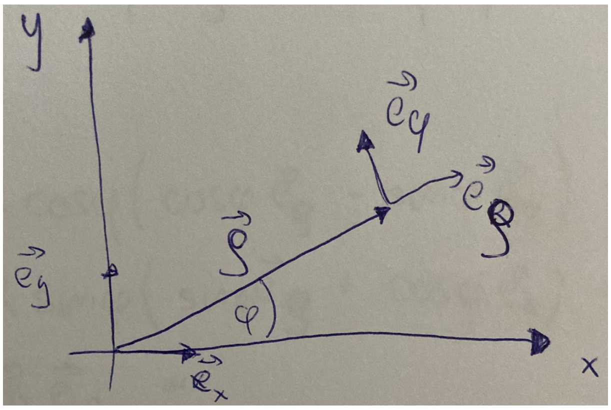
Układ współrzędnych biegunowych:

$\vec{\rho} = x\vec{e}_x + y\vec{e}_y = \rho\cos\varphi\vec{e}_x + \rho\sin\varphi\vec{e}_y$
$= [\rho\cos\varphi, \rho\sin\varphi] = A_\rho\vec{e}_\rho + A_\varphi\vec{e}_\varphi$



Wyznaczamy A_ρ i A_φ:

$\vec{e}_\rho = (\vec{e}_\rho \cdot \vec{e}_x)\,\vec{e}_x + (\vec{e}_\rho \cdot \vec{e}_y)\,\vec{e}_y =$
$= \cos\varphi\vec{e}_x + \cos(90^\circ - \varphi) \cdot \vec{e}_y =$
$= \cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y$

$\vec{e}_\varphi = (\vec{e}_\varphi \cdot \vec{e}_x)\,\vec{e}_x + (\vec{e}_\varphi \cdot \vec{e}_y)\,\vec{e}_y =$
$= \cos(90^\circ + \varphi)\,\vec{e}_x + \cos\varphi \cdot \vec{e}_y =$
$= -\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y$

$\vec{\rho} = \rho\cos\varphi\vec{e}_x + \rho\sin\varphi\vec{e}_y$

$$\begin{cases} \vec{e}_\rho = \cos\varphi\vec{e}_x + \sin\varphi\vec{e}_y \\ \vec{e}_\varphi = -\sin\varphi\vec{e}_x + \cos\varphi\vec{e}_y \end{cases}$$

$\cos\varphi\vec{e}_\rho = (\cos\varphi)^2\vec{e}_x + \sin\varphi \cdot \cos\varphi\vec{e}_y$
$-\sin\varphi\vec{e}_\varphi = (\sin\varphi)^2\vec{e}_x - \sin\varphi \cdot \cos\varphi\vec{e}_y$

$$\begin{cases} \vec{e}_x = \cos\varphi\vec{e}_\rho - \sin\varphi\vec{e}_\varphi \\ \vec{e}_y = \sin\varphi\vec{e}_\rho + \cos\varphi\vec{e}_\varphi \end{cases}$$

Ostatecrmie:

$$\begin{aligned}\vec{\rho} = \rho\cos\varphi\,(\cos\varphi\vec{e}_\rho - \sin\varphi\vec{e}_\varphi) + \\ + \rho\sin\varphi\,(\sin\varphi\vec{e}_\rho + \cos\varphi\vec{e}_\varphi) = \\ = \rho\vec{e}_\rho\end{aligned}$$

Prędkość i przyspieszenie we współrzędnych biegunowych:

$\vec{\rho} = x\vec{e}_x + y\vec{e}_y$

$\vec{v} = \frac{d\vec{\rho}}{dt} = \frac{dx}{dt}\vec{e}_x + \frac{dy}{dt}\vec{e}_y$

$\begin{cases} x = \rho\cos\varphi \\ y = \rho\sin\varphi \end{cases}$

Z powyższego liczymy pochodne:

$\begin{cases} \frac{dx}{dt} = \frac{d\rho}{dt}\cos\varphi - \rho\sin\varphi\cdot\frac{d\varphi}{dt} \\ \frac{dy}{dt} = \frac{d\rho}{dt}\sin\varphi + \rho\cos\varphi\cdot\frac{d\varphi}{dt} \end{cases}$

$\vec{v} = \left(\frac{d\rho}{dt}\cos\varphi - \rho\sin\varphi\frac{dy}{dt}\right)\vec{e}_x + \left(\frac{d\rho}{dt}\sin\varphi + \rho\cos\varphi\frac{d\varphi}{dt}\right)\vec{e}_y =$
$= \left(\frac{d\rho}{dt}\cos\varphi - \rho\sin\varphi\frac{d\varphi}{dt}\right)\cdot(\cos\varphi\vec{e}_\rho - \sin q\vec{e}_\varphi) +$
$+ \left(\frac{d\rho}{dt}\sin\varphi + \rho\cos\varphi\frac{d\varphi}{dt}\right)\cdot(\sin\varphi\vec{e}_\rho + \cos\varphi\vec{e}_\varphi) =$
$= \frac{d\rho}{dt}\vec{e}_\rho + \rho\frac{d\varphi}{dt}\vec{e}_\varphi$

$\boxed{\vec{v} = \frac{d\rho}{dt}\vec{e}_\rho + \rho\frac{d\varphi}{dt}\vec{e}_\varphi}$

Znajdujemy pochodne wersorów:

$\vec{\rho} = \rho\vec{e}_\rho$

$\vec{v} = \frac{d\vec{\rho}}{dt} = \frac{d\rho}{dt}\vec{e}_\rho + \rho\frac{d\vec{e}_\rho}{dt}, \qquad \rho\frac{d\vec{e}_\rho}{dt} = \rho\frac{d\varphi}{dt}\vec{e}_\varphi$

$\boxed{\frac{d\vec{e}_\rho}{dt} = \frac{d\varphi}{dt}\vec{e}_\varphi}$

Nalezy amaleźć jeszcze:

$$\frac{d\vec{e}_\varphi}{dt} = ?$$

$\vec{0} = \frac{d\vec{e}_x}{dt} =$
$= -\sin\varphi\cdot\frac{d\varphi}{dt}\vec{e}_\rho + \cos\varphi\frac{d\vec{e}_\rho}{dt} +$
$- \cos\varphi\cdot\frac{d\varphi}{dt}\vec{e}_\varphi - \sin\varphi\frac{d\vec{e}_\varphi}{dt} =$
$= -\sin\varphi\frac{d\varphi}{dt}\vec{e}_\rho + \cos\varphi\frac{d\varphi}{dt}\vec{e}_\varphi +$

$$
\begin{aligned}
&-\cos\varphi\tfrac{d\varphi}{dt}\vec{e}_\varphi - \sin\varphi\tfrac{d\vec{e}_\varphi}{dt} = \\
&= -\sin\varphi\tfrac{d\varphi}{dt}\vec{e}_\rho - \sin\varphi\tfrac{d\vec{e}_\varphi}{dt} = \\
&= -\sin\varphi\cdot\left(\tfrac{d\varphi}{dt}\vec{e}_\rho + \tfrac{d\vec{e}_\varphi}{dt}\right)
\end{aligned}
$$

$$\boxed{\tfrac{d\vec{e}_\varphi}{dt} = -\tfrac{d\varphi}{dt}\vec{e}_\rho}$$

Przyspieszenie:

$$
\begin{aligned}
&\vec{a} = \tfrac{d\vec{v}}{dt} = \tfrac{d^2\vec{\rho}}{dt^2} = \tfrac{d}{dt}\left(\tfrac{d\rho}{dt}\vec{e}_\rho + \rho\tfrac{d\vec{e}_\rho}{dt}\right) = \\
&= \tfrac{d^2\rho}{dt^2}\vec{e}_\rho + \tfrac{d\rho}{dt}\cdot\tfrac{d\vec{e}_\rho}{dt} + \tfrac{d\rho}{dt}\cdot\tfrac{d\vec{e}_\rho}{dt} + \rho\tfrac{d^2\vec{e}_\rho}{dt^2} = \\
&= \tfrac{d^2\rho}{dt^2}\vec{e}_\rho + 2\tfrac{d\rho}{dt}\cdot\tfrac{d\varphi}{dt}\vec{e}_\varphi + \rho\tfrac{d^2\varphi}{dt^2}\vec{e}_\varphi + \rho\tfrac{d\varphi}{dt}\left(-\tfrac{d\varphi}{dt}\vec{e}_\rho\right) = \\
&= \left[\tfrac{d^2\rho}{dt^2} - \rho\left(\tfrac{d\varphi}{dt}\right)^2\right]\vec{e}_\rho + \left[2\tfrac{d\rho}{dt}\cdot\tfrac{d\varphi}{dt} + \rho\tfrac{d^2\varphi}{dt^2}\right]\vec{e}_\varphi
\end{aligned}
$$