Construction of initial data with monotonous Geroch mass

István Rácz

istvan.racz@fuw.edu.pl & racz.istvan@wigner.mta.hu

Faculty of Physics, University of Warsaw, Warsaw, Poland

Wigner Research Center for Physics, Budapest, Hungary

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Motivations:

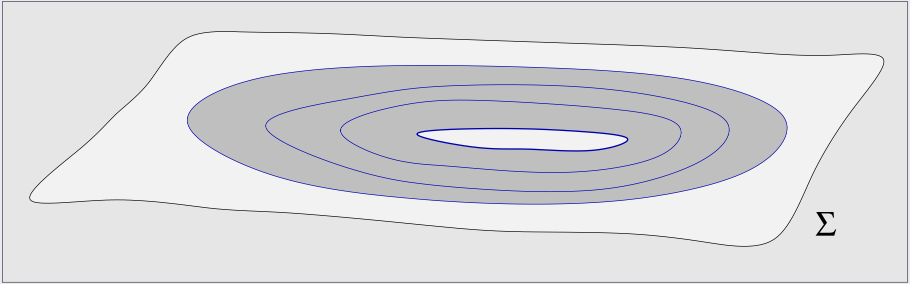
GR is a **metric theory of gravity**:

- it is highly non-trivial to assign, in a sensible way, mass, energy, linear or angular momenta to bounded spatial regions
- "... it is almost certain that we have to understand conserved (or quasi conserved) quantities which can control the field in a more local manner. In other words, we expect some concept of quasi-local mass will be useful."
- efforts to prove the **positive mass theorem** and the **Penrose inequalities** using quasi-local techniques Geroch (1973), Wald, Jang (1977), Jang (1978), Kijowski (1986), Chruściel (1986), Jezierski, Kijowski (1987), Huisken, Ilmanen (1997, 2001), Frauendiener (2001), Bray (2001), Malec, Mars, Simon (2002), Bray, Lee (2009),. . .

The aim is to outline:

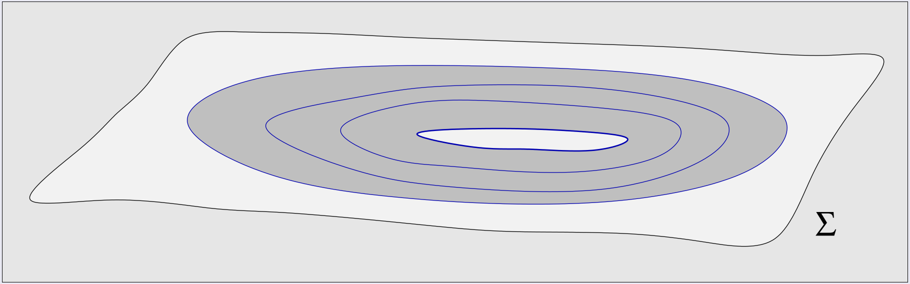
- a **simple construction** of a high variety of Riemannian three-spaces with prescribed, whence globally existing regular foliation and flow such that
  - the (quasi-local) **Geroch mass**—that can be evaluated on the leaves of the foliations—is **non-decreasing** with respect to the applied flow
  - the foliation gets to be **mean-convex** w.r.t. the constructed three-metric
- **construction** of initial data retaining all these preferable properties...

Foliations by topological two-spheres:



- consider a smooth 3-dimensional manifold Σ with a Riemannian metric h_{ij}
- assume
$$\boxed{\Sigma \approx \mathbb{R} \times \mathscr{S}} \qquad origin(s)(!)$$
i.e. Σ is smoothly foliated by a one-parameter family of top. two-spheres $\mathscr{S}_\rho$: $\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$
- $$\Longrightarrow \quad \boxed{\partial_i \rho \;\; \& \;\; h^{ij} \;\longrightarrow\; \widehat{n}_i\,, \widehat{n}^i = h^{ij}\widehat{n}_j \;\; \ldots \;\; \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i\widehat{n}_j}$$
- '$\widehat{\ }$' to distinguish quantities that could also be viewed as fields on the leaves

Mean-convex foliations:



- **the induced Riemannian metric** on the $\mathscr{S}_\rho$ level sets

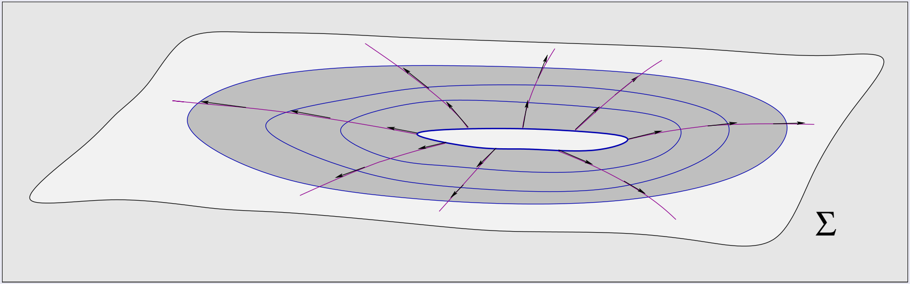
$$\boxed{\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j h_{kl}}$$

- **the extrinsic curvature** given by the symmetric tensor field

$$\boxed{\widehat{K}_{ij} = \widehat{\gamma}^l{}_i D_l\, \widehat{n}_j = \tfrac{1}{2}\, \mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij}\,,} \qquad D_i, \mathscr{L}_{\widehat{n}}$$

- a $\rho = const$ level surface is called to be **mean-convex if its mean curvature**, $\boxed{\widehat{K}^l{}_l = \widehat{\gamma}^{ij}\, \widehat{K}_{ij} = \widehat{\gamma}^{ij} D_i\, \widehat{n}_j}$, is positive on $\mathscr{S}_\rho$

Flows:



- a smooth vector field ρ^i on Σ **is a flow** ("evolution vector field") w.r.t. $\mathscr{S}_\rho$
  - if the integral curves of ρ^i **intersect each leaves precisely once**, and
  - if ρ^i **is scaled such that** $\boxed{\rho^i \partial_i \rho = 1}$ holds throughout Σ
- any smooth flow can be decomposed in terms of its '**lapse**' and '**shift**' as

$$\boxed{\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i} \qquad \boxed{\widehat{N} = \rho^i \widehat{n}_i = (\widehat{n}^i \partial_i \rho)^{-1}} \qquad \boxed{\widehat{N}^i = \widehat{\gamma}^i{}_j\, \rho^j}$$

  - the lapse **measures the normal separation of the surfaces** $\mathscr{S}_\rho$

Variation of the area:

- to any mean-convex foliation $\exists$ a (quasi-local) **orientation of the leaves** $\mathscr{S}_\rho$
- a flow ρ^i is called **outward pointing** if the area is increasing w.r.t. it
- variation of the area $\boxed{\mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \widehat{\epsilon}}$ of the $\rho = const$ level surfaces, w.r.t. ρ^i

$$\boxed{\mathscr{L}_\rho \mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \mathscr{L}_\rho \widehat{\boldsymbol{\epsilon}} = \int_{\mathscr{S}_\rho} \left\{ \widehat{N} (\widehat{K}^l{}_l) + (\widehat{D}_i \widehat{N}^i) \right\} \widehat{\boldsymbol{\epsilon}} = \int_{\mathscr{S}_\rho} \widehat{N} (\widehat{K}^l{}_l)\, \widehat{\boldsymbol{\epsilon}},}$$

  the relations $\mathscr{L}_{\widehat{n}} \widehat{\boldsymbol{\epsilon}} = (\widehat{K}^l{}_l)\, \widehat{\epsilon}$ and $\mathscr{L}_{\widehat{N}} \widehat{\boldsymbol{\epsilon}} = \frac{1}{2}\, \widehat{\gamma}^{ij} \mathscr{L}_{\widehat{N}} \widehat{\gamma}_{ij}\, \widehat{\boldsymbol{\epsilon}} = (\widehat{D}_i \widehat{N}^i)\, \widehat{\epsilon}$, along with the vanishing of the integral of the total divergence $\widehat{D}_i \widehat{N}^i$, were applied.
- $\widehat{N}$ does not vanish on Σ unless the Riemannian three-metric

$$\boxed{h^{ij} = \widehat{\gamma}^{ij} + \widehat{N}^{-2} (\rho^i - \widehat{N}^i)(\rho^j - \widehat{N}^j)}$$

  gets to be singular
- for **mean-convex foliations** $\boxed{\widehat{N} \widehat{K}^l{}_l > 0}$ $\Longrightarrow$ the **area is increasing** w.r.t. ρ^i
- the orientations by $\widehat{n}^i$ and ρ^i coincide

The Geroch mass:

- the (quasi-local) Geroch mass (equal to the Hawking mass only if $K^i{}_i = 0$)

$$m_{\mathcal{G}} = \frac{\mathscr{A}_\rho^{1/2}}{64\pi^{3/2}} \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2 \right] \widehat{\boldsymbol{\epsilon}}$$

  where $\widehat{R}$ is the scalar curvature of the metric $\widehat{\gamma}_{ij}$ on the leaves
- for mean-convex foliations the area $\mathscr{A}_\rho$ is monotonously increasing
- it suffices to investigate

$$W(\rho) = \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2 \right] \widehat{\boldsymbol{\epsilon}}$$

- **if both $\mathscr{A}_\rho$ and $W(\rho)$ were non-decreasing**, and for some specific ρ_* value, $W(\rho_*)$ was zero or positive then $\boxed{m_{\mathcal{G}} \geq 0}$ would hold to the exterior of $\mathscr{S}_{\rho_*}$ in Σ

The variation of $W(\rho)$:

- the **key equation** we shall use **relates the scalar curvatures** of h_{ij} and $\widehat{\gamma}_{ij}$

$$ {}^{(3)}R = \widehat{R} - \Big\{ 2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N} \Big\} \qquad (*) $$

$$ \begin{aligned} \mathscr{L}_\rho W &= -\int_{\mathscr{S}_\rho} \mathscr{L}_\rho\Big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{\boldsymbol\epsilon}\Big] = -\int_{\mathscr{S}_\rho}\Big\{\widehat{N}\,\mathscr{L}_{\widehat{n}}\Big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{\boldsymbol\epsilon}\Big] + \mathscr{L}_{\widehat{N}}\Big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{\boldsymbol\epsilon}\Big]\Big\} \\ &= -\int_{\mathscr{S}_\rho}\big(\widehat{N}\widehat{K}^l{}_l\big)\Big[2\,\mathscr{L}_{\widehat{n}}\,(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2\Big]\,\widehat{\boldsymbol\epsilon} - \int_{\mathscr{S}_\rho}\widehat{D}_i\Big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{N}^i\Big]\widehat{\boldsymbol\epsilon} \\ &= -\int_{\mathscr{S}_\rho}\big(\widehat{N}\widehat{K}^l{}_l\big)\Big[\big(\widehat{R} - {}^{(3)}R\big) - \widehat{K}_{kl}\widehat{K}^{kl} - 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}\Big]\,\widehat{\boldsymbol\epsilon} \end{aligned} $$

- where on 1^{st} line $\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i$ and the Gauss-Bonnet theorem
- on 2^{nd} line the relations $\mathscr{L}_{\widehat{n}}\,\widehat{\boldsymbol\epsilon} = (\widehat{K}^l{}_l)\,\widehat{\boldsymbol\epsilon}$ and $\mathscr{L}_{\widehat{N}}\,\widehat{\boldsymbol\epsilon} = (\widehat{D}_i\widehat{N}^i)\,\widehat{\boldsymbol\epsilon}$
- on 3^{rd} line (*) and the vanishing of the integral of $\widehat{D}_i\Big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{N}^i\Big]$ were used

The variation of $W(\rho)$:

- by the Leibniz rule

$$\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N} = \widehat{D}^l\big(\widehat{N}^{-1}\widehat{D}_l\widehat{N}\big) + \widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})$$

- and by introducing the trace-free part of $\widehat{K}_{ij}$

$$\overset{\circ}{\widehat{K}}_{ij} = \widehat{K}_{ij} - \tfrac{1}{2}\,\widehat{\gamma}_{ij}\,(\widehat{K}^l{}_l), \qquad \widehat{K}_{kl}\widehat{K}^{kl} = \overset{\circ}{\widehat{K}}_{kl}\overset{\circ}{\widehat{K}}{}^{kl} + \tfrac{1}{2}\,(\widehat{K}^l{}_l)^2$$

- and using the vanishing of the integral of the total divergence $\widehat{D}^l(\widehat{N}^{-1}\widehat{D}_l\widehat{N})$

$$\begin{aligned}\mathscr{L}_\rho W = &-\tfrac{1}{2}\int_{\mathscr{S}_\rho}(\widehat{N}\widehat{K}^l{}_l)\left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2\right]\widehat{\boldsymbol{\epsilon}} \\ &+\int_{\mathscr{S}_\rho}(\widehat{N}\,\widehat{K}^l{}_l)\left[{}^{(3)}R + \overset{\circ}{\widehat{K}}_{kl}\overset{\circ}{\widehat{K}}{}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})\right]\widehat{\boldsymbol{\epsilon}}\end{aligned}$$

Rigidity of the setup:

- **if** the product $\widehat{N}\widehat{K}^l{}_l$ **could be replaced** by its **mean value**
- $$\overline{\widehat{N}\widehat{K}^l{}_l} = \frac{\int_{\mathscr{S}_\rho} \widehat{N}\widehat{K}^l{}_l\, \widehat{\epsilon}}{\int_{\mathscr{S}_\rho} \widehat{\epsilon}} \qquad \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]$$

$$[(64\,\pi^{3/2})/(\mathscr{A}_\rho)^{1/2}]\cdot \mathscr{L}_\rho\, m_{\mathcal{G}} = \mathscr{L}_\rho W + \tfrac{1}{2}\,(\mathscr{L}_\rho \log[\mathscr{A}_\rho])\, W \geq 0$$

provided that $\int_{\mathscr{S}_\rho} \Big[\, {}^{(3)}\!R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}{}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})\Big]\,\widehat{\epsilon} \geq 0$

- once in addition to h_{ij} **a foliation and a flow are fixed** not only the **mean curvature** $\widehat{K}^l{}_l$ **BUT** the **lapse** $\widehat{N}$ and the **shift** $\widehat{N}^i$ get also to be fixed

$$\widehat{K}^l{}_l = \widehat{\gamma}^{ij}\,\widehat{K}_{ij} = \widehat{\gamma}^{ij}\,D_i\,\widehat{n}_j \qquad \widehat{N} = \rho^i\widehat{n}_i = (\widehat{n}^i\partial_i\rho)^{-1} \qquad \widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j$$

- the only "freedom" is a relabeling of the leaves by using a function $\overline{\rho} = \overline{\rho}(\rho)$ but this cannot yield more than a rescaling $\widehat{N} \to \widehat{N}(\mathrm{d}\rho/\mathrm{d}\overline{\rho})$ of the lapse
- **(!)** at best $\widehat{N}\widehat{K}^l{}_l$ is a smooth positive function on the leaves of the foliation

How to get control on the monotonicity?

What we have by hands: $\{\widehat{N}, \widehat{N}^A, \widehat{\gamma}_{AB}\,;\, \rho:\Sigma\to\mathbb{R},\ \rho^i = (\partial_\rho)^i\}$

- a Riemannian metric h_{ij} defined on a three-surface Σ
- Σ is foliated by topological two-spheres: $\Sigma \approx \mathbb{R}\times\mathbb{S}^2$ ..... $\rho:\Sigma\to\mathbb{R}$ is chosen
- a flow ρ^i was also fixed on Σ such that $\rho^i\partial_i\rho = 1$
- the later two can be used to introduce coordinates (ρ, x^A) adapted to the flow ρ^i such that $\boxed{\rho^i = (\partial_\rho)^i}$, and such that the shift $\widehat{N}^i$ and the metric $\widehat{\gamma}_{ij}$ can be given as a two-vector $\boxed{\widehat{N}^A}$ and a non-singular 2×2 matrix $\boxed{\widehat{\gamma}_{AB}}$ both smoothly depending on the coordinates ρ, x^A, where A takes the values $2, 3$
- line element of the Riemannian metric h_{ij} (can be given in its ADM form)

$$\boxed{\mathrm{d}s^2 = \widehat{N}^2\mathrm{d}\rho^2 + \widehat{\gamma}_{AB}\left(\mathrm{d}x^A + \widehat{N}^A\mathrm{d}\rho\right)\left(\mathrm{d}x^B + \widehat{N}^B\mathrm{d}\rho\right)}$$

The challenge is:

- choose a maximal subset of the fields $\{h_{ij}\,;\,\rho:\Sigma\to\mathbb{R},\ \rho^i = (\partial_\rho)^i\}$ such that

$$\boxed{\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]} \qquad \boxed{\int_{\mathscr{S}_\rho}\left[{}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,(\widehat{D}_k\widehat{N})(\widehat{D}^k\widehat{N})\right]\widehat{\epsilon}}$$

Solution 1°: using the inverse mean curvature flow (IMCF)

- choose a maximal subset of the fields $\{h_{ij}\,;\, \rho : \Sigma \to \mathbb{R},\ \rho^i = (\partial_\rho)^i\}$ such that

$$\boxed{\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]} \qquad \boxed{\int_{\mathscr{S}_\rho}\Big[\,{}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,(\widehat{D}_k\widehat{N})(\widehat{D}^k\widehat{N})\Big]\,\widehat{\epsilon}}$$

- what is if we keep (Σ, h_{ij}) but drop $\rho : \Sigma \to \mathbb{R}$ and the shift from $\rho^i = (\partial_\rho)^i$

The foliation and part of the flow is to be determined dynamically

- the **inverse mean curvature flow**

$$\boxed{\rho^i_{\{IMCF\}} = \big(\widehat{K}^l{}_l\big)^{-1}\,\widehat{n}^i + \widehat{N}^i_{\{IMCF\}}}$$

- as for the corresponding foliation $\widehat{N}\widehat{K}^l{}_l \equiv 1$ hold: if this flow existed globally the Geroch mass would be non-decreasing w.r.t it
- one can relax these condition by using a generalized IMCF

$$\boxed{\rho^i = \mathscr{L}_\rho(\log[\mathscr{A}_\rho])\,\rho^i_{\{IMCF\}}}$$

- **(!)** global existence and regularity remains a serious issue

Solution 2°: using a prescribed, globally existing foliation

- choose a maximal subset of the fields $\{h_{ij}\,;\, \rho:\Sigma\to\mathbb{R},\ \rho^i=(\partial_\rho)^i\}$ such that

$$\widehat N\widehat K^l{}_l=\overline{\widehat N\widehat K^l{}_l}=\mathscr{L}_\rho\log[\mathscr{A}_\rho]$$

$$\int_{\mathscr{S}_\rho}\Big[\,{}^{(3)}R+\mathring{\widehat K}_{kl}\mathring{\widehat K}^{kl}+2\,\widehat N^{-2}\,(\widehat D_k\widehat N)(\widehat D^k\widehat N)\Big]\,\widehat{\boldsymbol\epsilon}$$

- what is if we drop the three-metric h_{ij} BUT keep a globally well-defined foliation $\rho:\Sigma\to\mathbb{R}$, a flow ρ^i and the induced metric $\widehat\gamma_{ij}$ on the leaves: in coordinates (ρ,x^A) adapted to the flow $\rho^i=(\partial_\rho)^i$ the induced metric: $\widehat\gamma_{AB}$

Using prescribed foliation, flow, induced metric: $h_{ij}\leftrightarrow\widehat N,\ \widehat N^A,\ \widehat\gamma_{AB}$

- $\rho^i=\widehat N\,\widehat n^i+\widehat N^i$ however counterintuitive it is: we may always construct shift $\widehat N^i$ with desirable properties:

$$\widehat N\widehat K^l{}_l=\tfrac{1}{2}\,\widehat\gamma^{ij}\mathscr{L}_\rho\widehat\gamma_{ij}-\widehat D_i\widehat N^i$$

- as $\widehat N\widehat K^l{}_l=\overline{\widehat N\widehat K^l{}_l}=\mathscr{L}_\rho\log[\mathscr{A}_\rho]$ wished to be guaranteed,

$$\widehat D_A\widehat N^A=\mathscr{L}_\rho\log[\sqrt{\det(\widehat\gamma_{AB})}\,]-\mathscr{L}_\rho\log[\mathscr{A}_\rho]\qquad(**)$$

Solution 2°: using prescribed foliation, flow and $\widehat{\gamma}_{AB}$

Solving $\widehat{D}_A \widehat{N}^A = \mathscr{L}_\rho \log[\sqrt{\det(\widehat{\gamma}_{AB})}\,] - \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ (**) on $\mathscr{S}_\rho$

- on topological two-spheres using then the Hodge decomposition of the shift

$$\widehat{N}^A = \widehat{D}^A \chi + \widehat{\epsilon}^{AB} \widehat{D}_B \eta$$

χ and η are some smooth functions on $\mathscr{S}$, (**)

$$\widehat{D}^A \widehat{D}_A \chi = \mathscr{L}_\rho \log[\,\sqrt{\det(\widehat{\gamma}_{AB})}\,] - \mathscr{L}_\rho \log[\mathscr{A}_\rho]$$

- solubility of this elliptic equation with smooth coefficients and source terms on the succeeding individual topological two-spheres is guaranteed
- there is an inherent sphere-by-sphere constant value of ambiguity in χ which, however, does not effect the determination of the first term in $\widehat{N}^A$

We have not done yet (!) $\int_{\mathscr{S}_\rho} \Big[\, {}^{(3)}\!R + \mathring{\widehat{K}}_{kl} \mathring{\widehat{K}}^{kl} + 2\, \widehat{N}^{-2} \, (\widehat{D}_k \widehat{N})(\widehat{D}^k \widehat{N}) \Big]\, \widehat{\boldsymbol{\epsilon}}$

- in clearing up the picture let us have a glance again of the key equation

$${}^{(3)}\!R = \widehat{R} - \Big\{ 2\, \mathscr{L}_{\widehat{n}} (\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl} \widehat{K}^{kl} + 2\, \widehat{N}^{-1}\, \widehat{D}^l \widehat{D}_l \widehat{N} \Big\} \qquad (*)$$

A parabolic equation for $\widehat{N}$:

- as noticed first by Bartnik (1993), while applying quasi-spherical foliations (*) can be viewed as a parabolic equation for $\widehat{N}$
- remarkably, (*) **can always be seen to be a parabolic eqn** for $\widehat{N}$ **IF** ${}^{(3)}R$, $\widehat{\gamma}_{AB}$ and $\widehat{N}^A$ can be treated as prescribed fields
- introducing $\boxed{\overset{\star}{K}_{AB} = \widehat{N}\widehat{K}_{AB}}$ and $\boxed{\overset{\star}{K} = \frac{1}{2}\,\widehat{\gamma}^{AB}\mathscr{L}_\rho\widehat{\gamma}_{AB} - \widehat{D}_A\widehat{N}^A}$ to **eliminate hidden occurrence** of the lapse in (*) we get

$$\boxed{\overset{\star}{K}\,[\,(\partial_\rho\widehat{N}) - \widehat{N}^A(\hat{D}_A\widehat{N})\,] = \widehat{N}^2(\widehat{D}^A\widehat{D}_A\widehat{N}) + \mathcal{A}\,\widehat{N} - \tfrac{1}{2}\,(\,\widehat{R} - {}^{(3)}R\,)\,\widehat{N}^3}$$

where $\boxed{\mathcal{A} = \partial_\rho\overset{\star}{K} + \frac{1}{2}\,[\overset{\star}{K}{}^2 + \overset{\star}{K}_{AB}\overset{\star}{K}{}^{AB}]}$ with $\boxed{\overset{\star}{K} = \overline{\widehat{N}\widehat{K}^A{}_A} = \mathscr{L}_\rho \log[\mathscr{A}_\rho] > 0}$

- it is standard to obtain **existence of unique solutions to this (Bernoulli type) uniformly parabolic PDE** in a sufficiently small one-sided neighborhood of $\mathscr{S}$ in Σ

Theorem

Suppose that a choice had been made for a smooth real function ${}^{(3)}R : \Sigma \to \mathbb{R}$ *and also for the freely specifiable smooth distribution of two-metrics* $\widehat{\gamma}_{AB}$ *so that the constructed* $\widehat{N}^A$ *gets smooth and also that* $\overset{\star}{K} = \mathscr{L}_\rho \log[\mathscr{A}_\rho] > 0$ *is constant on the leaves* $\mathscr{S}_\rho$ *throughout* Σ*. Assume that smooth positive initial data* ${}_{(0)}\widehat{N}$ *had also been chosen to our Bernoulli type parabolic equation on one of the level surfaces, say on* $\mathscr{S}_{\rho_0}$*, in* Σ*. Then, for some suitable* $\varepsilon > 0$*, there exists a unique smooth solution* $\widehat{N}$ *to the parabolic equation in a one-sided neighborhood* $\mathscr{S}_{[\rho_0,\rho_0+\varepsilon)}$ *of* $\mathscr{S}_{\rho_0}$ *in* Σ *such that* $\widehat{N}|_{\mathscr{S}_{\rho_0}} = {}_{(0)}\widehat{N}$*.*

Global existence of unique solutions:

- our main concern is **global existence (!)**
- it should not come as a surprise that **an analogous parabolic equation** came up **in deriving the evolutionary form** of the Hamiltonian constraints in [Rácz I: *Constrains as evolutionary systems*, Class. Quant. Grav. **33** 015014 (2016)]

Theorem

Assume that a smooth real function ${}^{(3)}R : \Sigma \to \mathbb{R}$ *and a smooth distribution of two-metrics* $\widehat{\gamma}_{AB}$ *are chosen such that the inequality* ${}^{(3)}R \leq \widehat{R}$ *holds on each of the individual leaves* $\mathscr{S}_\rho$ *on* Σ*, and such that the integrals* $\int_{\rho_0}^{\rho} \mathcal{A}/\overset{\star}{K}\, \mathrm{d}\rho'$ *and* $\int_{\rho_0}^{\rho} \left(\frac{1}{2}\,({}^{(3)}R - \widehat{R}\,)/\overset{\star}{K}\right) \cdot \exp\left[\int_{\rho_0}^{\rho'} \mathcal{A}/\overset{\star}{K}\, \mathrm{d}\rho''\right] \mathrm{d}\rho'$ *exist and they are finite for all* $\rho \geq \rho_0$*. Then, upper and lower solutions to Bernoulli type parabolic equation exist such that—for any choice of a smooth strictly positive initial data* ${}_{(0)}\widehat{N}$ *on* $\mathscr{S}_{\rho_0}$*—they are both guaranteed to be positive and bounded away from zero and infinity. Then any global smooth solution to our Bernoulli type parabolic equation, with initial* ${}_{(0)}\widehat{N}$ *on* $\mathscr{S}_{\rho_0}$*, is also guaranteed to remain positive and bounded away from zero and infinity for all* $\rho \geq \rho_0$*.*

How to get initial data while retaining all the preferable properties?

- could the proposed new method also be used to get sensible initial data specifications for Einstein's equations such that
  - the initial data surface gets to be foliated by mean-convex topological two-spheres, and such that
  - the Geroch mass is non-decreasing with respect to the foliation and flow
- two alternative evolutionary methods were introduced to solve the constraints [Rácz I: *Constrains as evolutionary systems*, Class. Quant. Grav. **33** 015014 (2016)]
- in the algebraic-hyperbolic formulation, the entire Riemannian three-metric is part of the freely specifiable fields on Σ
- combine the new construction outlined in the previous part with solving the algebraic-hyperbolic form of the constraints
  - first a Riemannian three-metric h_{ij} should be constructed with all the aforementioned preferable properties and then,
  - using this as an input, try to solve the algebraic-hyperbolic form of the constraints for suitable parts of the other symmetric field K_{ij}

The constraints:

- the geometric part of the initial data can be represented by a pair of smooth fields (h_{ij}, K_{ij}) on Σ, where h_{ij} is a Riemannian metric while K_{ij} is a symmetric tensor field there

$$
{}^{(3)}\!R + \left(K^j{}_j\right)^2 - K_{ij}K^{ij} = 0\,, \qquad D_j K^j{}_i - D_i K^j{}_j = 0
$$

  where ${}^{(3)}\!R$ and D_i denote the scalar curvature and the covariant derivative operator associated with h_{ij}, respectively.

- the arena to set up the algebraic-hyperbolic form of the constraints had already been introduced
- the only missing ingredient is the $2+1$ decomposition of the symmetric tensor field K_{ij} given as

$$
K_{ij} = \boldsymbol{\kappa}\,\widehat{n}_i\widehat{n}_j + [\widehat{n}_i\,\mathbf{k}_j + \widehat{n}_j\,\mathbf{k}_i] + \mathbf{K}_{ij}
$$

  where $\boldsymbol{\kappa} = \widehat{n}^k\widehat{n}^l\,K_{kl}$, $\mathbf{k}_i = \widehat{\gamma}^k{}_i\,\widehat{n}^l\,K_{kl}$ and $\mathbf{K}_{ij} = \widehat{\gamma}^k{}_i\widehat{\gamma}^l{}_j\,K_{kl}$

- it is also essential to replace $\mathbf{K}_{ij}$ by its trace and trace free parts given as

$$
\mathbf{K}^l{}_l = \widehat{\gamma}^{kl}\,\mathbf{K}_{kl} \quad \text{and} \quad \mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{2}\,\widehat{\gamma}_{ij}\,\mathbf{K}^l{}_l
$$

The algebraic-hyperbolic system:

- the pair (h_{ij}, K_{ij}) is replaced by the fields $\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}, \mathring{\mathbf{K}}_{ij}, \boldsymbol{\kappa}, \mathbf{k}_i$ and $\mathbf{K}^l{}_l$
- whereas the algebraic-hyperbolic form of the constraints

$$\mathscr{L}_{\widehat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l \mathbf{k}_l + 2\,\dot{\widehat{n}}^l\,\mathbf{k}_l - [\,\boldsymbol{\kappa} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)\,]\,(\widehat{K}^l{}_l) + \mathring{\mathbf{K}}_{kl}\widehat{K}^{kl} = 0$$

$$\mathscr{L}_{\widehat{n}}\mathbf{k}_i + (\mathbf{K}^l{}_l)^{-1}[\,\boldsymbol{\kappa}\,\widehat{D}_i(\mathbf{K}^l{}_l) - 2\,\mathbf{k}^l\,\widehat{D}_i\mathbf{k}_l\,] + (2\,\mathbf{K}^l{}_l)^{-1}\,\widehat{D}_i\boldsymbol{\kappa}_0$$
$$+(\widehat{K}^l{}_l)\,\mathbf{k}_i + [\,\boldsymbol{\kappa} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)\,]\,\dot{\widehat{n}}_i - \dot{\widehat{n}}^l\,\mathring{\mathbf{K}}_{li} + \widehat{D}^l\mathring{\mathbf{K}}_{li} = 0$$

where $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}_0$ are given by the algebraic relations

$$\boldsymbol{\kappa} = (2\,\mathbf{K}^l{}_l)^{-1}\big[\,2\,\mathbf{k}^l\mathbf{k}_l - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 - \boldsymbol{\kappa}_0\,\big] \quad \text{and} \quad \boldsymbol{\kappa}_0 = {}^{(3)}\!R - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl}$$

- Hamiltonian constraint: solved algebraically for $\boldsymbol{\kappa}$
- symmetrizable hyperbolic system for $\mathbf{K}^l{}_l$ and $\mathbf{k}_i$ provided that $\boldsymbol{\kappa}\,\mathbf{K}^l{}_l < 0$
- the algebraic-hyperbolic equations for $\boldsymbol{\kappa}, \mathbf{K}^l{}_l$ and $\mathbf{k}_i$ are the Hamiltonian and momentum constrains, whereas the rest of the variables, $\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}$ and $\mathring{\mathbf{K}}_{ij}$,—among which the first three ones stand for the three metric h_{ij}—are freely specifiable throughout Σ

$\boldsymbol{\kappa}_0 \Leftarrow$ freely specifiable fields

- by choosing the initial data ${}_{(0)}\mathbf{K}^l{}_l$ and ${}_{(0)}\mathbf{k}_i$ suitably the sign condition $\boldsymbol{\kappa}\,\mathbf{K}^l{}_l < 0$ (at least locally) can always guaranteed to hold [Rácz I: CQG]
  - it holds globally for near Kerr configurations on Kerr-Schild time slices
- our basic equations comprise a symmetrizable hyperbolic system possessing (at least locally) a well-posed initial value problem

Theorem

Suppose that $\widehat{N}, \widehat{N}^i$ and $\widehat{\gamma}_{ij}$ are smooth and they are as they were constructed or given in constructing the Riemannian three-spaces above. Choose $\mathring{\mathbf{K}}_{ij}$ to be a smooth traceless field on Σ such that $\mathring{\mathbf{K}}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j\, \mathring{\mathbf{K}}_{ij}$. Assume that smooth initial data ${}_{(0)}\mathbf{K}^l{}_l$ and ${}_{(0)}\mathbf{k}_i$ had also been chosen to our hyperbolic system on one of the level surfaces $\mathscr{S}_{\rho_0}$ in Σ such that $\boldsymbol{\kappa}\,\mathbf{K}^l{}_l < 0$ holds in a neighborhood of $\mathscr{S}_{\rho_0}$. Then, in a suitable subset of this neighborhood, there exists a unique smooth solution, $\mathbf{K}^l{}_l$ and $\mathbf{k}_i$, to the symmetrizable hyperbolic equations such that $\mathbf{K}^l{}_l|_{\mathscr{S}_{\rho_0}} = {}_{(0)}\mathbf{K}^l{}_l$, $\mathbf{k}_i|_{\mathscr{S}_{\rho_0}} = {}_{(0)}\mathbf{k}_i$, and such that the Hamiltonian constraint also hold with the yielded $\boldsymbol{\kappa}$.

Summary:

a construction was introduced: which could be used to get a high variety of Riemannian three-spaces and initial data sets such that

1. the prescribed, whence globally existing regular foliation and flow: get to be **generalized inverse mean curvature foliation** : $\widehat{N}\widehat{K}^l{}_l = \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ & the flow gets to be a **generalized IMCF**
2. the **Geroch mass**—that can be evaluated on the leaves of the foliations—is **non-decreasing** w.r.t. the applied foliation and flow
3. the topology of Σ **could be**: $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{R}\times\mathbb{S}^2$, $\mathbb{S}^1\times\mathbb{S}^2$, $(1,2,0,0)$
4. the first part of our proposal, yielding Riemannian three-spaces, **applies to wide range of geometrized theories of gravity**
   - concerning the metric (on M or on Σ): no use of Einstein's equations or any other field equation had been applied anywhere in our construction
   - as only the Riemannian character of the metric on Σ was used the signature of the metric on the ambient space could be either Lor. or Euc.
5. **the playground is open:** questions of physical interest...