Fully constraint time evolution in Einstein theory

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How to get a fully constrained evolutionary scheme?

- one needs to uncover (intimate) relations between various parts of Einstein's equations
  - spherically symmetric case ✓
  - without symmetries ?✓?

Based on

- I. Rácz: *On the use of the Kodama vector field in spherically symmetric dynamical problems*, Class. Quantum Grav. **23**, 115-123 (2006)
- I. Rácz: *Is the Bianchi identity always hyperbolic?* Class. Quantum Grav. 31 (2014) 155004
- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'* Class. Quantum Grav. 32 (2015) 015006
- I. Rácz: *Dynamical determination of the gravitational degrees of freedom*, arXiv:1412.0667
- I. Rácz: *Constraints as evolutionary systems*, Class. Quantum Grav. **33** 015014 (2016)

Assumptions:

- **The primary space:** (M, g_{ab})
  - M : $n+1$-dim. ($n \geq 3$), smooth, paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian$_{(-,+,\dots,+)}$ or Riemannian$_{(+,\dots,+)}$ metric
- **Einsteinian space:** Einstein's equation restricting the geometry

$$G_{ab} - \mathscr{G}_{ab} = 0$$

with source term $\mathscr{G}_{ab}$ having a vanishing divergence, $\nabla^a \mathscr{G}_{ab} = 0$.

  - or, in a more familiarly looking setup

$$\left[\, R_{ab} - \tfrac{1}{2}\, g_{ab}\, R \,\right] + \Lambda\, g_{ab} = 8\pi\, T_{ab}$$

  - with matter fields satisfying their individual field equations with energy-momentum tensor T_{ab} and with cosmological constant Λ

$$\mathscr{G}_{ab} = 8\pi\, T_{ab} - \Lambda\, g_{ab}$$

Dynamics in the spherically symmetric case:

- ! (t, r, θ, ϕ) — r is the area radius: $\mathcal{A} = 4\pi r^2$
- the line element is $\mathrm{d}s^2 = -A\mathrm{d}t^2 + B\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin\theta^2\mathrm{d}\phi^2)$

  A and B are assumed to be smooth functions of the coordinates t and r
- $E_{ab} = G_{ab} - \mathscr{G}_{ab}$ with only 'four' non-trivial components: $E_{tt}, E_{tr}, E_{rr}, E_{\theta\theta}$
- $\nabla^a \mathscr{G}_{ab} = 0 \Rightarrow \nabla^a E_{ab} = 0$ (with only two non-trivial components: 't' and 'r')

$$
\begin{aligned}
&-2\,AB^2r\left(\frac{\partial}{\partial t}E_{tt}\right) + 2\,B^2rE_{tt}\frac{\partial}{\partial t}A + ABrE_{tr}\left(\frac{\partial}{\partial r}A\right) + 2\,A^2Br\left(\frac{\partial}{\partial r}E_{tr}\right) \\
&\quad - ABrE_{tt}\left(\frac{\partial}{\partial t}B\right) - A^2rE_{tr}\frac{\partial}{\partial r}B - A^2rE_{rr}\frac{\partial}{\partial t}B + 4A^2B\,E_{tr} = 0 \\
&2\,AB^2r^3\left(\frac{\partial}{\partial t}E_{tr}\right) - B^2r^3E_{tt}\frac{\partial}{\partial r}A - B^2r^3E_{tr}\frac{\partial}{\partial t}A + ABr^3E_{tr}\left(\frac{\partial}{\partial t}B\right) \\
&\quad - ABr^3E_{rr}\left(\frac{\partial}{\partial r}A\right) - 2\,A^2Br^3\left(\frac{\partial}{\partial r}E_{rr}\right) + 2\,A^2r^3E_{rr}\frac{\partial}{\partial r}B \\
&\qquad - 4\,A^2Br^2E_{rr} + 4\,A^2B^2E_{\theta\theta} = 0
\end{aligned}
$$

The standard argument:

- assume that the **evolution equations** hold: $E_{rr} = 0$ and $E_{\theta\theta} = 0$
- the **constraints** propagate: $E_{tt} = 0$ and $E_{tr} = 0$

$$-2\,AB^2r\left(\frac{\partial}{\partial t}E_{tt}\right) + 2\,B^2rE_{tt}\frac{\partial}{\partial t}A + ABrE_{tr}\left(\frac{\partial}{\partial r}A\right) + 2\,A^2Br\left(\frac{\partial}{\partial r}E_{tr}\right) - ABrE_{tt}\left(\frac{\partial}{\partial t}B\right) - A^2rE_{tr}\frac{\partial}{\partial r}B - \cancel{A^2rE_{rr}\frac{\partial}{\partial t}}B + 4A^2B\,E_{tr} = 0$$

$$2\,AB^2r^3\left(\frac{\partial}{\partial t}E_{tr}\right) - B^2r^3E_{tt}\frac{\partial}{\partial r}A - B^2r^3E_{tr}\frac{\partial}{\partial t}A + ABr^3E_{tr}\left(\frac{\partial}{\partial t}B\right) - \cancel{ABr^3E_{rr}\left(\frac{\partial}{\partial r}A\right)} - \cancel{2\,A^2Br^3\left(\frac{\partial}{\partial r}E_{rr}\right)} + \cancel{2\,A^2r^3E_{rr}\frac{\partial}{\partial r}}B - \cancel{4\,A^2Br^2E_{rr}} + \cancel{4\,A^2B^2E_{\theta\theta}} = 0$$

- a first-order strongly hyperbolic system for E_{tt}, E_{tr} **linear** & **homogeneous**
- if they vanish on one of the $t = const$ time-level surface they vanish on each

The non-standard argument:

- assume that **one of the constraints** $E_{tt} = 0$ and **one of the evolutionary equations** $E_{rr} = 0$ hold on each of the $t = const$ time-level surfaces

$$
\begin{aligned}
&-2\,AB^2r\left(\frac{\partial}{\partial t}E_{tt}\right)+2\,B^2rE_{tt}\frac{\partial}{\partial t}A+ABrE_{tr}\left(\frac{\partial}{\partial r}A\right)+2\,A^2Br\left(\frac{\partial}{\partial r}E_{tr}\right)\\
&\quad-ABrE_{tt}\left(\frac{\partial}{\partial t}B\right)-A^2rE_{tr}\frac{\partial}{\partial r}B-A^2rE_{rr}\frac{\partial}{\partial t}B+4A^2B\,E_{tr}=0\\
&2\,AB^2r^3\left(\frac{\partial}{\partial t}E_{tr}\right)-B^2r^3E_{tt}\frac{\partial}{\partial r}A-B^2r^3E_{tr}\frac{\partial}{\partial t}A+ABr^3E_{tr}\left(\frac{\partial}{\partial t}B\right)\\
&\quad-ABr^3E_{rr}\left(\frac{\partial}{\partial r}A\right)-2\,A^2Br^3\left(\frac{\partial}{\partial r}E_{rr}\right)+2\,A^2r^3E_{rr}\frac{\partial}{\partial r}B\\
&\quad-4\,A^2Br^2E_{rr}+4\,A^2B^2E_{\theta\theta}=0
\end{aligned}
$$

- the firts equation is a **first-order ODE** for E_{tr} **linear and homogeneous**
  $\Rightarrow$ $E_{tr} \equiv 0$ on any of the time-level surfaces if $E_{tr} = 0$ at the origin!!!
- the second equation implies that $E_{\theta\theta}$ **vanishes everywhere**

The case of a scalar field 'ala Choptuik':

- $$T_{ab} = \nabla_a\psi\nabla_b\psi - \tfrac{1}{2}g_{ab}\nabla^e\psi\nabla_e\psi$$

  and using the auxiliary variables

  $$\Phi = \frac{\partial}{\partial r}\psi, \qquad \Pi = \sqrt{\frac{B}{A}}\frac{\partial}{\partial t}\psi$$

- $\nabla^e\nabla_e\psi = 0 \quad \Rightarrow \quad \dfrac{\partial}{\partial t}\Phi = \dfrac{\sqrt{\frac{A}{B}}\left(r\left(\frac{\partial}{\partial r}\Pi\right) + \Pi(1 - B)\right)}{r}$

- $E_{tt} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial r}B = \frac{B(1-B)}{r} + 4\pi r B\left(\Pi^2 + \Phi^2\right) \qquad (B|_{r=0} = 1)$

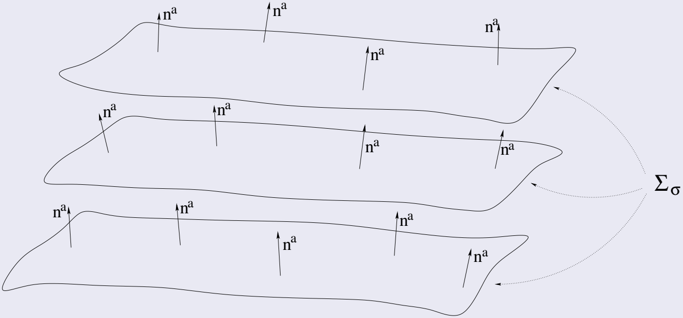
- $E_{tr} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial t}B = 8\pi r \Phi \Pi \sqrt{AB} \qquad (B|_{r=0} = 1 \text{ !!!})$

- $E_{rr} = 0 \quad \Rightarrow \quad \frac{\partial}{\partial r}A = -\frac{A(1-B)}{r} + 4\pi r A\left(\Pi^2 + \Phi^2\right).$

The primary $n+1$ splitting:

No restriction on the topology by Einstein's equations! (local PDEs)

- **Assume:** M is foliated by a one-parameter family of homologous hypersurfaces, i.e. $M \simeq \mathbb{R} \times \Sigma$, for some codimension-one manifold Σ.
  - known to **hold for globally hyperbolic spacetimes** (Lorentzian case)
  - **equivalent to** the existence of a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing gradient $\nabla_a \sigma$ such that the $\sigma = const$ level surfaces $\Sigma_\sigma = \{\sigma\} \times \Sigma$ comprise the one-parameter foliation of M.



The main creatures:

- n^a the 'unit norm' vector field that is normal to the Σ_σ level surfaces

$$\boxed{n^a n_a = \epsilon}$$

  - ϵ takes the value -1 or $+1$ for Lorentzian or Riemannian metric g_{ab}, resp.

- **the projection operator**

$$\boxed{h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b}$$

- **the metric induced**

$$\boxed{h_{ab} = h_a{}^e h_b{}^f\, g_{ef} = g_{ab} - \epsilon\, n_a n_b}$$

- the covariant derivative operator $\boxed{D_a}$ associated with h_{ab}: $\forall\ \omega_b$ on Σ

$$\boxed{D_a \omega_b := h_a{}^d h_b{}^e \nabla_d\, \omega_e}$$

- **the extrinsic curvature** and the **acceleration of** n^a on Σ

$$\boxed{K_{ab} = h^e{}_a \nabla_e n_b = \tfrac{1}{2}\, \mathscr{L}_n h_{ab}} \qquad \boxed{\dot{n}_a := n^e \nabla_e n_a}$$

Decompositions of various fields:

arbitrary symmetric tensor field P_{ab} on M can be decomposed

in terms of n^a and fields living on the $\sigma = const$ level surfaces as

$$P_{ab} = \boldsymbol{\pi}\, n_a n_b + [n_a\, \mathbf{p}_b + n_b\, \mathbf{p}_a] + \mathbf{P}_{ab}$$

where $$\boldsymbol{\pi} = n^e n^f\, P_{ef}\,, \quad \mathbf{p}_a = \epsilon\, h^e{}_a n^f\, P_{ef}\,, \quad \mathbf{P}_{ab} = h^e{}_a h^f{}_b\, P_{ef}$$

decomposition of the contraction $\nabla^a P_{ab}$:

$$\epsilon\,(\nabla^a P_{ae})\, n^e = \mathscr{L}_n \boldsymbol{\pi} + D^e \mathbf{p}_e + [\boldsymbol{\pi}\,(K^e{}_e) - \epsilon\, \mathbf{P}_{ef} K^{ef} - 2\,\epsilon\, \dot{n}^e \mathbf{p}_e]$$
$$(\nabla^a P_{ae})\, h^e{}_b = \mathscr{L}_n \mathbf{p}_b + D^e \mathbf{P}_{eb} + [(K^e{}_e)\, \mathbf{p}_b + \dot{n}_b\, \boldsymbol{\pi} - \epsilon\, \dot{n}^e \mathbf{P}_{eb}]$$

l.h.s. of Einstein's equation: $E_{ab} = G_{ab} - \mathscr{G}_{ab}$

$$E_{ab} = n_a n_b\, E^{(\mathcal{H})} + [n_a\, E^{(\mathcal{M})}_b + n_b\, E^{(\mathcal{M})}_a] + (E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})$$

$$E^{(\mathcal{H})} = n^e n^f\, E_{ef}\,, \quad E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f\, E_{ef}\,, \quad E^{(\mathcal{EVOL})}_{ab} = h^e{}_a h^f{}_b\, E_{ef} - h_{ab}\, E^{(\mathcal{H})}$$

Relations between various parts of Einstein's equations:

$$\mathscr{L}_n E^{(\mathcal{H})} + D^e E_e^{(\mathcal{M})} + [\, E^{(\mathcal{H})} (K^e{}_e) - 2\,\epsilon\, (\dot{n}^e\, E_e^{(\mathcal{M})}) - \epsilon\, K^{ae} (E_{ae}^{(\mathcal{EVOL})} + h_{ae}\, E^{(\mathcal{H})})\,] = 0$$

Div

$$\mathscr{L}_n E_b^{(\mathcal{M})} + D^a (E_{ab}^{(\mathcal{EVOL})} + h_{ab}\, E^{(\mathcal{H})}) + [\,(K^e{}_e)\, E_b^{(\mathcal{M})} + E^{(\mathcal{H})}\, \dot{n}_b - \epsilon\, (E_{ab}^{(\mathcal{EVOL})} + h_{ab}\, E^{(\mathcal{H})})\, \dot{n}^a\,] = 0$$

reverse the argument

If the constraint expressions $E^{(\mathcal{H})}$ and $E_a^{(\mathcal{M})}$ vanish on the $\sigma = const$ level surfaces then the relations

$$K^{ab}\, E_{ab}^{(\mathcal{EVOL})} = 0$$

$$D^a E_{ab}^{(\mathcal{EVOL})} - \epsilon\, \dot{n}^a\, E_{ab}^{(\mathcal{EVOL})} = 0$$

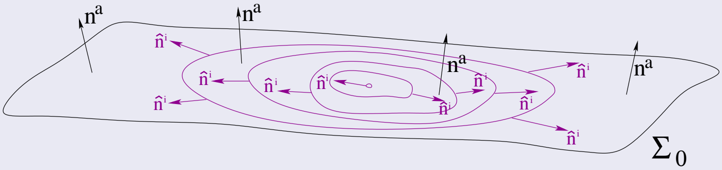
hold for the evolutionary expression $E_{ab}^{(\mathcal{EVOL})}$

$$h^e{}_a h^f{}_b\, E_{ef} = E_{ab}^{(\mathcal{EVOL})} + \cancel{h_{ab}\, E^{(\mathcal{H})}}$$

The secondary $[n-1]+1$ splitting:

Assume that on one of the $\sigma = const$ level surfaces—say on Σ_0, for some $\sigma = \sigma_0\,(\in \mathbb{R})$,—there exists a smooth function $\rho : \Sigma_0 \to \mathbb{R}$, with (a.e.—almost everywhere) non-vanishing gradient, $\partial_i \rho$, such that:

- the $\rho = const$ level surfaces $\mathscr{S}_\rho$ provide a one-parameter foliation of Σ_0



- the **metric** h_{ij} **on** Σ_0 can be decomposed as

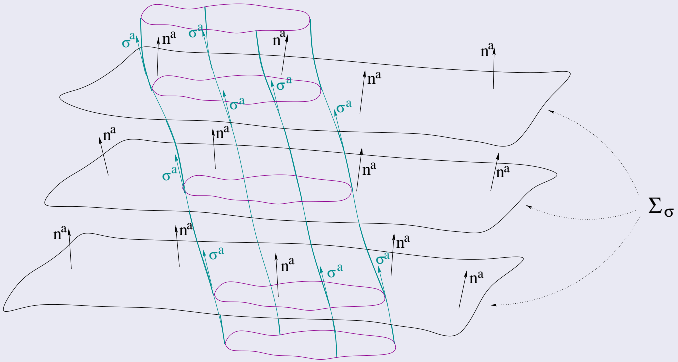
$$\boxed{h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i \widehat{n}_j} \qquad \text{where} \quad \boxed{\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j h_{ij}} \quad \text{with} \quad \boxed{\widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i \widehat{n}_j}$$

  in terms of the positive definite metric $\widehat{\gamma}_{ij}$, induced on the $\mathscr{S}_\rho$ hypersurfaces,
- **the secondary extrinsic curvature** and the **acceleration of** $\widehat{n}^a$ on the $\mathscr{S}_\rho$ hypersurfaces

$$\boxed{\widehat{K}_{ij} = \tfrac{1}{2}\,\mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij}} \qquad \boxed{\dot{\widehat{n}}_i := \widehat{n}^l D_l \widehat{n}_i}$$

The two-parameter family of foliations:

The Lie drag this foliation of Σ_0 along the integral curves of the vector field σ^a yields then a two-parameter family of foliating surfaces: $\mathscr{S}_{\sigma,\rho}$



- the fields $\widehat{n}^i$, $\widehat{\gamma}_{ij}$ and the projection $\boxed{\widehat{\gamma}^k{}_l = h^k{}_l - \widehat{n}^k \widehat{n}_l}$, to the codimension-two surfaces $\mathscr{S}_{\sigma,\rho}$, get to be well-defined on each of the individual $\sigma = const$ hypersurfaces

Recasting the reduced Einstein equations:

the kinematical background $\Longrightarrow$ $[n-1]+1$ decomposition of $E^{(\mathcal{EVOL})}_{ab}$

$$h_b{}^e h_d{}^f R_{ef} = {}^{(n)}R_{bd} + \epsilon\,\{-\mathscr{L}_n K_{bd} - K_{bd}K_e{}^e + 2K_b{}^e K_{de} - \epsilon\, N^{-1} D_b D_d N\}$$

$$R = {}^{(n)}R + \epsilon\,\{-2\,\mathscr{L}_n(K_{bd}h^{bd}) - (K_e{}^e)^2 - K_{ef}K^{ef} - 2\,\epsilon\, N^{-1} D^e D_e N\}$$

one gets

$$\boxed{h_b{}^e h_d{}^f E_{ef}} = h_b{}^e h_d{}^f\,\{[\,R_{ef} - \tfrac{1}{2}\,g_{ef}R\,] - \mathscr{G}_{bd}\} = h_b{}^e h_d{}^f\,\{[\,R_{ef} - \tfrac{1}{2}\,h_{ef}R\,] - \mathscr{G}_{bd}\}$$

$$= [\,{}^{(n)}R_{bd} - \tfrac{1}{2}\,h_{ef}\,{}^{(n)}R\,] - {}^{(n)}\mathscr{G}_{bd} = {}^{(n)}G_{bd} - {}^{(n)}\mathscr{G}_{bd} = \boxed{{}^{(n)}E_{bd}}$$

where

$$\begin{aligned}{}^{(n)}\mathscr{G}_{ab} = \mathfrak{S}_{ab} - \epsilon\Big\{ &- \mathscr{L}_n K_{ab} - (K^e{}_e)K_{ab} + 2\,K_{ae}K^e{}_b - \epsilon\,N^{-1}\,D_a D_b N \\ &+ h_{ab}\big[\,\mathscr{L}_n(K^e{}_e) + \tfrac{1}{2}\,(K^e{}_e)^2 + \tfrac{1}{2}\,K_{ef}K^{ef} + \epsilon\,N^{-1}\,D^e D_e N\,\big]\Big\}\end{aligned}$$

$${}^{(n)}E_{ij} = \widehat{E}^{(\mathcal{H})}\widehat{n}_i\widehat{n}_j + [\widehat{n}_i\widehat{E}^{(\mathcal{M})}_j + \widehat{n}_j\widehat{E}^{(\mathcal{M})}_i\,] + (\widehat{E}^{(\mathcal{EVOL})}_{ij} + \widehat{\gamma}_{ij}\widehat{E}^{(\mathcal{H})})$$

$$\widehat{E}^{(\mathcal{H})} = \widehat{n}^e\widehat{n}^f\,{}^{(n)}E_{ef}, \quad \widehat{E}^{(\mathcal{M})}_i = \widehat{\gamma}^e{}_j\widehat{n}^f\,{}^{(n)}E_{ef}, \quad \widehat{E}^{(\mathcal{EVOL})}_{ij} = \widehat{\gamma}^e{}_i\widehat{\gamma}^f{}_j\,{}^{(n)}E_{ef} - \widehat{\gamma}_{ij}\widehat{E}^{(\mathcal{H})}$$

Relations between various parts of the basic equations:

Substituting the $[n-1]+1$ splitting of ${}^{(n)}E_{ij}$:

$$K^{ab}\,{}^{(n)}E_{ab} = 0$$
$$D^a[{}^{(n)}E_{ab}] - \epsilon\,\dot{n}^a\,{}^{(n)}E_{ab} = 0$$

as

$${}^{(n)}E_{ab} = h^e{}_a h^f{}_b\, E_{ef} = E^{(\mathcal{EVOL})}_{ab} + \cancel{h_{ab}\, E^{(\mathcal{H})}}$$

$$K^{ab\,(n)}E_{ab} = \boldsymbol{\kappa}\,\widehat{E}^{(\mathcal{H})} + 2\,\mathbf{k}^e\,\widehat{E}^{(\mathcal{M})}_e + \mathbf{K}^{ef}\,\cancel{\widehat{E}^{(\mathcal{EVOL})}_{ef}} + (\mathbf{K}^e{}_e)\,\widehat{E}^{(\mathcal{H})}$$
$$\dot{n}^{a\,(n)}E_{ab} = [(\widehat{n}_a\dot{n}^a)\widehat{E}^{(\mathcal{H})} + (\dot{n}^a\,\widehat{E}^{(\mathcal{M})}_a)]\widehat{n}_b + (\widehat{n}_a\dot{n}^a)\widehat{E}^{(\mathcal{M})}_b + \dot{n}^a\,[\cancel{\widehat{E}^{(\mathcal{EVOL})}_{ab}} + \widehat{\gamma}_{ab}\,\widehat{E}^{(\mathcal{H})}]$$
$$\widehat{n}^e D^a[{}^{(n)}E_{ae}] = \mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{H})} + \widehat{D}^e\widehat{E}^{(\mathcal{M})}_e + (\widehat{K}^e{}_e)\,\widehat{E}^{(\mathcal{H})} - [\cancel{\widehat{E}^{(\mathcal{EVOL})}_{ef}} + \widehat{\gamma}_{ef}\,\widehat{E}^{(\mathcal{H})}]\,\widehat{K}^{ef} - 2\,\dot{\widehat{n}}^e\widehat{E}^{(\mathcal{M})}_e$$
$$\widehat{\gamma}^e{}_b D^a[{}^{(n)}E_{ae}] = \mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{M})}_b + \widehat{D}^e[\cancel{\widehat{E}^{(\mathcal{EVOL})}_{eb}} + \widehat{\gamma}_{eb}\,\widehat{E}^{(\mathcal{H})}] + (\widehat{K}^e{}_e)\,\widehat{E}^{(\mathcal{M})}_b - \dot{\widehat{n}}^e\,\cancel{\widehat{E}^{(\mathcal{EVOL})}_{eb}}$$

$$\mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{H})} + \widehat{\gamma}^{ef}\,\widehat{D}_e\widehat{E}^{(\mathcal{M})}_f = \widehat{\mathscr{E}}$$
$$\mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{M})}_b + \widehat{D}_b\widehat{E}^{(\mathcal{H})} = \widehat{\mathscr{E}}_b$$

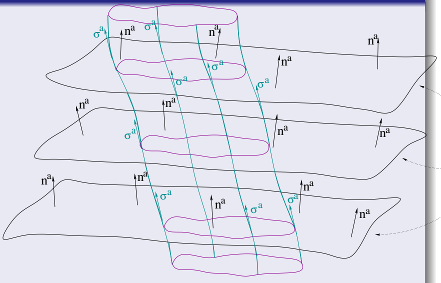
$\Longrightarrow$ **IF** $\widehat{E}^{(\mathcal{EVOL})}_{ef} = 0$ holds: a **linear and homogeneous FOSH** for $(\widehat{E}^{(\mathcal{H})}, \widehat{E}^{(\mathcal{M})}_i)^T$

A fully constrained evolutionary scheme?

Theorem

- *Assume that the primary constraint expressions* $E^{(\mathcal{H})}$ *and* $E_a^{(\mathcal{M})}$ *vanish on the* $\sigma = const$ *level surfaces, also that*
- *the secondary constraint expressions* $\widehat{E}^{(\mathcal{H})}$ *and* $\widehat{E}_a^{(\mathcal{M})}$ *vanish along the hypersurface yielded by the Lie dragging,* $\mathscr{W}_{\rho_0} = \Phi_\sigma[\mathscr{S}_{\rho_0}]$*, of one of the level surfaces* $\mathscr{S}_{\rho_0}$ *foliating* Σ_0*.*
- $\Longrightarrow$

  Then, to get solutions to the full set of Einstein's equations $G_{ab} - \mathscr{G}_{ab} = 0$ *it suffices*—**regardless whether the primary metric g_{ab} is Riemannian or Lorentzian**—*to solve, in addition, only the secondary reduced equations* $\widehat{E}_{ij}^{(\mathcal{EVOL})} = 0$*.*



Remark (i).: *the Lie dragging is done by using the one-parameter group of diffeomorphisms,* Φ_σ*, associated by the "time evolution vector field"* σ^a *— could be only a world-line*

Remark (ii): *if one wants to setup an initial-boundary value problem on either side of the hypersurface* $\mathscr{W}_{\rho_0}$ *the previous theorem provides a clear mean to identify the geometrical freedom we have on* $\mathscr{W}_{\rho_0}$