Construction of Riemannian three-geometries with monotonous Geroch mass

István Rácz

istvan.racz@fuw.edu.pl & racz.istvan@wigner.mta.hu

Faculty of Physics, University of Warsaw, Warsaw, Poland

Wigner Research Center for Physics, Budapest, Hungary

Supported by the POLONEZ programme of the National Science Centre of Poland (under the project No. 2016/23/P/ST1/04195) which has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No. 665778.

Department of Mathematics
Fudan University, Fudan, China
21 October 2019

Motivations:

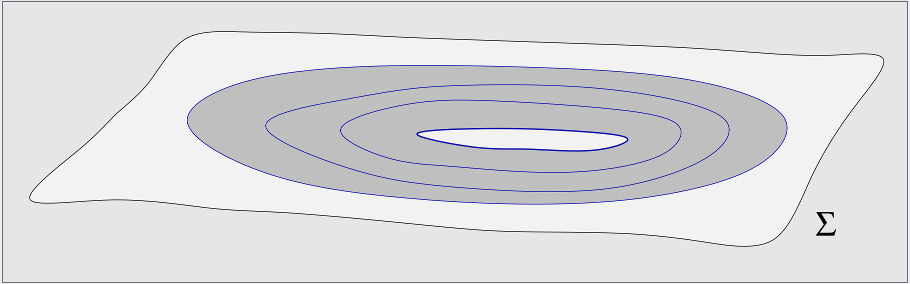
GR is a **metric theory of gravity**:

- it is highly non-trivial to assign, in a sensible way, mass, energy, linear or angular momenta to bounded spatial regions
- "... it is almost certain that we have to understand conserved (or quasi conserved) quantities which can control the field in a more local manner. In other words, we expect some concept of quasi-local mass will be useful."
- efforts to prove the **positive mass theorem** and the **Penrose inequalities** using quasi-local techniques Geroch (1973), Wald, Jang (1977), Jang (1978), Kijowski (1986), Chruściel (1986), Jezierski, Kijowski (1987), Huisken, Ilmanen (1997, 2001), Frauendiener (2001), Bray (2001), Malec, Mars, Simon (2002), Bray, Lee (2009),. . .

The aim is to outline:

- a **simple construction** of a high variety of Riemannian three-spaces with prescribed, whence globally existing regular foliation and flow such that
  - the (quasi-local) **Geroch mass**—that can be evaluated on the leaves of the foliations—is **non-decreasing** with respect to the applied flow
  - the foliation gets to be **mean-convex** w.r.t. the constructed three-metric

Foliations by topological two-spheres:

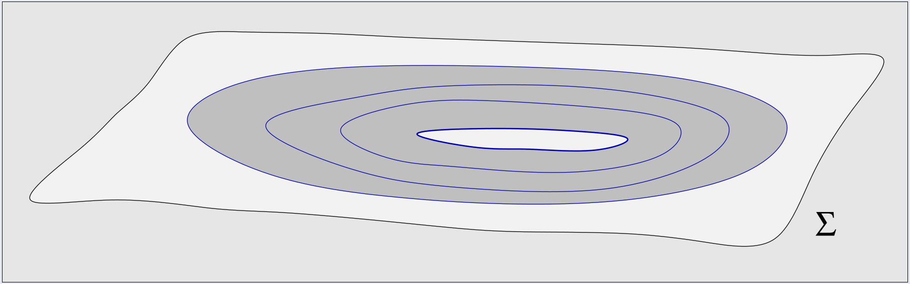


- consider a smooth 3-dimensional manifold Σ with a Riemannian metric h_{ij}
- assume $\boxed{\Sigma \approx \mathbb{R} \times \mathscr{S}}$ $\quad origin(s)(!)$

  i.e. Σ is smoothly foliated by a one-parameter family of top. two-spheres $\mathscr{S}_\rho$: $\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

- $\Longrightarrow$ $\boxed{\partial_i \rho \ \& \ h^{ij} \longrightarrow \widehat{n}_i \,, \widehat{n}^i = h^{ij}\widehat{n}_j \ \ldots \ \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i \widehat{n}_j}$
- '$\widehat{\ }$' to distinguish quantities that could also be viewed as fields on the leaves

Mean-convex foliations:



- **the induced Riemannian metric** on the $\mathscr{S}_\rho$ level sets

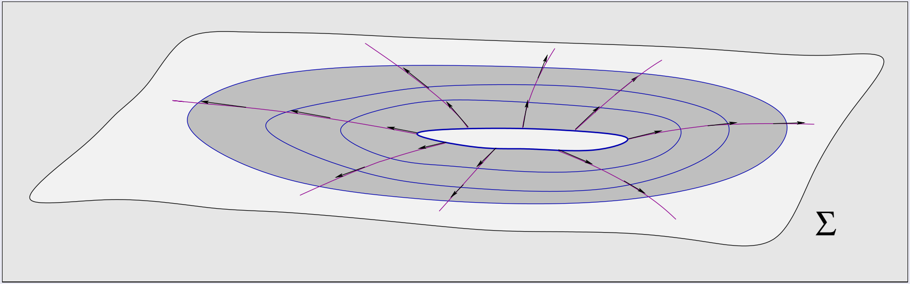
$$\boxed{\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j h_{kl}}$$

- **the extrinsic curvature** given by the symmetric tensor field

$$\boxed{\widehat{K}_{ij} = \widehat{\gamma}^l{}_i D_l \, \widehat{n}_j = \tfrac{1}{2} \, \mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij} \,,} \qquad D_i, \mathscr{L}_{\widehat{n}}$$

- a $\rho = const$ level surface is called to be **mean-convex if its mean curvature**, $\boxed{\widehat{K}^l{}_l = \widehat{\gamma}^{ij} \widehat{K}_{ij} = \widehat{\gamma}^{ij} D_i \, \widehat{n}_j}$, is positive on $\mathscr{S}_\rho$

Flows:



- a smooth vector field ρ^i on Σ **is a flow** ("evolution vector field") w.r.t. $\mathscr{S}_\rho$
  - if the integral curves of ρ^i **intersect each leaves precisely once**, and
  - if ρ^i **is scaled such that** $\boxed{\rho^i\partial_i\rho = 1}$ holds throughout Σ
- any smooth flow can be decomposed in terms of its '**lapse**' and '**shift**' as

$$\boxed{\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i} \qquad \boxed{\widehat{N} = \rho^i\widehat{n}_i = (\widehat{n}^i\partial_i\rho)^{-1}} \quad \boxed{\widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j}$$

  - the lapse **measures the normal separation of the surfaces** $\mathscr{S}_\rho$

Variation of the area:

- to any mean-convex foliation $\exists$ a (quasi-local) **orientation of the leaves** $\mathscr{S}_\rho$
- a flow ρ^i is called **outward pointing** if the area is increasing w.r.t. it
- variation of the area $\boxed{\mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \widehat{\epsilon}}$ of the $\rho = const$ level surfaces, w.r.t. ρ^i

$$\boxed{\mathscr{L}_\rho \mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \mathscr{L}_\rho \widehat{\boldsymbol\epsilon} = \int_{\mathscr{S}_\rho} \left\{ \widehat{N}(\widehat{K}^l{}_l) + (\widehat{D}_i \widehat{N}^i) \right\} \widehat{\boldsymbol\epsilon} = \int_{\mathscr{S}_\rho} \widehat{N}(\widehat{K}^l{}_l)\, \widehat{\boldsymbol\epsilon},}$$

  the relations $\mathscr{L}_{\widehat{n}} \widehat{\boldsymbol\epsilon} = (\widehat{K}^l{}_l)\, \widehat{\boldsymbol\epsilon}$ and $\mathscr{L}_{\widehat{N}} \widehat{\boldsymbol\epsilon} = \frac{1}{2} \widehat{\gamma}^{ij} \mathscr{L}_{\widehat{N}} \widehat{\gamma}_{ij} \widehat{\boldsymbol\epsilon} = (\widehat{D}_i \widehat{N}^i)\, \widehat{\boldsymbol\epsilon}$, along with the vanishing of the integral of the total divergence $\widehat{D}_i \widehat{N}^i$, were applied.
- $\widehat{N}$ does not vanish on Σ unless the Riemannian three-metric

$$\boxed{h^{ij} = \widehat{\gamma}^{ij} + \widehat{N}^{-2}(\rho^i - \widehat{N}^i)(\rho^j - \widehat{N}^j)}$$

  gets to be singular
- if $\widehat{N} > 0 \Longrightarrow$ the orientations by $\widehat{n}^i$ and ρ^i coincide: **assume** that $\widehat{N} > 0$
- for **mean-convex foliations** $\boxed{\widehat{N} \widehat{K}^l{}_l > 0} \Longrightarrow$ the **area is increasing** w.r.t. ρ^i

The Geroch mass:

- the (quasi-local) Geroch mass (equal to the Hawking mass only if $K^i{}_i = 0$)

$$\boxed{m_{\mathcal{G}} = \frac{\mathscr{A}_\rho^{1/2}}{64\pi^{3/2}} \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2 \right] \widehat{\boldsymbol{\epsilon}}}$$

  where $\widehat{R}$ is the scalar curvature of the metric $\widehat{\gamma}_{ij}$ on the leaves
- for mean-convex foliations the area $\mathscr{A}_\rho$ is monotonously increasing
- it suffices to investigate

$$\boxed{W(\rho) = \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2 \right] \widehat{\boldsymbol{\epsilon}}}$$

- **if both $\mathscr{A}_\rho$ and $W(\rho)$ were non-decreasing**, and for some specific ρ_* value, $W(\rho_*)$ was zero or positive then $\boxed{m_{\mathcal{G}} \geq 0}$ would hold to the exterior of $\mathscr{S}_{\rho_*}$ in Σ

The variation of $W(\rho)$:

- the **key equation** we shall use **relates the scalar curvatures** of h_{ij} and $\widehat{\gamma}_{ij}$

$$ {}^{(3)}R = \widehat{R} - \Big\{ 2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N} \Big\} \qquad (*) $$

$$ \begin{aligned} \mathscr{L}_\rho W &= -\int_{\mathscr{S}_\rho} \mathscr{L}_\rho\Big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{\boldsymbol\epsilon}\Big] = -\int_{\mathscr{S}_\rho} \Big\{ \widehat{N}\,\mathscr{L}_{\widehat{n}}\Big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{\boldsymbol\epsilon}\Big] + \mathscr{L}_{\widehat{N}}\Big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{\boldsymbol\epsilon}\Big] \Big\} \\ &= -\int_{\mathscr{S}_\rho} \big(\widehat{N}\widehat{K}^l{}_l\big)\Big[2\,\mathscr{L}_{\widehat{n}}\,(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2\Big]\,\widehat{\boldsymbol\epsilon} - \int_{\mathscr{S}_\rho} \widehat{D}_i\big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{N}^i\,\big]\widehat{\boldsymbol\epsilon} \\ &= -\int_{\mathscr{S}_\rho} \big(\widehat{N}\widehat{K}^l{}_l\big)\Big[\big(\,\widehat{R} - {}^{(3)}R\,\big) - \widehat{K}_{kl}\widehat{K}^{kl} - 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}\Big]\,\widehat{\boldsymbol\epsilon} \end{aligned} $$

- where on 1^{st} line $\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i$ and the Gauss-Bonnet theorem
- on 2^{nd} line the relations $\mathscr{L}_{\widehat{n}}\,\widehat{\boldsymbol\epsilon} = (\widehat{K}^l{}_l)\,\widehat{\boldsymbol\epsilon}$ and $\mathscr{L}_{\widehat{N}}\,\widehat{\boldsymbol\epsilon} = (\widehat{D}_i\widehat{N}^i)\,\widehat{\boldsymbol\epsilon}$
- on 3^{rd} line (*) and the vanishing of the integral of $\widehat{D}_i\big[\big(\widehat{K}^l{}_l\big)^2\,\widehat{N}^i\,\big]$ were used

The variation of $W(\rho)$:

- by the Leibniz rule

$$\boxed{\widehat N^{-1}\widehat D^l\widehat D_l\widehat N=\widehat D^l\big(\widehat N^{-1}\widehat D_l\widehat N\big)+\widehat N^{-2}\,\widehat\gamma^{kl}\,(\widehat D_k\widehat N)(\widehat D_l\widehat N)}$$

- and by introducing the trace-free part of $\widehat K_{ij}$

$$\boxed{\mathring{\widehat K}_{ij}=\widehat K_{ij}-\tfrac{1}{2}\,\widehat\gamma_{ij}\,(\widehat K^l{}_l),\qquad \widehat K_{kl}\widehat K^{kl}=\mathring{\widehat K}_{kl}\mathring{\widehat K}{}^{kl}+\tfrac{1}{2}\,(\widehat K^l{}_l)^2}$$

- and using the vanishing of the integral of the total divergence $\widehat D^l(\widehat N^{-1}\widehat D_l\widehat N)$

$$\boxed{\begin{aligned}\mathscr{L}_\rho W=&-\tfrac{1}{2}\int_{\mathscr{S}_\rho}(\widehat N\widehat K^l{}_l)\left[2\,\widehat R-(\widehat K^l{}_l)^2\right]\widehat{\boldsymbol\epsilon}\\ &+\int_{\mathscr{S}_\rho}(\widehat N\,\widehat K^l{}_l)\left[{}^{(3)}\!R+\mathring{\widehat K}_{kl}\mathring{\widehat K}{}^{kl}+2\,\widehat N^{-2}\,\widehat\gamma^{kl}\,(\widehat D_k\widehat N)(\widehat D_l\widehat N)\right]\widehat{\boldsymbol\epsilon}\end{aligned}}$$

Rigidity of the setup:

- **if** the product $\boxed{\widehat{N}\widehat{K}^l{}_l}$ **could be replaced** by its **mean value**

- $$\boxed{\overline{\widehat{N}\widehat{K}^l{}_l} = \frac{\int_{\mathscr{S}_\rho} \widehat{N}\widehat{K}^l{}_l\,\widehat{\epsilon}}{\int_{\mathscr{S}_\rho}\widehat{\epsilon}}} \qquad \boxed{\overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]}$$

$$\boxed{[(64\,\pi^{3/2})/(\mathscr{A}_\rho)^{1/2}]\cdot\mathscr{L}_\rho\, m_{\mathcal{G}} = \mathscr{L}_\rho W + \tfrac{1}{2}\,(\mathscr{L}_\rho \log[\mathscr{A}_\rho])\,W \geq 0}$$

provided that $\int_{\mathscr{S}_\rho}\Big[\,{}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}{}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})\Big]\,\widehat{\epsilon} \geq 0$

- once in addition to h_{ij} **a foliation and a flow are fixed** not only the **mean curvature** $\widehat{K}^l{}_l$ **BUT** the **lapse** $\widehat{N}$ and the **shift** $\widehat{N}^i$ get also to be fixed

$$\boxed{\widehat{K}^l{}_l = \widehat{\gamma}^{ij}\,\widehat{K}_{ij} = \widehat{\gamma}^{ij}\,D_i\,\widehat{n}_j} \qquad \boxed{\widehat{N} = \rho^i\widehat{n}_i = (\widehat{n}^i\partial_i\rho)^{-1}} \qquad \boxed{\widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j}$$

- the only "freedom" is a relabeling of the leaves by using a function $\overline{\rho} = \overline{\rho}(\rho)$ but this cannot yield more than a rescaling $\widehat{N} \to \widehat{N}(\mathrm{d}\rho/\mathrm{d}\overline{\rho})$ of the lapse
- **(!)** at best $\boxed{\widehat{N}\widehat{K}^l{}_l}$ is a smooth positive function on the leaves of the foliation

How to get control on the monotonicity?

What we have by hands: $\{\widehat{N}, \widehat{N}^A, \widehat{\gamma}_{AB}\,;\, \rho : \Sigma \to \mathbb{R},\ \rho^i = (\partial_\rho)^i\}$

- a Riemannian metric h_{ij} defined on a three-surface Σ
- Σ is foliated by topological two-spheres: $\Sigma \approx \mathbb{R} \times \mathbb{S}^2$ ..... $\rho : \Sigma \to \mathbb{R}$ is chosen
- a flow ρ^i was also fixed on Σ such that $\rho^i \partial_i \rho = 1$
- the later two can be used to introduce coordinates (ρ, x^A) adapted to the flow ρ^i such that $\boxed{\rho^i = (\partial_\rho)^i}$, and such that the shift $\widehat{N}^i$ and the metric $\widehat{\gamma}_{ij}$ can be given as a two-vector $\boxed{\widehat{N}^A}$ and a non-singular 2×2 matrix $\boxed{\widehat{\gamma}_{AB}}$ both smoothly depending on the coordinates ρ, x^A, where A takes the values $2, 3$
- line element of the Riemannian metric h_{ij} (can be given in its ADM form)

$$\boxed{\mathrm{d}s^2 = \widehat{N}^2 \mathrm{d}\rho^2 + \widehat{\gamma}_{AB} \left(\mathrm{d}x^A + \widehat{N}^A \mathrm{d}\rho\right) \left(\mathrm{d}x^B + \widehat{N}^B \mathrm{d}\rho\right)}$$

The challenge is:

- choose a maximal subset of the fields $\{h_{ij}\,;\, \rho : \Sigma \to \mathbb{R},\ \rho^i = (\partial_\rho)^i\}$ such that

$$\boxed{\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]} \qquad \boxed{\int_{\mathscr{S}_\rho} \left[{}^{(3)}\!R + \mathring{\widehat{K}}_{kl} \mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,(\widehat{D}_k \widehat{N})(\widehat{D}^k \widehat{N}) \right] \widehat{\boldsymbol{\epsilon}} \geq 0}$$

Solution 1°: using the inverse mean curvature flow (IMCF)

- choose a maximal subset of the fields $\{h_{ij}\,;\,\rho:\Sigma\to\mathbb{R},\ \rho^i=(\partial_\rho)^i\}$ such that

$$\boxed{\widehat{N}\widehat{K}^l{}_l=\overline{\widehat{N}\widehat{K}^l{}_l}=\mathscr{L}_\rho\log[\mathscr{A}_\rho]}\qquad \boxed{\int_{\mathscr{S}_\rho}\Big[\,{}^{(3)}\!R+\mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl}+2\,\widehat{N}^{-2}\,(\widehat{D}_k\widehat{N})(\widehat{D}^k\widehat{N})\Big]\,\widehat{\boldsymbol{\epsilon}}\geq 0}$$

- what is if we keep (Σ,h_{ij}) but drop $\rho:\Sigma\to\mathbb{R}$ and the shift from $\rho^i=(\partial_\rho)^i$

The foliation and part of the flow is to be determined dynamically

- the **inverse mean curvature flow**

$$\boxed{\rho^i_{\{IMCF\}}=\left(\widehat{K}^l{}_l\right)^{-1}\widehat{n}^i+\widehat{N}^i_{\{IMCF\}}}$$

- as for the corresponding foliation $\widehat{N}\widehat{K}^l{}_l\equiv 1$ hold: if this flow existed globally the Geroch mass would be non-decreasing w.r.t it
- one can relax these condition by using a **generalized IMCF**

$$\boxed{\rho^i=\mathscr{L}_\rho(\log[\mathscr{A}_\rho])\,\rho^i_{\{IMCF\}}}$$

- **(!)** global existence and regularity remains a serious issue

Solution 2°: using a prescribed, globally existing foliation

- choose a maximal subset of the fields $\{h_{ij}\,;\, \rho:\Sigma\to\mathbb{R},\ \rho^i=(\partial_\rho)^i\}$ such that

$$\boxed{\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]} \qquad \boxed{\int_{\mathscr{S}_\rho}\Big[\,{}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}{}^{kl} + 2\,\widehat{N}^{-2}\,(\widehat{D}_k\widehat{N})(\widehat{D}^k\widehat{N})\Big]\,\widehat{\boldsymbol{\epsilon}} \geq 0}$$

- what is if we drop the three-metric h_{ij} BUT keep a globally well-defined foliation $\rho:\Sigma\to\mathbb{R}$, a flow ρ^i and the induced metric $\widehat{\gamma}_{ij}$ on the leaves: in coordinates (ρ, x^A) adapted to the flow $\rho^i=(\partial_\rho)^i$ the induced metric: $\widehat{\gamma}_{AB}$

Using prescribed foliation, flow, induced metric: $h_{ij}\leftrightarrow \widehat{N},\ \widehat{N}^A,\ \widehat{\gamma}_{AB}$

- $\boxed{\rho^i=\widehat{N}\,\widehat{n}^i+\widehat{N}^i}$ however counterintuitive it is: we may always construct shift $\widehat{N}^i$ with desirable properties:

$$\boxed{\widehat{N}\widehat{K}^l{}_l = \tfrac{1}{2}\,\widehat{\gamma}^{ij}\,\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_i\widehat{N}^i}$$

- as $\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ wished to be guaranteed,

$$\boxed{\widehat{D}_A\widehat{N}^A = \mathscr{L}_\rho \log\big[\sqrt{\det(\widehat{\gamma}_{AB})}\,\big] - \mathscr{L}_\rho \log[\mathscr{A}_\rho] \qquad (**)}$$

Solution 2°: using prescribed foliation, flow and $\widehat{\gamma}_{AB}$

Solving $\widehat{D}_A\widehat{N}^A = \mathscr{L}_\rho \log[\sqrt{\det(\widehat{\gamma}_{AB})}\,] - \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ (**) on $\mathscr{S}_\rho$

- on topological two-spheres using then the Hodge decomposition of the shift

$$\boxed{\widehat{N}^A = \widehat{D}^A\chi + \widehat{\epsilon}^{AB}\widehat{D}_B\eta}$$

χ and η are some smooth functions on $\mathscr{S}$, (**)

$$\boxed{\widehat{D}^A\widehat{D}_A\chi = \mathscr{L}_\rho \log[\,\sqrt{\det(\widehat{\gamma}_{AB})}\,] - \mathscr{L}_\rho \log[\mathscr{A}_\rho]}$$

- solubility of this elliptic equation with smooth coefficients and source terms on the succeeding individual topological two-spheres is guaranteed
- on each sphere there is an inherent constant value of ambiguity in the solution for χ which, however, does not affect the first term in $\widehat{N}^A$
- as, by construction, **for any solution to (**)** $\boxed{\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]}$ holds the $\mathscr{S}_\rho$ foliation of Σ gets to be a **generalized inverse mean curvature foliation** with respect to the yielded three-metric:

$$\boxed{h_{ij} \quad \Longleftrightarrow \quad \widehat{N}, \widehat{N}^A, \widehat{\gamma}_{AB}}$$

A hyperbolic equation for $\widehat{N}$:

We have not done yet **(!)** $\int_{\mathscr{S}_\rho} \Big[{}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,(\widehat{D}_k\widehat{N})(\widehat{D}^k\widehat{N}) \Big]\,\widehat{\boldsymbol{\epsilon}} \geq 0$

- in clearing up the picture let us have a glance again of the key equation

$$\boxed{{}^{(3)}R = \widehat{R} - \Big\{ 2\,\mathscr{L}_{\hat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N} \Big\}}$$

- by exactly the same type of arrangements we used before one gets

$$\int_{\mathscr{S}_\rho} \Big[{}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,(\widehat{D}_k\widehat{N})(\widehat{D}^k\widehat{N}) \Big]\,\widehat{\boldsymbol{\epsilon}}$$
$$= \int_{\mathscr{S}_\rho} \Big[\widehat{R} - \Big\{ 2\,\mathscr{L}_{\hat{n}}(\widehat{K}^l{}_l) + \tfrac{3}{2}\,(\widehat{K}^l{}_l)^2 \Big\} \Big]\,\widehat{\boldsymbol{\epsilon}} = f \geq 0 \qquad (**)$$

- remarkably, (*) is **a first order hyperbolic equation** for $W = \widehat{N}^{-2}$

$$\boxed{\overset{\star}{K}\,[\,\partial_\rho W - \widehat{N}^A\hat{D}_A W\,] = [\,\partial_\rho \overset{\star}{K} - \widehat{N}^A\hat{D}_A\overset{\star}{K} + \tfrac{3}{2}\,\overset{\star}{K}{}^2\,]\,W + (\,\widehat{R} - F\,)}$$

- $\boxed{\overset{\star}{K} = \widehat{N}\widehat{K}^A{}_A = \tfrac{1}{2}\,\widehat{\gamma}^{AB}[\mathscr{L}_\rho\widehat{\gamma}_{AB} - \widehat{D}_{(A}\widehat{N}_{B)}] = \mathscr{L}_\rho \log[\mathscr{A}_\rho] \;\;\&\;\; \int_{\mathscr{S}_\rho} F\,\widehat{\boldsymbol{\epsilon}} = f \geq 0}$

$$\overset{\star}{K}\,[\,\partial_\rho W - \widehat{N}^A \hat{D}_A W\,] = [\,\partial_\rho \overset{\star}{K} - \widehat{N}^A \hat{D}_A \overset{\star}{K} + \tfrac{3}{2}\,\overset{\star}{K}^2\,]\,W + (\,\widehat{R} - F\,) \qquad (***)$$

- it is **a linear FOSH equation** for $w = \widehat{N}^{-2}$
- it is standard to obtain **global existence of unique solutions** to this **linear** first order (symmetric) hyperbolic equation

Theorem

Let $\widehat{N}^A$ be a smooth vector field that is constructed using the given distribution of two-metrics $\widehat{\gamma}_{AB}$ such that $\overset{\star}{K} = \overline{\widehat{N}\,\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ holds throughout Σ. Assume that smooth positive initial data ${}_{(0)}W$ had also been chosen to the linear first order (symmetric) hyperbolic equation (***) on a $\rho = \rho_0$ level surface in Σ. Then, for any choice of a smooth real function $F : \Sigma \to \mathbb{R}$ with non-negative integral $\int_{\mathscr{S}_\rho} F\,\widehat{\boldsymbol{\epsilon}} \geq 0$, there exists a smooth global unique solution $W = \widehat{N}^{-2}$ to (***) such that it remains positive and bounded away from zero and infinity for all admissible values of ρ, in particular, for all $\rho \geq \rho_0$.

Indirect control on the scalar curvature ${}^{(3)}R$:

- the scalar curvature ${}^{(3)}R$ indirectly affected by the choice of the non-negative function $f : \Sigma \to \mathbb{R}$, or, alternatively, by that of the function F, that is related to f via the relation

$$\boxed{\int_{\mathscr{S}_\rho} F\,\widehat{\boldsymbol{\epsilon}} = f \geq 0}$$

  to see this recall that ${}^{(3)}R$ is the scalar curvature of the metric

$$\boxed{\widehat{N}, \widehat{N}^A, \widehat{\gamma}_{AB}}$$

  and (***) is to determine the lapse $\widehat{N}$

- if, for instance, $F = 0$ by solving

$$\boxed{\overset{\star}{K}\left[\,\partial_\rho W - \widehat{N}^A \hat{D}_A W\,\right] = \left[\,\partial_\rho \overset{\star}{K} - \widehat{N}^A \hat{D}_A \overset{\star}{K} + \tfrac{3}{2}\,\overset{\star}{K}{}^2\,\right] W + \widehat{R}}$$

  for the scalar curvature ${}^{(3)}R \leq 0$ holds everywhere on Σ

The variety of three-spaces:

- in order to see the extent of the variance of three-spaces we may construct by the above outlined process we have to take into account the variance of freedom we have in fixing the lapse and shift

$$\boxed{\widehat{N}, \widehat{N}^A}$$

- for instance, in constructing the shift

$$\boxed{\widehat{N}^A = \widehat{D}^A\chi + \widehat{\epsilon}^{AB}\widehat{D}_B\eta}$$

  η could be chosen to be arbitrary on Σ

- in getting the actual form of (***) the choice for the function $F : \Sigma \to \mathbb{R}$ is essentially free (with $F \sim F + \Delta\mathcal{F}$: $\mathcal{F}$ is arbitrary!)
- the choice we have to make for the initial data ${}_0\widehat{N}$, that is a smooth positive function on a $\rho = \rho_0$ id. surface, is also free

Getting initial data with all the preferable properties?

- could the proposed new method also be used to get sensible initial data (h_{ij}, K_{ij}) for Einstein's equations such that
  - the initial data surface gets to be foliated by quasi-convex topological two-spheres, and such that
  - the Geroch mass is non-decreasing with respect to the foliation and flow
- two alternative evolutionary methods were introduced to solve the constraints

  [Rácz I: *Constrains as evolutionary systems*, Class. Quant. Grav. **33** 015014 (2016)]
- in the algebraic-hyperbolic formulation, the entire Riemannian three-metric is part of the freely specifiable fields on Σ
- combine the new construction outlined in the previous part with solving the algebraic-hyperbolic form of the constraints
  - first a Riemannian three-metric h_{ij} should be constructed with all the aforementioned preferable properties and then,
  - using this as an input, one has to solve the algebraic-hyperbolic form of the constraints for suitable parts of K_{ij}

Summary:

a construction was introduced: which could be used to get a high variety of Riemannian three-spaces (and initial data sets) such that

1. the prescribed, whence globally existing regular foliation and flow: get to be **generalized inverse mean curvature foliation** : $\widehat{N}\widehat{K}^{l}{}_{l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ & the flow gets to be a **generalized IMCF**
2. the **Geroch mass**—that can be evaluated on the leaves of the foliations—is **non-decreasing** w.r.t. the applied foliation and flow
3. the topology of Σ **could be**: $\mathbb{R}^3$, $\mathbb{S}^3$, $\mathbb{R}\times\mathbb{S}^2$, $\mathbb{S}^1\times\mathbb{S}^2$, $(1,2,0,0)$
4. our proposal, yielding Riemannian three-spaces, **applies to wide range of geometrized theories of gravity**
   - concerning the metric (on M or on Σ): no use of Einstein's equations or any other field equation had been applied anywhere in our construction
   - as only the Riemannian character of the metric on Σ was used the signature of the metric on the ambient space could be either Lor. or Euc.


謝謝
Thank You