The many faces of the constraints

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The main message:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

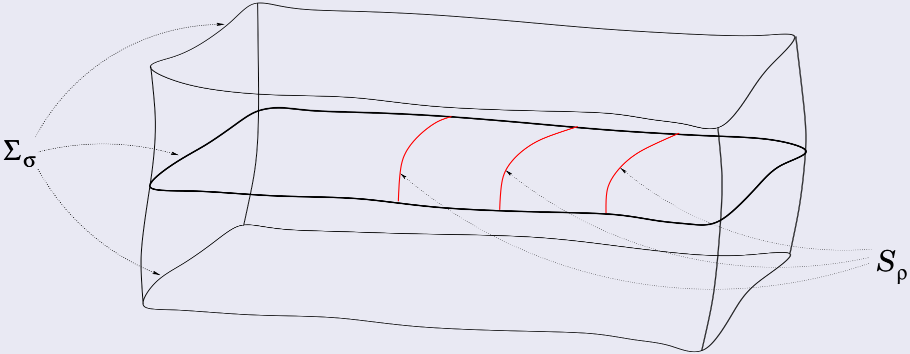
Based on some recent works:

- I. Rácz: *Is the Bianchi identity always hyperbolic?*, CQG **31** 155004 (2014)
- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, Class. Quantum Grav. **32** (2015) 015006
- I. Rácz: *Dynamical determination of the gravitational degrees of freedom*, arXiv:1412.0667 (2015)
- I. Rácz: *Constraints as evolutionary systems*, CQG **33** 015014 (2016)
- I. Rácz and J. Winicour: *Black hole initial data without elliptic equations*, Phys. Rev. D **91**, 124013 (2015)
- I. Rácz: *A simple method of constructing binary black hole initial data*, Astronomy Reports 62 953-958 (2018)
- I. Rácz: *On the ADM charges of multiple black holes*, arXiv:1608.02283
- I. Rácz and J. Winicour: *Toward computing gravitational initial data without elliptic solvers*, CQG **35** 135002 (2018)
- K. Csukás and I. Rácz: *On the asymptotics of solutions to the evolutionary form of the constraints*, to be submitted for publication (2019)

All the involved results are valid for arbitrary dimension: i.e. for $dim(M) = n \ (\geq 4)$. Nevertheless, for the sake of simplicity attention will be restricted to the case of $n = 4$.

Outline:

- **Einsteinian spaces:** (M, g_{ab})
  - First part
  - Second part



- in both cases metrics of Euclidean signature will be involved
- no gauge condition

  ... arbitrary choice of foliations & "evolutionary" vector field

The basic setup:

- **Einsteinian spaces:** (M, g_{ab})
  - M : 4-dimensional, smooth, paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian$_{(-,+,+,+)}$ or Riemannian$_{(+,+,+,+)}$ metric
- **Einstein's equations:**

$$\boxed{G_{ab} - \mathscr{G}_{ab} = 0} \quad \text{with source term:} \quad \boxed{\nabla^a \mathscr{G}_{ab} = 0}$$

  - $\boxed{\nabla_a}$ denotes the covariant derivative operator associated with g_{ab}.
  - in a more familiar setup: **Einstein's equations** with cosmological constant Λ

$$\boxed{[R_{ab} - \tfrac{1}{2}\, g_{ab}\, R] + \Lambda\, g_{ab} = 8\pi\, T_{ab}}$$

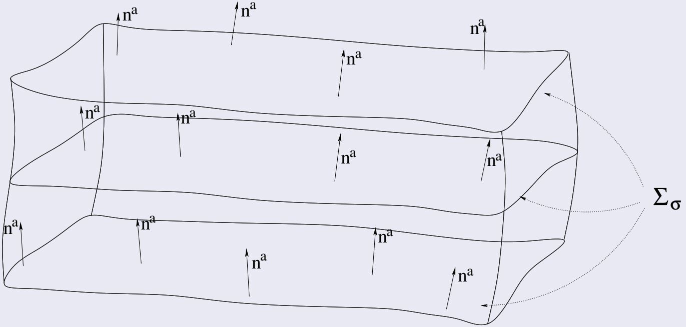
    with matter fields satisfying their Euler-Lagrange equations
  -

$$\boxed{\mathscr{G}_{ab} = 8\pi\, T_{ab} - \Lambda\, g_{ab}}$$

PART I:

The primary splitting

- **Assume:** M is smoothly foliated by a one-parameter family of homologous hypersurfaces, i.e. $M \simeq \mathbb{R} \times \Sigma$, for some three-dimensional manifold Σ.
  - known to **hold for globally hyperbolic spacetimes** (Lorentzian case)
  - **equivalent to** the existence of a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing gradient $\partial_a\sigma$ such that the $\sigma = const$ level surfaces $\Sigma_\sigma = \{\sigma\} \times \Sigma$ comprise the one-parameter foliation of M.
  - $$n_a \sim \partial_a\sigma \ \ldots\ \& \ldots\ g^{ab} \ \longrightarrow \ \ n^a = g^{ab} n_b$$



Projections:

The projection operator:

- n^a the 'unit norm' vector field that is normal to the Σ_σ level surfaces

$$n^a n_a = \epsilon$$

  - the sign is not fixed: ϵ takes the value -1 or $+1$ for Lorentzian or Riemannian metric g_{ab}, respectively

- **the projection operator**

$$h^a{}_b = \delta^a{}_b - \epsilon\, n^a n_b$$

  to the level surfaces of $\sigma : M \to \mathbb{R}$.

- **the induced metric** on the $\sigma = const$ level surfaces

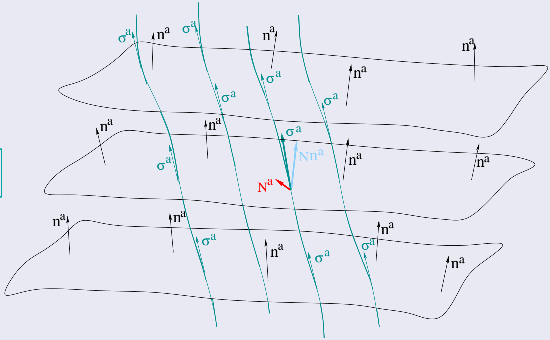
$$h_{ab} = h^e{}_a h^f{}_b\, g_{ef}$$

  - D_a denotes the covariant derivative operator associated with h_{ab}.

σ^a is "time evolution vector field" **if**:

- the integral curves of σ^a meet the $\sigma = const$ level surfaces precisely once
- $\sigma^e \nabla_e \sigma = 1$

$$\sigma^a = \sigma^a_\perp + \sigma^a_\parallel = N\, n^a + N^a$$



- where N and N^a denotes the **lapse** and **shift** of σ^a:

$$N = \epsilon\,(\sigma^e n_e) \quad \text{and} \quad N^a = h^a{}_e\,\sigma^e$$

- adopted coordinates & Lie derivatives: ! (x^2, x^3, x^4) local coordinates on Σ_0; extend them along the integral curves of σ^a onto a neighborhood of Σ_0 in M; (σ, x^2, x^3, x^4) local coordinates there & Lie derivative $\mathscr{L}_\sigma T$ is $\partial_\sigma T = \partial T / \partial\sigma$

Decompositions of various fields:

Any symmetric tensor field P_{ab} can be decomposed

in terms of n^a and fields intrinsic to the individual $\sigma = const$ level surfaces as

$$P_{ab} = \boldsymbol{\pi}\, n_a n_b + [n_a\, \mathbf{p}_b + n_b\, \mathbf{p}_a] + \mathbf{P}_{ab}$$

where $\boldsymbol{\pi} = n^e n^f\, P_{ef}\,, \quad \mathbf{p}_a = \epsilon\, h^e{}_a n^f\, P_{ef}\,, \quad \mathbf{P}_{ab} = h^e{}_a h^f{}_b\, P_{ef}$

It is also rewarding to inspect the decomposition of the cov. divergence $\nabla^a P_{ab}$:

$$\epsilon\, (\nabla^a P_{ae})\, n^e = \mathscr{L}_n \boldsymbol{\pi} + D^e \mathbf{p}_e + [\boldsymbol{\pi}\, (K^e{}_e) - \epsilon\, \mathbf{P}_{ef} K^{ef} - 2\, \epsilon\, \dot{n}^e \mathbf{p}_e]$$

$$(\nabla^a P_{ae})\, h^e{}_b = \mathscr{L}_n \mathbf{p}_b + D^e \mathbf{P}_{eb} + [(K^e{}_e)\, \mathbf{p}_b + \dot{n}_b\, \boldsymbol{\pi} - \epsilon\, \dot{n}^e \mathbf{P}_{eb}]$$

$$K_{ab} = h^e{}_a \nabla_e n_b = \tfrac{1}{2}\, \mathscr{L}_n h_{ab} \qquad \dot{n}_a := n^e \nabla_e n_a = -\epsilon\, D_a \ln N$$

$$\nabla^a P_{ae} = g^{ab} \nabla_a P_{be} = \left[\, h^{ab} - \epsilon\, n^a n^b \right] \nabla_a \left\{ \boldsymbol{\pi}\, n_b n_e + [\, n_b\, \mathbf{p}_e + n_b\, \mathbf{p}_e] + \mathbf{P}_{be} \right\}$$

back

Decompositions of various fields:

Examples:

- the metric

$$g_{ab} = \epsilon\, n_a n_b + h_{ab}$$

- the "source term"

$$\mathscr{G}_{ab} = n_a n_b\, \mathfrak{e} + [n_a\, \mathfrak{p}_b + n_b\, \mathfrak{p}_a] + \mathfrak{S}_{ab}$$

where $$\mathfrak{e} = n^e n^f\, \mathscr{G}_{ef}\,, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f\, \mathscr{G}_{ef}\,, \quad \mathfrak{S}_{ab} = h^e{}_a h^f{}_b\, \mathscr{G}_{ef}$$

- l.h.s. of Einstein's equation: $E_{ab} = G_{ab} - \mathscr{G}_{ab}$

$$E_{ab} = n_a n_b\, E^{(\mathcal{H})} + [n_a\, E^{(\mathcal{M})}_b + n_b\, E^{(\mathcal{M})}_a] + (E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})$$

$$E^{(\mathcal{H})} = n^e n^f\, E_{ef}\,, \quad E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f\, E_{ef}\,, \quad E^{(\mathcal{EVOL})}_{ab} = h^e{}_a h^f{}_b\, E_{ef} - h_{ab}\, E^{(\mathcal{H})}$$

The decomposition of the covariant divergence $\nabla^a E_{ab} = 0$ of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$:

$$\mathscr{L}_n E^{(\mathcal{H})} + D^e E^{(\mathcal{M})}_e + [\, E^{(\mathcal{H})} (K^e{}_e) - 2\,\epsilon\, (\dot n^e\, E^{(\mathcal{M})}_e) \quad - \epsilon\, K^{ae}\, (E^{(\mathcal{EVOL})}_{ae} + h_{ae}\, E^{(\mathcal{H})})\,] = 0$$

Div

$$\mathscr{L}_n E^{(\mathcal{M})}_b + D^a (E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})}) + [\, (K^e{}_e)\, E^{(\mathcal{M})}_b + E^{(\mathcal{H})}\, \dot n_b \quad - \epsilon\, (E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})\, \dot n^a\,] = 0$$

1st order symmetric hyperbolic system: linear and homogeneous in $(E^{(\mathcal{H})}, E^{(\mathcal{M})}_I)^T$:

- $N \times$ "(1)" and $N h^{ij} \times$ "(2)" in local coordinates (σ, x^1, x^2, x^3) adopted to an arbitrary flow field $\boxed{\sigma^a = N\, n^a + N^a: \quad \sigma^e \nabla_e \sigma = 1}$ and the foliation $\{\Sigma_\sigma\}$, read as

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & h^{IJ} \end{pmatrix} \partial_\sigma + \begin{pmatrix} -N^K & N\, h^{IK} \\ N\, h^{JK} & -N^K\, h^{IJ} \end{pmatrix} \partial_K \right\} \begin{pmatrix} E^{(\mathcal{H})} \\ E^{(\mathcal{M})}_I \end{pmatrix} = \begin{pmatrix} \mathscr{E} \\ \mathscr{E}^J \end{pmatrix}$$

  !!! the source terms $\mathscr{E}$ and $\mathscr{E}^J$ are linear and homogeneous in $E^{(\mathcal{H})}$ and $E^{(\mathcal{M})}_I$!!! ϵ

- $\boxed{\mathcal{A}^\mu\, \partial_\mu v + \mathcal{B}\, v = 0}$ **with** $v = (E^{(\mathcal{H})}, E^{(\mathcal{M})}_I)^T$ $\quad FOSH$!!! $v \equiv 0$

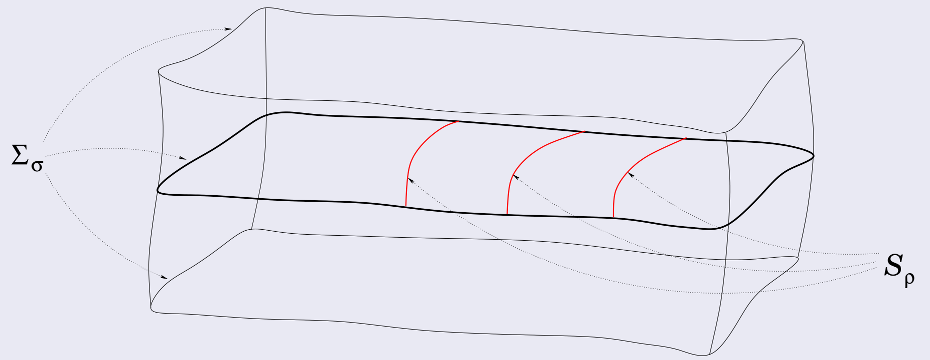
The main result of the first part:

Theorem

Let (M, g_{ab}) *be an Einsteinian space as specified and assume that the metric* h_{ab} *induced on the* $\sigma = const$ *level surfaces is Riemannian. Then,* **regardless whether** g_{ab} **is of Lorentzian or Euclidean signature**, *any solution to the reduced equations* $E^{(\mathcal{EVOL})}_{ab} = 0$ *is also a solution to the full set of field equations* $G_{ab} - \mathscr{G}_{ab} = 0$ *provided that the constraint expressions* $E^{(\mathcal{H})}$ *and* $E^{(\mathcal{M})}_{a}$ *vanish on one of the* $\sigma = const$ *level surfaces.*

- no gauge condition was used anywhere in the above analyze !
  - it applies regardless of the choice of the foliation, $\boxed{\Sigma_\sigma}$, of M and for any choice of the flow field, $\boxed{\sigma^a \rightleftarrows N, N^a}$

PART II:



The explicit form of the constraints:

The constraint expressions are projections of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$:

$$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{-\epsilon\, {}^{(3)}R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0$$
$$E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\,[D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a] = 0$$

- where D_a denotes the covariant derivative operator associated with h_{ab} and

$$\mathfrak{e} = n^e n^f \mathscr{G}_{ef}, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f \mathscr{G}_{ef}$$

- it is an underdetermined system: 4 equations for 12 variables

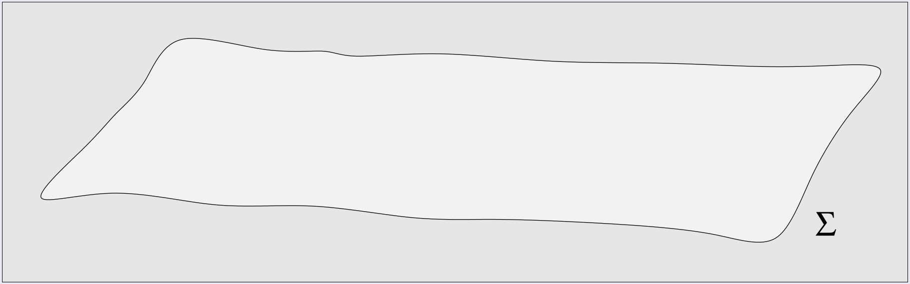
$$(h_{ij}, K_{ij})$$

A simple example:

Consider the underdetermined equation on $\Sigma \approx \mathbb{R}^2$ with some coordinates (χ, ξ)

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

- it is an equation for the four variables $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and $\boldsymbol{z}$ on Σ
- in advance of solving it three of these variables have to be fixed on Σ

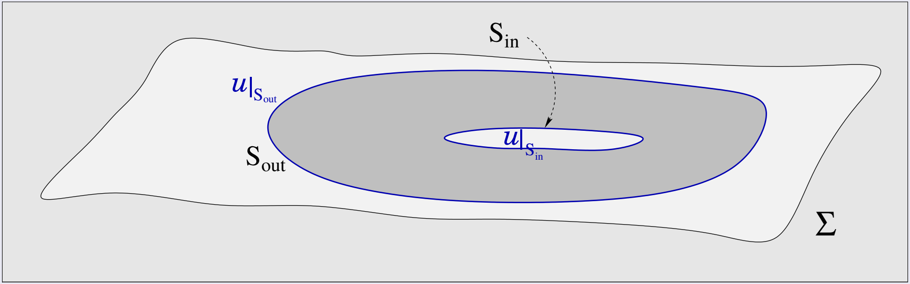


A simple example:

It is an elliptic equation for u on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{v}, \boldsymbol{w}$ and $\boldsymbol{z}$ have to be specified on $\mathbb{R}^2$
- the variable $\boldsymbol{u}$ has also to be fixed at the boundaries $\mathrm{S_{out}}$ and $\mathrm{S_{in}}$

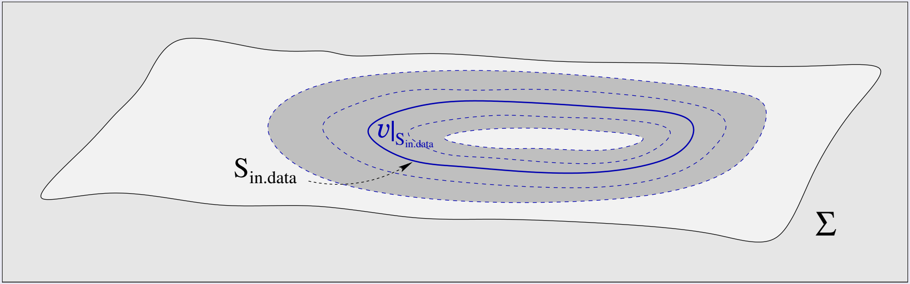


A simple example:

It is a hyperbolic equation for v on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}, \boldsymbol{w}$ and $\boldsymbol{z}$ have to be specified on $\mathbb{R}^2$
- the variable $\boldsymbol{v}$ has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$

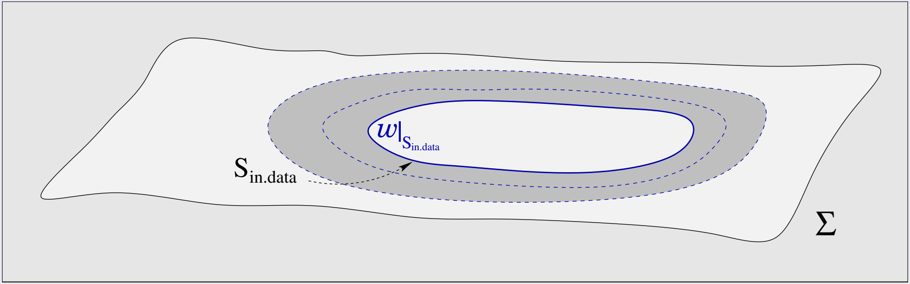


A simple example:

It is a parabolic equation for w on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ have to be fixed on $\mathbb{R}^2$: $a > 0$
- the variable w has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$

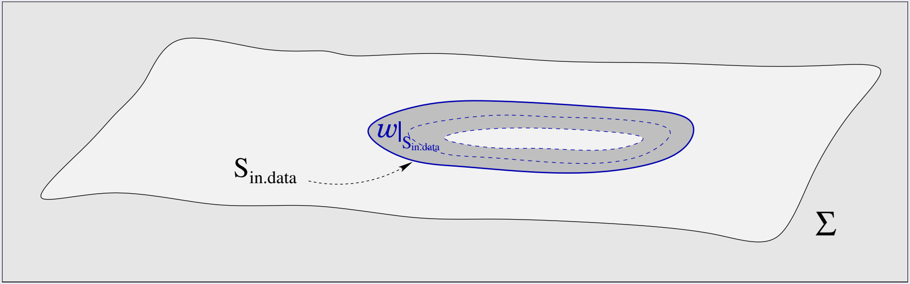


A simple example:

It is a parabolic equation for w on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ have to be fixed on $\mathbb{R}^2$: $\boxed{a < 0}$
- the variable $\boldsymbol{w}$ has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$



A simple example:

It is an algebraic equation for $\boldsymbol{z}$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi^2 - \partial_\xi^2)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- once the variables u, v, w are specified on $\mathbb{R}^2$ the solution is determined as

$$\boldsymbol{z} = -\left[(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi^2 - \partial_\xi^2)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w\right]$$

New variables by applying $2+1$ decompositions:

Splitting of the metric h_{ij}:

assume: $\Sigma \approx \mathbb{R} \times \mathscr{S}$

Σ is smoothly foliated by a one-parameter family of two-surfaces $\mathscr{S}_\rho$:
$\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

$$\Longrightarrow \quad \widehat{n}_i \sim \partial_i \rho \;\ldots\; \& \ldots \; h^{ij} \longrightarrow \widehat{n}^i = h^{ij}\widehat{n}_j \longrightarrow \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i \widehat{n}_j$$

- choose ρ^i to be a flow field on Σ: the integral curves... & $\rho^i \partial_i \rho = 1$
- 'lapse' and 'shift' of ρ^i

$$\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i \,, \quad \text{where} \quad \widehat{N} = \rho^j \widehat{n}_j \quad \text{and} \quad \widehat{N}^i = \widehat{\gamma}^i{}_j\, \rho^j$$

- induced metric, extrinsic curvature and acceleration of the $\mathscr{S}_\rho$ level surfaces:

$$\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i\, \widehat{\gamma}^l{}_j\, h_{kl} \qquad \widehat{K}_{ij} = \tfrac{1}{2}\,\mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij} \qquad \dot{\widehat{n}}_i := \widehat{n}^l D_l \widehat{n}_i = -\widehat{D}_i \ln \widehat{N}$$

- the metric h_{ij} can then be given as

$$h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i \widehat{n}_j \quad \Longleftrightarrow \quad \{\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}\}$$

$2+1$ decompositions:

Splitting of the symmetric tensor field K_{ij}:

- $$K_{ij} = \boldsymbol{\kappa}\,\widehat{n}_i\widehat{n}_j + [\widehat{n}_i\,\mathbf{k}_j + \widehat{n}_j\,\mathbf{k}_i] + \mathbf{K}_{ij}$$

  where

  $$\boldsymbol{\kappa} = \widehat{n}^k\widehat{n}^l\,K_{kl}\,, \quad \mathbf{k}_i = \widehat{\gamma}^k{}_i\widehat{n}^l\,K_{kl} \quad \text{and} \quad \mathbf{K}_{ij} = \widehat{\gamma}^k{}_i\widehat{\gamma}^l{}_j\,K_{kl}$$

- the **trace** and **trace free** parts of $\mathbf{K}_{ij}$

  $$\mathbf{K}^l{}_l = \widehat{\gamma}^{kl}\,\mathbf{K}_{kl} \quad \text{and} \quad \mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{2}\,\widehat{\gamma}_{ij}\mathbf{K}^l{}_l$$

The new variables:

- $$(h_{ij}, K_{ij}) \quad \Longleftrightarrow \quad (\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$$

- these variables retain the physically distinguished nature of h_{ij} and K_{ij}

The momentum constraint:

$\dot{\hat{n}}_i := \hat{n}^l D_l \hat{n}_i = -\widehat{D}_i \ln \widehat{N}$

$$D_e K^e{}_a - D_a K^e{}_e - \epsilon\, \mathfrak{p}_a = 0$$

$\widehat{K}_{ij} = \frac{1}{2} \mathscr{L}_{\hat{n}} \hat{\gamma}_{ij};\ \widehat{K}^l{}_l = \hat{\gamma}^{ij} \widehat{K}_{ij}$

$$\mathscr{L}_{\hat{n}} \mathbf{k}_i - \tfrac{1}{2}\, \widehat{D}_i(\mathbf{K}^l{}_l) - \widehat{D}_i \boldsymbol{\kappa} + \widehat{D}^l \mathring{\mathbf{K}}_{li} + (\widehat{K}^l{}_l)\, \mathbf{k}_i + \boldsymbol{\kappa}\, \dot{\hat{n}}_i - \dot{\hat{n}}^l\, \mathbf{K}_{li} - \epsilon\, \mathfrak{p}_l\, \hat{\gamma}^l{}_i = 0$$

back: str.hyp.sys.

$$\mathscr{L}_{\hat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l \mathbf{k}_l - \boldsymbol{\kappa}\, (\widehat{K}^l{}_l) + \mathbf{K}_{kl} \widehat{K}^{kl} + 2\, \dot{\hat{n}}^l\, \mathbf{k}_l + \epsilon\, \mathfrak{p}_l\, \hat{n}^l = 0$$

First order symmetric hyperbolic system:

- contract "(1)" with $2\, \widehat{N}\, \hat{\gamma}^{ij}$ and mult. "(2)" by $\widehat{N}$, when writing them out in coordinates (ρ, x^2, x^3), adopted to the foliation $\mathscr{S}_\rho$ and the vector field ρ^i,

$$\left\{ \begin{pmatrix} 2\, \hat{\gamma}^{AB} & 0 \\ 0 & 1 \end{pmatrix} \partial_\rho + \begin{pmatrix} -2\, \widehat{N}^K\, \hat{\gamma}^{AB} & -\widehat{N}\, \hat{\gamma}^{AK} \\ -\widehat{N}\, \hat{\gamma}^{BK} & -\widehat{N}^K \end{pmatrix} \partial_K \right\} \begin{pmatrix} \mathbf{k}_B \\ \mathbf{K}^E{}_E \end{pmatrix} + \begin{pmatrix} \mathscr{B}^{A}_{(\mathbf{k})} \\ \mathscr{B}_{(\mathbf{K})} \end{pmatrix} = 0$$

- a first order **symmetric hyperbolic** system for the vector valued variable

$$(\mathbf{k}_B, \mathbf{K}^E{}_E)^T$$

!!! ρ **plays the role of 'time'** regardless of the value of $\epsilon = \pm 1$

The coupled constraint system:

The Hamiltonian constraint in terms of the new variables:

- $$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\,\{-\epsilon\,{}^{(3)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0$$

- using $$^{(3)}\!R = \widehat{R} - \Big\{2\,\mathscr{L}_{\hat n}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\Big\}$$

  $\widehat{R}$ and $\widehat{K}_{kl}$ denote the scalar and extrinsic curvature of $\widehat{\gamma}_{kl}$, respectively

- $$-\epsilon\,\widehat{R} + \epsilon\,\Big\{2\,\mathscr{L}_{\hat n}(\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl}\,\widehat{K}^{kl} + 2\,\widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\Big\} + 2\,\boldsymbol{\kappa}\,\mathbf{K}^l{}_l + \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \mathring{\mathbf{K}}_{kl}\,\mathring{\mathbf{K}}^{kl} - 2\,\mathfrak{e} = 0$$

Alternative choices yielding evolutionary systems:

- **parabolic equation** for $\widehat{N}$ — **parabolic-hyperbolic** for the coupled system
- **algebraic equation** for $\boldsymbol{\kappa}$ — **algebraic-hyperbolic** for the coupled system

The take home message:

On contrary to the folklore, in the considered two explicit examples, **evolutionary methods can be applied in spaces with metric of Euclidean signature where, in principle, there is no room for ‘time’**