On the construction of Riemannian three-spaces with smooth generalized inverse mean curvature flows

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General Relativity, Geometry and Analysis:
beyond the first 100 years after Einstein
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Motivations:

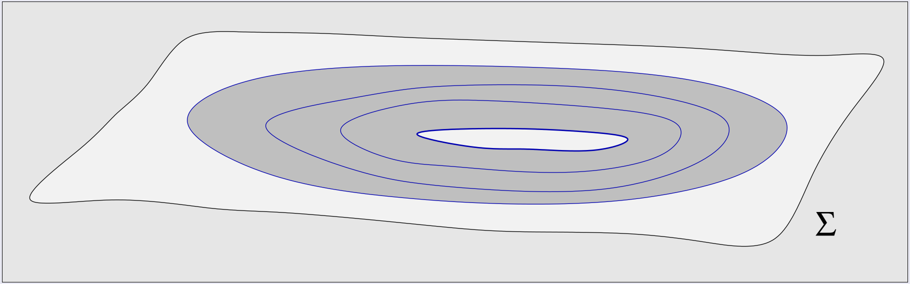
GR is a **metric theory of gravity**:

- it is highly non-trivial to assign, in a sensible way, mass, energy, linear and angular momenta to bounded spatial regions
- "... it is almost certain that we have to understand conserved (or quasi conserved) quantities which can control the field in a more local manner. In other words, we expect some concept of quasi-local mass will be useful."
- efforts to prove the **positive energy theorem** and the **Penrose inequalities** using quasi-local techniques Geroch (1973), Wald, Jang (1977), Jang (1978), Kijowski (1986), Chruściel (1986), Jezierski, Kijowski (1987), Huisken, Ilmanen (1997, 2001), Frauendiener (2001), Bray (2001), Malec, Mars, Simon (2002), Bray, Lee (2009),. . .

The aim is to outline:

- a **simple construction** of a high variety of Riemannian three-spaces with prescribed, whence globally existing regular foliation and flow such that
  - the (quasi-local) **Geroch energy**—that can be evaluated on the leaves of the foliations—is **non-decreasing** with respect to the applied flow, and
  - the foliation gets to be **a generalized** inverse mean curvature one

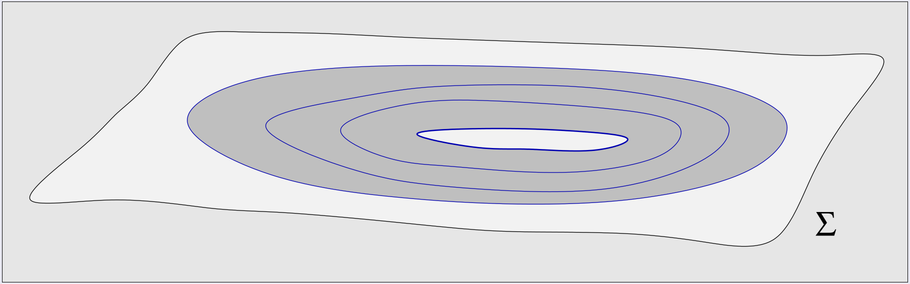
Foliations by topological two-spheres:



- consider a smooth 3-dimensional manifold Σ with a Riemannian metric h_{ij}
- assume $$\boxed{\Sigma \approx \mathbb{R} \times \mathscr{S}}$$ $origin(s)(!)$

  i.e. Σ is smoothly foliated by a one-parameter family of top. two-spheres $\mathscr{S}_\rho$: $\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$
- $$\Longrightarrow \quad \boxed{\partial_i \rho \;\;\&\;\; h^{ij} \;\longrightarrow\; \widehat{n}_i \,, \widehat{n}^i = h^{ij} \widehat{n}_j \;\;\ldots\;\; \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i \widehat{n}_j}$$
- '$\widehat{\ }$' to distinguish quantities that could also be viewed as fields on the leaves

The first and second fundamental forms:



- **the induced Riemannian metric** on the $\mathscr{S}_\rho$ level sets

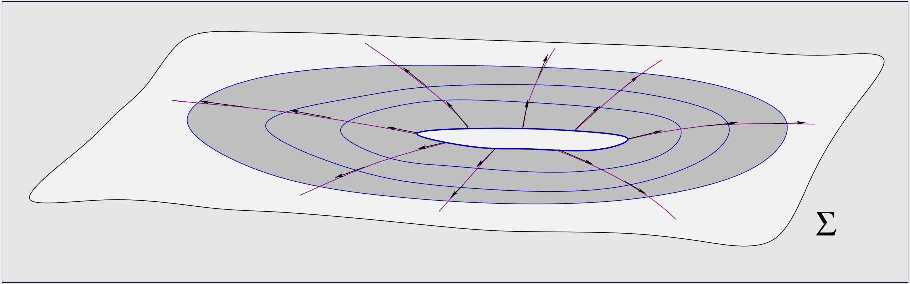
$$\boxed{\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j h_{kl}}$$

- **the extrinsic curvature** given by the symmetric tensor field

$$\boxed{\widehat{K}_{ij} = \widehat{\gamma}^l{}_i D_l \, \widehat{n}_j = \tfrac{1}{2} \mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij} ,} \qquad D_i, \mathscr{L}_{\widehat{n}}$$

- a $\rho = const$ level surface is called to be **mean-convex** if its mean curvature, $\boxed{\widehat{K}^l{}_l = \widehat{\gamma}^{ij} \widehat{K}_{ij} = \widehat{\gamma}^{ij} D_i \, \widehat{n}_j}$, is positive on $\mathscr{S}_\rho$

Flows:



- a smooth vector field ρ^i on Σ **is a flow** ("evolution vector field") w.r.t. $\mathscr{S}_\rho$
  - if the integral curves of ρ^i **intersect each leaves precisely once**, and
  - if ρ^i **is scaled such that** $\boxed{\rho^i \partial_i \rho = 1}$ holds throughout Σ
- any smooth flow can be decomposed in terms of its '**lapse**' and '**shift**' as

$$\boxed{\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i} \qquad \boxed{\widehat{N} = \rho^i \widehat{n}_i = (\widehat{n}^i \partial_i \rho)^{-1}} \quad \boxed{\widehat{N}^i = \widehat{\gamma}^i{}_j\, \rho^j}$$

  - the lapse **measures the normal separation of the surfaces** $\mathscr{S}_\rho$

Variation of the area:

- in various cases we shall assume that the area is strictly increasing
- for this reason it may be rewarding to inspect the variation of the area

$$\mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \widehat{\boldsymbol\epsilon}$$

of the $\rho = const$ level surfaces, w.r.t. a generic $\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i$

$$\mathscr{L}_\rho \mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \mathscr{L}_\rho \widehat{\boldsymbol\epsilon} = \int_{\mathscr{S}_\rho} \left\{ \widehat{N}(\widehat{K}^l{}_l) + (\widehat{D}_i \widehat{N}^i) \right\} \widehat{\boldsymbol\epsilon} = \int_{\mathscr{S}_\rho} \widehat{N}(\widehat{K}^l{}_l)\, \widehat{\boldsymbol\epsilon},$$

the relations $\mathscr{L}_{\widehat{n}}\,\widehat{\boldsymbol\epsilon} = (\widehat{K}^l{}_l)\,\widehat{\boldsymbol\epsilon}$ and $\mathscr{L}_{\widehat{N}}\,\widehat{\boldsymbol\epsilon} = \frac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_{\widehat{N}}\widehat{\gamma}_{ij}\,\widehat{\boldsymbol\epsilon} = (\widehat{D}_i\widehat{N}^i)\,\widehat{\boldsymbol\epsilon}$, along with the vanishing of the integral of the total divergence $\widehat{D}_i\widehat{N}^i$, were applied.

- $\widehat{N}$ does not vanish on Σ unless the Riemannian three-metric

$$h^{ij} = \widehat{\gamma}^{ij} + \widehat{N}^{-2}(\rho^i - \widehat{N}^i)(\rho^j - \widehat{N}^j)$$

gets to be singular

- the area is strictly increasing if $\int_{\mathscr{S}_\rho} \widehat{N}(\widehat{K}^l{}_l)\,\widehat{\boldsymbol\epsilon} > 0$
  - for mean-convex foliations $\widehat{N}\widehat{K}^l{}_l > 0 \Longrightarrow$ the area is strictly increasing

The Geroch energy:

- the (quasi-local) Geroch energy (equal to the Hawking energy only if $\widehat{\gamma}^{ij} K_{ij} = 0$)

$$E_{\mathcal{G}} = \frac{\mathscr{A}_\rho^{1/2}}{64\pi^{3/2}} \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2 \right] \widehat{\boldsymbol{\epsilon}}$$

  where $\widehat{R}$ is the scalar curvature of the metric $\widehat{\gamma}_{ij}$ on the leaves
- for mean-convex foliations the area $\mathscr{A}_\rho$ is monotonously increasing
- it suffices to investigate

$$W(\rho) = \int_{\mathscr{S}_\rho} \left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2 \right] \widehat{\boldsymbol{\epsilon}}$$

- **if both $\mathscr{A}_\rho$ and $W(\rho)$ were non-decreasing**, and for some specific ρ_* value, $W(\rho_*)$ was zero or positive then $\boxed{E_{\mathcal{G}} \geq 0}$ would hold to the exterior of $\mathscr{S}_{\rho_*}$ in Σ

The variation of $W(\rho)$:

- the **key equation relates the scalar curvatures** of h_{ij} and $\widehat{\gamma}_{ij}$

$$ {}^{(3)}R = \widehat{R} - \Big\{ 2\, \mathscr{L}_{\widehat{n}} (\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\, \widehat{N}^{-1}\, \widehat{D}^l \widehat{D}_l \widehat{N} \Big\} \qquad (*) $$

$$ \begin{aligned} \mathscr{L}_\rho W = & -\int_{\mathscr{S}_\rho} \mathscr{L}_\rho \Big[\big(\widehat{K}^l{}_l\big)^2 \widehat{\boldsymbol{\epsilon}} \Big] = -\int_{\mathscr{S}_\rho} \Big\{ \widehat{N}\, \mathscr{L}_{\widehat{n}} \Big[\big(\widehat{K}^l{}_l\big)^2 \widehat{\boldsymbol{\epsilon}} \Big] + \mathscr{L}_{\widehat{N}} \Big[\big(\widehat{K}^l{}_l\big)^2 \widehat{\boldsymbol{\epsilon}} \Big] \Big\} \\ = & -\int_{\mathscr{S}_\rho} \big(\widehat{N}\widehat{K}^l{}_l\big) \Big[2\, \mathscr{L}_{\widehat{n}} \,(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 \Big] \widehat{\boldsymbol{\epsilon}} - \int_{\mathscr{S}_\rho} \widehat{D}_i \Big[\big(\widehat{K}^l{}_l\big)^2 \widehat{N}^i \Big] \widehat{\boldsymbol{\epsilon}} \\ = & -\int_{\mathscr{S}_\rho} \big(\widehat{N}\widehat{K}^l{}_l\big) \Big[\big(\widehat{R} - {}^{(3)}R\big) - \widehat{K}_{kl}\widehat{K}^{kl} - 2\, \widehat{N}^{-1}\, \widehat{D}^l \widehat{D}_l \widehat{N} \Big] \widehat{\boldsymbol{\epsilon}} \end{aligned} $$

- where on 1^{st} line $\rho^i = \widehat{N}\, \widehat{n}^i + \widehat{N}^i$ and the Gauss-Bonnet theorem
- on 2^{nd} line the relations $\mathscr{L}_{\widehat{n}}\, \widehat{\boldsymbol{\epsilon}} = (\widehat{K}^l{}_l)\, \widehat{\boldsymbol{\epsilon}}$ and $\mathscr{L}_{\widehat{N}}\, \widehat{\boldsymbol{\epsilon}} = (\widehat{D}_i \widehat{N}^i)\, \widehat{\boldsymbol{\epsilon}}$
- on 3^{rd} line (*) and the vanishing of the integral of $\widehat{D}_i \Big[\big(\widehat{K}^l{}_l\big)^2 \widehat{N}^i \Big]$ were used

The variation of $W(\rho)$:

- by the Leibniz rule

$$\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N} = \widehat{D}^l\big(\widehat{N}^{-1}\widehat{D}_l\widehat{N}\big) + \widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})$$

- and by introducing the trace-free part of $\widehat{K}_{ij}$

$$\overset{\circ}{\widehat{K}}_{ij} = \widehat{K}_{ij} - \tfrac{1}{2}\,\widehat{\gamma}_{ij}\,(\widehat{K}^l{}_l), \qquad \widehat{K}_{kl}\widehat{K}^{kl} = \overset{\circ}{\widehat{K}}_{kl}\overset{\circ}{\widehat{K}}{}^{kl} + \tfrac{1}{2}\,(\widehat{K}^l{}_l)^2$$

- and using the vanishing of the integral of the total divergence $\widehat{D}^l(\widehat{N}^{-1}\widehat{D}_l\widehat{N})$

$$\begin{aligned}\mathscr{L}_\rho W = & -\tfrac{1}{2}\int_{\mathscr{S}_\rho}(\widehat{N}\widehat{K}^l{}_l)\left[2\,\widehat{R} - (\widehat{K}^l{}_l)^2\right]\widehat{\boldsymbol{\epsilon}} \\ & + \int_{\mathscr{S}_\rho}(\widehat{N}\,\widehat{K}^l{}_l)\left[{}^{(3)}R + \overset{\circ}{\widehat{K}}_{kl}\overset{\circ}{\widehat{K}}{}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N})\right]\widehat{\boldsymbol{\epsilon}}\end{aligned}$$

Rigidity of the setup:

- **if** the product $\widehat{N}\widehat{K}^l{}_l$ **could be replaced** by its **mean value**
- $$\overline{\widehat{N}\widehat{K}^l{}_l} = \frac{\int_{\mathscr{S}_\rho} \widehat{N}\widehat{K}^l{}_l \,\widehat{\epsilon}}{\int_{\mathscr{S}_\rho} \widehat{\epsilon}} \qquad \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]$$

$$[(64\,\pi^{3/2})/(\mathscr{A}_\rho)^{1/2}] \cdot \mathscr{L}_\rho\, E_{\mathcal{G}} = \mathscr{L}_\rho W + \tfrac{1}{2}\,(\mathscr{L}_\rho \log[\mathscr{A}_\rho])\, W \geq 0$$

provided that $\int_{\mathscr{S}_\rho} \Big[\, {}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,\widehat{\gamma}^{kl}\,(\widehat{D}_k\widehat{N})(\widehat{D}_l\widehat{N}) \Big]\,\widehat{\epsilon} \geq 0$

- once in addition to h_{ij} **a foliation and a flow are fixed** not only the **mean curvature** $\widehat{K}^l{}_l$ **BUT** the **lapse** $\widehat{N}$ and the **shift** $\widehat{N}^i$ get also to be fixed

$$\widehat{K}^l{}_l = \widehat{\gamma}^{ij}\widehat{K}_{ij} = \widehat{\gamma}^{ij} D_i\,\widehat{n}_j \qquad \widehat{N} = \rho^i\widehat{n}_i = (\widehat{n}^i\partial_i\rho)^{-1} \qquad \widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j$$

- the only "freedom" is a relabeling of the leaves by using a function $\overline{\rho} = \overline{\rho}(\rho)$ but this cannot yield more than a rescaling $\widehat{N} \to \widehat{N}(\mathrm{d}\rho/\mathrm{d}\overline{\rho})$ of the lapse
- **(!)** at best $\widehat{N}\widehat{K}^l{}_l$ is merely a smooth function on the leaves of the foliation

How to get control on the monotonicity?

What we have by hands: $\{\widehat{N}, \widehat{N}^A, \widehat{\gamma}_{AB}\,;\, \rho : \Sigma \to \mathbb{R},\, \rho^i = (\partial_\rho)^i\}$

- a Riemannian metric h_{ij} defined on a three-surface Σ
- Σ is foliated by topological two-spheres: $\Sigma \approx \mathbb{R} \times \mathbb{S}^2$ ..... $\rho : \Sigma \to \mathbb{R}$ is chosen
- a flow ρ^i was also fixed on Σ such that $\rho^i \partial_i \rho = 1$
- the later two can be used to introduce coordinates (ρ, x^A) adapted to the flow ρ^i such that $\boxed{\rho^i = (\partial_\rho)^i}$, and such that the shift $\widehat{N}^i$ and the metric $\widehat{\gamma}_{ij}$ can be given as a two-vector $\boxed{\widehat{N}^A}$ and a non-singular 2×2 matrix $\boxed{\widehat{\gamma}_{AB}}$ both smoothly depending on the coordinates ρ, x^A, where A takes the values $2, 3$
- line element of the Riemannian metric h_{ij} (can be given in its 'ADM' form)

$$\boxed{\mathrm{d}s^2 = \widehat{N}^2 \mathrm{d}\rho^2 + \widehat{\gamma}_{AB} \left(\mathrm{d}x^A + \widehat{N}^A \mathrm{d}\rho\right)\left(\mathrm{d}x^B + \widehat{N}^B \mathrm{d}\rho\right)}$$

The challenge is:

- choose a maximal subset of the fields $\{h_{ij}\,;\, \rho : \Sigma \to \mathbb{R},\, \rho^i = (\partial_\rho)^i\}$ such that

$$\boxed{\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]} \qquad \boxed{\int_{\mathscr{S}_\rho} \left[{}^{(3)}\!R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,(\widehat{D}_k\widehat{N})(\widehat{D}^k\widehat{N}) \right] \widehat{\boldsymbol{\epsilon}} \geq 0}$$

Solution 1°: using the inverse mean curvature flow (IMCF)

- choose a maximal subset of the fields $\{h_{ij}\,;\,\rho:\Sigma\to\mathbb{R},\ \rho^i=(\partial_\rho)^i\}$ such that

$$\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho] \qquad \int_{\mathscr{S}_\rho}\Big[{}^{(3)}R + \mathring{\widehat{K}}_{kl}\mathring{\widehat{K}}^{kl} + 2\,\widehat{N}^{-2}\,(\widehat{D}_k\widehat{N})(\widehat{D}^k\widehat{N})\Big]\,\widehat{\boldsymbol{\epsilon}} \geq 0$$

- what is if we keep (Σ, h_{ij}) but drop $\rho:\Sigma\to\mathbb{R}$ and the shift from $\rho^i=(\partial_\rho)^i$

The foliation and part of the flow are to be determined dynamically

- the **inverse mean curvature flow**

$$\rho^i_{\{IMCF\}} = \big(\widehat{K}^l{}_l\big)^{-1}\,\widehat{n}^i + \widehat{N}^i_{\{IMCF\}}$$

- as for the corresponding foliation $\widehat{N}\widehat{K}^l{}_l \equiv 1$ holds: if this flow existed globally the Geroch energy would be non-decreasing
- one can relax these conditions by using a **generalized IMCF**

$$\rho^i = \mathscr{L}_\rho(\log[\mathscr{A}_\rho])\,\rho^i_{\{IMCF\}}$$

- **(!)** global existence and regularity remain a serious issue

Solution 2°: using a prescribed, globally existing foliation

- choose a maximal subset of the fields $\{h_{ij}\,;\, \rho:\Sigma\to\mathbb{R},\ \rho^i=(\partial_\rho)^i\}$ such that

$$\widehat N \widehat K^l{}_l = \overline{\widehat N \widehat K^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho] \qquad \int_{\mathscr{S}_\rho}\Big[\,{}^{(3)}\!R + \mathring{\widehat K}_{kl}\mathring{\widehat K}^{kl} + 2\,\widehat N^{-2}\,(\widehat D_k \widehat N)(\widehat D^k \widehat N)\Big]\,\widehat{\boldsymbol\epsilon} \geq 0$$

- what is if we drop the three-metric h_{ij} BUT keep a globally well-defined foliation $\rho:\Sigma\to\mathbb{R}$, a flow ρ^i and the induced metric $\widehat\gamma_{ij}$ on the leaves: in coordinates (ρ, x^A) adapted to the flow $\rho^i=(\partial_\rho)^i$ the induced metric: $\widehat\gamma_{AB}$

Using prescribed foliation, flow, induced metric: $h_{ij} \leftrightarrow \widehat N,\ \widehat N^A,\ \widehat\gamma_{AB}$

- $\rho^i = \widehat N\,\widehat n^i + \widehat N^i$ however counter-intuitive it is: we may always construct shift $\widehat N^i$ with desirable properties:

$$\widehat N \widehat K^l{}_l = \tfrac{1}{2}\,\widehat\gamma^{ij}\,\mathscr{L}_\rho \widehat\gamma_{ij} - \widehat D_i \widehat N^i$$

- as $\widehat N \widehat K^l{}_l = \overline{\widehat N \widehat K^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ wished to be guaranteed,

$$\widehat D_A \widehat N^A = \mathscr{L}_\rho \log\big[\sqrt{\det(\widehat\gamma_{AB})}\,\big] - \mathscr{L}_\rho \log[\mathscr{A}_\rho] \qquad (**)$$

Solution 2°: using prescribed foliation, flow and $\widehat{\gamma}_{AB}$

Solving $\widehat{D}_A \widehat{N}^A = \mathscr{L}_\rho \log[\sqrt{\det(\widehat{\gamma}_{AB})}\,] - \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ (**) on $\mathscr{S}_\rho$

- on topological two-spheres using then the Hodge decomposition of the shift

$$\widehat{N}^A = \widehat{D}^A \chi + \widehat{\epsilon}^{AB} \widehat{D}_B \eta$$

where χ and η are some smooth functions on $\mathscr{S}$, (**)

$$\widehat{D}^A \widehat{D}_A \chi = \mathscr{L}_\rho \log[\,\sqrt{\det(\widehat{\gamma}_{AB})}\,] - \mathscr{L}_\rho \log[\mathscr{A}_\rho]$$

- solubility of this elliptic equation with smooth coefficients and source terms on the succeeding individual topological two-spheres is guaranteed
- on each sphere there is an inherent constant value of ambiguity in the solution for χ which, however, does not affect the first term in $\widehat{N}^A$
- as, by construction, **for any solution to (**)** $\boxed{\widehat{N}\widehat{K}^l{}_l = \overline{\widehat{N}\widehat{K}^l{}_l} = \mathscr{L}_\rho \log[\mathscr{A}_\rho]}$ holds the $\mathscr{S}_\rho$ foliation of Σ gets to be a **generalized inverse mean curvature foliation** irrespective the choice of $\widehat{N}$:

$$h_{ij} \quad \Longleftrightarrow \quad \widehat{N}, \widehat{N}^A, \widehat{\gamma}_{AB}$$

Choosing the lapse:

We have not done yet **(!)** $\int_{\mathscr{S}_\rho}\Big[\,{}^{(3)}\!R + \hat{\mathring{K}}_{kl}\hat{\mathring{K}}^{kl} + 2\,\widehat{N}^{-2}\,(\widehat{D}_k\widehat{N})(\widehat{D}^k\widehat{N})\Big]\,\widehat{\boldsymbol{\epsilon}} \geq 0$

- in clearing up the picture let us have a glance again of the key equation

$$ {}^{(3)}\!R = \widehat{R} - \Big\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\,\widehat{D}^l\widehat{D}_l\widehat{N}\Big\} $$

- by exactly the same type of arrangements we used before one gets

$$ \int_{\mathscr{S}_\rho}\Big[\,\widehat{R} - \Big\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + \tfrac{3}{2}\,(\widehat{K}^l{}_l)^2\Big\}\Big]\,\widehat{\boldsymbol{\epsilon}} \geq 0 $$

-

$$ \overset{\star}{K} = \widehat{N}\,\widehat{K}^A{}_A = \widehat{\gamma}^{AB}\big[\tfrac{1}{2}\,\mathscr{L}_\rho\widehat{\gamma}_{AB} - \widehat{D}_{(A}\widehat{N}_{B)}\big] = \mathscr{L}_\rho\log[\mathscr{A}_\rho] $$

- using the replacement $(\widehat{K}^l{}_l)^2 = \overset{\star}{K}{}^2\,\widehat{N}^{-2}$ we get

$$ 2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + \tfrac{3}{2}\,(\widehat{K}^l{}_l)^2 = \overset{\star}{K}{}^{-1}\Big\{\mathscr{L}_\rho\big[\overset{\star}{K}{}^2\widehat{N}^{-2}\big] - \widehat{N}^A\widehat{D}_A\big[\overset{\star}{K}{}^2\widehat{N}^{-2}\big]\Big\} + \tfrac{3}{2}\,\big[\overset{\star}{K}{}^2\widehat{N}^{-2}\big] $$

Choosing the lapse: $\int_{\mathscr{S}_\rho}\Big[\,{}^{(3)}\!R + \hat{\mathring{K}}_{kl}\hat{\mathring{K}}^{kl} + 2\,\widehat{N}^{-2}\,(\widehat{D}_k\widehat{N})(\widehat{D}^k\widehat{N})\Big]\,\widehat{\boldsymbol{\epsilon}} \geq 0$

- assuming from now on that ρ is the area "radial coordinate", i.e. $\mathscr{A}_\rho = 4\pi\rho^2$, we get

$$\overset{\star}{K} = \mathscr{L}_\rho \log[\mathscr{A}_\rho] = \tfrac{2}{\rho} \quad\Longrightarrow\quad \mathscr{L}_\rho \overset{\star}{K} = -\tfrac{2}{\rho^2}$$

$$2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + \tfrac{3}{2}\,(\widehat{K}^l{}_l)^2 = \Big\{\mathscr{L}_\rho\big(\log\big[\tfrac{2}{\rho}\,\widehat{N}^{-2}\big]\big) + \mathscr{L}_\rho\big(\log\big[\sqrt{\det(\widehat{\gamma}_{AB})}\big]\big)\Big\}\,\big[\tfrac{2}{\rho}\,\widehat{N}^{-2}\big]$$

- $$\int_{\mathscr{S}_\rho}\Big[\widehat{R} - \Big\{2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + \tfrac{3}{2}\,(\widehat{K}^l{}_l)^2\Big\}\Big]\,\widehat{\boldsymbol{\epsilon}} \geq 0$$

$\Longleftrightarrow$

$$8\pi \geq \int_{\mathscr{S}_\rho}\Big[2\,\mathscr{L}_{\widehat{n}}(\widehat{K}^l{}_l) + \tfrac{3}{2}\,(\widehat{K}^l{}_l)^2\Big]\,\widehat{\boldsymbol{\epsilon}} = \mathscr{L}_\rho\Big[\int_{\mathscr{S}_\rho}\tfrac{2}{\rho}\,\widehat{N}^{-2}\,\widehat{\boldsymbol{\epsilon}}\Big]$$

- $$\int_{\mathscr{S}_\rho}\widehat{N}^{-2}\,\widehat{\boldsymbol{\epsilon}} \leq 4\pi\,\rho\,[\rho - \rho_0] = \mathscr{A}_\rho - \sqrt{\mathscr{A}_{\rho_0}\cdot\mathscr{A}_\rho}$$

The "quasi-local" Penrose inequality:

- using $\boxed{\mathscr{A}_\rho = \int_{\mathscr{S}_\rho} \widehat{\boldsymbol\epsilon}}$ & $\boxed{\int_{\mathscr{S}_\rho} \widehat{N}^{-2}\,\widehat{\boldsymbol\epsilon} \leq \mathscr{A}_\rho - \sqrt{\mathscr{A}_{\rho_0}\cdot\mathscr{A}_\rho}}$ $\Longrightarrow$

$$\boxed{(\mathscr{A}_{\rho_0}\cdot\mathscr{A}_\rho)^{1/2} \leq \int_{\mathscr{S}_\rho}\big[1-\widehat{N}^{-2}\big]\,\widehat{\boldsymbol\epsilon}}$$

- $$\begin{aligned} E_{\mathcal{G}} &= \frac{\mathscr{A}_\rho^{1/2}}{64\pi^{3/2}}\int_{\mathscr{S}_\rho}\Big[2\,\widehat{R}-(\widehat{K}^l{}_l)^2\Big]\,\widehat{\boldsymbol\epsilon} = \frac{\mathscr{A}_\rho^{1/2}}{64\pi^{3/2}}\Big[16\pi - \int_{\mathscr{S}_\rho}\overset{\star}{K}{}^2\,\widehat{N}^{-2}\widehat{\boldsymbol\epsilon}\Big] \\ &= \frac{\mathscr{A}_\rho^{1/2}}{64\pi^{3/2}}\Big[\big(\tfrac{2}{\rho}\big)^2(4\pi\rho^2) - \int_{\mathscr{S}_\rho}\big(\tfrac{2}{\rho}\big)^2\,\widehat{N}^{-2}\widehat{\boldsymbol\epsilon}\Big] = \frac{1}{4\,\pi^{1/2}\,\mathscr{A}_\rho^{1/2}}\int_{\mathscr{S}_\rho}[1-\widehat{N}^{-2}]\widehat{\boldsymbol\epsilon} \end{aligned}$$

$\Longrightarrow$

$$\boxed{E_{\mathcal{G}} \geq \tfrac{1}{4\,\pi^{1/2}}\,\mathscr{A}_{\rho_0}^{1/2}}$$

valid for any $\rho > \rho_0$

- the "quasi-local" Penrose inequality

$$\boxed{\mathscr{A}_\rho \leq 16\pi\, E_{\mathcal{G}}^2}$$

The variety of three-spaces:

- in order to see the extent of the variance of the constructed three-spaces we have to take into account the freedom we have in fixing the lapse and shift

$$\boxed{\widehat{N}, \widehat{N}^A}$$

- for instance, in constructing the shift

$$\boxed{\widehat{N}^A = \widehat{D}^A\chi + \widehat{\epsilon}^{AB}\widehat{D}_B\eta}$$

  η could be chosen to be arbitrary on Σ

- the choice of the lapse is limited only by the integral inequality

$$\boxed{\int_{\mathscr{S}_\rho}\widehat{N}^{-2}\,\widehat{\boldsymbol{\epsilon}} \leq \mathscr{A}_\rho - \sqrt{\mathscr{A}_{\rho_0}\cdot\mathscr{A}_\rho}}$$

Summary:

a construction was introduced: which could be used to get a high variety of Riemannian three-spaces such that

1. the prescribed, whence globally existing regular foliation and flow: get to be **generalized inverse mean curvature foliation** : $\widehat{N}\widehat{K}^l{}_l = \mathscr{L}_\rho \log[\mathscr{A}_\rho]$ & the flow gets to be a **generalized IMCF**
2. the **Geroch energy** is **non-decreasing** & the **quasi-local form of the Penrose inequality** holds
3. if the metric we started with is asymptotically flat the **Penrose inequality** & the **positive energy theorem** holds
4. our proposal, yielding Riemannian three-spaces, **applies to wide range of geometrized theories of gravity**
   - as for the metric (on M or on Σ): **no use of** Einstein's equations or any other **field equations** had been applied anywhere in our construction
   - as **only the Riemannian character** of the metric on Σ was used the signature of the metric on the ambient space could be either Lor. or Euc.

Tack!