Towards a new proof of the positive mass theorem

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The positive mass theorems:

- as GR is a **metric theory of gravity**
  - it is highly non-trivial to talk about, for instance, the mass—combining the gravitational and matter contributions—of bounded spatial regions

- **asymptotically flat spacetimes**: sensible notion of **total mass**

- **the proof of the positivity** of this global or ADM mass
  - the first attempt to prove by Geroch was quasi-local in its basic character
  - neither of the known generic proofs are so

- **the first complete proof** of the positive mass theorem by Schoen and Yau (1979 - 1981):
  - minimal surface theory and global existence of solutions to Jang’s eqn.

- **Witten’s proof** (1981):
  - inspired by positivity of energy in the context of supergravity, reduces the problem to proving solubility of the Dirac equation in asymptotically flat configurations
Motivations:

The exiting proofs:

- though they are generic the involved technicalities are considerable

The aims:

- outline a relatively simple alternative proof of the positive mass theorem
- try to restore its original quasi-local character
  - demonstrating that to a hypersurface with non-negative scalar curvature flows can be constructed such that the (quasi-local) Geroch mass—that can be evaluated on the leaves of the generated foliations—is non-decreasing with respect to these flows
- the proposed procedure does not require asymptotic flatness: applies to any subregion admitting non-negative scalar curvature (and suitable quasi-convex foliations)
- the ultimate aim is to show—in case of an asymptotically flat time-slices with non-negative scalar curvature—that the desired flows exist globally
consider a smooth 3-dimensional manifold $\Sigma$ with a Riemannian metric $h_{ij}$

assume

$$\Sigma \approx \mathbb{R} \times \mathcal{I}$$

i.e. $\Sigma$ is smoothly foliated by a one-parameter family of two-surfaces $\mathcal{I}_\rho$:

$\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

$$\begin{align*}
\partial_i \rho & \quad \& \quad h^{ij} \quad \to \quad \hat{n}_i, \hat{n}^i = h^{ij} \hat{n}_j \quad \ldots \quad \hat{\gamma}^i_j = \delta^i_j - \hat{n}^i \hat{n}_j
\end{align*}$$
Quasi-convex foliations:

- The induced Riemannian metric on the $\mathcal{I}_\rho$ level sets
  \[ \tilde{\gamma}_{ij} = \tilde{\gamma}^k_i \tilde{\gamma}^l_j h_{kl} \]

- The extrinsic curvature given by the symmetric tensor field
  \[ \tilde{K}_{ij} = \tilde{\gamma}^l_i D_l \tilde{n}_j = \frac{1}{2} \mathcal{L}_{\tilde{n}} \tilde{\gamma}_{ij}, \quad D_i, \mathcal{L}_{\tilde{n}} \]

- A $\rho = \text{const}$ level surface is called quasi-convex if its mean curvature, \[ \tilde{K}^l_i = \tilde{\gamma}^{ij} \tilde{K}_{ij}, \] is positive on $\mathcal{I}_\rho$
a smooth vector field $\rho^i$ on $\Sigma$ is a flow, w.r.t. $\mathcal{I}_\rho$

- if the integral curves of $\rho^i$ intersect each leaves precisely once, and
- if $\rho^i$ is scaled such that $\rho^i \partial_i \rho = 1$ holds throughout $\Sigma$

any smooth flow can be decomposed in terms of its ‘lapse’ and ‘shift’ as

$$\rho^i = \hat{N} \hat{n}^i + \hat{N}^i$$

$$\hat{N} = \rho^i \hat{n}_i, \quad \hat{N}^i = \hat{\gamma}^i_j \rho^j$$

the lapse measures the normal separation of the surfaces $\mathcal{I}_\rho$
The variation of the area:

- to any quasi-convex foliation ∃ a (quasi-local) orientation of the leaves $\mathcal{L}_\rho$
- a flow $\rho^i$ is called **outward pointing** if the area is increasing w.r.t. it
- variation of the area $\mathcal{A}_\rho = \int_{\mathcal{L}_\rho} \hat{\epsilon}$ of the $\rho = \text{const}$ level surfaces, w.r.t. $\rho^i$

$$\mathcal{L}_\rho \mathcal{A}_\rho = \int_{\mathcal{L}_\rho} \mathcal{L}_\rho \hat{\epsilon} = \int_{\mathcal{L}_\rho} \left\{ \hat{N}(\hat{K}^l_l) + (\hat{D}_i \hat{N}^i) \right\} \hat{\epsilon} = \int_{\mathcal{L}_\rho} \hat{N}(\hat{K}^l_l) \hat{\epsilon},$$

the relations $\mathcal{L}_{\hat{n}} \hat{\epsilon} = (\hat{K}^l_l) \hat{\epsilon}$ and $\mathcal{L}_{\hat{N}} \hat{\epsilon} = \frac{1}{2} \hat{\gamma}^{ij} \mathcal{L}_{\hat{N}} \hat{\gamma}_{ij} \hat{\epsilon} = (\hat{D}_i \hat{N}^i) \hat{\epsilon}$, along with the vanishing of the integral of the total divergence $\hat{D}_i \hat{N}^i$, were applied.

- $\hat{N}$ does not vanish on $\Sigma$ unless the Riemannian three-metric

$$h^{ij} = \hat{\gamma}^{ij} + \hat{N}^{-2}(\rho^i - \hat{N}^i)(\rho^j - \hat{N}^j)$$

gets to be singular

- for quasi-convex foliations $\hat{N}\hat{K}^l_l > 0 \implies$ the area is increasing w.r.t. $\rho^i$
- the orientations by $\hat{n}^i$ and $\rho^i$ coincide
Attempts to provide a quasi-local proof of the PMT:

- attempts all using the Geroch or the Hawking mass (are equal if $K^i_i = 0$)
  - Geroch (1973)
  - Wald & Jang (1977)
  - Jang (1978)
  - Kijowski (1986)
  - Jezieski & Kijowski (1987)
  - Huisken & Ilmanen (1997, 2001)
  - Frauendiener (2001)
  - Bray (2001), Bray & Lee (2009)
  - ...

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The Geroch mass:

- the (quasi-local) Geroch mass

\[ m_G = \frac{\mathcal{A}_\rho^{1/2}}{64\pi^{3/2}} \int_{\mathcal{I}_\rho} \left[ 2 \hat{R} - (\hat{K}_l)^2 \right] \hat{\epsilon} \]

where \( \hat{R} \) is the scalar curvature of the metric \( \hat{\gamma}_{ij} \) on the leaves

- for quasi-convex foliations the area \( \mathcal{A}_\rho \) is monotonously increasing

- it suffices to investigate

\[ W(\rho) = \int_{\mathcal{I}_\rho} \left[ 2 \hat{R} - (\hat{K}_l)^2 \right] \hat{\epsilon} \]

- if \( W(\rho) \) was non-decreasing, and for some specific \( \rho_* \) value, \( W(\rho_*) \) was zero or positive then \( m_G \geq 0 \) would hold to the exterior of \( \mathcal{I}_{\rho_*} \) in \( \Sigma \)
The variation of $W(\rho)$:

- the **key equation** we shall use relates the scalar curvatures of $h_{ij}$ and $\gamma_{ij}$

\[ (3) R = \hat{R} - \left\{ 2 \mathcal{L}_{\hat{n}}(\hat{K}^l_l) + (\hat{K}^l_l)^2 + \hat{K}_{kl} \hat{K}^{kl} + 2 \hat{N}^{-1} \hat{D}^l \hat{D}_l \hat{N} \right\} \]  

\[ \mathcal{L}_\rho W = - \int_{\mathcal{I}_\rho} \mathcal{L}_\rho \left[ (\hat{K}^l_l)^2 \hat{\epsilon} \right] = - \int_{\mathcal{I}_\rho} \left\{ \hat{N} \mathcal{L}_{\hat{n}} \left[ (\hat{K}^l_l)^2 \hat{\epsilon} \right] + \mathcal{L}_{\hat{N}} \left[ (\hat{K}^l_l)^2 \hat{\epsilon} \right] \right\} \]

\[ = - \int_{\mathcal{I}_\rho} (\hat{N} \hat{K}^l_l) \left[ 2 \mathcal{L}_{\hat{n}} (\hat{K}^l_l) + (\hat{K}^l_l)^2 \right] \hat{\epsilon} - \int_{\mathcal{I}_\rho} \hat{D}_i \left[ (\hat{K}^l_l)^2 \hat{N}^i \right] \hat{\epsilon} \]

\[ = - \int_{\mathcal{I}_\rho} (\hat{N} \hat{K}^l_l) \left[ (\hat{R} - (3) R) - \hat{K}_{kl} \hat{K}^{kl} - 2 \hat{N}^{-1} \hat{D}^l \hat{D}_l \hat{N} \right] \hat{\epsilon} \]

- where on 1\textsuperscript{st} line $\rho^i = \hat{N} \hat{n}^i + \hat{N}^i$ and the Gauss-Bonnet theorem
- on 2\textsuperscript{nd} line the relations $\mathcal{L}_{\hat{n}} \hat{\epsilon} = (\hat{K}^l_l) \hat{\epsilon}$ and $\mathcal{L}_{\hat{N}} \hat{\epsilon} = (\hat{D}_i \hat{N}^i) \hat{\epsilon}$
- on 3\textsuperscript{rd} line (*) and the vanishing of the integral of $\hat{D}_i \left[ (\hat{K}^l_l)^2 \hat{N}^i \right] \hat{\epsilon}$ were used
The variation of $W(\rho)$:

- by the Leibniz rule

\[
\hat{N}^{-1} \hat{D}^l \hat{D}_l \hat{N} = \hat{D}^l (\hat{N}^{-1} \hat{D}_l \hat{N}) + \hat{N}^{-2} \hat{\gamma}^{kl} (\hat{D}_k \hat{N})(\hat{D}_l \hat{N})
\]

- and by using the trace-free part of $\hat{K}_{ij}$

\[
\hat{\mathring{K}}_{ij} = \hat{K}_{ij} - \frac{1}{2} \hat{\gamma}_{ij} (\hat{K}_l^l), \quad \hat{K}_{kl} \hat{K}^{kl} = \hat{\mathring{K}}_{kl} \hat{\mathring{K}}^{kl} + \frac{1}{2} (\hat{K}_l^l)^2
\]

- and using the vanishing of the integral of the total divergence $\hat{D}^l (\hat{N}^{-1} \hat{D}_l \hat{N})$

\[
\mathcal{L}_\rho W = - \frac{1}{2} \int_{\mathcal{S}} (\hat{N} \hat{K}_l^l) \left[ 2 \hat{R} - (\hat{K}_l^l)^2 \right] \hat{\epsilon} \\
+ \int_{\mathcal{S}} (\hat{N} \hat{K}_l^l) \left[ (3) R + \hat{\mathring{K}}_{kl} \hat{\mathring{K}}^{kl} + 2 \hat{N}^{-2} \hat{\gamma}^{kl} (\hat{D}_k \hat{N})(\hat{D}_l \hat{N}) \right] \hat{\epsilon}
\]
A desired type flow:

- **once a foliation is fixed**, by specifying the function \( \rho : \Sigma \rightarrow \mathbb{R} \), **not only** the mean curvature, \( \hat{K}^l l \), **but the lapse** \( \hat{N} \), as well, gets to be fixed

\[
\hat{n}_i = \hat{N} \left( \partial_i \rho \right)
\]

- the **inverse mean curvature flow**

\[
\rho^i = (\hat{K}^l l)^{-1} \hat{n}^i
\]

If this flow existed globally the Geroch mass would be non-decreasing w.r.t it

- **but what is if only** the product \( \hat{N} \hat{K}^l l \) **is replaced** by its **mean value**

\[
\underline{\hat{N} \hat{K}^l l} = \frac{\int_{\mathcal{A}_\rho} \hat{N} \hat{K}^l l \, \hat{\epsilon}}{\int_{\mathcal{A}_\rho} \hat{\epsilon}}
\]

\[
\underline{\hat{N} \hat{K}^l l} = \mathcal{L}_\rho \log[\mathcal{A}_\rho]
\]

\[
[(64 \pi^{3/2})/(\mathcal{A}_\rho)^{1/2}] \cdot \mathcal{L}_\rho \, m_g = \mathcal{L}_\rho \, W + \frac{1}{2} \left( \mathcal{L}_\rho \log[\mathcal{A}_\rho] \right) \, W \geq 0
\]
We can also adjust the shift

\[ \rho^i = \hat{N} \hat{n}^i + \hat{N}^i \] : we have a freedom in choosing the shift \( \hat{N}^i \)

\[ \hat{N} \hat{K}^l_l = \frac{1}{2} \hat{\gamma}^{ij} \mathcal{L}_\rho \hat{\gamma}_{ij} - \hat{D}_i \hat{N}^i \]

or equivalently, once \( \hat{N} \hat{K}^l_l = \bar{\hat{N} \hat{K}^l_l} = \mathcal{L}_\rho \log[A_\rho] \) is guaranteed to hold

\[ \hat{D}_A \hat{N}^A = \mathcal{L}_\rho \sqrt{\det(\hat{\gamma}_{ij})} - \mathcal{L}_\rho \log[A_\rho] \quad (**) \]

on topological two-spheres using then the Hodge decomposition of the shift

\[ \hat{N}^A = \hat{D}^A \chi + \hat{\epsilon}^{AB} \hat{D}_B \eta \], \( \chi \) and \( \eta \) are some smooth functions on \( \mathcal{S} \), (**) 

\[ \hat{D}^A \hat{D}_A \chi = \mathcal{L}_\rho \sqrt{\det(\hat{\gamma}_{ij})} - \mathcal{L}_\rho \log[A_\rho] \]

solubility in terms of spherical harmonics presumes that some standard polar coordinates \((\vartheta, \varphi)\) given on the unit sphere \( \mathbb{S}^2 \) are transferred to \( \mathcal{S} \)
The construction of a flow:

1. Start by choosing a topological two-sphere $\mathcal{S}$ in $\Sigma$, with induced metric $\hat{\gamma}_{ij}$, such that it is quasi-convex, $\hat{K}_l > 0$, and also $W \geq 0$ holds on $\mathcal{S}$.
2. Choose a small positive real number $A > 0$ and set $\hat{N} = A \cdot (\hat{K}_l)^{-1}$ on $\mathcal{S}$.
3. Construct an infinitesimally close two-surface $\mathcal{S}'$ simply by Lie dragging the points of $\mathcal{S}$ along the (auxiliary) flow $\rho^i = \hat{N} \hat{n}^i$ in $\Sigma$.
4. By comparing the metric induced on $\mathcal{S}$ and $\mathcal{S}'$, respectively, both terms $L_\rho \sqrt{\det(\hat{\gamma}_{ij})}$ and $L_\rho \log[\mathcal{A}_\rho]$ can be evaluated on $\mathcal{S}'$. 

The desired flow on the given Riemannian three-surface $\Sigma$, with the metric $h_{ij}$.
The construction of a flow:

in the succeeding steps we have to update both the lapse and the shift such that the relation $\hat{N}\hat{K}^l_l = \hat{N}\hat{K}^l_l$ gets to be maintained in each of these steps

5 update first lapse on $\mathcal{I}'$ by setting $\hat{N} = \mathcal{L}_\rho \log[\mathcal{A}_\rho] \cdot (\hat{K}^l_l)^{-1}$, where $\mathcal{L}_\rho \log[\mathcal{A}_\rho] > 0$ is determined in the previous infinitesimal step

6 the key point here is that the shift can also be updated on $\mathcal{I}'$—such that $\hat{N}\hat{K}^l_l = \hat{N}\hat{K}^l_l$ holds there—simply by solving (***) for $\hat{N}^A$

7 the succeeding infinitesimal step: by Lie dragging the points of $\mathcal{I}'$ to $\mathcal{I}''$ . . .
by performing analogous sequences of infinitesimal steps ultimately we get a one-parameter family of two-surfaces $\mathcal{I}_\rho$ foliating (at least) a one-sided neighborhood of $\mathcal{I}$ in $\Sigma$ such that the product $\hat{N}\hat{K}^l_l$ is guaranteed to be positive and constant on each of the individual leaves

the vanishing of $\mathcal{L}_\rho \log[A_\rho]$ could get on the way of the applicability, i.e. minimal or maximal surfaces represent natural limits of applicability

the bifurcation surface of the Schwarzschild spacetime is a minimal surface on the $t_{\text{Schw}} = \text{const}$ time-slices, the Kerr-Schild $t_{KS} = \text{const}$ time-slices of the same spacetime can be foliated by metric spheres with area radius ranging from zero to infinity, and they do not contain any minimal surface

it is also of obvious interest to know if the desired type of foliation would exist or could be constructed globally

by inspecting the proposed construction it gets clear that all the steps are “safe” as far as the lapse $\hat{N}$ is bounded and it is regular throughout $\Sigma$

in clearing up the picture let us have a glance again of the key equation

$$^{(3)}R = \hat{R} - \left\{ 2\mathcal{L}_{\hat{n}}(\hat{K}^l_l) + (\hat{K}^l_l)^2 + \hat{K}_{kl}\hat{K}^{kl} + 2\hat{N}^{-1}\hat{D}^l\hat{D}_l\hat{N} \right\}$$ (**
The parabolic equation governing the evolution of $\hat{N}$:

- as noticed first by Bartnik (1993), while applying quasi-spherical foliations, (*) can be viewed as a parabolic equation for $\hat{N}$

- remarkably, (*) can always be put to be a parabolic equation for the lapse provided that $(3)^R \geq 0$, $\hat{\gamma}_{ij}$ and $\hat{N}^i$ can be treated as prescribed fields

- with applying the notation $\hat{K}_{ij} = \hat{N} \hat{K}_{ij}$ and

\[
\hat{K} = \frac{1}{2} \hat{\gamma}^{ij} \mathcal{L}_\rho \hat{\gamma}_{ij} - \hat{D}_l \hat{N}^i
\]

we can eliminate hidden presence of the lapse in (*) and get

\[
\hat{K} \left[ (\partial_\rho \hat{N}) - \hat{N}^l (\hat{D}_l \hat{N}) \right] = \hat{N}^2 (\hat{D}_l \hat{N}_l \hat{N}) + \mathcal{A} \hat{N} - \frac{1}{2} (\hat{R} - (3)^R) \hat{N}^3
\]

where $\mathcal{A} = \partial_\rho \hat{K} + \frac{1}{2} [\hat{K}^2 + \hat{K}_{kl} \hat{K}^{kl}]$, with

\[
\hat{K} = \hat{N} \hat{K}^l_l = \mathcal{L}_\rho \log[\mathcal{A}_\rho] > 0
\]

- it is standard to obtain existence of unique solutions to this uniformly parabolic PDE in a sufficiently small one-sided neighborhood of $\mathcal{S}$ in $\Sigma$
The global existence of solutions to this parabolic equation:

- our main concern is **global existence (!)**
- it should not be a surprise that an analogous parabolic equation came up in deriving the evolutionary form of the Hamiltonian constraints in [Rácz I: Constrains as evolutionary systems, Class. Quant. Grav. 33 015014 (2016)]

(slightly generalizing Bartnik’s results) **global existence of solutions** to the parabolic PDEs

\[
\dot{K} \left[ (\partial_\rho \hat{N}) - \hat{N}^l (\hat{D}_l \hat{N}) \right] = \hat{N}^2 (\hat{D}^l \hat{N}_l \hat{N}) + A \hat{N} - \frac{1}{2} (\hat{R} - (3) \hat{R}) \hat{N}^3
\]

could be derived

- assume now that \( \rho \) is the area radius such that \( \mathcal{A}_\rho = 4\pi \rho^2 \)
- the condition guaranteeing that for some positive and bounded initial data for \( 0 \hat{N} \) on \( \mathcal{I} \) the solution \( \hat{N} \) remains positive and bounded away from infinity for all \( \rho \geq \rho_0 \) ultimately can be given by referring to

\[
\mathcal{K} = \sup_{\rho \in [\rho_0, \infty)} \left\{ \frac{1}{4 \sqrt{\rho_0}} \int_{\rho_0}^{\rho} \rho^{3/2} \cdot \left[ \max_{\mathcal{I}_{\rho'}} \left( (3) \hat{R} - \hat{R} \right) \right] d\rho' \right\}
\]

1. if \( \mathcal{K} \leq 0 \) then any smooth positive bounded initial data \( 0 \hat{N} \) is fine
2. if \( \mathcal{K} > 0 \) then \( 0 \hat{N} \) has to be chosen such that \( 0 \hat{N} < 1/\sqrt{\mathcal{K}} \) [but choosing \( A > 0 \) small (!)]

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A relatively simple method is proposed to generate a flow on any three-dimensional Riemannian hypersurface, with non-negative scalar curvature in a four-dimensional ambient space.

1. it is far more flexible than the inverse mean curvature flow
2. this flow can be used to construct quasi-convex foliations
3. the (quasi-local) Geroch mass—associated with the foliating level surfaces—is non-decreasing w.r.t the proposed flow
4. hints on the global existence and regularity were provided
5. the construction applies to wide range of geometrized theories of gravity

- no use of Einstein’s equations or any other field equation on the metric of the ambient space had been applied anywhere in our construction
- as only the Riemannian character of the metric on $\Sigma$ was used the signature of the metric on the ambient space could be either Lor. or Euc.