain:

$$g_{\mu\nu}\Gamma^\mu_{aB}\Gamma^\nu_{Ab} X^a X^b = \tilde{\tilde{g}}^{ED} \Gamma_{EAb} X^b \Gamma_{DBa} X^a \ , \tag{6.6}$$

and, consequently:

$$\begin{aligned} & g_{\mu\nu}\left(\Gamma^\mu_{aB}\Gamma^\nu_{Ab} - \Gamma^\mu_{ab}\Gamma^\nu_{AB}\right) X^a X^b \\ &= \tilde{\tilde{g}}^{ED}\left(\Gamma_{EAb}\Gamma_{DBa} - \Gamma_{DAB}\Gamma_{Eab}\right) X^a X^b + w_a X^a l_{AB} \ . \end{aligned} \tag{6.7}$$

Moreover, one can check the following "internal" identity on S:

$$\begin{aligned} & \tilde{\tilde{g}}^{AB}\left[\frac{1}{2}\left(g_{bA,aB} + g_{aB,bA} - g_{AB,ab} - g_{ab,AB}\right) + \tilde{\tilde{g}}^{ED}\left(\Gamma_{EAb}\Gamma_{DBa} - \Gamma_{DAB}\Gamma_{Eab}\right)\right] X^a X^b \\ &= X^a l_{,a} - \tilde{\tilde{g}}^{ED} l_{EA} l_{DB} \tilde{\tilde{g}}^{AB} \ . \end{aligned} \tag{6.8}$$

Consequently, expression (6.4) reads:

$$\tilde{\tilde{g}}^{AB} R_{AaBd} X^a X^d = w_a X^a \cdot l + X^a l_{,a} - \tilde{\tilde{g}}^{ED} \tilde{\tilde{g}}^{AB} l_{EA} l_{DB} \ . \tag{6.9}$$

Substituting the above into (4.1), we obtain (4.3).

To prove formula (4.2), we use again (6.3) in the following configuration:

$$R^\lambda{}_{a\lambda B} X^a = \tilde{\tilde{g}}^{AC} R_{CaAB} X^a + R^0{}_{acB} X^c X^a \ , \tag{6.10}$$

and use expansion (6.5). This gives us:

$$\begin{aligned} R_{AaBD} X^a &= \frac{1}{2}\left(g_{AD,Ba} - g_{AB,Da} + g_{aB,DA} - g_{aD,AB}\right) X^a \\ &\quad + \tilde{\tilde{g}}^{EF}\left(\Gamma_{FBa}\Gamma_{EAD} - \Gamma_{FDa}\Gamma_{EAB}\right) X^a + w_D l_{AB} - w_B l_{AD} \, . \end{aligned} \tag{6.11}$$

Similarly as in (6.8), we obtain

$$\begin{aligned} & \frac{1}{2}\left(g_{AD,Ba} - g_{AB,Da} + g_{aB,DA} - g_{aD,AB}\right) X^a \\ & + \tilde{\tilde{g}}^{EF} X^a \left(\Gamma_{FBa}\Gamma_{EAD} - \Gamma_{FDa}\Gamma_{EAB}\right) = l_{AB\|D} - l_{AD\|B} \ , \end{aligned} \tag{6.12}$$

which implies:

$$R_{AaBD}X^a = l_{AB\|D} - l_{AD\|B} + w_D l_{AB} - w_B l_{AD} \ . \tag{6.13}$$

Similarly, the second term in expression (6.10) may be rewritten as:

$$\begin{aligned} R^0{}_{abD}X^aX^b &= \left(\Gamma^0_{aD,b} - \Gamma^0_{ab,D} + \Gamma^0_{\mu b}\Gamma^\mu_{aD} - \Gamma^0_{\mu D}\Gamma^\mu_{ab}\right) X^aX^b \\ &= \left(w_{a,D} - w_{D,a}\right) X^a + w_A \tilde{\tilde{g}}^{AB} l_{BD} \ . \end{aligned} \tag{6.14}$$

Substituting this expression into (4.2), we obtain (4.5).

Proof of identity (5.28)

To prove (5.28) or, equivalently, (5.29) and (5.30), observe that for $\mathbf{G}^a{}_b = s\lambda X^a[w_b]$ we have:

$$\mathbf{G}_{AB} = 0 \quad \mathbf{G}^0{}_B = s\lambda[w_B] \ .$$

This implies that (5.31) is automatically fulfilled. To prove (5.29), we observe that:

$$\begin{aligned} 0 = s\overline{\nabla}_a\mathbf{G}^a{}_B &= \partial_0\left(\lambda[w_B]\right) - \left(\lambda n^A[w_B]\right)_{||A} - \lambda[w_A]n^A{}_{||B} \\ &= \lambda\left\{-l[w_B] + \partial_0[w_B] - n^A[w_B]_{||A} - [w_A]n^A{}_{||B}\right\} \end{aligned} \tag{6.15}$$

is equivalent to the jump of (4.5) between the two sides of S. On the other hand, the jump of (4.3) vanishes identically, which ends the proof.

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