On the use of evolutionary methods in spaces of Euclidean signature

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The main message:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

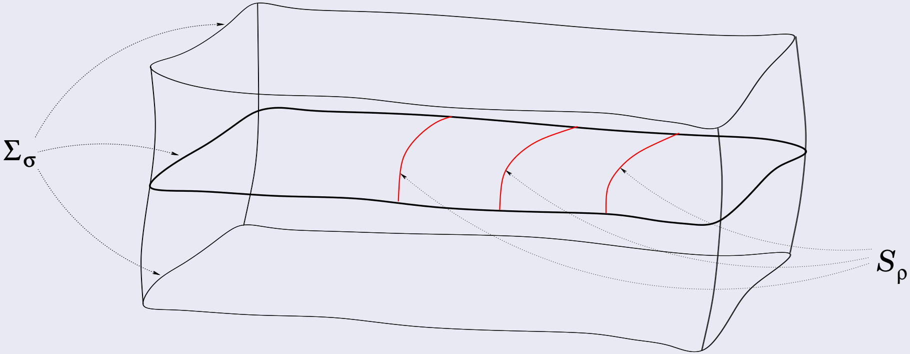
Based on some recent works:

- I. Rácz: *Is the Bianchi identity always hyperbolic?*, CQG **31** 155004 (2014)
- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, Class. Quantum Grav. **32** (2015) 015006
- I. Rácz: *Dynamical determination of the gravitational degrees of freedom*, arXiv:1412.0667 (2015)
- I. Rácz: *Constraints as evolutionary systems*, CQG **33** 015014 (2016)
- I. Rácz and J. Winicour: *Black hole initial data without elliptic equations*, Phys. Rev. D **91**, 124013 (2015)
- I. Rácz: *A simple method of constructing binary black hole initial data*, Astronomy Reports 62 953-958 (2018)
- I. Rácz: *On the ADM charges of multiple black holes*, arXiv:1608.02283
- I. Rácz and J. Winicour: *Toward computing gravitational initial data without elliptic solvers*, CQG **35** 135002 (2018)
- K. Csukás and I. Rácz: *On the asymptotics of solutions to the evolutionary form of the constraints*, to be submitted for publication (2019)

All the involved results are valid for arbitrary dimension: i.e. for $dim(M) = n\ (\geq 4)$. Nevertheless, for the sake of simplicity attention will be restricted to the case of $n = 4$.

Outline:

- **Einsteinian spaces:** (M, g_{ab})
  - First part
  - Second part



- in both cases metrics of Euclidean signature will be involved
- no gauge condition

  ... arbitrary choice of foliations & "evolutionary" vector field

The basic setup:

- **Einsteinian spaces:** (M, g_{ab})
  - M : 4-dimensional, smooth, paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian$_{(-,+,+,+)}$ or Riemannian$_{(+,+,+,+)}$ metric
- **Einstein's equations:**

$$\boxed{G_{ab} - \mathscr{G}_{ab} = 0} \quad \text{with source term:} \quad \boxed{\nabla^a \mathscr{G}_{ab} = 0}$$

  - $\boxed{\nabla_a}$ denotes the covariant derivative operator associated with g_{ab}.
  - in a more familiar setup: **Einstein's equations** with cosmological constant Λ

$$\boxed{[R_{ab} - \tfrac{1}{2}\, g_{ab}\, R] + \Lambda\, g_{ab} = 8\pi\, T_{ab}}$$

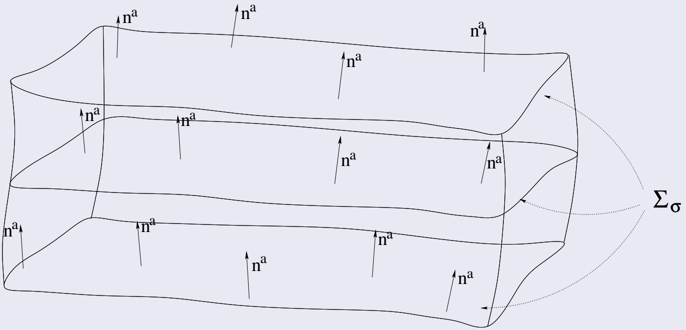
    with matter fields satisfying their Euler-Lagrange equations
  -

$$\boxed{\mathscr{G}_{ab} = 8\pi\, T_{ab} - \Lambda\, g_{ab}}$$

PART I:

The primary splitting

- **Assume:** M is smoothly foliated by a one-parameter family of homologous hypersurfaces, i.e. $M \simeq \mathbb{R} \times \Sigma$, for some three-dimensional manifold Σ.
  - known to **hold for globally hyperbolic spacetimes** (Lorentzian case)
  - **equivalent to** the existence of a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing gradient $\partial_a \sigma$ such that the $\sigma = const$ level surfaces $\Sigma_\sigma = \{\sigma\} \times \Sigma$ comprise the one-parameter foliation of M.
  - $$n_a \sim \partial_a \sigma \ \ldots \ \& \ldots \ g^{ab} \ \longrightarrow \ n^a = g^{ab} n_b$$



Projections:

The projection operator:

- n^a the 'unit norm' vector field that is normal to the Σ_σ level surfaces

$$n^a n_a = \epsilon$$

  - the sign is not fixed: ϵ takes the value -1 or $+1$ for Lorentzian or Riemannian metric g_{ab}, respectively

- **the projection operator**

$$h^a{}_b = \delta^a{}_b - \epsilon\, n^a n_b$$

  to the level surfaces of $\sigma : M \to \mathbb{R}$.

- **the induced metric** on the $\sigma = const$ level surfaces

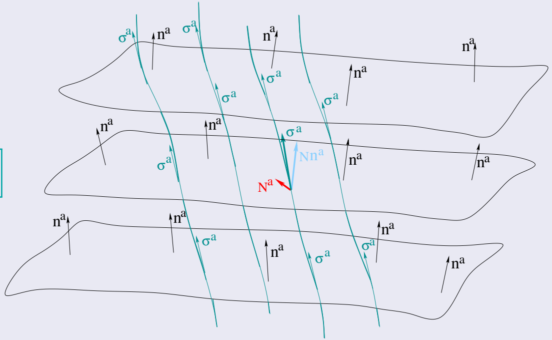
$$h_{ab} = h^e{}_a h^f{}_b\, g_{ef}$$

  - D_a denotes the covariant derivative operator associated with h_{ab}.

σ^a is "time evolution vector field" **if**:

- the integral curves of σ^a meet the $\sigma = const$ level surfaces precisely once
- $\sigma^e \nabla_e \sigma = 1$

$$\sigma^a = \sigma^a_\perp + \sigma^a_\parallel = N\, n^a + N^a$$



- where N and N^a denotes the **lapse** and **shift** of σ^a:

$$N = \epsilon\, (\sigma^e n_e) \quad \text{and} \quad N^a = h^a{}_e\, \sigma^e$$

Decompositions of various fields:

Any symmetric tensor field P_{ab} can be decomposed

in terms of n^a and fields intrinsic to the individual $\sigma = const$ level surfaces as

$$P_{ab} = \boldsymbol{\pi}\, n_a n_b + [n_a\, \mathbf{p}_b + n_b\, \mathbf{p}_a] + \mathbf{P}_{ab}$$

where $$\boldsymbol{\pi} = n^e n^f\, P_{ef}\,, \quad \mathbf{p}_a = \epsilon\, h^e{}_a n^f\, P_{ef}\,, \quad \mathbf{P}_{ab} = h^e{}_a h^f{}_b\, P_{ef}$$

It is also rewarding to inspect the decomposition of the cov. divergence $\nabla^a P_{ab}$:

$$\epsilon\, (\nabla^a P_{ae})\, n^e = \mathscr{L}_n \boldsymbol{\pi} + D^e \mathbf{p}_e + [\boldsymbol{\pi}\, (K^e{}_e) - \epsilon\, \mathbf{P}_{ef} K^{ef} - 2\, \epsilon\, \dot{n}^e \mathbf{p}_e]$$
$$(\nabla^a P_{ae})\, h^e{}_b = \mathscr{L}_n \mathbf{p}_b + D^e \mathbf{P}_{eb} + [(K^e{}_e)\, \mathbf{p}_b + \dot{n}_b\, \boldsymbol{\pi} - \epsilon\, \dot{n}^e \mathbf{P}_{eb}]$$

$$K_{ab} = h^e{}_a \nabla_e n_b = \tfrac{1}{2}\, \mathscr{L}_n h_{ab}$$ $$\dot{n}_a := n^e \nabla_e n_a = -\epsilon\, D_a \ln N$$

back

Decompositions of various fields:

Examples:

- the metric

$$g_{ab} = \epsilon\, n_a n_b + h_{ab}$$

- the "source term"

$$\mathscr{G}_{ab} = n_a n_b\, \mathfrak{e} + [n_a\, \mathfrak{p}_b + n_b\, \mathfrak{p}_a] + \mathfrak{S}_{ab}$$

where $\mathfrak{e} = n^e n^f\, \mathscr{G}_{ef}, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f\, \mathscr{G}_{ef}, \quad \mathfrak{S}_{ab} = h^e{}_a h^f{}_b\, \mathscr{G}_{ef}$

- l.h.s. of Einstein's equation: $E_{ab} = G_{ab} - \mathscr{G}_{ab}$

$$E_{ab} = n_a n_b\, E^{(\mathcal{H})} + [n_a\, E^{(\mathcal{M})}_b + n_b\, E^{(\mathcal{M})}_a] + (E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})$$

$$E^{(\mathcal{H})} = n^e n^f\, E_{ef}, \quad E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f\, E_{ef}, \quad E^{(\mathcal{EVOL})}_{ab} = h^e{}_a h^f{}_b\, E_{ef} - h_{ab}\, E^{(\mathcal{H})}$$

The decomposition of the covariant divergence $\nabla^a E_{ab} = 0$ of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$:

$$\mathscr{L}_n E^{(\mathcal{H})} + D^e E^{(\mathcal{M})}_e + [\, E^{(\mathcal{H})} (K^e{}_e) - 2\,\epsilon\,(\dot{n}^e E^{(\mathcal{M})}_e) - \epsilon\, K^{ae} (E^{(\mathcal{EVOL})}_{ae} + h_{ae} E^{(\mathcal{H})})\,] = 0$$

Div

$$\mathscr{L}_n E^{(\mathcal{M})}_b + D^a (E^{(\mathcal{EVOL})}_{ab} + h_{ab} E^{(\mathcal{H})}) + [\,(K^e{}_e) E^{(\mathcal{M})}_b + E^{(\mathcal{H})}\,\dot{n}_b - \epsilon\,(E^{(\mathcal{EVOL})}_{ab} + h_{ab} E^{(\mathcal{H})})\,\dot{n}^a\,] = 0$$

1st order symmetric hyperbolic system: linear and homogeneous in $(E^{(\mathcal{H})}, E^{(\mathcal{M})}_I)^T$:

- $N \times$ "(1)" and $N h^{ij} \times$ "(2)" in local coordinates (σ, x^1, x^2, x^3) adopted to an arbitrary flow field $\sigma^a = N\, n^a + N^a\!: \quad \sigma^e \nabla_e \sigma = 1$ and the foliation $\{\Sigma_\sigma\}$, read as

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & h^{IJ} \end{pmatrix} \partial_\sigma + \begin{pmatrix} -N^K & N\, h^{IK} \\ N\, h^{JK} & -N^K\, h^{IJ} \end{pmatrix} \partial_K \right\} \begin{pmatrix} E^{(\mathcal{H})} \\ E^{(\mathcal{M})}_I \end{pmatrix} = \begin{pmatrix} \mathscr{E} \\ \mathscr{E}^J \end{pmatrix}$$

- !!! the source terms $\mathscr{E}$ and $\mathscr{E}^J$ are linear and homogeneous in $E^{(\mathcal{H})}$ and $E^{(\mathcal{M})}_I$!!! ϵ

$$\mathcal{A}^\mu\, \partial_\mu v + \mathcal{B}\, v = 0 \quad \text{with} \quad v = (E^{(\mathcal{H})}, E^{(\mathcal{M})}_I)^T \qquad FOSH \;!!!\; v \equiv 0$$

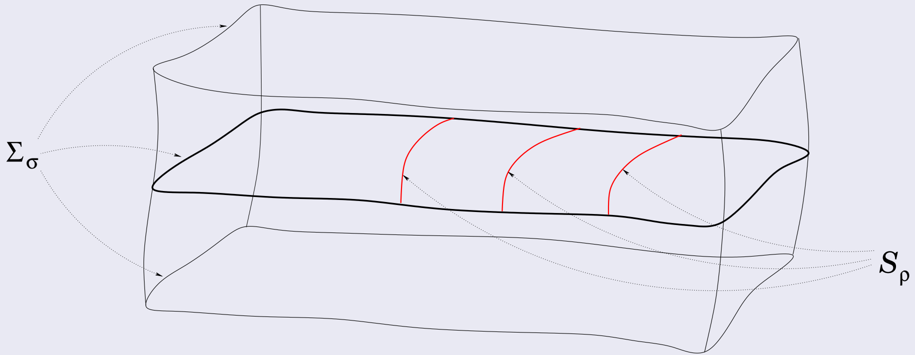
The main result of the first part:

Theorem

Let (M, g_{ab}) be an Einsteinian space as specified and assume that the metric h_{ab} induced on the $\sigma = const$ level surfaces is Riemannian. Then, **regardless whether g_{ab} is of Lorentzian or Euclidean signature**, *any solution to the reduced equations $E^{(\mathcal{EVOL})}_{ab} = 0$ is also a solution to the full set of field equations $G_{ab} - \mathscr{G}_{ab} = 0$ provided that the constraint expressions $E^{(\mathcal{H})}$ and $E^{(\mathcal{M})}_{a}$ vanish on one of the $\sigma = const$ level surfaces.*

- no gauge condition was used anywhere in the above analyze !
  - it applies regardless of the choice of the foliation, $\boxed{\Sigma_\sigma}$, of M and for any choice of the flow field, $\boxed{\sigma^a \rightleftarrows N, N^a}$

PART II:



The explicit form of the constraints:

The constraint expressions are projections of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$:

$$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\, {}^{(3)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0$$

$$E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\,[D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a] = 0$$

- where D_a denotes the covariant derivative operator associated with h_{ab} and

$$\mathfrak{e} = n^e n^f \mathscr{G}_{ef}, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f \mathscr{G}_{ef}$$

- it is an underdetermined system: 4 equations for 12 variables

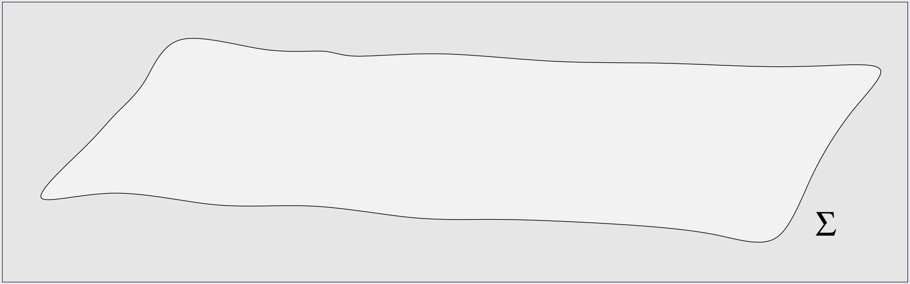
$$(h_{ij}, K_{ij})$$

A simple example:

Consider the underdetermined equation on $\Sigma \approx \mathbb{R}^2$ with some coordinates (χ, ξ)

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

- it is an equation for the four variables u, v, w and z on Σ
- in advance of solving it three of these variables have to be fixed on Σ

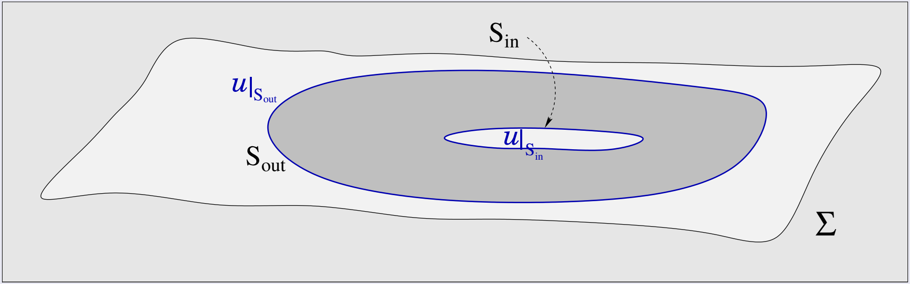


A simple example:

It is an elliptic equation for u on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

- in solving this equation the variables v, w and z have to be specified on $\mathbb{R}^2$
- the variable u has also to be fixed at the boundaries S_{out} and S_{in}

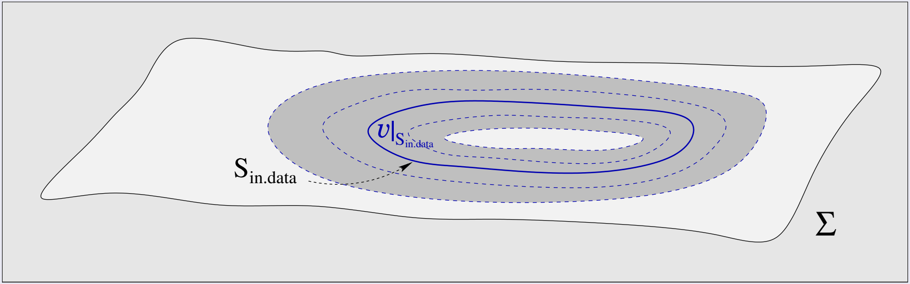


A simple example:

It is a hyperbolic equation for v on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}$, w and $\boldsymbol{z}$ have to be specified on $\mathbb{R}^2$
- the variable $\boldsymbol{v}$ has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$

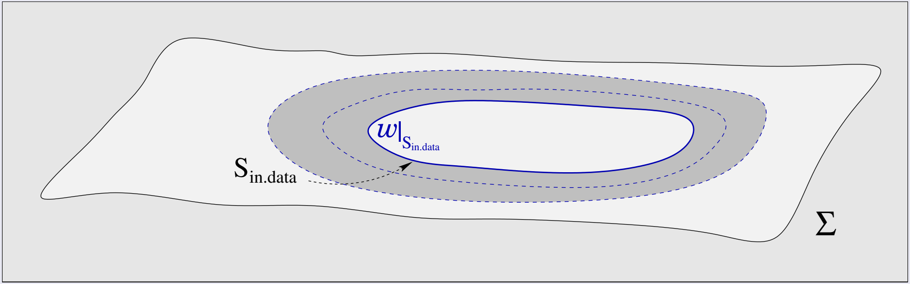


A simple example:

It is a parabolic equation for w on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ have to be fixed on $\mathbb{R}^2$: $a > 0$
- the variable w has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$

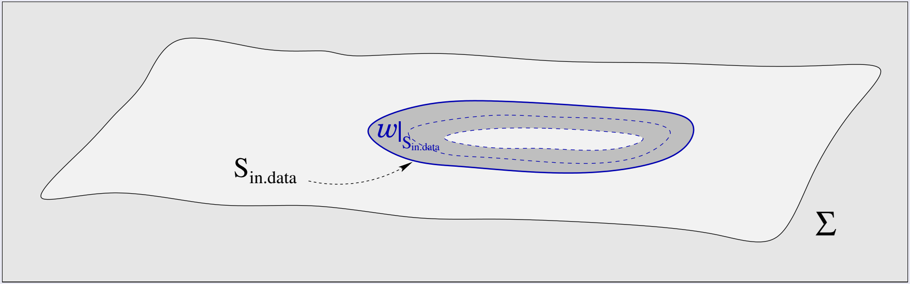


A simple example:

It is a parabolic equation for w on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ have to be fixed on $\mathbb{R}^2$: $\boxed{a < 0}$
- the variable w has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$



A simple example:

It is an algebraic equation for $\boldsymbol{z}$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi^2 - \partial_\xi^2)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- once the variables u, v, w are specified on $\mathbb{R}^2$ the solution is determined as

$$\boldsymbol{z} = -\left[(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi^2 - \partial_\xi^2)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w\right]$$

New variables by applying $2+1$ decompositions:

Splitting of the metric h_{ij}:

assume: $\Sigma \approx \mathbb{R} \times \mathscr{S}$

Σ is smoothly foliated by a one-parameter family of two-surfaces $\mathscr{S}_\rho$:
$\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

$$\Longrightarrow \quad \widehat{n}_i \sim \partial_i \rho \;\ldots\; \& \ldots \; h^{ij} \;\longrightarrow\; \widehat{n}^i = h^{ij}\widehat{n}_j \;\longrightarrow\; \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i\widehat{n}_j$$

- choose ρ^i to be a flow field on Σ: the integral curves... & $\rho^i\partial_i\rho = 1$
- 'lapse' and 'shift' of ρ^i

$$\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i\,, \quad \text{where} \quad \widehat{N} = \rho^j\widehat{n}_j \quad \text{and} \quad \widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j$$

- induced metric, extrinsic curvature and acceleration of the $\mathscr{S}_\rho$ level surfaces:

$$\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i\,\widehat{\gamma}^l{}_j\,h_{kl} \qquad \widehat{K}_{ij} = \tfrac{1}{2}\,\mathscr{L}_{\widehat{n}}\widehat{\gamma}_{ij} \qquad \dot{\widehat{n}}_i := \widehat{n}^l D_l \widehat{n}_i = -\widehat{D}_i \ln \widehat{N}$$

- the metric h_{ij} can then be given as

$$h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i\widehat{n}_j \quad \Longleftrightarrow \quad \{\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}\}$$

$2+1$ decompositions:

Splitting of the symmetric tensor field K_{ij}:

- $$K_{ij} = \boldsymbol{\kappa}\, \widehat{n}_i \widehat{n}_j + [\widehat{n}_i\, \mathbf{k}_j + \widehat{n}_j\, \mathbf{k}_i] + \mathbf{K}_{ij}$$

  where

  $$\boldsymbol{\kappa} = \widehat{n}^k \widehat{n}^l\, K_{kl}\,, \quad \mathbf{k}_i = \widehat{\gamma}^k{}_i \widehat{n}^l\, K_{kl} \quad \text{and} \quad \mathbf{K}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j\, K_{kl}$$

- the **trace** and **trace free** parts of $\mathbf{K}_{ij}$

  $$\mathbf{K}^l{}_l = \widehat{\gamma}^{kl}\, \mathbf{K}_{kl} \quad \text{and} \quad \mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{2}\, \widehat{\gamma}_{ij} \mathbf{K}^l{}_l$$

The new variables:

- $$(h_{ij}, K_{ij}) \quad \Longleftrightarrow \quad (\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$$

- these variables retain the physically distinguished nature of h_{ij} and K_{ij}

The momentum constraint:

$\dot{\hat{n}}_i := \hat{n}^l D_l \hat{n}_i = -\widehat{D}_i \ln \widehat{N}$

$$D_e K^e{}_a - D_a K^e{}_e - \epsilon\, \mathfrak{p}_a = 0$$

$\widehat{K}_{ij} = \frac{1}{2}\mathscr{L}_{\hat{n}}\hat{\gamma}_{ij};\ \widehat{K}^l{}_l = \hat{\gamma}^{ij}\widehat{K}_{ij}$

$$\mathscr{L}_{\hat{n}}\mathbf{k}_i - \tfrac{1}{2}\widehat{D}_i(\mathbf{K}^l{}_l) - \widehat{D}_i\boldsymbol{\kappa} + \widehat{D}^l\mathring{\mathbf{K}}_{li} + (\widehat{K}^l{}_l)\,\mathbf{k}_i + \boldsymbol{\kappa}\,\dot{\hat{n}}_i - \dot{\hat{n}}^l\,\mathbf{K}_{li} - \epsilon\,\mathfrak{p}_l\,\hat{\gamma}^l{}_i = 0$$

back: str.hyp.sys.

$$\mathscr{L}_{\hat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l - \boldsymbol{\kappa}\,(\widehat{K}^l{}_l) + \mathbf{K}_{kl}\widehat{K}^{kl} + 2\,\dot{\hat{n}}^l\,\mathbf{k}_l + \epsilon\,\mathfrak{p}_l\,\hat{n}^l = 0$$

First order symmetric hyperbolic system:

- contract "(1)" with $2\,\widehat{N}\,\hat{\gamma}^{ij}$ and mult. "(2)" by $\widehat{N}$, when writing them out in coordinates (ρ, x^2, x^3), adopted to the foliation $\mathscr{S}_\rho$ and the vector field ρ^i,

$$\left\{\begin{pmatrix} 2\,\hat{\gamma}^{AB} & 0 \\ 0 & 1 \end{pmatrix}\partial_\rho + \begin{pmatrix} -2\,\widehat{N}^K\,\hat{\gamma}^{AB} & -\widehat{N}\,\hat{\gamma}^{AK} \\ -\widehat{N}\,\hat{\gamma}^{BK} & -\widehat{N}^K \end{pmatrix}\partial_K\right\}\begin{pmatrix} \mathbf{k}_B \\ \mathbf{K}^E{}_E \end{pmatrix} + \begin{pmatrix} \mathscr{B}^{A}_{(\mathbf{k})} \\ \mathscr{B}_{(\mathbf{K})} \end{pmatrix} = 0$$

- a first order **symmetric hyperbolic** system for the vector valued variable

$$(\mathbf{k}_B, \mathbf{K}^E{}_E)^T$$

!!! ρ **plays the role of 'time'** regardless of the value of $\epsilon = \pm 1$

The Hamiltonian constraint:

The Hamiltonian constraint in terms of the new variables:

- $$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\,{}^{(3)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0$$
- using $$ {}^{(3)}\!R = \widehat{R} - \left\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\right\}$$

  $\widehat{R}$ and $\widehat{K}_{kl}$ denote the scalar and extrinsic curvature of $\widehat{\gamma}_{kl}$, respectively
- $$-\epsilon\,\widehat{R} + \epsilon\left\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\,\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\right\} + 2\,\boldsymbol{\kappa}\,\mathbf{K}^l{}_l + \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e} = 0$$

Alternative choices yielding evolutionary systems:

- it is a **parabolic equation** for $\widehat{N}$ (the sign of $\widehat{K}^l{}_l$ plays a role)
- it is an **algebraic equation** for $\boldsymbol{\kappa}$ (what is if $\mathbf{K}^l{}_l$ vanishes somewhere?)

The parabolic-hyperbolic system:

The Hamiltonian constraint as a parabolic equation for $\widehat{N}$:

$$-\epsilon\,\widehat{R} + \epsilon\left\{2\,\boxed{\mathscr{L}_{\hat n}(\widehat{K}^l{}_l)} + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\,\widehat{K}^{kl} + 2\,\boxed{\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}}\right\} + 2\,\boldsymbol{\kappa}\,\mathbf{K}^l{}_l + \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e} = 0$$

- $\widehat{K}^l{}_l = \widehat{\gamma}^{ij}\,\widehat{K}_{ij} = \widehat{N}^{-1}[\,\tfrac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_j\widehat{N}^j\,] = \widehat{N}^{-1}\overset{\star}{K}$ as $\widehat{n}^i = \widehat{N}^{-1}[\,\rho^i - \widehat{N}^i\,]$
- $\mathscr{L}_{\hat n}(\widehat{K}^l{}_l) = -\widehat{N}^{-3}\overset{\star}{K}\,[\,(\partial_\rho\widehat{N}) - (\widehat{N}^l\widehat{D}_l\widehat{N})\,] + \widehat{N}^{-2}[\,(\partial_\rho\overset{\star}{K}) - (\widehat{N}^l\widehat{D}_l\overset{\star}{K})\,]$
- using $$\mathcal{A} = 2\,[\,(\partial_\rho\overset{\star}{K}) - \widehat{N}^l(\widehat{D}_l\overset{\star}{K})\,] + \overset{\star}{K}{}^2 + \overset{\star}{K}_{kl}\overset{\star}{K}{}^{kl}$$ $$\mathcal{B} = -\widehat{R} + \epsilon\,[\,2\,\boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) + \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e}\,]$$
- it gets to be a **Bernoulli-type parabolic partial differential equation** provided that $\overset{\star}{K}$ ...
- $$2\,\overset{\star}{K}\,[\,(\partial_\rho\widehat{N}) - \widehat{N}^l(\widehat{D}_l\widehat{N})\,] = 2\,\widehat{N}^2(\widehat{D}^l\widehat{D}_l\widehat{N}) + \mathcal{A}\,\widehat{N} + \mathcal{B}\,\widehat{N}^3$$ & momentum constr.
- in highly specialized cases of "quasi-spherical" foliations with $\widehat{\gamma}_{ij} = r^2\,\mathring{\gamma}_{ij}$ and with time symmetric initial data $K_{ij} \equiv 0$ R. Bartnik (1993), G. Weinstein & B. Smith (2004)

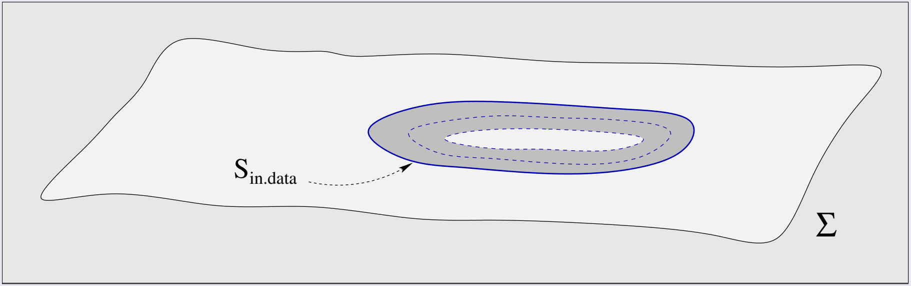
Constraints as evolutionary systems I.

The parabolic-hyperbolic system:

- (h_{ij}, K_{ij}) represented by the variables $(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$
- the constraints comprise a **parabolic-hyperbolic** system for $(\widehat{N}, \mathbf{k}_i, \mathbf{K}^l{}_l)$
  - with freely specifiable variables on Σ and on $\mathrm{S}_{\mathrm{in.data}}$:

$$(\widehat{N}|_{\mathrm{S}_{\mathrm{in.data}}}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i|_{\mathrm{S}_{\mathrm{in.data}}}, \mathbf{K}^l{}_l|_{\mathrm{S}_{\mathrm{in.data}}}, \mathring{\mathbf{K}}_{ij})$$

  - a fixed $(+/-)$ sign of $\overset{\star}{K} = \frac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_j\widehat{N}^j$ can be guaranteed



The strongly hyperbolic system:

The Hamiltonian constraint as an algebraic equation for $\boldsymbol{\kappa}$:

$$-\epsilon\,\widehat{R} + \epsilon\left\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\,\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\,\widehat{D}^l\,\widehat{D}_l\widehat{N}\right\} + 2\,\boxed{\boldsymbol{\kappa}}\,\mathbf{K}^l{}_l + \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e} = 0$$

whence $\boldsymbol{\kappa} = (2\,\mathbf{K}^l{}_l)^{-1}[\,2\,\mathbf{k}^l\mathbf{k}_l - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 - \boldsymbol{\kappa}_0\,]\,, \quad \boldsymbol{\kappa}_0 = -\epsilon\,{}^{(3)}R - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e}$

- by eliminating $\widehat{D}_i\boldsymbol{\kappa}$ from the momentum constraint [mom. constr.] one gets

$$\mathscr{L}_{\widehat{n}}\mathbf{k}_i + (\mathbf{K}^l{}_l)^{-1}[\,\boldsymbol{\kappa}\,\widehat{D}_i(\mathbf{K}^l{}_l) - 2\,\mathbf{k}^l\,\widehat{D}_i\mathbf{k}_l\,] + (2\,\mathbf{K}^l{}_l)^{-1}\,\widehat{D}_i\boldsymbol{\kappa}_0 + (\widehat{K}^l{}_l)\,\mathbf{k}_i + [\,\boldsymbol{\kappa} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)\,]\,\dot{\widehat{n}}_i - \dot{\widehat{n}}^l\,\mathring{\mathbf{K}}_{li} + \widehat{D}^l\mathring{\mathbf{K}}_{li} - \epsilon\,\mathfrak{p}_l\,\widehat{\gamma}^l{}_i = 0\,,$$

$$\mathscr{L}_{\widehat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l - \boldsymbol{\kappa}\,(\widehat{K}^l{}_l) + \mathbf{K}_{kl}\widehat{K}^{kl} + 2\,\dot{\widehat{n}}^l\,\mathbf{k}_l + \epsilon\,\mathfrak{p}_l\,\widehat{n}^l = 0$$

- the above system is a **strongly hyperbolic** one for $(\mathbf{k}_i, \mathbf{K}^l{}_l)^T$ provided that $\boldsymbol{\kappa}\cdot\mathbf{K}^l{}_l < 0$
- $\boldsymbol{\kappa}$ is determined algebraically once $\mathbf{k}_i$ and $\mathbf{K}^l{}_l$ are known !!!
- the entire three-metric $h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i\widehat{n}_j$ is freely specifiable. !!!

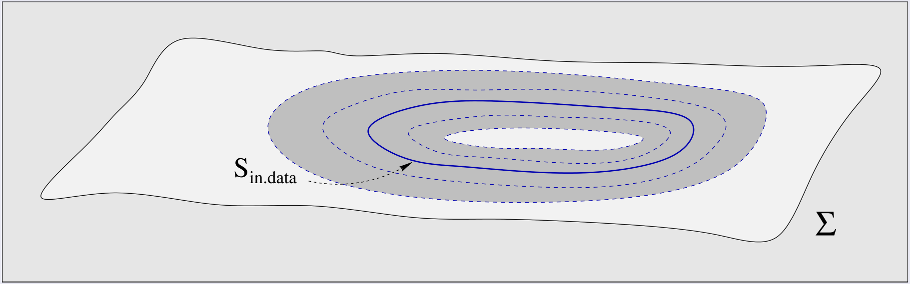
Constraints as evolutionary systems II:

The strongly hyperbolic system:

- (h_{ij}, K_{ij}) represented by the variables $(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$
- the constraints form a **strongly hyperbolic** system for $(\mathbf{k}_i, \mathbf{K}^l{}_l)$ (alg.for $\boldsymbol{\kappa}$)
  - with freely specifiable variables on Σ and on $\mathrm{S}_{\mathrm{in.data}}$:

$$(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boxed{\boldsymbol{\kappa}}, \mathbf{k}_i|_{\mathrm{S}_{\mathrm{in.data}}}, \mathbf{K}^l{}_l|_{\mathrm{S}_{\mathrm{in.data}}}, \mathring{\mathbf{K}}_{ij})$$

  - by choosing the free data properly $\boldsymbol{\kappa} \cdot \mathbf{K}^l{}_l < 0$ can be guaranteed (globally**?**)



Summary:

4-dimensional **Riemannian and Lorentzian spaces** satisfying Einstein's equations, and some mild topological assumptions, were considered. **!!!** $[n(\geq 4)]$

1. it was shown that **the constraint expressions** satisfy a **FOSH system** that is linear and homogeneous $\Longrightarrow$ (the constraints propagate)
2. concerning the constraint equations in Einstein's theory it was shown:
   - **momentum constraint** as a **first order symmetric hyperbolic system**
   - **the Hamiltonian constraint** as a **parabolic** or an **algebraic** equation
   - **in either case** the coupled constraint equations comprise **a well-posed evolutionary system:** a **parabolic-hyperbolic** or a **strongly hyperbolic**,
   - in C^∞ setting **(local) existence and uniqueness** of solutions are guaranteed
3. **!!! regardless** whether the primary space is Riemannian or Lorentzian
4. **!!! no use** of gauge conditions

The take home message:

On contrary to the folklore, in the considered two explicit examples, **evolutionary methods can be applied in spaces with metric of Euclidean signature where, in principle, there is no room for 'time'**


謝謝
Thank You

The roots of the evolutionary aspects

The first order symmetric hyperbolic system for $(E^{(\mathcal{H})}, E_i^{(\mathcal{M})})^T$

- Characteristic directions associated with the FOSH system governing the evolution of the constraint expressions are determined as

$$[\, h^{ij} - n^i n^j \,]\, \xi_i \xi_j = [\, g^{ij} - (1+\epsilon)\, n^i n^j \,]\, \xi_i \xi_j = 0$$

The momentum constraint: first order symmetric hyperbolic system

- with characteristic cone given as

$$\left[\, \widehat{\gamma}^{ij} - 2\, \widehat{n}^i \widehat{n}^j \,\right]\, \xi_i \xi_j = \left[\, h^{ij} - 3\, \widehat{n}^i \widehat{n}^j \,\right]\, \xi_i \xi_j = 0$$

Deriving a Lorentzian metric from a Riemannian one

- ... given a Riemannian metric $\mathfrak{g}_{ij}$, a unit form field $\mathfrak{n}_i$ and a positive real function α $\Longrightarrow$ a metric of Lorentzian signature can be defined as

$$\check{\mathfrak{g}}_{ij} = \mathfrak{g}_{ij} - (1+\alpha)\, \mathfrak{n}_i \mathfrak{n}_j$$

The conformal (elliptic) method:

Lichnerowicz A (1944) and York J W (1972):

- replace

$$h_{ij} = \phi^4 \, \widetilde{h}_{ij} \quad \text{and} \quad K_{ij} - \tfrac{1}{3}\, h_{ij}\, K^l{}_l = \phi^{-2}\, \widetilde{K}_{ij}$$

- using these variables the constraints are put into a **semilinear elliptic system**

$$\widetilde{D}^l \widetilde{D}_l \phi + \epsilon\, \tfrac{1}{8}\, \widetilde{R}\, \phi + \tfrac{1}{8}\, \widetilde{K}_{ij} \widetilde{K}^{ij}\, \phi^{-7} - \left[\tfrac{1}{12}\, (K^l{}_l)^2 - \tfrac{1}{4}\, \mathfrak{e}\right] \phi^5 = 0$$

where $\widetilde{D}_l$, $\widetilde{R}$, ... $\widetilde{h}_{ij}$

- $\widetilde{K}_{ij} = \widetilde{K}^{[L]}_{ij} + \widetilde{K}^{[TT]}_{ij}$, where $\widetilde{K}^{[L]}_{ij} = \left(\widetilde{D}_i X_j + \widetilde{D}_j X_i - \tfrac{2}{3}\, \widetilde{h}_{ij} \widetilde{D}^l X_l\right)$

$$\widetilde{D}^l \widetilde{D}_l X_i + \tfrac{1}{3}\, \widetilde{D}_i(\widetilde{D}^l X_l) + \widetilde{R}_i{}^l X_l - \tfrac{2}{3}\, \phi^6\, \widetilde{D}_i(K^l{}_l) + \epsilon\, \phi^{10} \mathfrak{p}_i = 0$$

- $(h_{ij}, K_{ij}) \quad \longleftrightarrow \quad \left(\phi, \widetilde{h}_{ij}; K^l{}_l, X_i, \widetilde{K}^{[TT]}_{ij}\right)$