# On the use of evolutionary methods in spaces of Euclidean signature

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1 July, 2019

# The main message:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

## Based on some recent works:

- I. Rácz: *Is the Bianchi identity always hyperbolic?*, CQG **31** 155004 (2014)
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All the involved results are valid for arbitrary dimension: i.e. for $dim(M) = n\ (\geq 4)$. Nevertheless, for the sake of simplicity attention will be restricted to the case of $n = 4$.

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# Outline:

- **Einsteinian spaces:** $(M, g_{ab})$
  - First part
  - Second part
  - in both cases metrics of Euclidean signature will be involved
  - no gauge condition

    ... arbitrary choice of foliations & "evolutionary" vector field

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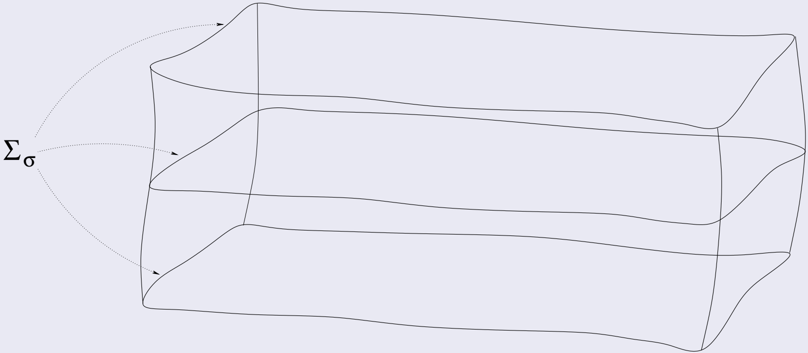


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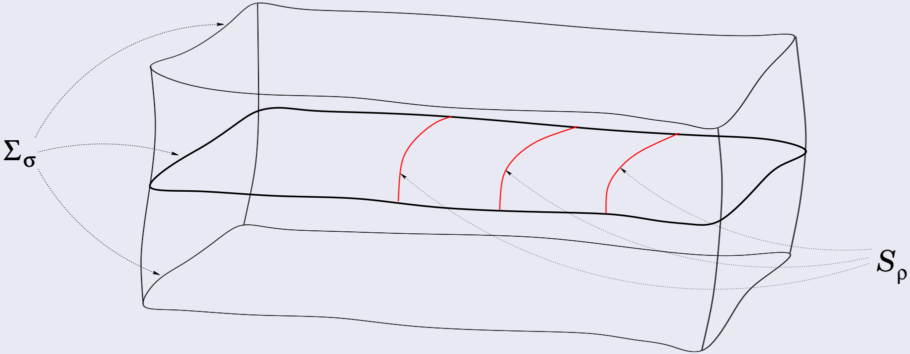


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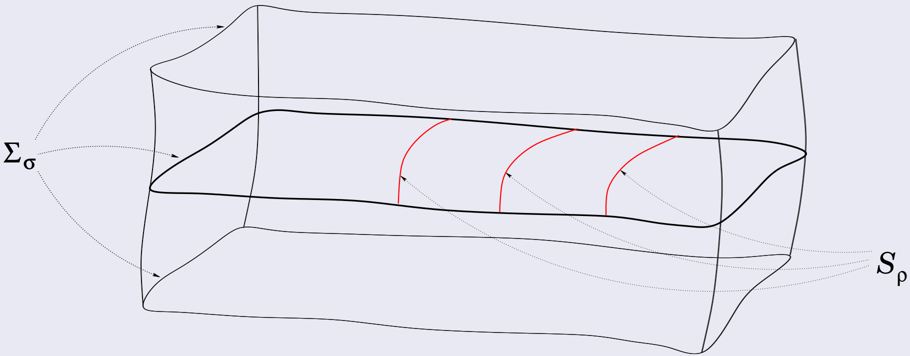


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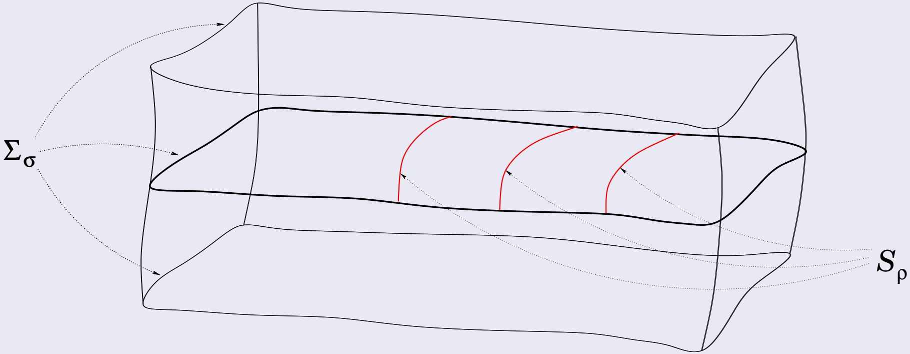


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# The basic setup:

- **Einsteinian spaces:** $(M, g_{ab})$
  - $M$ : 4-dimensional, smooth, paracompact, connected, orientable manifold
  - $g_{ab}$: smooth Lorentzian$_{(-,+,+,+)}$ or Riemannian$_{(+,+,+,+)}$ metric
- **Einstein's equations:**

$$\boxed{G_{ab} - \mathscr{G}_{ab} = 0} \quad \text{with source term:} \quad \boxed{\nabla^a \mathscr{G}_{ab} = 0}$$

  - $\boxed{\nabla_a}$ denotes the covariant derivative operator associated with $g_{ab}$.
  - in a more familiar setup: **Einstein's equations** with cosmological constant $\Lambda$

$$\boxed{[R_{ab} - \tfrac{1}{2}\, g_{ab}\, R] + \Lambda\, g_{ab} = 8\pi\, T_{ab}}$$

    with matter fields satisfying their Euler-Lagrange equations
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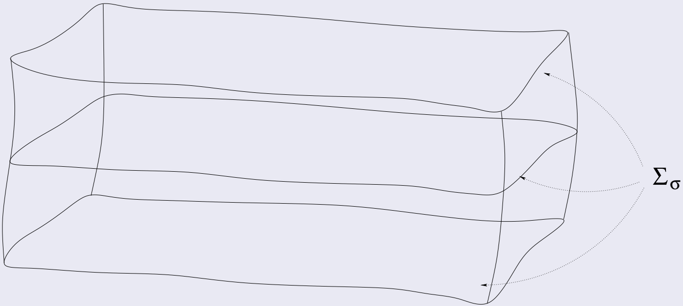
## The primary splitting

- **Assume:** $M$ is smoothly foliated by a one-parameter family of homologous hypersurfaces, i.e. $M \simeq \mathbb{R} \times \Sigma$, for some three-dimensional manifold $\Sigma$.
  - known to **hold for globally hyperbolic spacetimes** (Lorentzian case)
  - **equivalent to** the existence of a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing gradient $\partial_a \sigma$ such that the $\sigma = const$ level surfaces $\Sigma_\sigma = \{\sigma\} \times \Sigma$ comprise the one-parameter foliation of $M$.
  - $n_a \sim \partial_a \sigma \; \dots \; \& \dots \; g^{ab} \; \longrightarrow \; n^a = g^{ab} n_b$



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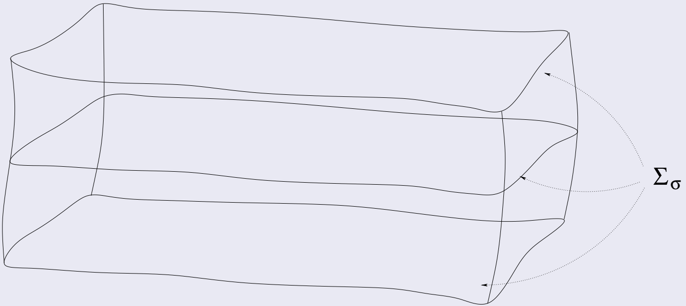

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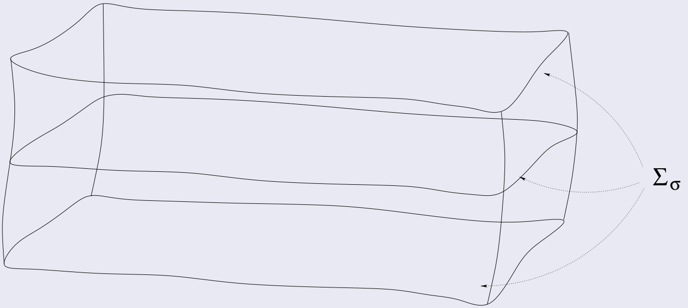

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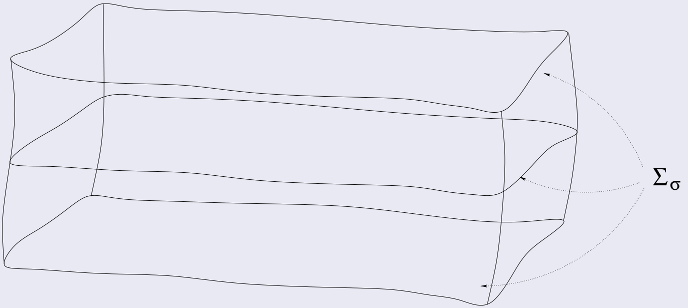

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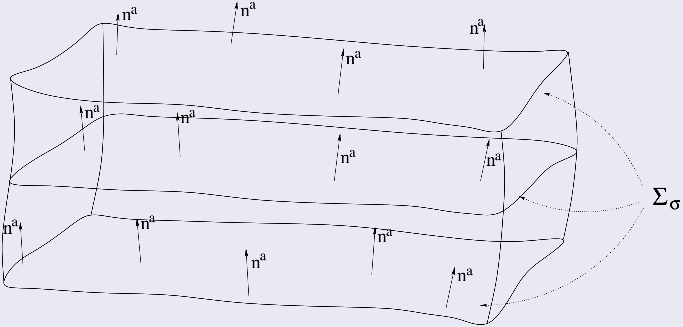

# Projections:

## The projection operator:

- $n^a$ the 'unit norm' vector field that is normal to the $\Sigma_\sigma$ level surfaces

- **the projection operator**

$$h^a{}_b = \delta^a{}_b - \epsilon\, n^a n_b$$

  to the level surfaces of $\sigma : M \to \mathbb{R}$.

- **the induced metric** on the $\sigma = const$ level surfaces

$$h_{ab} = h^e{}_a h^f{}_b\, g_{ef}$$

  - $D_a$ denotes the covariant derivative operator associated with $h_{ab}$.

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$$\boxed{n^a n_a = \epsilon}$$

  - the sign is not fixed: $\epsilon$ takes the value $-1$ or $+1$ for Lorentzian or Riemannian metric $g_{ab}$, respectively

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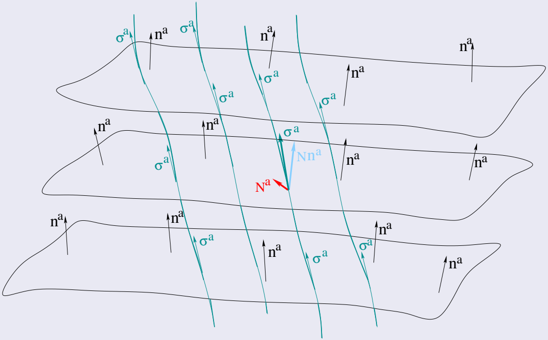
# $\sigma^a$ is "time evolution vector field" **if**:

- the integral curves of $\sigma^a$ meet the $\sigma = const$ level surfaces precisely once
- $\boxed{\sigma^e \nabla_e \sigma = 1}$




- where $N$ and $N^a$ denotes the **lapse** and **shift** of $\sigma^a$:

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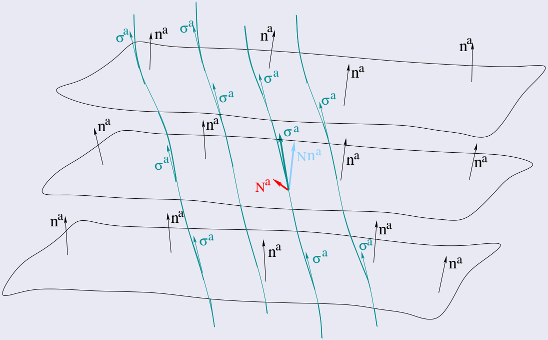


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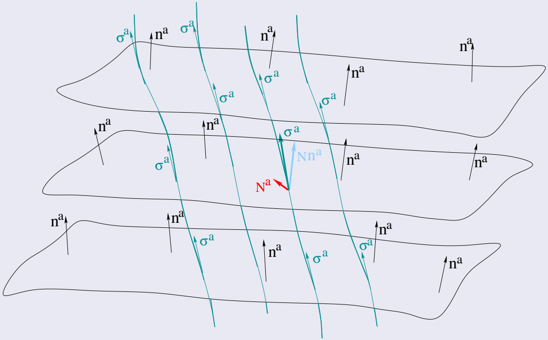


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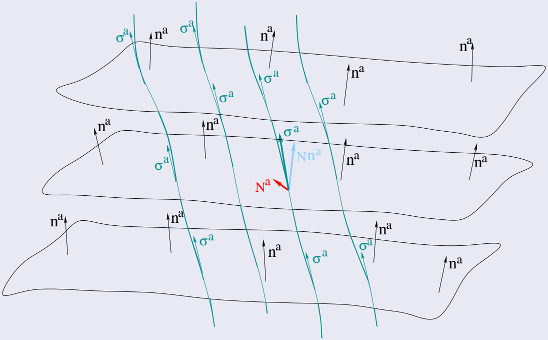


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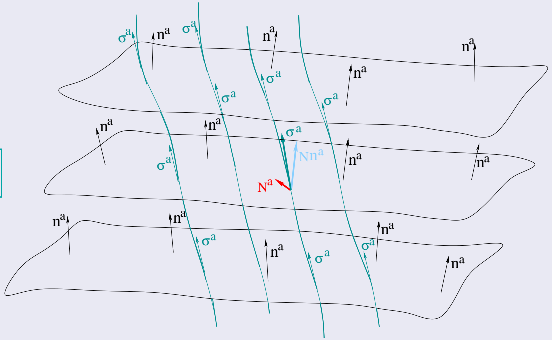


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# Decompositions of various fields:

## Any symmetric tensor field $P_{ab}$ can be decomposed

in terms of $n^a$ and fields intrinsic to the individual $\sigma = const$ level surfaces as

$$P_{ab} = \boldsymbol{\pi}\, n_a n_b + [n_a\, \mathbf{p}_b + n_b\, \mathbf{p}_a] + \mathbf{P}_{ab}$$

## It is also rewarding to inspect the decomposition of the cov. divergence $\nabla^a P_{ab}$:

$$\epsilon\,(\nabla^a P_{ae})\, n^e = \mathscr{L}_n \boldsymbol{\pi} + D^e \mathbf{p}_e + [\boldsymbol{\pi}\,(K^e{}_e) - \epsilon\, \mathbf{P}_{ef} K^{ef} - 2\,\epsilon\, \dot{n}^e \mathbf{p}_e]$$

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where $$\boldsymbol{\pi} = n^e n^f\, P_{ef}\,, \quad \mathbf{p}_a = \epsilon\, h^e{}_a n^f\, P_{ef}\,, \quad \mathbf{P}_{ab} = h^e{}_a h^f{}_b\, P_{ef}$$

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## Any symmetric tensor field $P_{ab}$ can be decomposed

in terms of $n^a$ and fields intrinsic to the individual $\sigma = const$ level surfaces as

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# The main result of the first part:

## Theorem

*Let* $(M, g_{ab})$ *be an Einsteinian space as specified and assume that the metric* $h_{ab}$ *induced on the* $\sigma = const$ *level surfaces is Riemannian. Then,* **regardless whether** $g_{ab}$ **is of Lorentzian or Euclidean signature**, *any solution to the reduced equations* $E^{(\mathcal{EVOL})}_{ab} = 0$ *is also a solution to the full set of field equations* $G_{ab} - \mathscr{G}_{ab} = 0$ *provided that the constraint expressions* $E^{(\mathcal{H})}$ *and* $E^{(\mathcal{M})}_{a}$ *vanish on one of the* $\sigma = const$ *level surfaces.*

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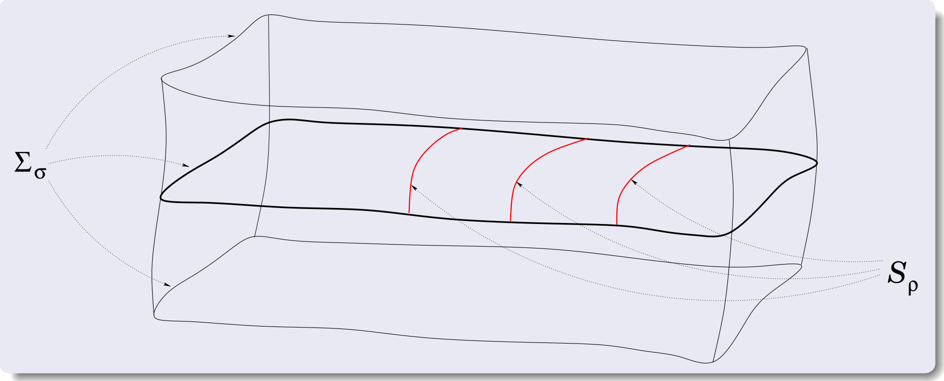
# PART II:



# The explicit form of the constraints:

## The constraint expressions are projections of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$:

$$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\, {}^{(3)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0$$
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- it is an underdetermined system: 4 equations for 12 variables

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Consider the underdetermined equation on $\Sigma \approx \mathbb{R}^2$ with some coordinates $(\chi, \xi)$

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

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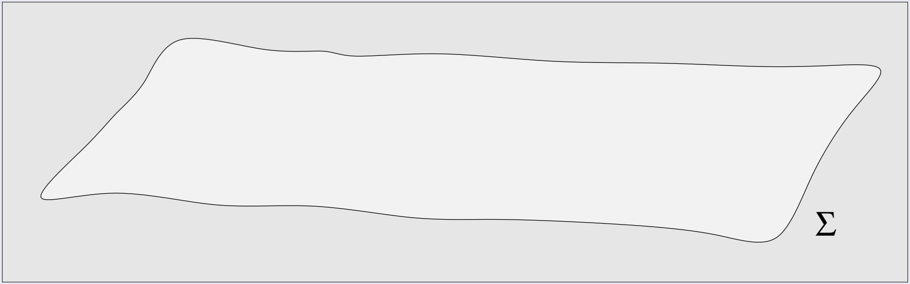

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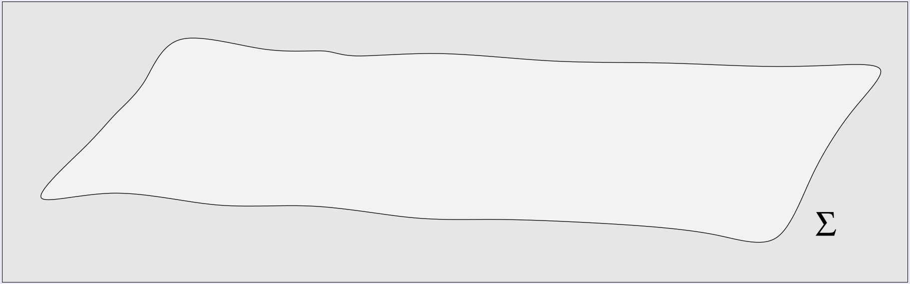

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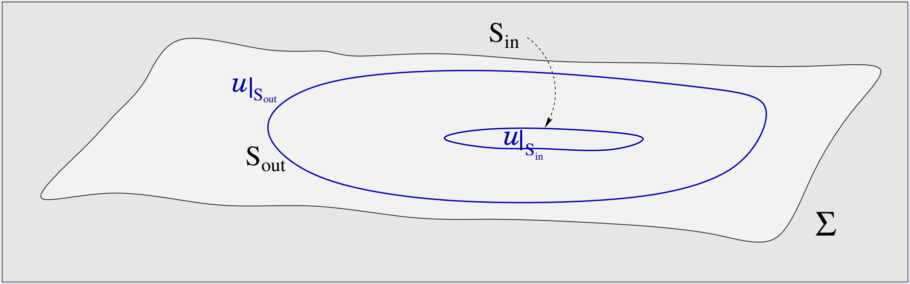

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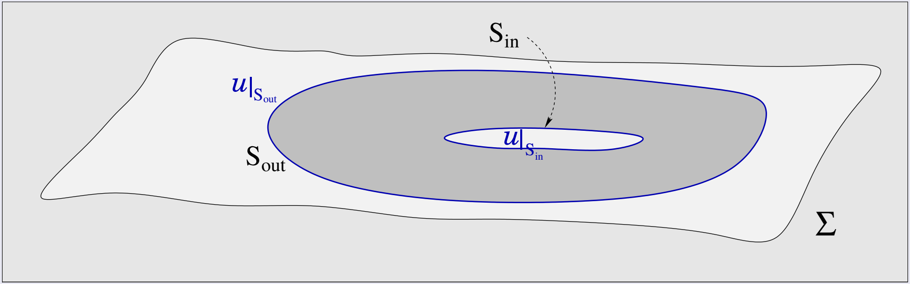

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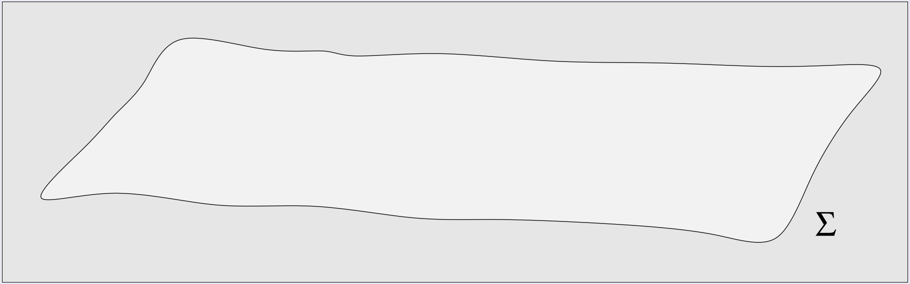

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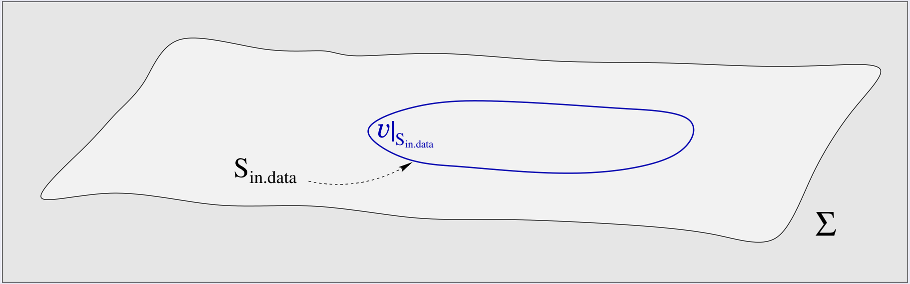

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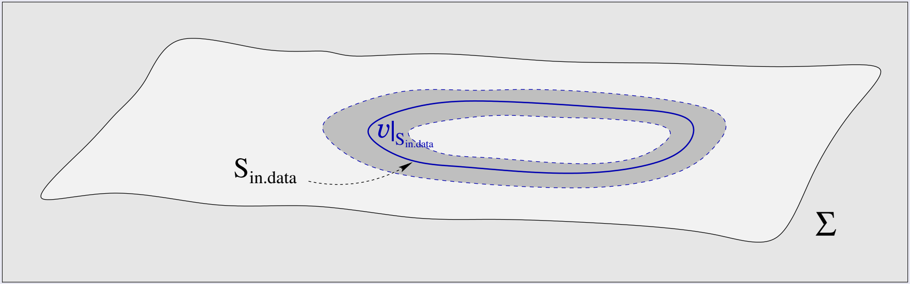

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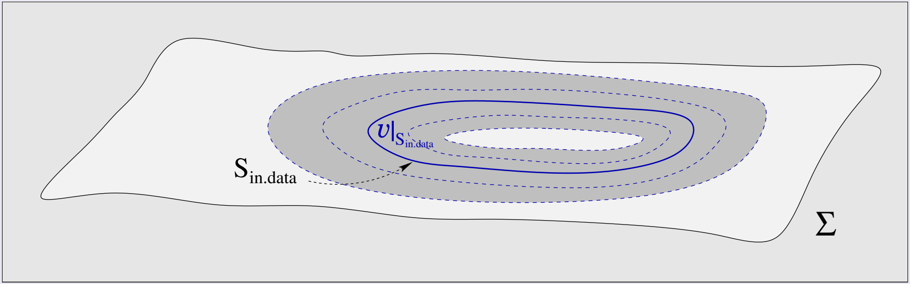

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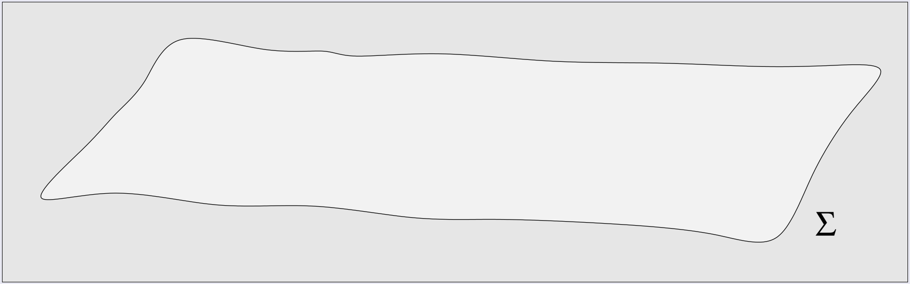

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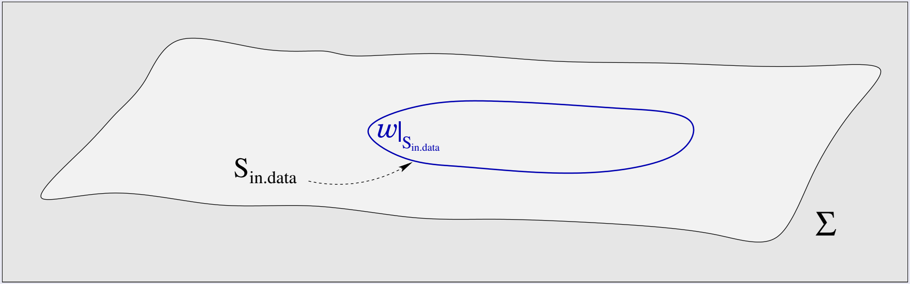

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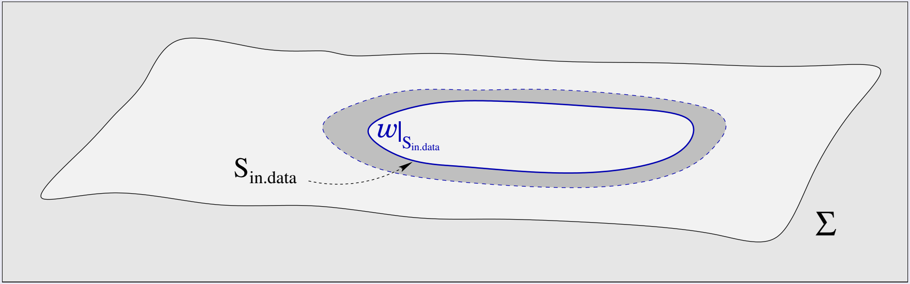

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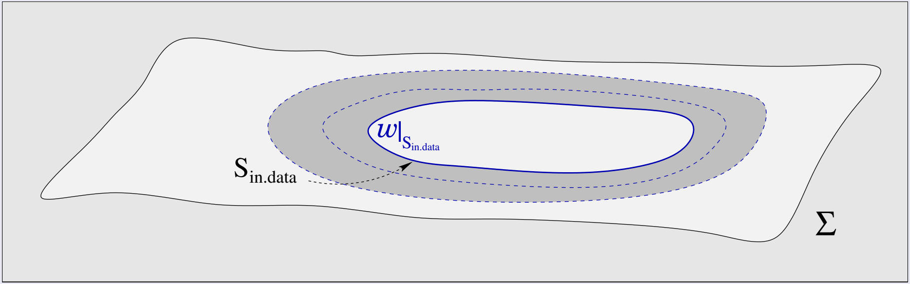

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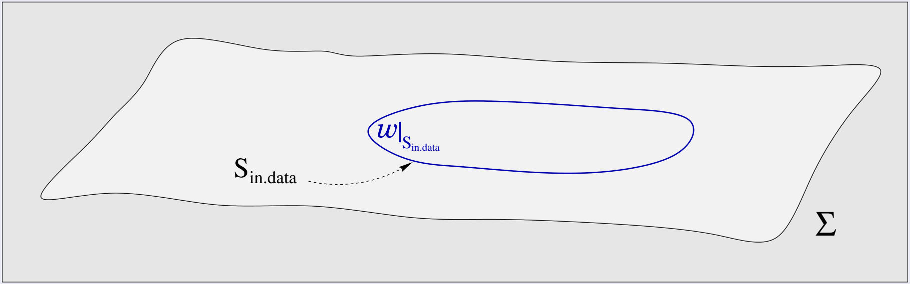

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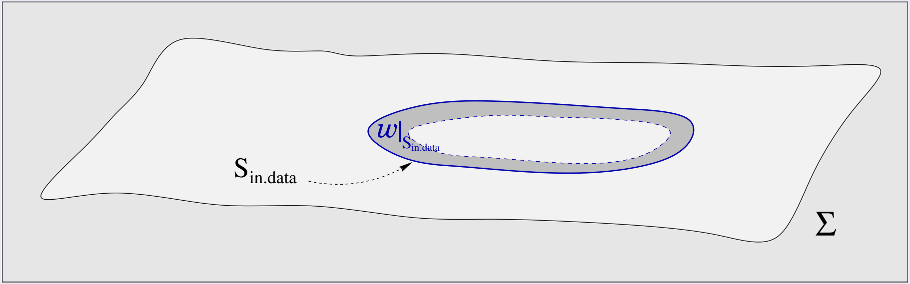

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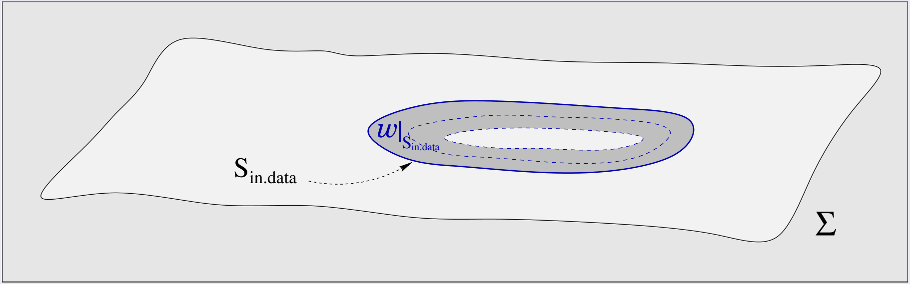

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$$(\partial_\chi^2 + \partial_\xi^2)u + (\partial_\chi^2 - \partial_\xi^2)v + (a\,\partial_\chi - \partial_\xi^2)w + z = 0$$

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# New variables by applying $2+1$ decompositions:

## Splitting of the metric $h_{ij}$:

- choose $\rho^i$ to be a flow field on $\Sigma$: the integral curves... & $\rho^i\partial_i\rho = 1$
- 'lapse' and 'shift' of $\rho^i$

$$\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i\,, \quad \text{where} \quad \widehat{N} = \rho^j\widehat{n}_j \quad \text{and} \quad \widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j$$

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# $2+1$ decompositions:

## Splitting of the symmetric tensor field $K_{ij}$:

- 
$$K_{ij} = \boldsymbol{\kappa}\, \widehat{n}_i \widehat{n}_j + [\widehat{n}_i\, \mathbf{k}_j + \widehat{n}_j\, \mathbf{k}_i] + \mathbf{K}_{ij}$$

  where

$$\boldsymbol{\kappa} = \widehat{n}^k \widehat{n}^l\, K_{kl}\,, \quad \mathbf{k}_i = \widehat{\gamma}^k{}_i \widehat{n}^l\, K_{kl} \quad \text{and} \quad \mathbf{K}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j\, K_{kl}$$

- the **trace** and **trace free** parts of $\mathbf{K}_{ij}$

$$\mathbf{K}^l{}_l = \widehat{\gamma}^{kl}\, \mathbf{K}_{kl} \quad \text{and} \quad \mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{2}\, \widehat{\gamma}_{ij} \mathbf{K}^l{}_l$$

## The new variables:

- 
$$(h_{ij}, K_{ij}) \iff (\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$$

- these variables retain the physically distinguished nature of $h_{ij}$ and $K_{ij}$

# $2+1$ decompositions:

## Splitting of the symmetric tensor field $K_{ij}$:

- $$K_{ij} = \boldsymbol{\kappa}\,\widehat{n}_i\widehat{n}_j + [\widehat{n}_i\,\mathbf{k}_j + \widehat{n}_j\,\mathbf{k}_i] + \mathbf{K}_{ij}$$

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- the **trace** and **trace free** parts of $\mathbf{K}_{ij}$

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# The momentum constraint:

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# The Hamiltonian constraint:

## The Hamiltonian constraint in terms of the new variables:

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- it is a **parabolic equation** for $\widehat{N}$ (the sign of $\widehat{K}^l{}_l$ plays a role)
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- $$-\epsilon\,\widehat{R} + \epsilon\left\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\,\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\right\} + 2\,\boldsymbol{\kappa}\,\mathbf{K}^l{}_l + \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e} = 0$$

## Alternative choices yielding evolutionary systems:

- it is a **parabolic equation** for $\widehat{N}$ (the sign of $\widehat{K}^l{}_l$ plays a role)
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# The Hamiltonian constraint:

## The Hamiltonian constraint in terms of the new variables:

- $$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\, {}^{(3)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0$$

- using $$ {}^{(3)}\!R = \widehat{R} - \left\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\right\}$$

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- $(h_{ij}, K_{ij})$ represented by the variables $(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$
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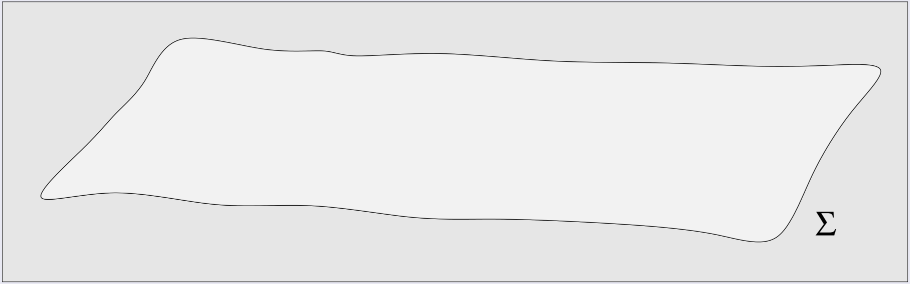

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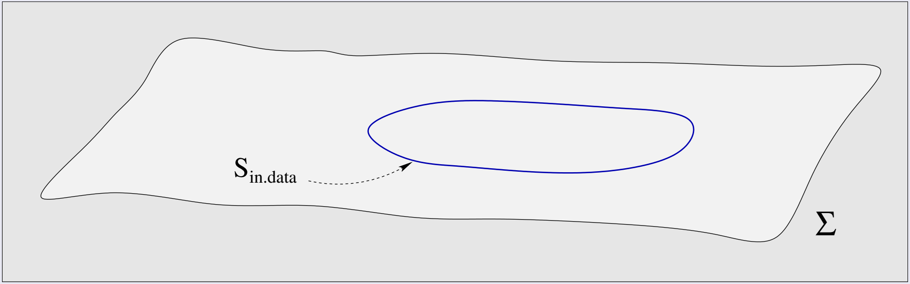

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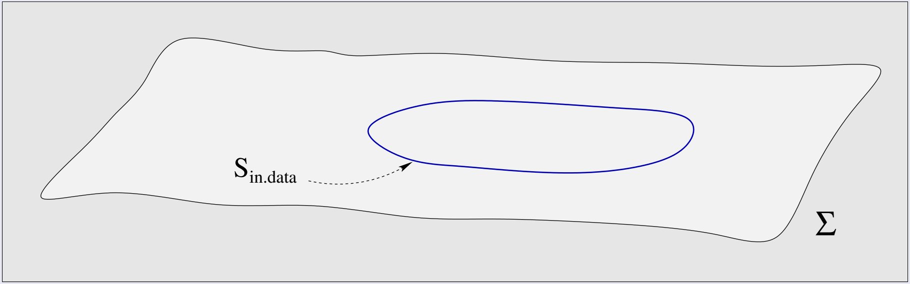

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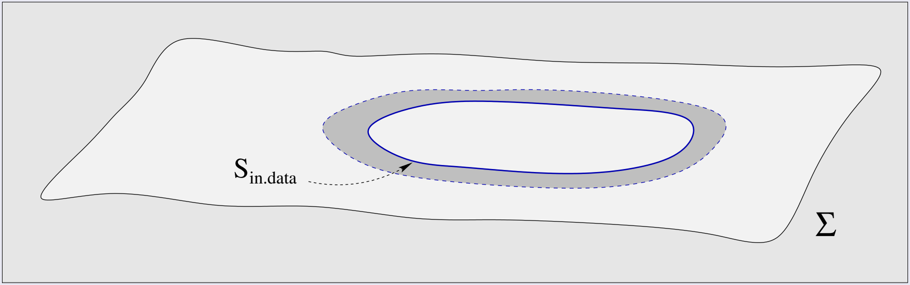

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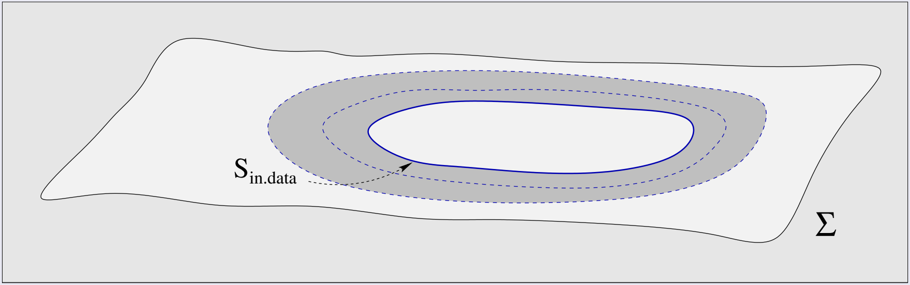

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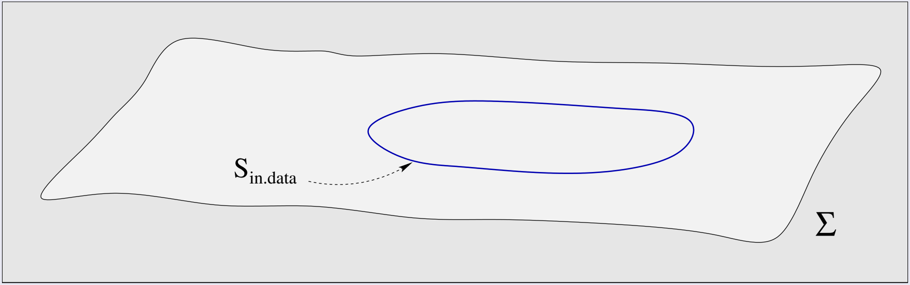

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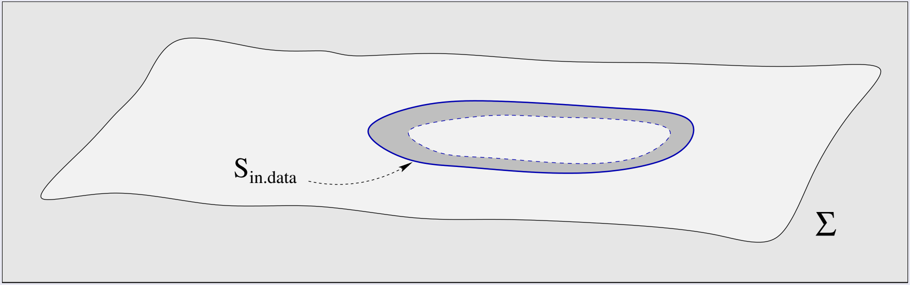

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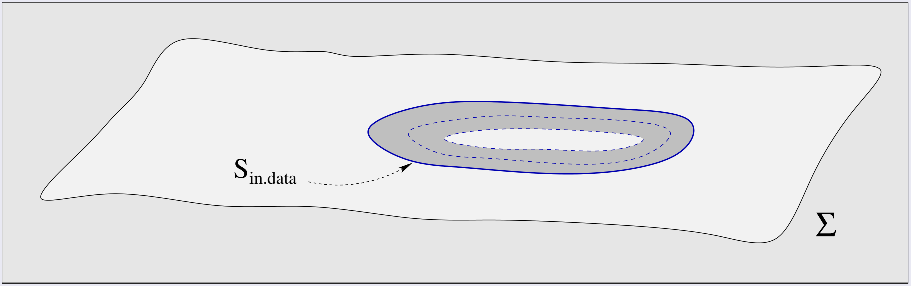

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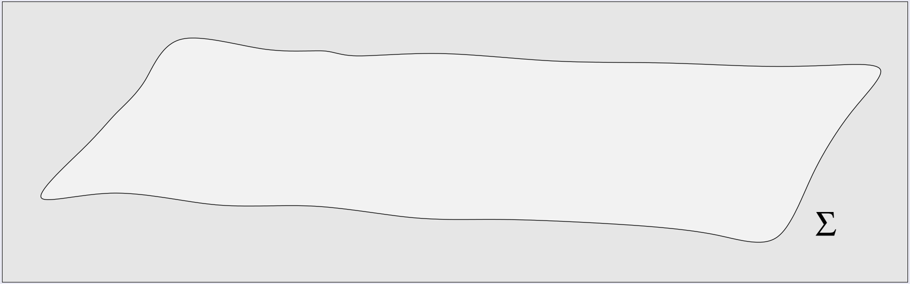

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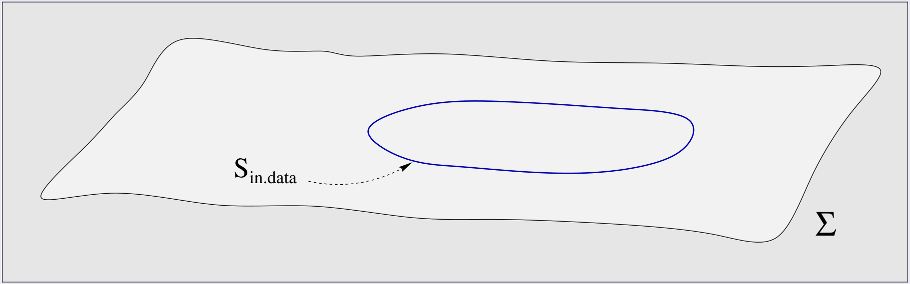

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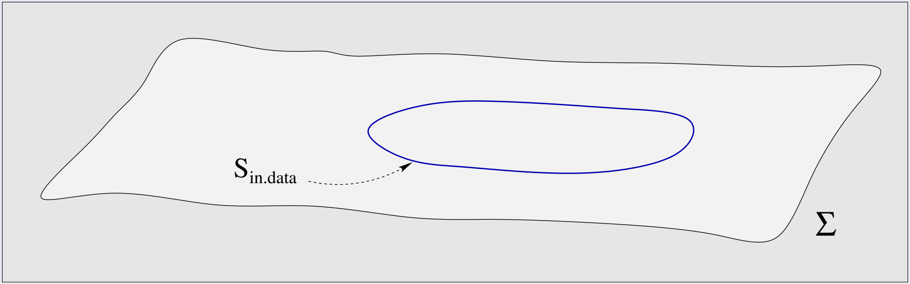

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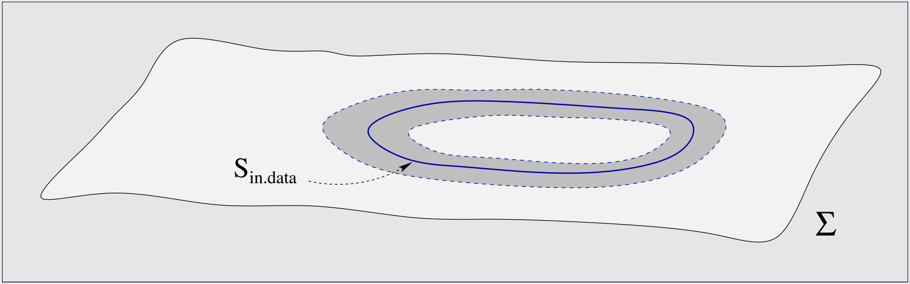

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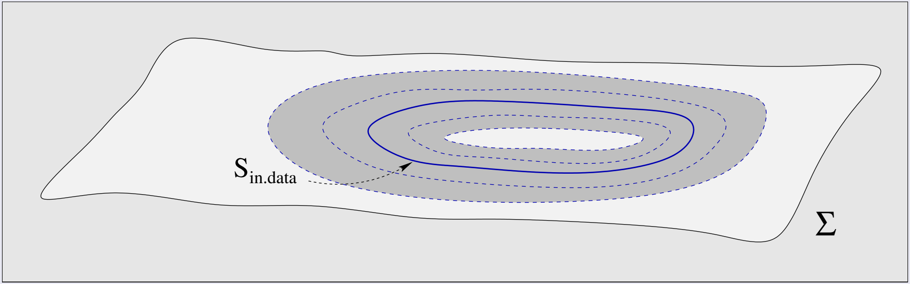

# Summary:

4-dimensional **Riemannian and Lorentzian spaces** satisfying Einstein's equations, and some mild topological assumptions, were considered. **!!!** $[n(\geq 4)]$

1. it was shown that **the constraint expressions** satisfy a **FOSH system** that is linear and homogeneous $\Longrightarrow$ (the constraints propagate)
2. concerning the constraint equations in Einstein's theory it was shown:
   - momentum constraint as a first order symmetric hyperbolic system
   - the Hamiltonian constraint as a parabolic or an algebraic equation
   - in either case the coupled constraint equations comprise a well-posed evolutionary system: a parabolic-hyperbolic or a strongly hyperbolic
   - in $C^\omega$ setting (local) existence and uniqueness of solutions are guaranteed
3. **!!! regardless** whether the primary space is Riemannian or Lorentzian
4. **!!! no use** of gauge conditions

## The take home message:

On contrary to the folklore, in the considered two explicit examples, **evolutionary methods can be applied in spaces with metric of Euclidean signature where, in principle, there is no room for 'time'**

# Summary:

4-dimensional **Riemannian and Lorentzian spaces** satisfying Einstein's equations, and some mild topological assumptions, were considered. **!!!** $[n(\geq 4)]$

1. it was shown that **the constraint expressions** satisfy a **FOSH system** that is linear and homogeneous $\Longrightarrow$ (the constraints propagate)
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## The first order symmetric hyperbolic system for $(E^{(\mathcal{H})}, E_i^{(\mathcal{M})})^T$

- Characteristic directions associated with the FOSH system governing the evolution of the constraint expressions are determined as

$$[\,h^{ij} - n^i n^j\,]\,\xi_i\xi_j = [\,g^{ij} - (1+\epsilon)\,n^i n^j\,]\,\xi_i\xi_j = 0$$

## The momentum constraint: first order symmetric hyperbolic system

- with characteristic cone given as

$$[\,\widehat{\gamma}^{ij} - 2\,\widehat{n}^i\widehat{n}^j\,]\;\xi_i\xi_j = [\,h^{ij} - 3\,\widehat{n}^i\widehat{n}^j\,]\;\xi_i\xi_j = 0$$

## Deriving a Lorentzian metric from a Riemannian one

- ... given a Riemannian metric $g_{ij}$, a unit form field $n_i$ and a positive real function $\alpha$

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# The conformal (elliptic) method:

## Lichnerowicz A (1944) and York J W (1972):

- replace

$$h_{ij} = \phi^4 \, \widetilde{h}_{ij} \quad \text{and} \quad K_{ij} - \tfrac{1}{3}\, h_{ij}\, K^l{}_l = \phi^{-2}\, \widetilde{K}_{ij}$$

- using these variables the constraints are put into a **semilinear elliptic system**

$$\widetilde{D}^l \widetilde{D}_l \phi + \epsilon\, \tfrac{1}{8}\, \widetilde{R}\, \phi + \tfrac{1}{8}\, \widetilde{K}_{ij} \widetilde{K}^{ij}\, \phi^{-7} - \left[\tfrac{1}{12}\, (K^l{}_l)^2 - \tfrac{1}{4}\, \mathfrak{e}\right] \phi^5 = 0$$

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