Stationary black holes as holographs

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Black hole physics

- significant part of our knowledge about black holes originates from what we learnt about the well-known Schwarzschild or Kerr black holes.
  - *the only* asymptotically flat stationary black hole solutions to the vacuum Einsteins equations
  - asymptotically flatness $\Longrightarrow$ black holes completely isolated in space
- it is of obvious interest to know how these isolated black holes might be distorted by external matter distributions
- it may also be tempting to determine all the possible distorted black hole solutions and (at least) to provide a clear characterization of them
- a key result in the black holes uniqueness proofs is the black hole rigidity theorem by **Hawking**:
  - the event horizon of an !*analytic*! stationary asymptotically flat electrovac black hole spacetime is necessarily a Killing horizon, i.e., the spacetime must possess a Killing field (possibly distinct from the stationary Killing field) which is normal to the event horizon
  - in an asymptotically flat stationary (non-static) black hole spacetime there exists an additional axial Killing field, i.e., a stationary black hole spacetime is either static or stationary axisymmetric.

Generic stationary distorted black holes:

- spacetimes admitting a one-parameter family of isometry actions and an associated Killing horizon (not as general as spacetimes with isolated horizon!)
- in studying distorted stationary black holes the assumption on the asymptotic flatness should be relaxed
  - „a priory" we do not assume any sort of asymptotic behavior $\Longrightarrow$ whenever a regular asymptotic region exists the relevant asymptotic properties should be deduced by using the field equations
- static distorted black hole solutions were considered to be relevant only locally by representing a black hole solution yielded by the distortion of the Schwarzschild solution by certain external mass distributions: Israel & Khan 64', Mysak & Szekeres 66', Geroch & Hartle 82' (all static & axially symmetric solutions are given)
- V. Frolov,.... distorted static black hole spacetimes may also play important role in context of four (or higher) dimensional theories whenever one (or some) of the spacelike dimensions is (or are) compactified

Motivation: To have a framework capable to include all the "stationary" distorted black hole spacetimes

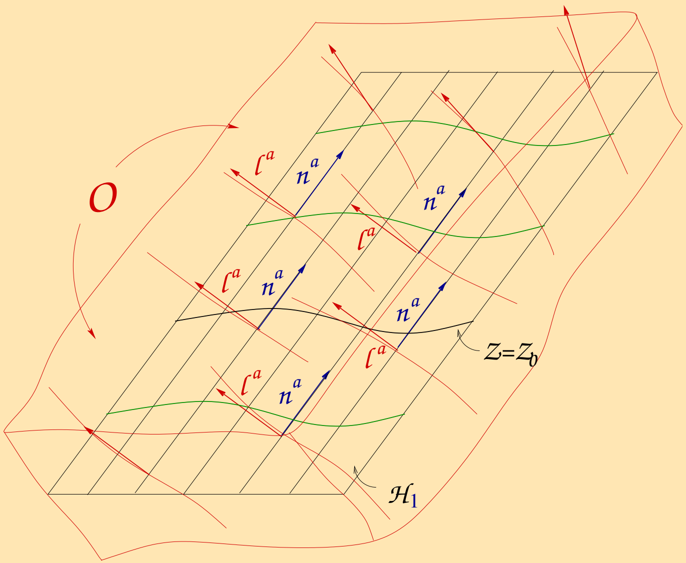
- Rácz, I. (2007): *Stationary black holes as holographs*, Classical and Quantum Gravity **24**, 5541-5571
- Rácz, I. (2014): *Stationary black holes as holographs II.*, Classical and Quantum Gravity **31**, 035006
  - Here we shall present some results relevant for the pure vacuum case. We would like to emphasize that the associated techniques do extend to spacetimes with a source free electromagnetic field and for the inclusion of a non-zero cosmological constant.
- Cole, M.J., Rácz, I., Valiente Kroon, J.A. (2018): *Killing spinor data on distorted black hole horizons and the uniqueness of stationary vacuum black holes*, arXiv:1804.10287, submitted to Class. and Quantum Grav.

- **Spacetime:** (M, g_{ab})
  - M : smooth, 4-dim., paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian metric with signature $(+,-,-,-)$
  - (M, g_{ab}) is time orientable; a time orientation has been chosen.

$$\boxed{R_{ab} = 0}$$

Gaussian null coordinates

- we shall use the Newman-Penrose formalism which refer to the Gaussian null coordinates (u, r, x^3, x^4)



- n^a: $\quad n^e \nabla_e n^a = 0$ on $\mathcal{H}_1 = \chi_u[\mathcal{Z}_0]$
- l^a: $\quad l^e n_e|_{\mathcal{Z}_u} = 1 \quad \& \quad l^e \nabla_e l^a = 0$ on $\mathscr{O}$
- $\exists$ an open subset of $\mathcal{Z}_0$ on which local coordinates (x^3, x^4) can be defined

The most general form of the spacetime metric

- in $\mathcal{O}$ and (u, r, x^3, x^4) the spacetime metric takes the form

$$\boxed{\mathrm{d}s^2 = g_{uu}\,\mathrm{d}u^2 + 2\,\mathrm{d}r\,\mathrm{d}u + 2\,g_{uA}\mathrm{d}u\mathrm{d}x^A + g_{AB}\,\mathrm{d}x^A\mathrm{d}x^B}$$

  where g_{uu}, g_{uA} and g_{AB} are smooth functions of the coordinates u, r, x^3, x^4 in $\mathcal{O}$ such that g_{uu} and g_{uA} vanish on $\mathscr{H}_1$, and g_{AB} is a negative definite 2×2 matrix.

- By patching domains where Gaussian null coordinates can be defined one can always extend results derived in one of them to the entire of the underlying „elementary spacetime region" $\mathcal{O}$.

The Newman-Penrose formalism:

- Choose now real-valued functions U, X^A and complex-valued functions ω, ξ^A on $\mathcal{O}$, with Gaussian null coordinates (u, r, x^3, x^4) such that
  - the functions U, X^A, ω vanish on $\mathscr{H}_1$
- we obtain a **complex null tetrad:** $\{l^a, n^a, m^a, \overline{m}^a\}$ in $\mathcal{O}$ by setting

$$\boxed{l^\mu = \delta^\mu{}_r, \quad n^\mu = \delta^\mu{}_u + U\delta^\mu{}_r + X^A\delta^\mu{}_A, \quad m^\mu = \omega\delta^\mu{}_r + \xi^A\delta^\mu{}_A}$$

- the contravariant form of the metric in $\mathcal{O}$ can then be given as

$$g^{ab} = l^a n^b + l^b n^a - m^a\overline{m}^b - \overline{m}^a m^b\,, \qquad g^{\alpha\beta} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & g^{rr} & g^{rB} \\ 0 & g^{Ar} & g^{AB} \end{pmatrix}$$

$$g^{rr} = 2(U - \omega\bar{\omega}), \quad g^{rA} = X^A - (\bar{\omega}\xi^A + \omega\bar{\xi}^A), \quad g^{AB} = -(\xi^A\bar{\xi}^B + \bar{\xi}^A\xi^B)$$

Spin coefficients

$\kappa = l^a m^b \nabla_a l_b$	$\varepsilon = \frac{1}{2} l^a (n^b \nabla_a l_b - \overline{m}^b \nabla_a m_b)$	$\pi = -l^a \overline{m}^b \nabla_a n_b$
$\rho = \overline{m}^a m^b \nabla_a l_b$	$\alpha = \frac{1}{2} \overline{m}^a (n^b \nabla_a l_b - \overline{m}^b \nabla_a m_b)$	$\lambda = -\overline{m}^a \overline{m}^b \nabla_a n_b$
$\sigma = m^a m^b \nabla_a l_b$	$\beta = \frac{1}{2} m^a (n^b \nabla_a l_b - \overline{m}^b \nabla_a m_b)$	$\mu = -m^a \overline{m}^b \nabla_a n_b$
$\tau = n^a m^b \nabla_a l_b$	$\gamma = \frac{1}{2} n^a (n^b \nabla_a l_b - \overline{m}^b \nabla_a m_b)$	$\nu = -n^a \overline{m}^b \nabla_a n_b$

Weyl spinor components

$\Psi_0 = -C_{abcd} l^a m^b l^c m^d$
$\Psi_1 = -C_{abcd} l^a n^b l^c m^d$
$\Psi_2 = -\frac{1}{2} C_{abcd} (l^a n^b l^c n^d - l^a n^b m^c \overline{m}^d)$
$\Psi_3 = -C_{abcd} n^a l^b n^c \overline{m}^d$
$\Psi_4 = -C_{abcd} n^a \overline{m}^b n^c \overline{m}^d$

- The Newman-Penrose equations relates derivatives of the spin coefficients and Weyl spinor components in the direction of the frame vectors defined above and denote the corresponding operators in $\mathcal{O}$ by

$$\mathrm{D} = \partial/\partial r,\ \ \Delta = \partial/\partial u + U\cdot\partial/\partial r + X^A\cdot\partial/\partial x^A,\ \ \delta = \omega\cdot\partial/\partial r + \xi^A\cdot\partial/\partial x^A$$

- To simplify the NP equations a part of the gauge freedom can be fixed by assuming that the tetrad $\{l^a, n^a, m^a, \overline{m}^a\}$ is parallelly propagated along the null geodesics with tangent $l^a = (\partial/\partial r)^a$ in $\mathcal{O}$.
- These assumptions guarantee that for the spin coefficients, corresponding to this specific choice of complex null tetrad, $\kappa = \pi = \varepsilon = 0$, $\rho = \overline{\rho}$, $\tau = \overline{\alpha} + \beta$ hold everywhere in $\mathcal{O}$.
- $n^e\nabla_e n^a = 0$ on $\mathscr{H}_1 \Longrightarrow \nu \equiv 0$ on $\mathscr{H}_1$
- u is an affine par. along the generators of $\mathscr{H}_1 \Longrightarrow \gamma + \overline{\gamma} \equiv 0$ on $\mathscr{H}_1$
- $\gamma = \frac{1}{2}n^a(n^b\nabla_a l_b - \overline{m}^b\nabla_a m_b) \Rightarrow \exists\ \phi : \mathcal{H}_1 \to \mathbb{R}$ real function: by performing the rotation $m^a \to e^{i\phi}m^a$: $\gamma \equiv 0$ on $\mathscr{H}_1$ can be ensured.

These gauge fixing simplifies the equations in great extent.

The Newman - Penrose equations:

$$
\begin{aligned}
\mathrm{D}(\rho) &= \rho^2 + \sigma\bar{\sigma} && \text{(NP.1)}\\
\mathrm{D}(\sigma) &= 2\rho\sigma + \Psi_0 && \text{(NP.2)}\\
\mathrm{D}(\tau) &= \tau\rho + \bar{\tau}\sigma + \Psi_1 && \text{(NP.3)}\\
\mathrm{D}(\alpha) &= \rho\alpha + \beta\bar{\sigma} && \text{(NP.4)}\\
\mathrm{D}(\beta) &= \alpha\sigma + \rho\beta + \Psi_1 && \text{(NP.5)}\\
\mathrm{D}(\gamma) &= \tau\alpha + \bar{\tau}\beta + \Psi_2 && \text{(NP.6)}\\
\mathrm{D}(\lambda) &= \rho\lambda + \bar{\sigma}\mu && \text{(NP.7)}\\
\mathrm{D}(\mu) &= \rho\mu + \sigma\lambda + \Psi_2 && \text{(NP.8)}\\
\mathrm{D}(\nu) &= \bar{\tau}\mu + \tau\lambda + \Psi_3 && \text{(NP.9)}\\
\Delta(\lambda) - \bar{\delta}(\nu) &= (\bar{\gamma} - 3\gamma - \mu - \bar{\mu})\lambda + (3\alpha + \bar{\beta} - \bar{\tau})\nu - \Psi_4 && \text{(NP.10)}\\
\delta(\rho) - \bar{\delta}(\sigma) &= (\bar{\alpha} + \beta)\rho - (3\alpha - \bar{\beta})\sigma - \Psi_1 && \text{(NP.11)}\\
\delta(\alpha) - \bar{\delta}(\beta) &= \rho\mu - \sigma\lambda + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta - \Psi_2 && \text{(NP.12)}\\
\delta(\lambda) - \bar{\delta}(\mu) &= (\alpha + \bar{\beta})\mu + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 && \text{(NP.13)}\\
\delta(\nu) - \Delta(\mu) &= \mu^2 + \lambda\bar{\lambda} + (\gamma + \bar{\gamma})\mu + (\tau - \bar{\alpha} - 3\beta)\nu && \text{(NP.14)}\\
\delta(\gamma) - \Delta(\beta) &= \mu\tau - \sigma\nu - (\gamma - \bar{\gamma} - \mu)\beta + \alpha\bar{\lambda} && \text{(NP.15)}\\
\delta(\tau) - \Delta(\sigma) &= \mu\sigma + \bar{\lambda}\rho + (\tau - \bar{\alpha} + \beta)\tau - (3\gamma - \bar{\gamma})\sigma && \text{(NP.16)}\\
\Delta(\rho) - \bar{\delta}(\tau) &= -\rho\bar{\mu} - \sigma\lambda + (\gamma + \bar{\gamma})\rho - (\bar{\tau} + \alpha - \bar{\beta})\tau - \Psi_2 && \text{(NP.17)}\\
\Delta(\alpha) - \bar{\delta}(\gamma) &= \rho\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \bar{\mu})\alpha + (\bar{\beta} - \bar{\tau})\gamma - \Psi_3 && \text{(NP.18)}
\end{aligned}
$$

The Bianchi identities:

$$\begin{aligned}
\mathrm{D}(\Psi_1) - \bar{\delta}(\Psi_0) &= -4\alpha\Psi_0 + 4\rho\Psi_1 && \text{(B.1)}\\
\Delta(\Psi_0) - \delta(\Psi_1) &= (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 && \text{(B.2)}\\
\mathrm{D}(\Psi_2) - \bar{\delta}(\Psi_1) &= -\lambda\Psi_0 - 2\alpha\Psi_1 + 3\rho\Psi_2 && \text{(B.3)}\\
\Delta(\Psi_1) - \delta(\Psi_2) &= \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 && \text{(B.4)}\\
\mathrm{D}(\Psi_3) - \bar{\delta}(\Psi_2) &= -2\lambda\Psi_1 + 2\rho\Psi_3 && \text{(B.5)}\\
\Delta(\Psi_2) - \delta(\Psi_3) &= 2\nu\Psi_1 - 3\mu\Psi_2 - 2\overline{\alpha}\Psi_3 + \sigma\Psi_4 && \text{(B.6)}\\
\mathrm{D}(\Psi_4) - \bar{\delta}(\Psi_3) &= -3\lambda\Psi_2 + 2\alpha\Psi_3 + \rho\Psi_4 && \text{(B.7)}\\
\Delta(\Psi_3) - \delta(\Psi_4) &= 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4 && \text{(B.8)}
\end{aligned}$$

The metric equations:

$$\begin{aligned}
\mathrm{D}(\xi^A) &= \rho\xi^A + \sigma\bar{\xi}^A && \text{(M.1)}\\
\mathrm{D}(\omega) &= \rho\omega + \sigma\overline{\omega} - \tau && \text{(M.2)}\\
\mathrm{D}(X^A) &= \tau\bar{\xi}^A + \bar{\tau}\xi^A && \text{(M.3)}\\
\mathrm{D}(U) &= \tau\overline{\omega} + \bar{\tau}\omega - (\gamma + \bar{\gamma}) && \text{(M.4)}\\
\delta(X^A) - \Delta(\xi^A) &= (\mu + \bar{\gamma} - \gamma)\xi^A + \bar{\lambda}\bar{\xi}^A && \text{(M.5)}\\
\delta(\bar{\xi}^A) - \bar{\delta}(\xi^A) &= (\overline{\beta} - \alpha)\xi^A + (\overline{\alpha} - \beta)\bar{\xi}^A && \text{(M.6)}\\
\delta(\overline{\omega}) - \bar{\delta}(\omega) &= (\overline{\beta} - \alpha)\omega + (\overline{\alpha} - \beta)\overline{\omega} + (\mu - \overline{\mu}) && \text{(M.7)}\\
\delta(U) - \Delta(\omega) &= (\mu + \bar{\gamma} - \gamma)\omega + \bar{\lambda}\overline{\omega} - \overline{\nu} && \text{(M.8)}
\end{aligned}$$

One of the motivations

- is to give the generic argument using the Newman-Penrose variables is rooted in remarks made by Chandrasekhar when discussing the role of the full set of the Newman-Penrose equations in Sections 7 and 8 of Chapter 1 in his brilliant book „The Mathematical Theory of Black Holes" (1983).
- right after he made an excellent (perhaps the best) introduction of the NP formalism in his book he ended up with the following comments:
- *„It is not clear how many of these equations are independent, how they are to be ordered or used and, indeed, what they are for."*
- the following part is to answer these questions by making use of a suitable adaptation hyperbolic reduction techniques

The characteristic initial value formulation

- Recall first that the Newman-Penrose equations, taking them as first order PDEs, with respect to Gaussian null coordinates, (u, r, x^3, x^4) in $\mathcal{O}$ for the vector valued variable

$$\boxed{\mathbb{V} = (\xi^A, \omega, X^A, U; \rho, \sigma, \tau, \alpha, \beta, \gamma, \lambda, \mu, \nu; \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4)}$$

  are overdetermined simply because we have more equations, (NP.1)-(NP.18),(B.1)-(B.8),(M.1)-(M.8), than unknowns.

- But some of the Newman-Penrose equations are „interior equations" on $\mathcal{Z}$, $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively

- the generic hyperbolic reduction procedures: by taking aside some of the Newman-Penrose equations and taking linear combinations some other ones we get a suitable „reduced set of vacuum field equations"

The reduced set of vacuum field equations

$$\mathrm{D}\,\xi^A = \rho\,\xi^A + \sigma\,\overline{\xi}^A \tag{RE.1}$$
$$\mathrm{D}\,\omega = \rho\,\omega + \sigma\,\overline{\omega} - \tau \tag{RE.2}$$
$$\mathrm{D}X^A = \tau\,\overline{\xi}^A + \overline{\tau}\,\xi^A \tag{RE.3}$$
$$\mathrm{D}\,U = \tau\,\overline{\omega} + \overline{\tau}\,\omega - (\gamma+\overline{\gamma}) \tag{RE.4}$$
$$\mathrm{D}\,\rho = \rho^2 + \sigma\,\overline{\sigma} \tag{RE.5}$$
$$\mathrm{D}\,\sigma = 2\rho\,\sigma + \Psi_0 \tag{RE.6}$$
$$\mathrm{D}\,\tau = \tau\,\rho + \overline{\tau}\,\sigma + \Psi_1 \tag{RE.7}$$
$$\mathrm{D}\,\alpha = \rho\,\alpha + \beta\,\overline{\sigma} \tag{RE.8}$$
$$\mathrm{D}\,\beta = \alpha\,\sigma + \rho\,\beta + \Psi_1 \tag{RE.9}$$
$$\mathrm{D}\,\gamma = \tau\,\alpha + \overline{\tau}\,\beta + \Psi_2 \tag{RE.10}$$
$$\mathrm{D}\,\lambda = \rho\,\lambda + \overline{\sigma}\,\mu \tag{RE.11}$$
$$\mathrm{D}\,\mu = \rho\,\mu + \sigma\,\lambda + \Psi_2 \tag{RE.12}$$
$$\mathrm{D}\,\nu = \overline{\tau}\,\mu + \tau\,\lambda + \Psi_3 \tag{RE.13}$$
$$\Delta\,\Psi_0 - \delta\,\Psi_1 = (4\,\gamma - \mu)\,\Psi_0 - 2\,(2\,\tau + \beta)\,\Psi_1 + 3\,\sigma\,\Psi_2 \tag{RE.14}$$
$$\Delta\,\Psi_1 + \mathrm{D}\,\Psi_1 - \delta\,\Psi_2 - \overline{\delta}\,\Psi_0 = (\nu - 4\,\alpha)\,\Psi_0 - 2\,(\mu - \gamma - 2\,\rho)\,\Psi_1 - 3\,\tau\,\Psi_2 - 2\,\sigma\,\Psi_3 \tag{RE.15}$$
$$\Delta\,\Psi_2 + \mathrm{D}\,\Psi_2 - \delta\,\Psi_3 - \overline{\delta}\,\Psi_1 = -\lambda\,\Psi_0 - 2\,(\alpha - \nu)\,\Psi_1 + 3\,(\rho - \mu)\,\Psi_2 - 2\,(\tau - \beta)\,\Psi_3 + \sigma\,\Psi_4 \tag{RE.16}$$
$$\Delta\,\Psi_3 + \mathrm{D}\,\Psi_3 - \delta\,\Psi_4 - \overline{\delta}\,\Psi_2 = -2\,\lambda\,\Psi_1 + 3\,\nu\,\Psi_2 + 2\,(\rho - \gamma - 2\,\mu)\,\Psi_3 + (4\,\beta - \tau)\,\Psi_4 \tag{RE.17}$$
$$\mathrm{D}\,\Psi_4 - \overline{\delta}\,\Psi_3 = -3\,\lambda\,\Psi_2 + 2\,\alpha\,\Psi_3 + \rho\,\Psi_4 \tag{RE.18}$$

- These equations, besides comprising a determined system for the vector variable, $\mathbb{V}$, (from evolutionary point of view) are equivalent the complete set of the Newman-Penrose equations. More precisely:

Theorem:

- Denote by $\mathbb{V}_0$ an initial data set, satisfying the „inner" Newman-Penrose equations on the initial data surface comprised **by the pair of intersecting null hypersurfaces** $\mathcal{H}_1$ and $\mathcal{H}_2$.

- If $\mathbb{V}$ is the solution on the domain of dependence $D[\mathcal{H}_1 \cup \mathcal{H}_2]$ to the reduced vacuum field equations, (RE.1)-(RE.18), with $\mathbb{V}|_{\mathcal{H}_1 \cup \mathcal{H}_2} = \mathbb{V}_0$, then $\mathbb{V}$ is also a solution to the full set of the Newman-Penrose equations.

- Moreover, the metric, the connection and the curvature tensor determined by $\mathbb{V}$ are so that the connection will be metric and torsion free, as well as, the curvature tensor which can be built from the Weyl spinor components is the curvature tensor associated with this torsion free connection.

!*! Note that the condition requiring the initial data $\mathbb{V}_0$ satisfies the „inner" Newman-Penrose equations on $\mathcal{H}_1 \cup \mathcal{H}_2$ is not as restrictive as it seems to be.

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- … some of the Newman-Penrose equations are „interior equations” on $\mathcal{Z}$, $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively …
- Therefore, we may start, instead of

$$\boxed{\mathbb{V}_0 = \{\xi^A, \omega, X^A, U; \rho, \sigma, \tau, \alpha, \beta, \gamma, \lambda, \mu, \nu; \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4\}|_{\mathcal{H}_1 \cup \mathcal{H}_2}}$$

with a „reduced initial data set”, $\mathbb{V}_0^{red}$, which consists of the specification of the Weyl spinor components Ψ_4 on $\mathcal{H}_1$ and Ψ_0 on $\mathcal{H}_2$, moreover, it is required to include the specification of the spin-coefficients $\rho, \sigma, \tau, \mu, \lambda$, along with a vector field ξ^A such that $g^{AB} = -(\xi^A \overline{\xi}^B + \overline{\xi}^A \xi^B)$ is a negative definite metric, on $\mathcal{Z}$.

$$\boxed{\mathbb{V}_0^{red} = \{\rho, \sigma, \mu, \lambda, \tau\,;\, \xi^A\}|_{\mathcal{Z}} \cup \{\Psi_4\}|_{\mathcal{H}_1} \cup \{\Psi_0\}|_{\mathcal{H}_2}}$$

  - the inner equations on $\mathcal{Z}$ can be solved algebraically for the rest of the variables listed in $\mathbb{V}$: $\{\xi^A, \cancel{\omega}, \cancel{X}^A, \cancel{U}; \rho, \sigma, \tau, \alpha, \beta, \cancel{\gamma}, \lambda, \mu, \cancel{\nu}; \Psi_0, \Psi_1, \Psi_2, \Psi_3, \Psi_4\}|_{\mathcal{Z}}$
  - once the components of $\mathbb{V}$ are known on $\mathcal{Z}$ the desired initial data $\mathbb{V}_0$ can be determined on $\mathcal{H}_1$ and $\mathcal{H}_2$ by integrating a sequence of ordinary differential equations.
  - $\mathbb{V}_0$, yielded by this construction, satisfies all the inner equations as it was required in the above Theorem.
- $\mathbb{V}_0^{red}$, along with the Newman-Penrose equations, determines uniquely the initial data set $\mathbb{V}_0$ on $\mathcal{H}_1 \cup \mathcal{H}_2$

- In addition to the fact that the reduced vacuum field equations, (RE.1)-(RE.18), comprise a determined system—by inspection of the particular form they have when written out in the Gaussian null coordinates (u, r, x^3, x^4)—it can also be verified that they possess the form

$$\boxed{\mathbb{A}^\mu \cdot \partial_\mu \mathbb{V} + \mathbb{B} = 0}$$

  where the matrices $\mathbb{A}^\mu$ and $\mathbb{B}$ smoothly depend on $\mathbb{V}$ and on its complex conjugate $\overline{\mathbb{V}}$.

- Moreover, it can also be seen that the matrices $\mathbb{A}^\mu$ are Hermitian, i.e., $\boxed{\overline{\mathbb{A}}^{\mu T} = \mathbb{A}^\mu}$ and the combination $\boxed{\mathbb{A}^\mu(n_\mu + l_\mu)}$ is positive definite. Thereby, the system comprised by (RE.1)-(RE.18) is a **quasilinear symmetric hyperbolic system** for which the existence and uniqueness of solutions to the characteristic initial value problem is guaranteed.

As a summary we have (for its proof see the Appendix of BH holograph II):

Theorem: In the characteristic initial value problem to any 'reduced initial data set' there always exists a unique solution to the vacuum Einstein's equations.

Stationary distorted black hole spacetimes

- These are definitely not the most generic configurations to which the above general results are known to apply**(!)**

- the bifurcate **Killing horizon** $\mathcal{H}^*$ is necessarily expansion and shear free $\Rightarrow$ λ and μ vanish on $\mathcal{H}_1$, while σ and ρ is identically zero on $\mathcal{H}_2$; moreover, Ψ_3 and Ψ_4 vanish on $\mathcal{H}_1$ and Ψ_0 and Ψ_1, vanish on $\mathcal{H}_2$.

- a reduced initial data set, $\mathbb{V}_0^{red}$, is given as

$$\boxed{\mathbb{V}_0^{red} = \{\, \not\rho,\, \not\sigma,\, \not\mu,\, \not\lambda,\, \tau\,;\, \xi^A\}|_{\mathcal{Z}} \cup \{\, \not\Psi_4\}|_{\mathcal{H}_1} \cup \{\, \not\Psi_0\}|_{\mathcal{H}_2}}$$

- $\Longrightarrow$ the only variables which can „yet" be freely specified as our initial data are the spin coefficient τ and the vector field ξ^A on $\mathcal{Z}$

Corollary: Consider a 4-dimensional stationary distorted vacuum black hole spacetime (M, g_{ab}) with a pair of intersecting expansion and shear free null hypersurfaces $\mathcal{H}_1 \cup \mathcal{H}_2$. Then the spacetime metric g_{ab} is uniquely determined in the black hole region once the spin coefficient τ and the vector field ξ^A are specified on the intersection $\mathcal{Z} = \mathcal{H}_1 \cap \mathcal{H}_2$.

The determination of a full initial data set $\mathbb{V}_0$, on $\mathcal{H}_1 \cup \mathcal{H}_2$

- The „inner equations” on $\mathcal{Z}$:
  - (M.6), along with our gauge condition $\tau = \overline{\alpha} + \beta$ in $\mathcal{O}$, yields that

$$\boxed{\delta\overline{\xi}^A - \overline{\delta}\xi^A = (2\,\overline{\beta} - \overline{\tau})\,\xi^A + (\tau - 2\,\beta)\,\overline{\xi}^A}$$

  - τ and ξ^A are known thus this equation can be solved algebraically for β and, in turn, for $\alpha = \overline{\tau} - \overline{\beta}$.
  - (NP.12) fixes the value of Ψ_2 on $\widetilde{\mathcal{Z}}$ as

$$\boxed{\Psi_2 = -\delta\alpha + \overline{\delta}\beta + \alpha\,\overline{\alpha} - 2\,\alpha\,\beta + \beta\,\overline{\beta}}$$

- The „inner equations” on $\mathcal{H}_2$: $(\Psi_0 \equiv 0)$
  - In virtue of $\mathrm{D}(\Psi_1) - \overline{\delta}(\Psi_0) = -4\alpha\Psi_0 + 4\rho\Psi_1$ & $\Psi_1|_{\mathcal{Z}} \equiv 0$ we have $\Psi_1|_{\mathcal{H}_2} \equiv 0$.
  - Similarly, $\rho|_{\mathcal{Z}} \equiv 0$ and $\sigma|_{\mathcal{Z}} \equiv 0$, along with $\mathrm{D}(\rho) = \rho^2 + \sigma\bar{\sigma}$ and $\mathrm{D}(\sigma) = 2\rho\sigma + \Psi_0$, yield $\rho \equiv 0$ and $\sigma \equiv 0$ on $\mathcal{H}_2$
  - Then $\mathrm{D}(\tau) = \tau\rho + \bar{\tau}\sigma + \Psi_1$, $\mathrm{D}(\alpha) = \rho\alpha + \beta\bar{\sigma}$, $\mathrm{D}(\beta) = \alpha\sigma + \rho\beta + \Psi_1$

$$\boxed{\mathrm{D}\alpha = \mathrm{D}\beta = \mathrm{D}\tau = 0 \quad \text{on} \quad \mathcal{H}_2} \qquad \ldots\ldots\ldots \quad \mathrm{D}\Psi_2 = 0 \quad \ldots\ldots\ldots$$

The full initial data set $\mathbb{V}_0$ on $\mathcal{H}_1 \cup \mathcal{H}_2$ determining stationary distorted vacuum black hole configurations

$\mathcal{H}_1$	$\mathcal{Z}$	$\mathcal{H}_2$
$\rho = u \cdot (\bar{\delta}\tau - 2\,\alpha\,\tau - \Psi_2)$	$\rho = 0$	$\rho = 0$
$\sigma = u \cdot (\delta\,\tau - 2\,\beta\,\tau)$	$\sigma = 0$	$\sigma = 0$
$\mu = 0$	$\mu = 0$	$\mu = r \cdot \Psi_2$
$\lambda = 0$	$\lambda = 0$	$\lambda = 0$
$\Delta\alpha = \Delta\beta = \Delta\tau = 0$	$\alpha, \beta : \ \tau = \overline{\alpha} + \beta$	$\mathrm{D}\alpha = \mathrm{D}\beta = \mathrm{D}\tau = 0$
$\Delta\Psi_2 = 0$	$\xi^A, \tau \ \rightarrow \ \alpha, \beta, \Psi_2$	$\mathrm{D}\Psi_2 = 0$
$\Psi_0 = \frac{1}{2}u^2 \cdot \left(\delta^2\Psi_2 - (7\tau + 2\beta)\cdot\delta\Psi_2 + 12\tau^2\Psi_2\right)$	$\Psi_0 = 0$	$\Psi_0 = 0$
$\Psi_1 = u \cdot (\delta\Psi_2 - 3\tau\Psi_2)$	$\Psi_1 = 0$	$\Psi_1 = 0$
$\Psi_3 = 0$	$\Psi_3 = 0$	$\Psi_3 = r \cdot \bar{\delta}\Psi_2$
$\Psi_4 = 0$	$\Psi_4 = 0$	$\Psi_4 = \frac{1}{2}\,r^2 \cdot (\bar{\delta}^2\Psi_2 + 2\alpha \cdot \bar{\delta}\Psi_2)$
$(\text{gauge}) \ \nu = 0 \quad \rightarrow$	$\nu = 0 \quad \rightarrow$	$\nu = \frac{1}{2}\,r^2 \cdot (\bar{\delta}\Psi_2 + \overline{\tau}\,\Psi_2)$
$(\text{gauge}) \ \gamma = 0 \quad \rightarrow$	$\gamma = 0 \quad \rightarrow$	$\gamma = r \cdot (\tau\,\alpha + \overline{\tau}\,\beta + \Psi_2)$

How to identify the Kerr solution?

The Kerr solution admits a non-trivial Killing spinor

Killing spinor:

- a symmetric rank 2 spinor $\kappa_{AB} = \kappa_{(AB)}$ satisfying the *Killing spinor equation*

$$\nabla_{A'(A}\kappa_{BC)} = 0$$

- Given a Killing spinor κ_{AB}, the spinor

$$\xi_{AA'} \equiv \nabla^{P}{}_{A'}\kappa_{AP}$$

  is the spinorial counterpart of a (possibly complex) Killing vector

- it satisfies the equation

$$\nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} = 0$$

The Killing form and the Ernst potential

- the Killing form of a KVF ξ^a

$$H_{ab} \equiv \nabla_{[a}\xi_{b]} = \nabla_a \xi_b$$

- its spinorial counterpart $H_{AA'BB'}$ and its self-dual part $\mathcal{H}_{AA'BB'}$ read as

$$H_{AA'BB'} \equiv \nabla_{AA'}\xi_{BB'}\,, \qquad \mathcal{H}_{AA'BB'} \equiv H_{AA'BB'} + \mathrm{i}H^*_{AA'BB'}$$

-

$$\mathcal{H}^2 \equiv \mathcal{H}_{ab}\mathcal{H}^{ab}$$

- Then, *Ernst form* of the Killing vector ξ^a is defined as

$$\chi_a = 2\xi^b \mathcal{H}_{ba}$$

- It is well-known that in vacuum, the Ernst form closed
  $\Rightarrow$ (locally) there exists a (complex) function, the *Ernst potential* χ:

$$\chi_a = \nabla_a \chi$$

- remarkably

$$\chi_{AA'} = 3\kappa^{CF}\Psi_{ABCF}\nabla_{DA'}\kappa^{DB}$$

How to find the Kerr black hole?

Theorem: [Marc Mars (2000)]

- Let $(\mathcal{M}, g)$ denote a smooth vacuum spacetime
  - endowed with a Killing spinor κ_{AB} satisfying $\kappa_{AB}\kappa^{AB} \neq 0$,
  - such that the spinor
  $$\xi_{AA'} \equiv \nabla^B{}_{A'}\kappa_{AB}$$
  is Hermitian.
- Then there exist two complex constants $\mathfrak{l}$ and $\mathfrak{c}$ such that
$$\mathcal{H}^2 = -\mathfrak{l}(\mathfrak{c} - \chi)^4$$
- If, in addition, $\mathfrak{c} = 1$ and $\mathfrak{l}$ is real positive, then $(\mathcal{M}, g)$ is locally isometric to the Kerr spacetime.

How to construct a Killing spinor field?

- wave equation for the Killing spinor:

$$\Box\kappa_{AB} + \Psi_{ABCD}\,\kappa^{CD} = 0$$

where Ψ_{ABCD} denotes the Weyl spinor.

- if κ_{AB} is a solution to this wave equation then the spinor fields

$$\begin{aligned} H_{A'ABC} &\equiv 3\nabla_{A'(A}\kappa_{BC)} \\ S_{AA'BB'} &\equiv \nabla_{AA'}\xi_{BB'} + \nabla_{BB'}\xi_{AA'} \end{aligned}$$

satisfy the system of wave equations

$$\begin{aligned} \Box H_{AA'BC} &= 4\big(\Psi_{(AB}{}^{PQ}H_{C)PQA'} + \nabla_{(A}{}^{Q'}S_{BC)Q'A'}\big), \\ \Box S_{AA'BB'} &= -\nabla_{AA'}(\Psi_{B}{}^{PQR}H_{B'PQR}) - \nabla_{BB'}(\Psi_{A}{}^{PQR}H_{A'PQR}) \\ &\quad +2\Psi_{AB}{}^{PQ}S_{PA'QB'} + 2\overline{\Psi}_{A'B'}{}^{P'Q'}S_{AP'BQ'} \end{aligned}$$

- *As the above equations constitute a system of homogeneous linear wave equations for the fields $H_{A'ABC}$ and $S_{AA'BB'}$, there vanishing are the conditions for the existence of a Killing spinor in the development*

The basic variables

- denote by $\{o^A, \iota^A\}$ a spin dyad normalised according to $o_A\iota^A = 1$
- NP null tetrad $\{l^a, n^a, m^a, \overline{m}^a\}$ —if $\{l^{AA'}, n^{AA'}, m^{AA'}, \overline{m}^{AA'}\}$ denote the spinorial counterparts of the null tetrad, one has the correspondences

$$l^{AA'} = o^A\overline{o}^{A'}, \qquad n^{AA'} = \iota^A\overline{\iota}^{A'}, \qquad m^{AA'} = o^A\overline{\iota}^{A'}, \qquad \overline{m}^{AA'} = \iota^A\overline{o}^{A'}$$

- The spinor κ_{AB} can be written as

$$\kappa_{AB} = \kappa_2 o_A o_B - 2\kappa_1 o_{(A}\iota_{B)} + \kappa_0 \iota_A\iota_B \,,$$

where
$$\kappa_0 \equiv \kappa_{AB}o^A o^B, \qquad \kappa_1 \equiv \kappa_{AB}o^A\iota^B, \qquad \kappa_2 \equiv \kappa_{AB}\iota^A\iota^B$$

- It can be readily verified that the scalars κ_2, κ_1 and κ_0 have, respectively, spin weights -1, $0, 1$ — i.e. they transform as

$$\kappa_j \mapsto e^{-2(j-1)\mathrm{i}\vartheta}\kappa_j$$

under a rotation $\{o^A, \iota^A\} \mapsto \{e^{\mathrm{i}\vartheta}o^A,\ e^{-\mathrm{i}\vartheta}\iota^A\}$.

The wave equation for the Killing spinor:

$$\Box\kappa_{AB} + \Psi_{ABCD}\,\kappa^{CD} = 0,$$

in completely generic context.

$$
\begin{aligned}
&D\Delta\kappa_2 + \Delta D\kappa_2 - \delta\bar\delta\kappa_2 - \bar\delta\delta\kappa_2\\
&\qquad +(\mu+\bar\mu+3\gamma-\bar\gamma)D\kappa_2 - (\rho+\bar\rho)\Delta\kappa_2 + (\bar\tau-3\alpha-\bar\beta)\delta\kappa_2 + (\bar\alpha-5\beta+\tau)\bar\delta\kappa_2\\
&\qquad +(\Psi_2+2\alpha\bar\alpha-8\alpha\beta-2\beta\bar\beta-2\gamma\rho+2\mu\rho-2\gamma\bar\rho+2\lambda\sigma+2\alpha\tau+2\beta\bar\tau+2D\gamma-2\delta\alpha-2\bar\delta\beta)\kappa_2\\
&\qquad +(\Psi_4-4\lambda\mu)\kappa_0 = 0\\
&D\Delta\kappa_1 + \Delta D\kappa_1 - \delta\bar\delta\kappa_1 - \bar\delta\delta\kappa_1\\
&\qquad -2\tau D\kappa_2 + (\mu+\bar\mu-\gamma-\bar\gamma)D\kappa_1 + 2\nu D\kappa_0 - (\rho+\bar\rho)\Delta\kappa_1 + 2\rho\delta\kappa_2 + (\alpha-\bar\beta+\bar\tau)\delta\kappa_1\\
&\qquad -2\lambda\delta\kappa_2 + 2\sigma\bar\delta\kappa_2 + (\bar\alpha-\beta+\tau)\bar\delta\kappa_1 - 2\mu\bar\delta\kappa_0\\
&\qquad +(-\Psi_1-\bar\alpha\rho+3\beta\rho+\alpha\sigma+\bar\beta\sigma\bar\rho\tau-\sigma\bar\tau-D\tau+\delta\rho\bar\delta\sigma)\kappa_2\\
&\qquad +(-\Psi_3+\bar\alpha\lambda+\beta\lambda+3\alpha\mu-\bar\beta\mu-\nu\rho-\nu\bar\rho+\lambda\tau+\mu\bar\tau+D\nu-\delta\lambda-\bar\delta\mu)\kappa_0 = 0\\
&D\Delta\kappa_0 + \Delta D\kappa_0 - \delta\bar\delta\kappa_0 - \bar\delta\delta\kappa_0\\
&\qquad +(\mu+\bar\mu-5\gamma-\bar\gamma)D\kappa_0 - (\rho+\bar\rho)\Delta\kappa_0 + (5\alpha-\bar\beta+\bar\tau)\delta\kappa_0 + (\bar\alpha+3\beta+\tau)\bar\delta\kappa_0\\
&\qquad +(\Psi_2-2\alpha\bar\alpha-8\alpha\beta+2\beta\bar\beta+2\gamma\rho+2\mu\rho+2\gamma\bar\rho+2\lambda\sigma-2\alpha\tau-2\beta\bar\tau-2D\gamma+2\delta\alpha+2\bar\delta\beta)\kappa_0\\
&\qquad +(\Psi_0-4\rho\sigma)\kappa_2 = 0
\end{aligned}
$$

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The transport equations on $\mathcal{H}_1$

$$\begin{aligned}
&2D\Delta\kappa_0 - \delta\overline{\delta}\kappa_0 - \overline{\delta}\delta\kappa_0 + (\overline{\alpha} + 3\beta)\overline{\delta}\kappa_0 + (5\alpha - \overline{\beta})\delta\kappa_0 + (\mu + \overline{\mu} - 4\gamma)D\kappa_0 + 4\tau D\kappa_1 \\
&\qquad + 2\kappa_1 D\tau + (\Psi_2 - 2\alpha\overline{\alpha} - 8\alpha\beta + 2\beta\overline{\beta} - 2\alpha\tau - 2\beta\overline{\tau} - 2D\gamma + 2\delta\alpha + 2\overline{\delta}\beta)\kappa_0 = 0 \\
&2D\Delta\kappa_1 - \delta\overline{\delta}\kappa_1 - \overline{\delta}\delta\kappa_1 - 2\nu D\kappa_0 + (\mu + \overline{\mu})D\kappa_1 + 2\tau D\kappa_2 + (\alpha - \overline{\beta})\delta\kappa_1 + 2\mu\overline{\delta}\kappa_0 + (\overline{\alpha} - \beta)\overline{\delta}\kappa_1 \\
&\qquad + (\Psi_3 - 3\alpha\mu + \overline{\beta}\mu - \mu\overline{\tau} - D\nu + \overline{\delta}\mu)\kappa_0 - 2\Psi_2\kappa_1 + \kappa_2 D\tau = 0 \\
&2D\Delta\kappa_2 - \delta\overline{\delta}\kappa_2 - \overline{\delta}\delta\kappa_2 - 4\nu D\kappa_1 + (4\gamma + \mu + \overline{\mu})D\kappa_2 - (3\alpha + \overline{\beta})\delta\kappa_2 + 4\mu\overline{\delta}\kappa_1 + (\overline{\alpha} - 5\beta)\overline{\delta}\kappa_2 \\
&\qquad + (\Psi_2 + 2\alpha\overline{\alpha} - 8\alpha\beta - 2\beta\overline{\beta} + 2\alpha\tau + 2\beta\overline{\tau} + 2D\gamma - 2\delta\alpha - 2\overline{\delta}\beta)\kappa_2 \\
&\qquad + (2\alpha\mu - 2\Psi_3 + 2\overline{\beta}\mu - 2\mu\overline{\tau} - 2D\nu + 2\overline{\delta}\mu)\kappa_1 + \Psi_4\kappa_0 = 0
\end{aligned}$$

If the value of the components κ_0, κ_1, κ_2 are known on $\mathcal{H}_1$, then the above equations can be read as a system of ordinary differential equations for the transversal derivatives

$$\Delta\kappa_0, \quad \Delta\kappa_1, \quad \Delta\kappa_2$$

along the null generators of $\mathcal{H}_1$. Initial data for these transport equations is naturally prescribed on $\mathcal{Z}$.

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The transport equations on $\mathcal{H}_2$

$$
\begin{aligned}
&2\Delta D\kappa_0 - \delta\overline{\delta}\kappa_0 - \overline{\delta}\delta\kappa_0 - (\rho+\overline{\rho})\Delta\kappa_0 + 4\tau D\kappa_1 + (5\alpha - \overline{\beta} + 2\overline{\tau})\delta\kappa_0 + (\overline{\alpha} + 3\beta + 2\tau)\overline{\delta}\kappa_0 \\
&\qquad +4\sigma\overline{\delta}\kappa_1 - 4\rho\delta\kappa_1 + (\Psi_2 - 2\alpha\overline{\alpha} - 8\alpha\beta + 2\beta\overline{\beta} - 2\alpha\tau - 2\beta\overline{\tau} + 2\delta\alpha + 2\overline{\delta}\beta)\kappa_0 \\
&\qquad +(2\overline{\alpha}\rho + 2\beta\rho + 6\alpha\sigma - 2\overline{\beta}\sigma - 2\overline{\rho}\tau + 2\sigma\overline{\tau} + 2D\tau - 2\delta\rho - 2\overline{\delta}\sigma - 2\Psi_1)\kappa_1 \\
&\qquad +(\Psi_0 - 4\rho\sigma)\kappa_2 = 0 \\
&2\Delta D\kappa_1 - \delta\overline{\delta}\kappa_1 - \overline{\delta}\delta\kappa_1 - (\rho+\overline{\rho})\Delta\kappa_1 + 2\tau D\kappa_2 + (\alpha - \overline{\beta} + 2\overline{\tau})\delta\kappa_1 + (\overline{\alpha} - \beta + 2\tau)\overline{\delta}\kappa_1 \\
&\qquad -2\rho\delta\kappa_2 + 2\sigma\overline{\delta}\kappa_2 - 2\Psi_2\kappa_1 \\
&\qquad +(\Psi_1 - \overline{\alpha}\rho - 3\beta\rho - \alpha\sigma - \overline{\beta}\sigma - \overline{\rho}\tau + \sigma\overline{\tau} + D\tau - \delta\rho - \overline{\delta}\sigma)\kappa_2 = 0 \\
&2\Delta D\kappa_2 - \delta\overline{\delta}\kappa_2 - \overline{\delta}\delta\kappa_2 - (\rho+\overline{\rho})\Delta\kappa_2 + (2\overline{\tau} - 3\alpha - \overline{\beta})\delta\kappa_2 + (\overline{\alpha} - 5\beta + 2\tau)\overline{\delta}\kappa_2 \\
&\qquad +(\Psi_2 + 2\alpha\overline{\alpha} - 8\alpha\beta - 2\beta\overline{\beta} + 2\alpha\tau + 2\beta\overline{\tau} - 2\delta\alpha - 2\overline{\delta}\beta)\kappa_2 = 0
\end{aligned}
$$

If the values of κ_0, κ_1, κ_2 are known on $\mathcal{H}_2$ then the above equations can be read as a system of ordinary differential equations for the transversal derivatives

$$
D\kappa_0, \qquad D\kappa_1, \qquad D\kappa_2
$$

along the null generators of $\mathcal{H}_2$. Initial data for these transport equations is naturally prescribed on $\mathcal{Z}$.

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The necessary and sufficient conditions for the existence of a Killing spinor field on a distorted black hole

The components of the Killing spinor field κ_{AB} on the null hypersurface $\mathcal{H}_1 \cup \mathcal{H}_2$.

$\mathcal{H}_1$	$\mathcal{Z}$	$\mathcal{H}_2$
$\kappa_0 = 0$	$\kappa_0 = 0$	$\kappa_0 = -2\,u\,(\eth\kappa_1 + \tau\,\kappa_1)$
$\kappa_1 = \kappa_1\|_{\mathcal{Z}}$	$\kappa_1 :\ \ \eth^2\kappa_1 = \overline{\eth}^2\kappa_1 = 0$	$\kappa_1 = \kappa_1\|_{\mathcal{Z}}$
$\kappa_2 = -2\,r\,\overline{\eth}\kappa_1$	$\kappa_2 = 0$	$\kappa_2 = 0$

The components of the Killing vector field $\xi_{AA'}$ on the null hypersurface $\mathcal{H}_1 \cup \mathcal{H}_2$.

$\mathcal{H}_1$	$\mathcal{Z}$	$\mathcal{H}_2$
$\xi_{11'} = -3\,r\,(\tau\,\overline{\eth}\kappa_1 - \overline{\tau}\,\eth\kappa_1)$	$\xi_{11'} = 0$	$\xi_{11'} = 0$
$\xi_{10'} = -3\,\overline{\eth}\kappa_1$	$\xi_{10'} = -3\,\overline{\eth}\kappa_1$	$\xi_{10'} = -3\,\overline{\eth}\kappa_1$
$\xi_{10'} = 3\,\eth\kappa_1$	$\xi_{10'} = 3\,\eth\kappa_1$	$\xi_{10'} = 3\,\eth\kappa_1$
$\xi_{00'} = 0$	$\xi_{00'} = 0$	$\xi_{00'} = -3\,u\,(\tau\,\overline{\eth}\kappa_1 - \overline{\tau}\,\eth\kappa_1)$

As due to **Hawking's** black hole topology theorem if $\mathcal{Z}$ compact without boundary then it has to be spherical.

the KVF $\xi_{AA'}$ is an **axial KVF** (it may be complex) on $\mathcal{Z}$

- It is known that if a generic Killing spinor field κ_{AB} is admitted by a vacuum spacetime (generic, if $\kappa_{AB} = \alpha_{(A}\beta_{B)}$, for some $\alpha_A \neq \beta_A$ spinors) then $\Psi_{ABCD} = \psi\, \kappa_{(AB}\kappa_{CD)}$ is of **Petrov type D**.

- axisymmetry: $\Rightarrow$
$$\kappa_1 = \mathfrak{c} + \mathfrak{d}\cos\theta$$
with $\mathfrak{c}, \mathfrak{d} \in \mathbb{C}$

- the integrability condition for the Killing spinor: $\Rightarrow$
$$\kappa_1^3\, \Psi_2 = \mathfrak{M}$$
with $\mathfrak{M} \in \mathbb{C}$

- $\Rightarrow$
$$\Psi_2 = \frac{\mathfrak{M}}{(\mathfrak{c} + \mathfrak{d}\cos\theta)^3}$$

- one parameter is eliminated by the Gauss-Bonnet formula
$$\int_{\mathcal{Z}} \Psi_2 \mathrm{d}S = -2\pi$$

- Those distorted black hole configurations which admit a Killing spinor form a five (real) parameter family of solutions. Each is of Petrov type D and possesses an axial symmetry everywhere in the Cauchy development of $\mathcal{H}_1 \cup \mathcal{H}_2$.