A new method of constructing binary black hole initial data

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Outline:

Motivations:

GW observations:

- inspiral and merger of binary black holes is of distinguished importance for the emerging field of gravitational wave astronomy
- non-linearities necessitate the use of accurate numerical approaches in determining the emitted waveforms
- precision of these simulations—in particular, their initializations—is of critical importance in enhancing the detection of gravitational wave signals

Based on:

- I. Rácz: *Constraints as evolutionary systems*, Class. Quantum Grav. **33** 015014 (2016); [arXiv:1508.01810]
- I. Rácz: *A simple method of constructing binary black hole initial data*, Astronomy Reports **62** 953-958 (2018); [arXiv:1605.01669]
- I. Rácz: *Supplemental Material* (2016) : http://www.kfki.hu/∼iracz/ SM-BH-data.pdf
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The constraints:

- vacuum initial data: (h_{ij}, K_{ij}) on a 3-dimensional manifold Σ
  - evolution equations $\mathscr{L}_n h_{ij} = ...$ & $\mathscr{L}_n K_{ij} = ...$ (in analogy $\dot{\mathbf{x}} = \mathbf{v}$ & $\dot{\mathbf{v}} = \mathbf{f}$)

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The conformal (elliptic) method:

Lichnerowicz A (1944) and York J W (1972):

- replace

$$h_{ij} = \phi^4 \, \widetilde{h}_{ij} \quad \text{and} \quad K_{ij} - \tfrac{1}{3}\, h_{ij}\, K^l{}_l = \phi^{-2}\, \widetilde{K}_{ij}$$

- using these variables the constraints are put into the **semilinear elliptic system**

$$\widetilde{D}^l \widetilde{D}_l \phi - \tfrac{1}{8}\, \widetilde{R}\, \phi + \tfrac{1}{8}\, \widetilde{K}_{ij} \widetilde{K}^{ij}\, \phi^{-7} - \tfrac{1}{12}\, (K^l{}_l)^2\, \phi^5 = 0$$

where $\widetilde{D}_l$, $\widetilde{R}$, ... $\widetilde{h}_{ij}$

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Impressive mathematical developments since 1944 but ...

- either "constancy" of $K^l{}_l$ or "smallness" of the TT part of $\widetilde{K}_{ij}$ is required
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  - no direct control on the physical parameters of the initial data specifications
- **boundary conditions:**
  - are known to influence solutions everywhere in their domains
  - the inner boundary conditions—they are applied with **excision** in the black hole interior—cannot simply be supported by intuition (**trumpet data** ...)
  - Bowen-York type initial data: $\widetilde{h}_{ij}$ is flat $\widetilde{h}_{ij} = \delta_{ij}$ and $K^l{}_l = 0$

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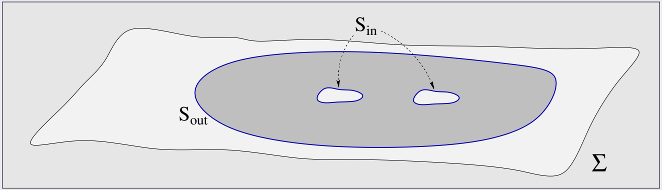

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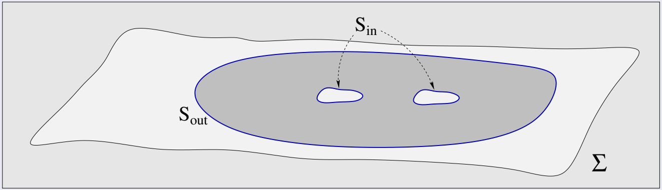

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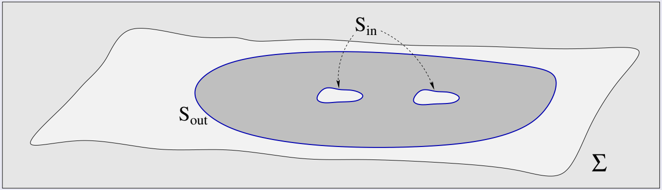

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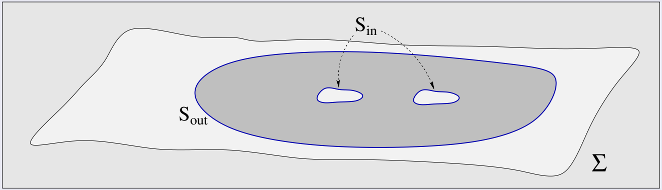

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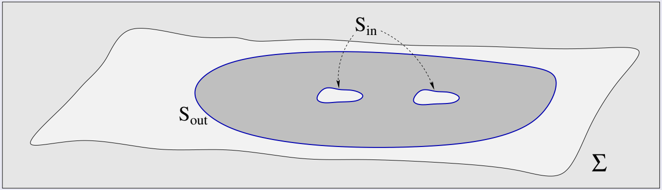

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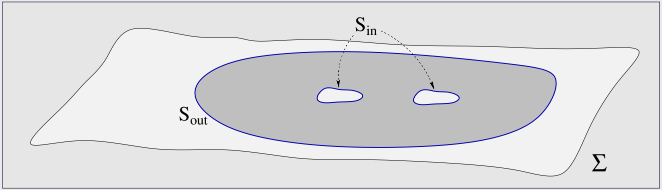

New variables by applying $2+1$ decompositions:

Splitting of the metric h_{ij}:

- 'lapse' and 'shift' of ρ^i
- induced metric, extrinsic curvature and acceleration of the $\mathscr{S}_\rho$ level surfaces:

$$\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i \, \widehat{\gamma}^l{}_j \, h_{kl} \qquad \widehat{K}_{ij} = \tfrac{1}{2} \mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij} \qquad \dot{\widehat{n}}_i := \widehat{n}^e \nabla_e \widehat{n}_i = -\widehat{D}_i \ln \widehat{N}$$

- the metric h_{ij} can then be given as

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Σ is smoothly foliated by a one-parameter family of two-surfaces $\mathscr{S}_\rho$:
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The parabolic-hyperbolic form of the constraints:

An evolutionary system for the constrained fields $\widehat{N}, \mathbf{k}_i$ and $\mathbf{K}^l{}_l$:

$$\overset{\star}{K}\,[\,(\partial_\rho\widehat{N}) - \widehat{N}^l(\widehat{D}_l\widehat{N})\,] - \widehat{N}^2(\widehat{D}^l\widehat{D}_l\widehat{N}) - \mathcal{A}\,\widehat{N} - \mathcal{B}\,\widehat{N}^3 = 0$$

$$\mathscr{L}_{\widehat{n}}\mathbf{k}_i - \tfrac{1}{2}\,\widehat{D}_i(\mathbf{K}^l{}_l) - \widehat{D}_i\boldsymbol{\kappa} + \widehat{D}^l\overset{\circ}{\mathbf{K}}_{li} + \widehat{N}\overset{\star}{K}\,\mathbf{k}_i + [\,\boldsymbol{\kappa} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)\,]\,\dot{\widehat{n}}_i - \dot{\widehat{n}}^l\,\overset{\circ}{\mathbf{K}}_{li} = 0$$

$$\mathscr{L}_{\widehat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l - \widehat{N}\overset{\star}{K}\,[\,\boldsymbol{\kappa} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)\,] + \widehat{N}\,\overset{\circ}{\mathbf{K}}_{kl}\overset{\star}{K}{}^{kl} + 2\,\dot{\widehat{n}}^l\,\mathbf{k}_l = 0\,,$$

where $\widehat{D}_i$ denotes the covariant derivative operator associated with $\widehat{\gamma}_{ij}$

$$\overset{\star}{K} = \tfrac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_j\widehat{N}^j$$

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$$\mathcal{A} = (\partial_\rho\overset{\star}{K}) - \widehat{N}^l(\widehat{D}_l\overset{\star}{K}) + \tfrac{1}{2}[\,\overset{\star}{K}{}^2 + \overset{\star}{K}_{kl}\overset{\star}{K}{}^{kl}\,]$$

$$\mathcal{B} = -\tfrac{1}{2}\,\big[\widehat{R} + 2\,\boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) + \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \overset{\circ}{\mathbf{K}}_{kl}\,\overset{\circ}{\mathbf{K}}{}^{kl}\,\big]$$

The parabolic-hyperbolic system:

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- no restriction on $\widehat{N}^i, \widehat{\gamma}_{ij}, \boldsymbol{\kappa}$ and $\mathring{\mathbf{K}}_{ij}$ $\Longrightarrow$ they are freely specifiable on Σ
- the parabolic equation is uniformly parabolic in those subregions of Σ, where $\overset{\star}{K}$ is either positive or negative
- $\overset{\star}{K}$ depends exclusively on the freely specifiable fields $\widehat{\gamma}_{ij}$ and $\widehat{N}^i$ $\Longrightarrow$ its sign can be tailored according to the desire of the investigated problem
- the combined evolutionary system is (locally) well-posed
  - if suitable initial values for the constrained fields $\widehat{N}, \mathbf{k}_i$ and $\mathbf{K}^l{}_l$ are given, on some level surface $\mathscr{S}_0$ in Σ, then, in the domain of dependence of $\mathscr{S}_0$, unique solution exists to the evolutionary system
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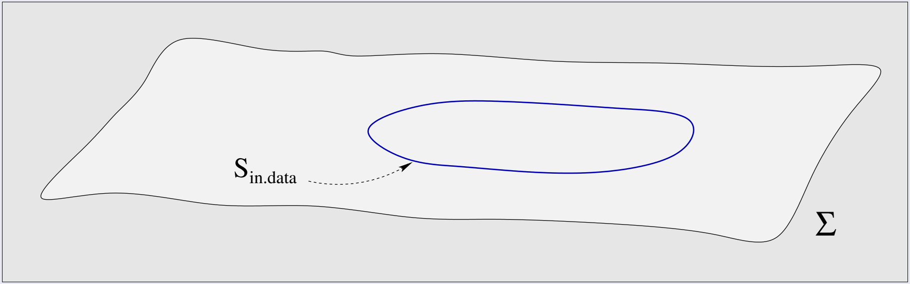

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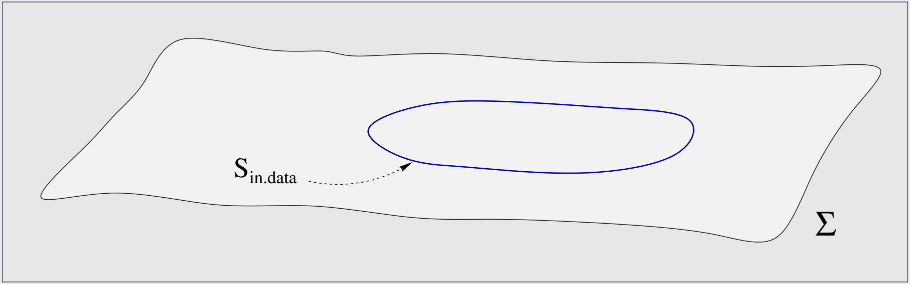

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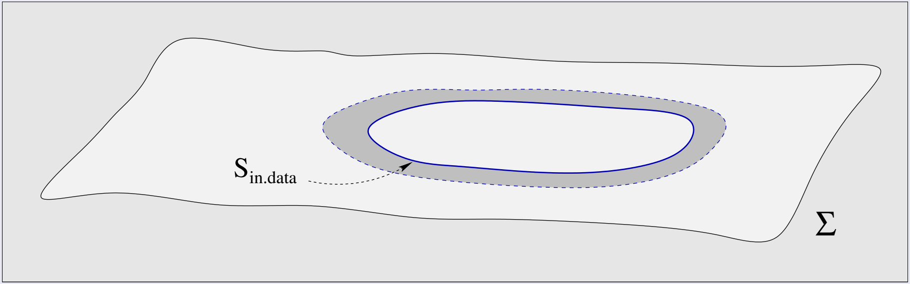

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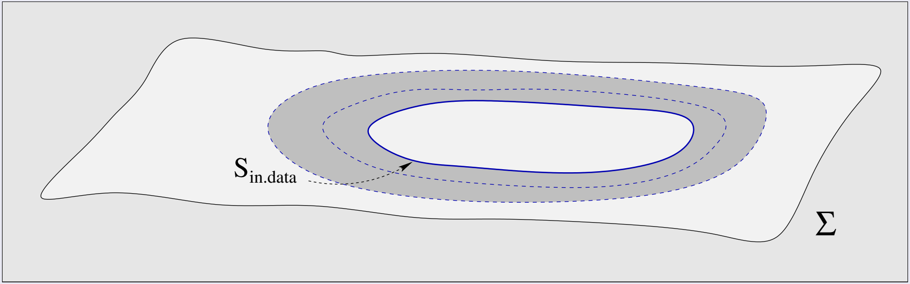

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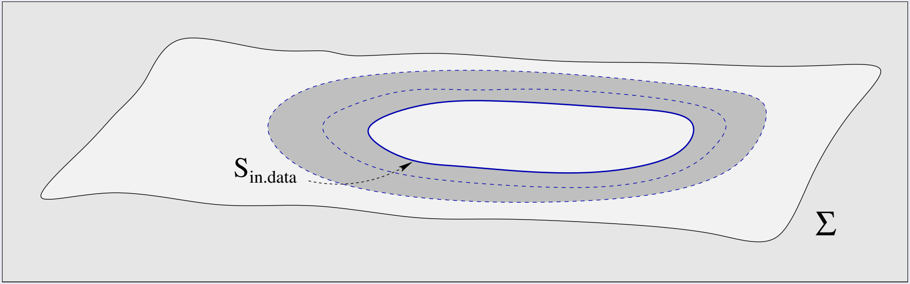

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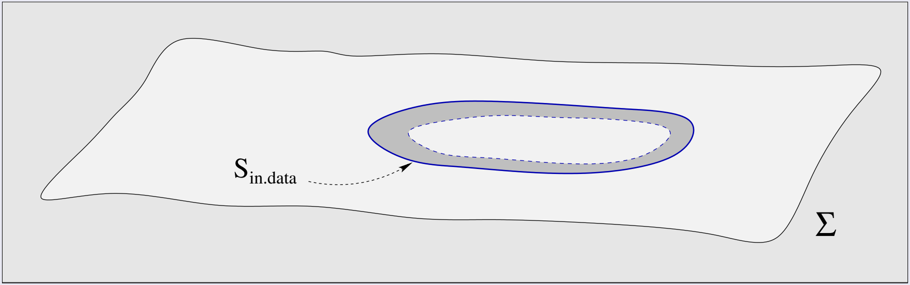

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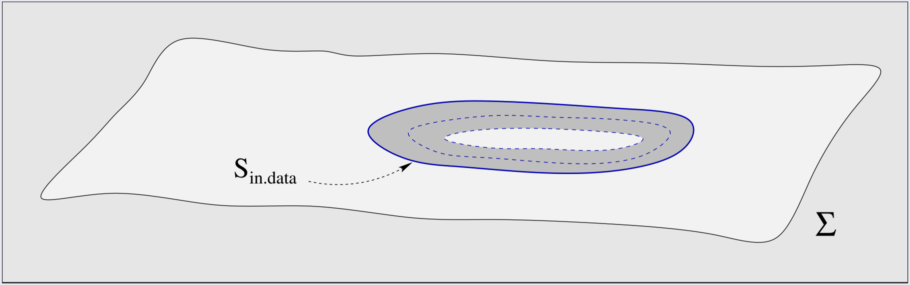

The Kerr black hole:

In Kerr-Schild form:

- $$g_{\alpha\beta} = \eta_{\alpha\beta} + 2H\ell_\alpha\ell_\beta$$
- inertial coordinates (t, x, y, z) adapted to the Minkowski background $\eta_{\alpha\beta}$

$$H = \frac{r^3 M}{r^4 + a^2 z^2}$$

$$\ell_\alpha = \left(1, \frac{r\,x + a\,y}{r^2 + a^2}, \frac{r\,y - a\,x}{r^2 + a^2}, \frac{z}{r}\right)$$

- the Boyer-Lindquist radial coordinate r is related to the spatial part of the inertial coordinates as

$$r^4 - (x^2 + y^2 + z^2 - a^2)\,r^2 - a^2\,z^2 = 0 \qquad \frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1$$

- the $r = const$ surfaces are "ellipsoids"
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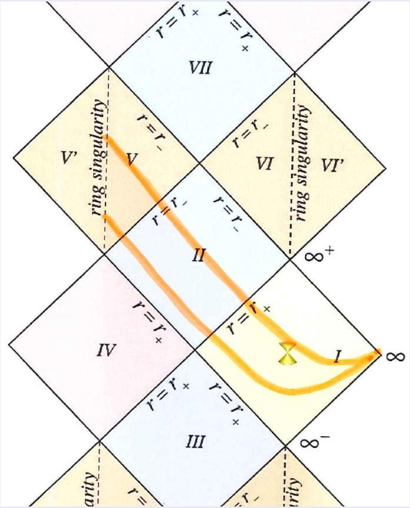

the $\theta = \frac{\pi}{2}$ section

$$\Sigma \approx \mathbb{R}^3 \setminus \{\text{“ring singularity”}\}$$

Generic Kerr-Schild black holes:

- the Kerr-Schild metrics are form-invariant under Lorentz transformations
  - if a Lorentz transformation $x'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, x^{\beta}$ is performed
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$$g'_{\alpha\beta} = \eta_{\alpha\beta} + 2H'\ell'_{\alpha}\ell'_{\beta}$$

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Superposed Kerr-Schild black holes:

We are looking for suitable free(ly specifiable) data:

- $$g_{\alpha\beta} = \eta_{\alpha\beta} + 2H^{[1]}\ell_\alpha{}^{[1]}\ell_\beta{}^{[1]} + 2H^{[2]}\ell_\alpha{}^{[2]}\ell_\beta{}^{[2]} \qquad (*)$$
  - $H^{[n]}$ and $\ell_\alpha{}^{[n]}$ correspond to the Kerr-Schild data for individual black holes
- (*) does not satisfy Einstein's equations
- good approximation close to the individual black holes
- Einstein tensor falls off at the rate $\mathcal{O}(|\vec{x}|^{-4})$, where $|\vec{x}| = \sqrt{x^2+y^2+z^2}$

Choice for the free data: Take a foliation of the $t_{Mink} = 0$ time-slice of (*)

- $\widehat{N}^i, \widehat{\gamma}_{ij}, \boldsymbol{\kappa}$ and $\mathring{\mathbf{K}}_{ij}$ as if (*) solved the Einstein equations
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the $\mathscr{S}_\rho$ surfaces have tacitly been assumed to be compact without boundary:

- in numerical approaches Σ is chosen to be a large but bounded subset of $\mathbb{R}^3$
- the product structure $\Sigma \approx \mathbb{R} \times \mathscr{S}$ can be guaranteed by choosing the $\mathscr{S}_\rho$ leaves to be diffeomorphic to a closed disk in $\mathbb{R}^2$

choose Σ to be a cubical region centered at the origin in $\mathbb{R}^3$:

- for large enough value of A ...
- boundary of Σ:
  six squares each with edges of size $2A$
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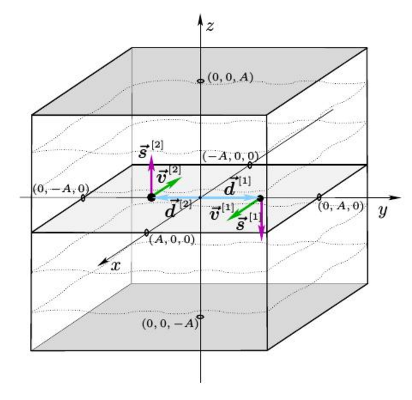

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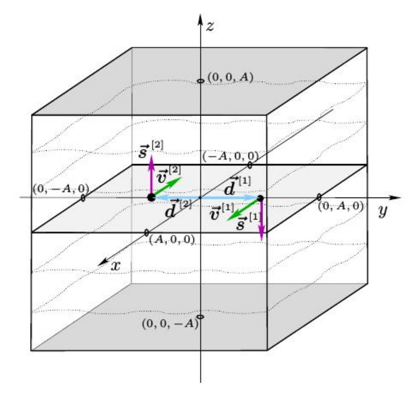

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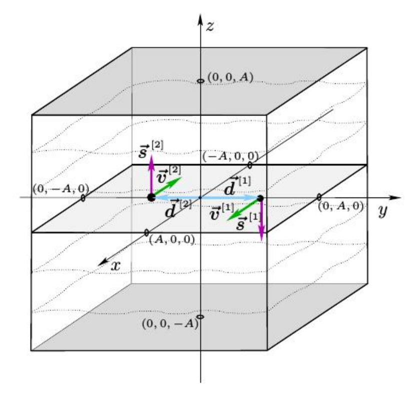

- for large enough value of A ...
- boundary of Σ:
  six squares each with edges of size $2A$
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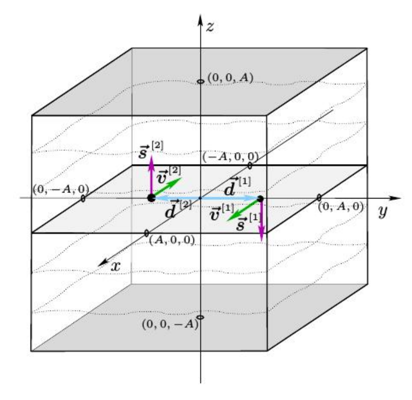

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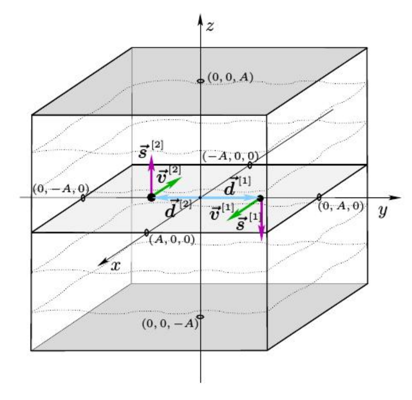

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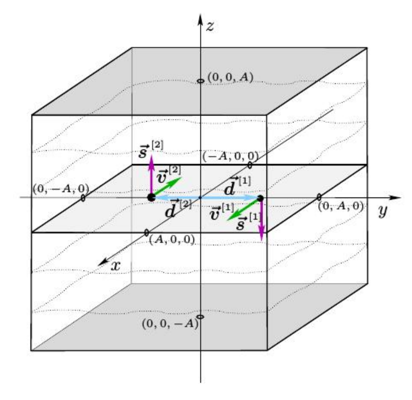

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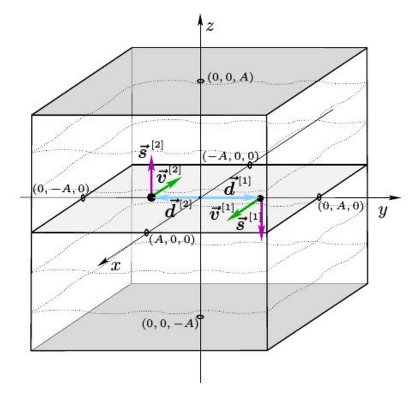

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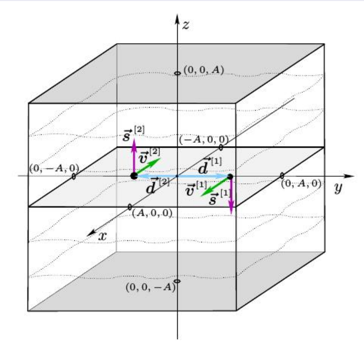

- the sign of $\overset{\star}{K}$ decides whether the parabolic-hyperbolic system evolves in the positive or negative ρ-direction

$$\overset{\star}{K}\,[\,(\partial_\rho \widehat{N}) - \widehat{N}^l(\widehat{D}_l\widehat{N})\,] = \widehat{N}^2(\widehat{D}^l\widehat{D}_l\widehat{N}) + \mathcal{A}\,\widehat{N} + \mathcal{B}\,\widehat{N}^3$$

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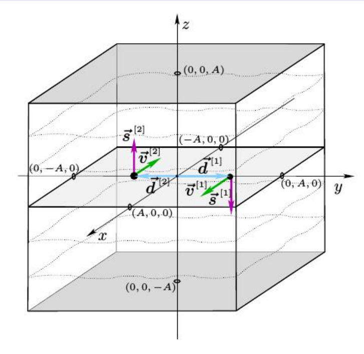

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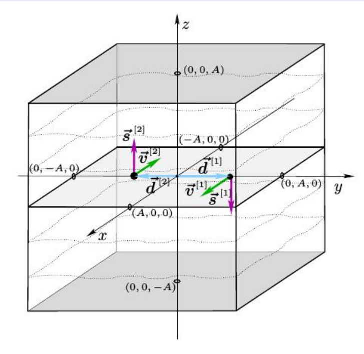

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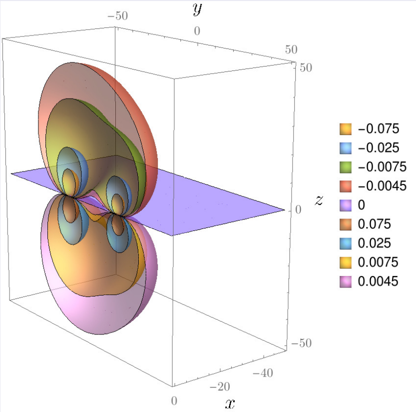
$\overset{\star}{K} = const$ level surfaces:

$$x < 0, \ 2A = 100$$
$$M^{[1]} = 1, \ \vec{d}^{\,[1]} = 20\,\vec{e}_y$$
$$\vec{v}^{\,[1]} = 0.5\,\vec{e}_x, \ a^{[1]} = 0.6$$
$$M^{[2]} = 2, \ \vec{d}^{\,[2]} = -10\,\vec{e}_y$$
$$\vec{v}^{\,[2]} = -0.25\,\vec{e}_x, \ a^{[2]} = -0.8$$



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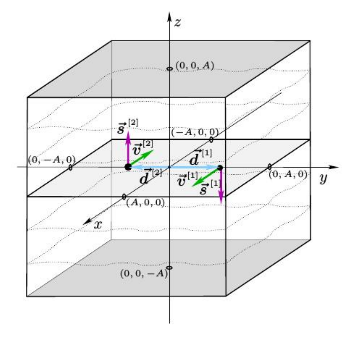

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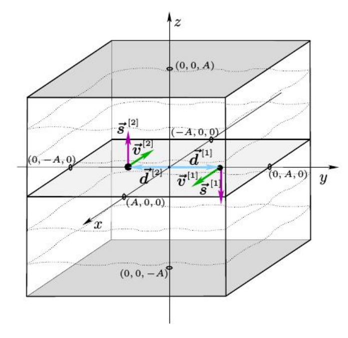

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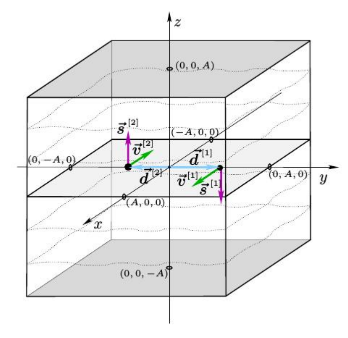

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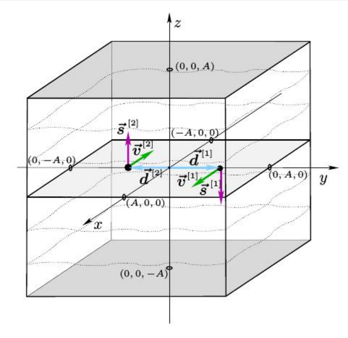

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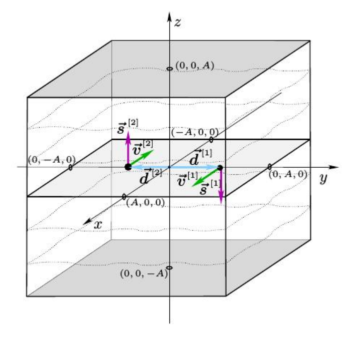

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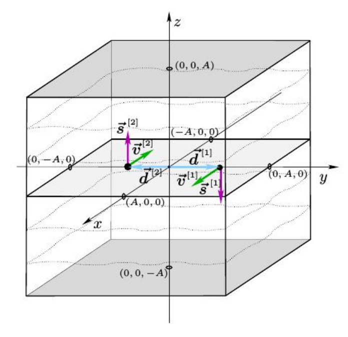

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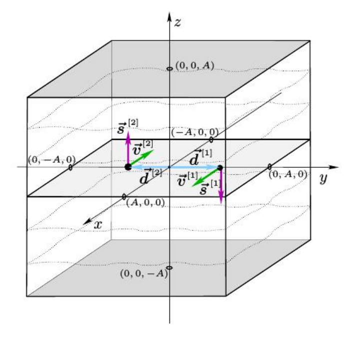

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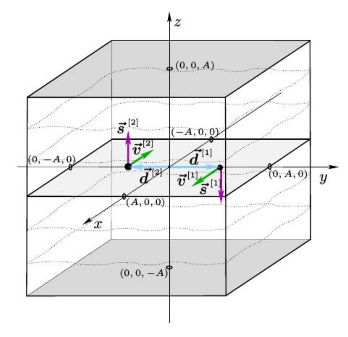

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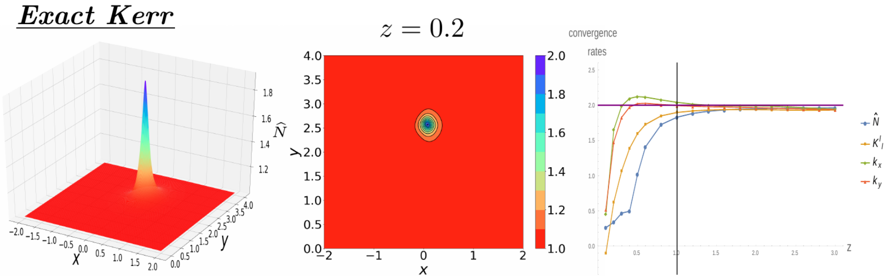
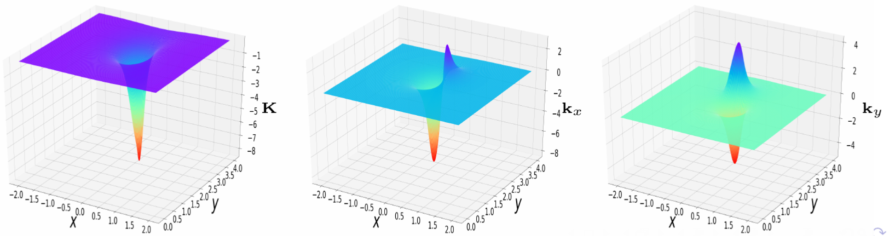
Exact Kerr: with input parameters $M = 0.5, a = 0.3M, v = 0.6, d = 5M$

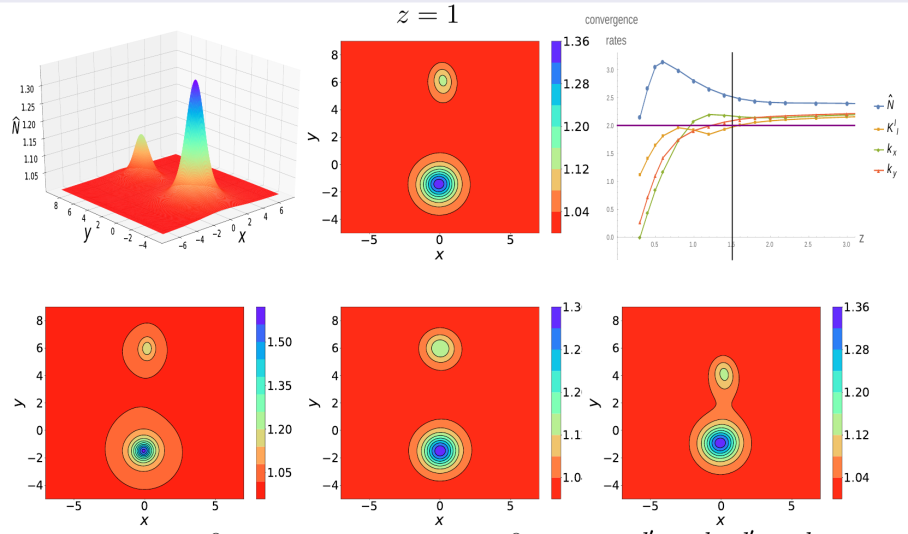


$$h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i \widehat{n}_j, \quad K_{ij} = \boldsymbol{\kappa}\, \widehat{n}_i \widehat{n}_j + \widehat{n}_i \mathbf{k}_j + \widehat{n}_j \mathbf{k}_i + \mathbf{K}_{ij}$$



A binary system: $M_1 = 0.25, a_1 = 0.7, v_1 = 0.68, d_1 = 6$

$M_2 = 3M_1, a_2 = -0.9, v_2 = v_1/4, d_2 = d_1/4$



$a_1 = a_2 = 0$ $\quad$ $v_1 = v_2 = 0$ $\quad$ $d_1' < d_1, d_2' < d_2$

video: the evolution of $\widehat{N}$

Input parameters and global ADM charges:

- **Input parameters:** the rest masses $M^{[n]}$, displacements $\vec{d}^{\,[n]}$, speeds $\vec{v}^{\,[n]}$ and spins $M^{[n]}a^{[n]}\vec{s}_\circ^{[n]}$ of the involved black holes
  - essentially the same as used in post-Newtonian description of binaries **!!!**
- **Global ADM charges:** in terms of the input parameters
  - though (*) does not satisfy Einstein's equations it is asymptotically flat
  - constructed by adding contributions of individual black hole metrics to a Minkowski background
  - the ADM quantities are linear in deviation from flat Euclidean space at infinity

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$$\begin{aligned} M^{ADM} &= \gamma^{[1]}M^{[1]} + \gamma^{[2]}M^{[2]} \\ M^{ADM}\vec{d}^{\,ADM} &= \gamma^{[1]}M^{[1]}\vec{d}^{\,[1]} + \gamma^{[2]}M^{[2]}\vec{d}^{\,[2]} \\ \vec{P}^{\,ADM} &= \gamma^{[1]}M^{[1]}\vec{v}^{\,[1]} + \gamma^{[2]}M^{[2]}\vec{v}^{\,[2]} \\ \vec{J}^{\,ADM} &= \gamma^{[1]}\left\{M^{[1]}\vec{d}^{\,[1]}\times\vec{v}^{\,[1]} + M^{[1]}a^{[1]}\vec{s}_{\circ}^{[1]}\right\} \\ &\quad + \gamma^{[2]}\left\{M^{[2]}\vec{d}^{\,[2]}\times\vec{v}^{\,[2]} + M^{[2]}a^{[2]}\vec{s}_{\circ}^{[2]}\right\} \end{aligned}$$

Input parameters and global ADM charges:

- **Input parameters:** the rest masses $M^{[n]}$, displacements $\vec{d}^{\,[n]}$, speeds $\vec{v}^{\,[n]}$ and spins $M^{[n]}a^{[n]}\vec{s}_\circ^{[n]}$ of the involved black holes
  - essentially the same as used in post-Newtonian description of binaries **!!!**
- **Global ADM charges:** in terms of the input parameters
  - though (*) does not satisfy Einstein's equations it is asymptotically flat
  - constructed by adding contributions of individual black hole metrics to a Minkowski background
  - the ADM quantities are linear in deviation from flat Euclidean space at infinity

$$
\begin{aligned}
M^{ADM} &= \gamma^{[1]}M^{[1]} + \gamma^{[2]}M^{[2]} \\
M^{ADM}\vec{d}^{\,ADM} &= \gamma^{[1]}M^{[1]}\vec{d}^{\,[1]} + \gamma^{[2]}M^{[2]}\vec{d}^{\,[2]} \\
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&\quad + \gamma^{[2]}\left\{M^{[2]}\vec{d}^{\,[2]}\times\vec{v}^{\,[2]} + M^{[2]}a^{[2]}\vec{s}_\circ^{[2]}\right\}
\end{aligned}
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Summary:

1. a new method to initialize time evolution of binary black hole systems by applying
   - a parabolic-hyperbolic formulation of constraint equations
   - superposing Kerr-Schild black holes
2. the parabolic-hyperbolic equations solved as an initial-boundary value problem
3. existence of unique (at least) C^2 solutions is guaranteed (apart from singularities)
4. construct initial data by integrating numerically the parabolic-hyperbolic form of the constraints: !!! paper is coming out soon
5. the input parameters—the rest masses, speeds, spins and displacements—are essentially the same as used in PN **!!!**
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