Symmetries of vacuum spacetimes with a compact Cauchy horizon of constant non-zero surface gravity

István Rácz
(joint work with Oliver Lindblad Petersen)
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istvan.racz@fuw.edu.pl & racz.istvan@wigner.mta.hu

Faculty of Physics, University of Warsaw, Warsaw, Poland
Wigner Research Center for Physics, Budapest, Hungary

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Singularities

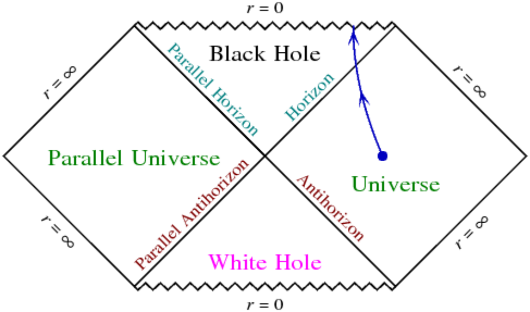
Incompleteness???

- tacitly it is always assumed that the base manifold of a spacetime represents all the events compatible with the history of the investigated physical system
- in the late 60's the "singularity theorems" of Penrose and Hawking provided a big challenge by predicting the existence of incomplete causal geodesics
  - spacetimes describing the expanding universe and the gravitational collapse of stars, under very plausible conditions, are causal geodesically incomplete
  - singularity $\Longleftrightarrow$ synonym of causal geodesic incompleteness

The weak cosmic censor conjecture

Singularity $\iff$ synonym of causal geodesic incompleteness

- it is anticipated that geodesic incompleteness should always come together with unbounded growth of ...
- BUT **no way to control** physics at "edges of the spacetime" (**!**) at best they should not influence observations far away from them
  - they are not visible, synonym
  - they are not naked (dressed/covered by event horizons)

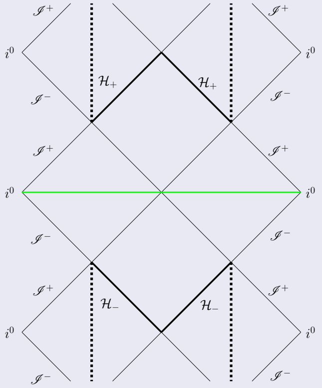


The strong cosmic censor conjecture

Are there bounds on the predictive power of GR ???

- Penrose came up with **a more philosophical question**:
  Could the predictive power of Einstein's theory be limited ???
- $\exists$ maximal spacelike hypersurfaces such that the Cauchy development of the data deduced on them is part of the original spacetime with incomplete geodesic in $D(\Sigma)$

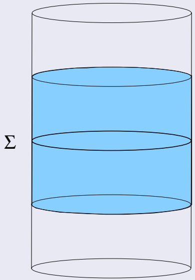
Kerr spacetime



Taub-NUT spacetime

$$M \approx \mathbb{R} \times \mathbb{S}^3$$

$$ds^2 = 4\,dt\sigma_1 + 4\,\frac{1-t^2}{1+t^2}\,\sigma_1^2 + (1+t^2)\,(\sigma_2^2 + \sigma_3^2)$$



The strong cosmic censor conjecture

SCCH of Penrose and a conjecture by Moncrief and Isenberg '83

- **Penrose's strong cosmic censorship conjecture:** Sufficiently generic spacetimes are maximal globally hyperbolic developments and they are never part of a larger spacetime.
- in the early 80's Moncrief carried out comprehensive investigations of various cosmological spacetimes
- spacetimes with a compact Cauchy horizon always have a Killing vector field that is null on the horizon $\mathcal{H}$ and spacelike on $D(\Sigma)$
- as spacetimes admitting Killing vector fields are non-generic—if the conjecture of Moncrief and Isenberg is true—we have an indirect verification of the SCCH in this special case

"Minkowski" spacetime

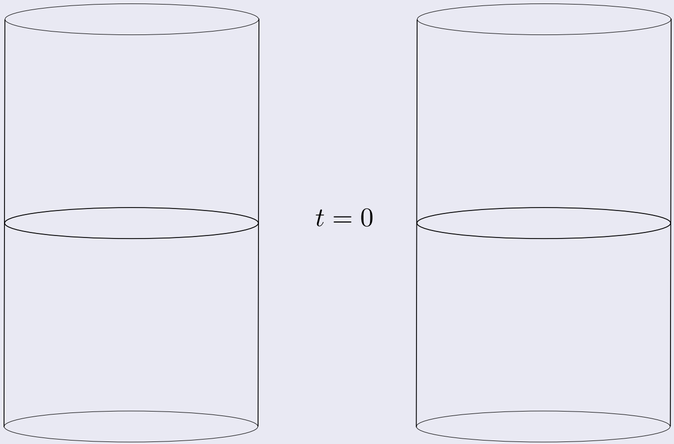
$M \approx \mathbb{R} \times \mathbb{S}^1$

$ds^2 = -dt^2 + dx^2$

Misner spacetime

$M \approx \mathbb{R} \times \mathbb{S}^1$

$ds^2 = 2\,dt\,dx + t\,dx^2$



Spacetimes with compact Cauchy surfaces

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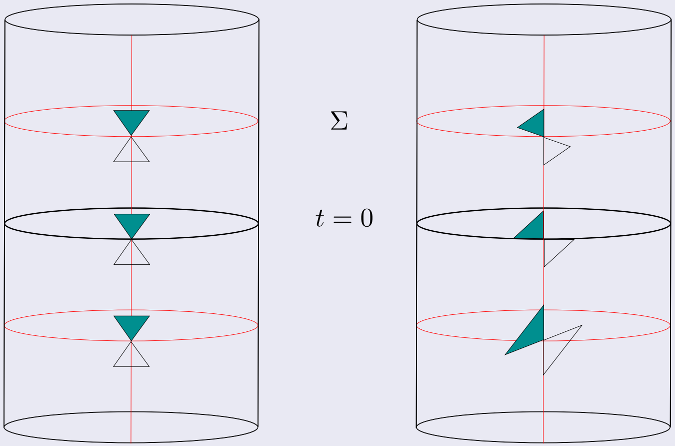

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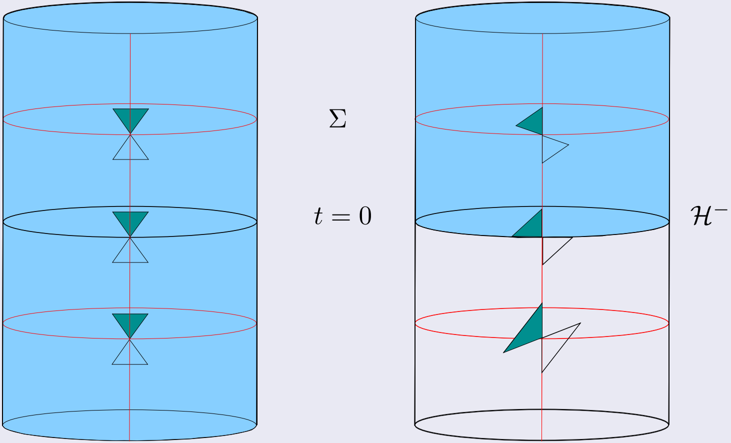

Assumptions (I)

- spacetimes: $\boxed{(M,g)}$ C^∞ connected time-oriented dim. $n+1(\geq 2)$
- Σ is a closed acausal topological hypersurface in M: with **Cauchy development** $D(\Sigma)$: a globally hyperbolic submanifold in (M,g)
- $\boxed{\mathcal{H}^\pm := \overline{D^\pm(\Sigma)} \setminus D^\pm(\Sigma)}$: future and past **Cauchy horizon**, resp.
  $\boxed{\partial D(\Sigma) = \mathcal{H}^+ \sqcup \mathcal{H}^-}$ **non-empty** and **compact**
- **Hawking, Larsson, Minguzzi:** assume $R_{ab}L^aL^b \geq 0$ for all null vectors L^a then $\mathcal{H}$ is a smooth **totally geodesic** submanifold
- $\exists$ a C^∞ fn. κ and a C^∞ non-vanishing vector field V, tangent to the generators of $\mathcal{H}$, $\boxed{\nabla_V V = \kappa V}$
  - (M,g) is time-oriented $\Longrightarrow \exists$ a nowhere vanishing timelike vector field T
  - T transversal to $\mathcal{H}$
    $\Longrightarrow \exists$ a one-form β on $\mathcal{H}$: $\beta(T) = 1$ & $\beta(X) = 0\ \forall\, X \in T\mathcal{H}$
  - $\exists$ a nowhere vanishing lightlike vector field V tangent to the generators $\mathcal{H}$: $\beta = g(V,.)$ on $\mathcal{H}$

Assumptions (II)

- **assumption:** $\exists$ a non-vanishing lightlike tangent vector field V, tangent to $\mathcal{H}$, and a **non-zero constant** κ $\boxed{\nabla_V V = \kappa\, V}$
  Def.: if this happen it is said that surface gravity is **normalised**
- in all known examples (horizons with closed generators or densely filling T^2's): κ **is always a non-zero constant**
- $\mathcal{H}$ is tot. geod. $\Longrightarrow$ $\boxed{g(\nabla_X V, Y) = 0}$ $\forall\, X, Y \in T\mathcal{H} \Longrightarrow \exists\, \omega$ on $\mathcal{H}$:

$$\boxed{\nabla_X V = \omega(X)\, V}$$

- ω is nowhere vanishing as for tangent field V $\boxed{\nabla_V V = \kappa\, V}$

$$\boxed{\omega(V) = \kappa = const \neq 0}$$

- since ω is non-vanishing $E := \ker(\omega)$ is a sub-vector bundle:
  (g is pos.def. on E)

$$\boxed{T\mathcal{H} = \mathbb{R}V \oplus E}$$

The main result:

Theorem: [Existence of a Killing vector field]

Assume that $R_{ab} = 0$ and that $\mathcal{H}$ is a compact Cauchy horizon in (M, g) such that its surface gravity can be normalised to a non-zero constant. Then (*regardless of the structure of the space of generators*) there exists a smooth non-trivial Killing vector field W^a on $\mathcal{H} \sqcup D(\Sigma)$, i.e.

$$\boxed{\mathcal{L}_W g_{ab} = 0}$$

W^a is lightlike on $\mathcal{H}$ and spacelike in $D(\Sigma)$ near $\mathcal{H}$, and any smooth extension of W^a across $\mathcal{H}$ to the complement of $\overline{D(\Sigma)}$ is timelike near the horizon $\mathcal{H}$.

The main steps:

(1) the Killing equation can be solved up to any order on $\mathcal{H}$

(2) wave equations with initial data on a compact Cauchy horizon $\Longrightarrow$ this Killing vector field extends to the globally hyperbolic region

The key technical elements (I.)

Lemma:

For any smooth vector field V^a on an n-dimensional differentiable manifold M endowed with a metric g_{ab} the following identities hold

$$\nabla^e\nabla_e V^a - \nabla^f\big[\mathcal{L}_V g_{fh} + (\nabla_e V^e)\, g_{fh}\big]g^{ha} = -\,R^a{}_f V^f$$

$$\nabla^e\nabla_e(\mathcal{L}_V g_{ab}) + 2\,R_a{}^e{}_b{}^f(\mathcal{L}_V g_{ef}) + \mathcal{L}_{[-\nabla^e\nabla_e V]}g_{ab} = -\,2\,\mathcal{L}_V R_{ab}$$

$$\mathcal{L}_{[R^e{}_f V^f]}g_{ab} + 2\,R_{(a|}{}^f\mathcal{L}_V g_{f|b)}$$

Remarks:

- no use of Einstein's equations or any other filed equation
- hold for any sufficiently regular vector field V^a
- the signature of g_{ab} does not play any role so it could be arbitrary
- But in the Lorentzian signature case on the horizon ...
  $\nabla^e\nabla_e \ldots = -2\,\nabla_V\nabla_t \ldots + c\,\nabla_t \ldots + \{\text{l.o.lin.diff.op. acting along } \mathcal{H}\}$

The key technical elements (II.)

Lemma:

Assume that $\mathfrak{a}$ is a smooth symmetric 2-tensor field on $\mathcal{H}$ (e.g. $\mathfrak{a}_{ab} = (\nabla_t)^k [\mathcal{L}_V g_{ab}]$) and β is a nowhere vanishing function such that

$$\boxed{\nabla_V \mathfrak{a}(X,Y) + \beta\, \mathfrak{a}(X,Y) = 0}$$

for all $X, Y \in E$. Then $\mathfrak{a}(X,Y) = 0$ for all $X, Y \in E$.

Remarks:

- g is positive definite on $E \quad \Longrightarrow \quad \mathfrak{g}(\mathfrak{a},\mathfrak{a}) := g^{ik} g^{jl}\, \mathfrak{a}_{ij}\mathfrak{a}_{kl}$ is a positive definite metric on $E^* \otimes_{sym} E^* (\subset E^* \otimes E^*)$
-
$$\boxed{\partial_V \mathfrak{g}(\mathfrak{a},\mathfrak{a}) + 2\beta\, \mathfrak{g}(\mathfrak{a},\mathfrak{a}) = 0}$$
- $\mathcal{H}$ is compact, the function $\mathfrak{g}(\mathfrak{a},\mathfrak{a})$ must attain its max. and min.
- $\partial_V \mathfrak{g}(\mathfrak{a},\mathfrak{a}) = 0$ at these locations $\&$ $\beta \neq 0 \Longrightarrow \mathfrak{g}(\mathfrak{a},\mathfrak{a}) = 0$ **!!!** no need to refer to the individual generators **!!!**

The key technical elements (III.)

Lemma: [The case $m = 0$]

Assume that $R_{ab}|_{t=0} = 0$. Then

$$[\,\mathcal{L}_V g_{ab}\,]|_{t=0} = 0$$
$$[\,\nabla^e\nabla_e V^a\,]|_{t=0} = 0\,.$$

Lemma: [The induction step]

Let $m \in \mathbb{N}$. Assume that $(\nabla_t)^k R_{ab}|_{t=0} = 0$ for all $k \leq m+1$, and that for all $k \leq m$

$$(\nabla_t)^k\,[\,\mathcal{L}_V g_{ab}\,]|_{t=0} = 0$$
$$(\nabla_t)^k\,[\,\nabla^e\nabla_e V^a\,]|_{t=0} = 0$$

Then

$$(\nabla_t)^{m+1}\,[\,\mathcal{L}_V g_{ab}\,]|_{t=0} = 0$$
$$(\nabla_t)^{m+1}\,[\,\nabla^e\nabla_e V^a\,]|_{t=0} = 0\,.$$

Summary:

- we proved that any smooth vacuum spacetime containing a compact Cauchy horizon with surface gravity that can be normalised to a non-zero constant

  (*regardless of the structure of the space of generators*)

  admits a Killing vector field
- this proves a conjecture by Moncrief and Isenberg from 1983: the maximal globally hyperbolic vacuum development of generic initial data cannot be extended across a compact Cauchy horizon

  (provided that surface gravity can be normalised to a non-zero constant)
- our result supports, thereby, the validity of the strong cosmic censorship conjecture of Penrose in the considered special case