Evolution in spaces of Euclidean signature?

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What does the title mean?

Hey, may the claim fit to Newtonian physics?

- Since 5 June 1686, when Newton's "Philosophiae Naturalis Principia Mathematica" was published the concepts of
  - **absolute space** and **absolute time** had been around (disagreements from Leibnitz)
  - "**absolute space**" is nothing but the 3-dimensional Euclidean space $\mathbb{E}^3$ with its metric structure; to be determined by means of observations
  - Trautman, A. (1964) *In Brandeis Summer Institute in Theoretical Physics, Prentice Hall*: the **Newtonian spacetime** is a **fibre bundle** with base the time line $\mathbb{R}$ and fiber $\mathbb{E}^3$.

Does the title have anything to with this?

- one could claim: in Newtonian physics **evolution** (some changes in absolute time) **occurs** in a three dimensional Euclidean space

Does this fit to our setup?

NO!

In what sense then the word evolution is used?

In our basic courses we learn that evolution equations are either:

- **parabolic**: a prototype is the **heat equation** with some source $\mathbf{f}$

$$\boxed{\partial_t \mathbf{u} = \Delta \mathbf{u} + \mathbf{f}}$$

- **hyperbolic**: a prototype is the **wave equation**

$$\boxed{\partial_t^2 \mathbf{u} = \Delta \mathbf{u}}$$

- the latter, once an obvious first order reduction is made, by introducing new variables

$$\mathbf{u}_{[t]} := \partial_t \mathbf{u}$$
$$\mathbf{u}_{[i]} := \partial_{x^i} \mathbf{u}$$

  takes the form of a **first order strongly hyperbolic** system (symmetrizable)

$$\boxed{\begin{aligned} \partial_t \mathbf{u} &= \mathbf{u}_{[t]} \\ \partial_t \mathbf{u}_{[t]} &= \partial_1 \mathbf{u}_{[1]} + \cdots + \partial_n \mathbf{u}_{[n]} \\ \partial_t \mathbf{u}_{[i]} &= \partial_i \mathbf{u}_{[t]} \end{aligned}}$$

  with trivial **constraints** $\partial_i \mathbf{u} = \mathbf{u}_{[i]}$

Why these type of equations play central role in physics?

PDE courses: for these type of evolution equations the **initial-boundary value problems** get to be **well-posed**.

- **parabolic** systems could be used e.g. in describing *varying of temperature of a body occupying a certain region with some heat source and some fixed temperature boundary*, or
- **hyperbolic** systems are used to describe various waves such as (*acoustic, elastic, electromagnetic or gravitational waves*)

In either cases method of **energy estimates** can be applied to show:

- **existence of weak (and regular) solutions**
- **uniqueness of weak (and regular) solutions**
- **continuous dependence on the initial data**
- for hyperbolic equations **finite speed (or causal) propagation**
- in some cases even the **asymptotic behavior** of solutions

Some of the equations are constraints!

Maxwell's equations in various form:

- $$\begin{aligned} \partial_t \mathbf{B} &= -\,curl\,\mathbf{E} && \text{(Faraday's law)} \\ \partial_t \mathbf{E} &= curl\,\mathbf{B} - \mathbf{j} && \text{(Ampère-Maxwell's law)} \\ div\,\mathbf{E} &= q && \text{(Coulomb's law)} \\ div\,\mathbf{B} &= 0 && \text{(Gauss's law)} \end{aligned}$$

  — first order evolutionary system **but with constraints !!!**

- by introducing the **Maxwell potential** $A_a = (\phi, A_i)$, and the parametrization

  $$E_i = -\partial_t A_i - \partial_i \phi \quad \& \quad B_i = \epsilon_{ijk}\,\partial_j A_k$$

  the constraint $div\,\mathbf{B} = 0$ gets to be solved automatically

- if we impose **temporal gauge** $\phi = A_0 = 0$ the 3-vector $\mathbf{A} = (A_i)$ satisfies

  $$\partial_t^2 \mathbf{A} - \Delta \mathbf{A} = \mathbf{j}$$

  — but the **initial data is not free** as it has to satisfy the **constraints**

  $$div\,\mathbf{A} = 0 \quad \& \quad div\,(\partial_t \mathbf{A}) = -q$$

- other gauge choices lead to similar constraints **!!!**

Einstein's theory of gravity comes also with constraints:

The Einstein equations: (lhs,rhs:-)

- one could say spacetime: (M, g_{ab}) with
  - M : 4-dimensional, smooth, paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian$_{(-,+,+,+)}$ metric
- $$\boxed{[R_{ab} - \tfrac{1}{2}\, g_{ab}\, R] + \Lambda\, g_{ab} = 8\pi\, T_{ab}}$$

  with cosmological constant Λ and with (some unspecified) matter fields satisfying their Euler-Lagrange equations
- let us make it simpler

  $$\boxed{G_{ab} - \mathscr{G}_{ab} = 0} \quad \text{with source term:} \quad \boxed{\nabla^a \mathscr{G}_{ab} = 0}$$
- we shall only refer to the lhs

  $$\boxed{E_{ab} := G_{ab} - \mathscr{G}_{ab}}$$

Vacuum equations $R_{\alpha\beta} = 0$ as non-linear wave equations:

- the components $g_{\alpha\beta}$ of the metric are our basic unknowns and the Ricci tensor in some local coordinates x^α reads (after inspecting it for a while) as

$$R_{\alpha\beta} = -\tfrac{1}{2} g^{\mu\nu} \partial_\mu \partial_\nu g_{\alpha\beta} + g_{\delta(\alpha} \nabla_{\beta)} \Gamma^\delta + H_{\alpha\beta}(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho}) = 0 \qquad (*)$$

where $\Gamma^\delta = g^{\alpha\beta} \Gamma^\mu{}_{\alpha\beta} = \Gamma^\delta(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho})$, [contraction of the Christoffel symbol]

- if functions Γ^δ were known $(*)$ would immediately give rise to a quasilinear, diagonal second-order hyperbolic system for the metric, with well-posed initial value problem
- key observation: **Yvone Choquet Bruhat** (1952) **!!!** replace the functions Γ^δ in $(*)$ by four arbitrarily chosen "gauge source functions" $f^\delta = f^\delta(x^\alpha)$ on $\mathbb{R}^4$ and solve the yielded "reduced system" $(*)'$
- do we really get a solution to $(*)$?
- it is then an interesting question: in which local coordinates the metric acquires those components subject to the "reduced system" $(*)'$?

Vacuum equations $R_{\alpha\beta} = 0$ as non-linear wave equations:

- **!!!** in any coordinates $$\Gamma^{\delta} = -g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\, x^{\delta}.$$
- the choice made for the functions $f^{\delta} = f^{\delta}(x^{\alpha})$ corresponds to the determination of the distinguished coordinates **!!!** x^{δ} by solving
$$g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\, x'^{\delta} = -f^{\delta}$$
with **initial data** such that differentials $\{dx^{\alpha}\}$ are linearly independent on the initial data surface. **!!!**
- the difference $\Gamma^{\delta} - f^{\delta}$ satisfies, in virtue of the twice-contracted Bianchi identity,
$$g^{\mu\nu}\nabla_{\mu}\nabla_{\nu}\,(\Gamma^{\delta} - f^{\delta}) + R_{\lambda}{}^{\delta}(\Gamma^{\lambda} - f^{\lambda}) = 0$$
- given the **initial data** $\{g_{\alpha\beta}, \partial_t\, g_{\alpha\beta}\}$ for $(*)'$, the vanishing of the **initial data for this subsidiary equation**, consisting of $\{\Gamma^{\delta} - f^{\delta}, \partial_t(\Gamma^{\delta} - f^{\delta})\}$, follows
- solutions to $(*)'$ are solutions to the original vacuum Einstein's equations $(*)$
- the argument works for the inclusion of matter $\rightarrow$ Einstein's equations can be put into the form of quasilinear wave equations

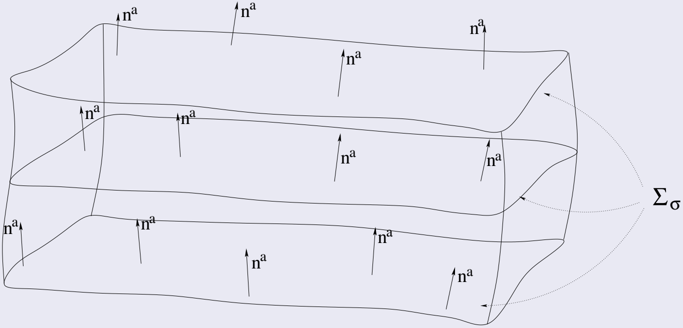
NOT YET!

Is this our setup?

Tacitly we have already used

Foliation by time

- **Assume:** M is foliated by a one-parameter family of homologous hypersurfaces, i.e. $M \simeq \mathbb{R} \times \Sigma$, for some three-dimensional manifold Σ.
- known to hold for any **globally hyperbolic spacetimes**
    - **equivalent to** the existence of a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing gradient $\nabla_a \sigma$ such that the $\sigma = const$ level surfaces $\Sigma_\sigma = \{\sigma\} \times \Sigma$ comprise the one-parameter foliation of M.
    - $$n_a \sim \nabla_a \sigma \;\; \ldots \; \& \ldots \; g^{ab} \;\longrightarrow\; n^a = g^{ab} n_b$$



Where are then the constraints in Einstein's theory?

- $$R_{\alpha\beta} = -\tfrac{1}{2} g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} + g_{\delta(\alpha}\nabla_{\beta)}\Gamma^\delta + H_{\alpha\beta}(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho}) = 0 \qquad (*)$$

  requires **initial data** consisting of $\{g_{\alpha\beta}, \partial_t\, g_{\alpha\beta}\}$
- in 4-dimension this pair of symmetric tensors possess (in general) 20 independent components.
- 4 are affected by the choice for f^δ
- another 4 by choice for (the "lapse" and "shift" part of) the time evolution 4-vector field $(\partial_t)^a$ connecting the succeeding time level surfaces.
- $\Longrightarrow$ apparently $12 = 20 - 4 - 4$ of the variables remain intact
- these can be represented by two symmetric tensor fields (h_{ab}, K_{ab}) on the initial data surface
- but we have 4 more relations: **the constraints**

$$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{{}^{(3)}R + (K^e{}_e)^2 - K_{ef}K^{ef}\} = 0$$
$$E^{(\mathcal{M})}_a = -\, h^e{}_a n^f E_{ef} = D_a K^e{}_e - D_e K^e{}_a = 0$$

where D_a denotes the covariant derivative operator associated with h_{ab}

The nature of constraints:

- in the **temporal gauge** the Maxwell potential $\mathbf{A} = (A_i)$ satisfying the wave equation $\partial_t^2 \mathbf{A} - \Delta \mathbf{A} = \mathbf{j}$ is also subject to the **initial data constraints**

$$div\, \mathbf{A} = 0 \quad \& \quad div\, (\partial_t \mathbf{A}) = -q$$

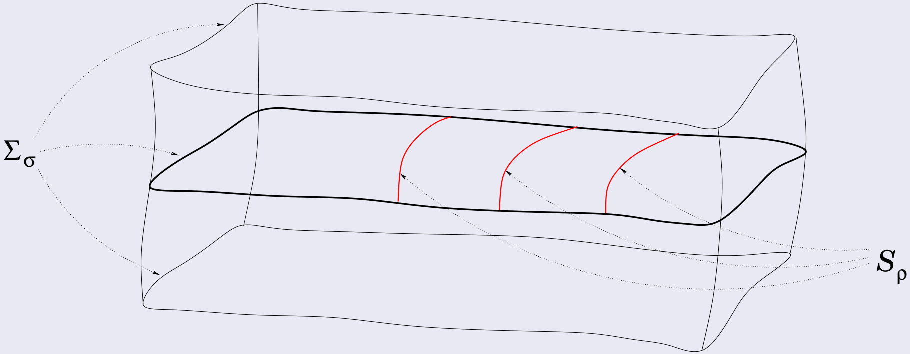
- the initial data (h_{ij}, K_{ij}) for the vacuum Einstein's equations is not free it has to satisfy the **Hamiltonian and momentum constraints**

$$E^{(\mathcal{H})} = n^e n^f E_{ef} = \tfrac{1}{2}\{ {}^{(3)}R + (K^e{}_e)^2 - K_{ef}K^{ef} \} = 0$$
$$E^{(\mathcal{M})}_a = -\, h^e{}_a n^f E_{ef} = D_a K^e{}_e - D_e K^e{}_a = 0$$

- the **constraint systems are always underdetermined**:
  - $div\, \mathbf{A} = 0 \quad \& \quad div\, (\partial_t \mathbf{A}) = -q$ $\longleftarrow$ 2 equations for 6 unknowns
  - $E^{(\mathcal{H})} = 0 \quad \& \quad E^{(\mathcal{M})}_a = 0$ $\longleftarrow$ 4 equations for 12 unknowns
- "typically involve solving of **elliptic boundary value problems** with consequent non-local complications" [e.g. with non-compact domains]

The outline of the rest of the talk:

- **Constrained systems:**



- on the time-level surfaces the metric is of Euclidean signature **!!!**
- no gauge condition

  arbitrary choice of foliations & 'evolutionary' vector field

Is this setup we looked for?

YES, IT IS

The main message:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

Based on some recent works;

- I. Rácz: *Is the Bianchi identity always hyperbolic?*, CQG **31** 155004 (2014)
- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, Class. Quantum Grav. **32** (2015) 015006
- I. Rácz: *Dynamical determination of the gravitational degrees of freedom*, arXiv:1412.0667 (2015)
- I. Rácz: *Constraints as evolutionary systems*, CQG **33** 015014 (2016)
- I. Rácz and J. Winicour: *Black hole initial data without elliptic equations*, Phys. Rev. D **91**, 124013 (2015)
- I. Rácz and J. Winicour: *On solving the constraints by integrating a strongly hyperbolic system*, arXiv:1601.05386
- I. Rácz: *A simple method of constructing binary black hole initial data*, arXiv:1605.01669
- I. Rácz: *On the ADM charges of multiple black holes*, arXiv:1608.02283
- A. Nakonieczna, L. Nakonieczny and I. Rácz: *Black hole initial data by numerical integration of the parabolic-hyperbolic form of the constraints*, arXiv:1712.00607
- I. Rácz and J. Winicour: *On computing black hole initial data without elliptic solvers*, arXiv:1712.03294

— constraints are almost exclusively referred to as elliptic equations in textbooks
— under some mild topological assumptions the constraints can also be solved as evolutionary systems

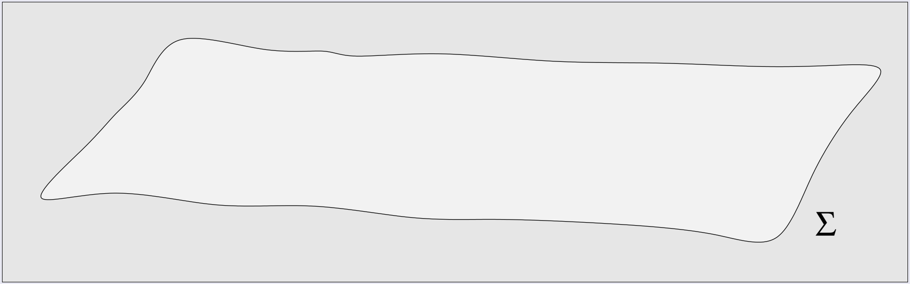
All the results are valid for arbitrary $dim(M) = n \ (\geq 4)$ dimension involving matter

A simple example:

Consider the underdetermined equation on $\Sigma \approx \mathbb{R}^2$ with some coordinates (χ, ξ)

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

- it is an equation for the four variables $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and $\boldsymbol{z}$ on Σ
- in advance of solving it three of these variables have to be fixed on Σ

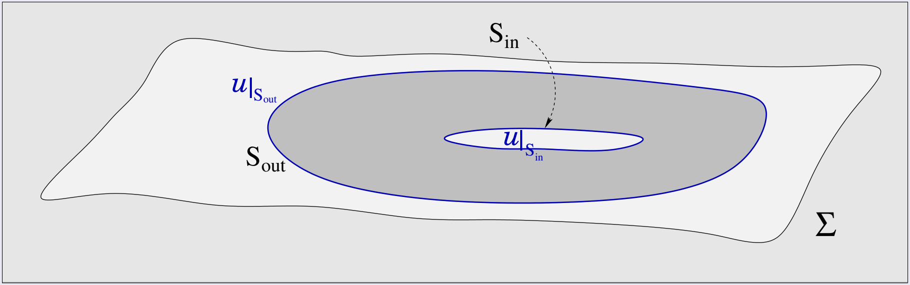


A simple example:

It is an elliptic equation for u on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

- in solving this equation the variables v, w and z have to be specified on $\mathbb{R}^2$
- the variable u has also to be fixed at the boundaries $\mathrm{S_{out}}$ and $\mathrm{S_{in}}$

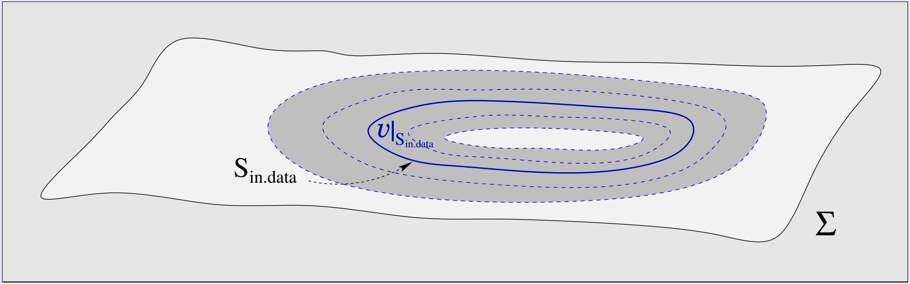


A simple example:

It is a hyperbolic equation for v on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}, w$ and $\boldsymbol{z}$ have to be specified on $\mathbb{R}^2$
- the variable $\boldsymbol{v}$ has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$

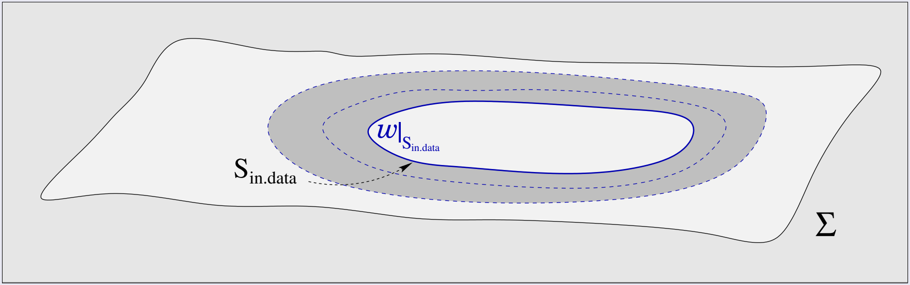


A simple example:

It is a parabolic equation for w on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ have to be fixed on $\mathbb{R}^2$: $a > 0$
- the variable w has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$

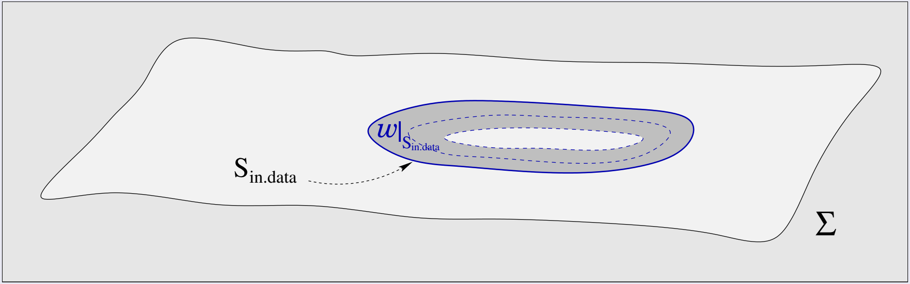


A simple example:

It is a parabolic equation for w on $\Sigma \approx \mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{z}$ have to be fixed on $\mathbb{R}^2$: $a < 0$
- the variable w has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$



A simple example:

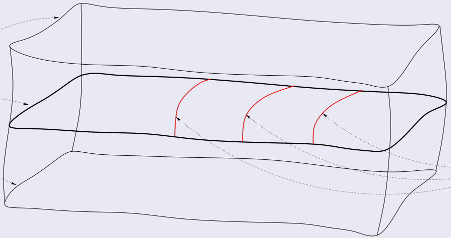
It is an algebraic equation for $\boldsymbol{z}$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi^2 - \partial_\xi^2)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- once the variables u, v, w are specified on $\mathbb{R}^2$ the solution is determined as

$$\boldsymbol{z} = -\left[(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi^2 - \partial_\xi^2)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w\right]$$

New variables by applying $2+1$ decompositions:



assume: $\boxed{\Sigma \approx \mathbb{R} \times \mathscr{S}}$

Σ is smoothly foliated by a one-parameter family of two-surfaces $\mathscr{S}_\rho$:
$\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

$$\Longrightarrow \quad \boxed{\widehat{n}_i = \widehat{N}\,\partial_i\rho \;\ldots\; \& \ldots\; h^{ij} \;\longrightarrow\; \widehat{n}^i = h^{ij}\widehat{n}_j \;\longrightarrow\; \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i\widehat{n}_j}$$

- choose ρ^i to be a vector field on Σ: the integral curves... & $\boxed{\rho^i \partial_i \rho = 1}$
- 'lapse' and 'shift' of ρ^i

$$\boxed{\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i\,, \quad \text{where} \quad \widehat{N} = \rho^j \widehat{n}_j \quad \text{and} \quad \widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j}$$

Solving the constraints by using a $2+1$ decompositions:

Solve $div\, \mathbf{L} = 0$!

- start with a co-vector field L_i on Σ foliated by the $\rho = const$ hypersurfaces
- L_i can be decomposed in terms of $\widehat{n}^i$ and fields living on the $\rho = const$ level surfaces as

$$L_i = \delta^j{}_i\, L_j = (\widehat{\gamma}^j{}_i + \widehat{n}^j \widehat{n}_i)\, L_j = \boldsymbol{\lambda}\, \widehat{n}_i + \mathbf{L}_i$$

  where

$$\boldsymbol{\lambda} = \widehat{n}^l\, L_l \quad \text{and} \quad \mathbf{L}_i = \widehat{\gamma}^j{}_i\, L_j$$

- in an analogous process by expanding $D_i L_j$ we get

$$D_i L_j = \delta^k{}_i \delta^l{}_j D_k\, [\, \delta^p{}_l\, L_p\,] = (\widehat{\gamma}^k{}_i + \widehat{n}^k \widehat{n}_i)(\widehat{\gamma}^l{}_j + \widehat{n}^l \widehat{n}_j) D_k\, [\, (\widehat{\gamma}^p{}_l + \widehat{n}^p \widehat{n}_l)\, L_p\,]$$

- using then the metric induced on the $\mathscr{S}_\rho$ level surfaces is

$$\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i\, \widehat{\gamma}^l{}_j\, h_{kl}$$

  and the unique covariant derivative operator $\widehat{D}_i$, compatible with this metric that is known to act on $\mathbf{L}_i = \widehat{\gamma}^j{}_i\, L_j$ as

$$\widehat{D}_i \mathbf{L}_j = \widehat{\gamma}^k{}_i\, \widehat{\gamma}^l{}_j D_k\, [\, \widehat{\gamma}^p{}_l\, L_p\,]$$

Solving the constraint $div\, \mathbf{L} = 0$:

- defining then the extrinsic curvature and acceleration of the foliation $\mathscr{S}_\rho$ as

$$\widehat{K}_{ij} = \tfrac{1}{2}\, \mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij} \qquad \dot{\widehat{n}}_i := \widehat{n}^l D_l \widehat{n}_i = -\widehat{D}_i \ln \widehat{N}$$

- by combining them with our previous observations we get

$$D_i L_j = \left[\widehat{D}_i \boldsymbol{\lambda} + \widehat{n}_i\, \mathscr{L}_{\widehat{n}} \boldsymbol{\lambda} \right] \widehat{n}_j + \boldsymbol{\lambda}\, (\widehat{K}_{ij} + \widehat{n}_i \dot{\widehat{n}}_j) + \widehat{D}_i \mathbf{L}_j - \widehat{n}_i \widehat{n}_j\, (\dot{\widehat{n}}^l \mathbf{L}_l) \\ + \left\{ \widehat{n}_i\, \mathscr{L}_{\widehat{n}} \mathbf{L}_j - \widehat{n}_i\, \mathbf{L}_l \widehat{K}^l{}_j - \widehat{n}_j\, \mathbf{L}_l \widehat{K}^l{}_i \right\}$$

- finally, by contracting the inverse of the metric

$$h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i \widehat{n}_j$$

we get

$$D^l L_l = h^{ij} D_i L_j = (\widehat{\gamma}^{ij} + \widehat{n}^i \widehat{n}^j)\, D_i L_j = \mathscr{L}_{\widehat{n}} \boldsymbol{\lambda} + \boldsymbol{\lambda}\, (\widehat{K}^l{}_l) + \widehat{D}^l \mathbf{L}_l + \dot{\widehat{n}}^l \mathbf{L}_l$$

- thus $div\, \mathbf{L} = 0 \;\longleftrightarrow\; D^l L_l = 0$ is equivalent to

$$\mathscr{L}_{\widehat{n}} \boldsymbol{\lambda} + \boldsymbol{\lambda}\, (\widehat{K}^l{}_l) + \widehat{D}^l \mathbf{L}_l + \dot{\widehat{n}}^l \mathbf{L}_l = 0$$

Solving the constraint $div\,\mathbf{L} = 0$:

- when writing this out in (local) coordinates (ρ, x^2, x^3), adopted to the foliation $\mathscr{S}_\rho$ and the vector field ρ^i, we get

$$\boxed{\partial_\rho \boldsymbol{\lambda} - \widehat{N}^K \partial_K \boldsymbol{\lambda} + \boldsymbol{\lambda}\, \widehat{N}\, (\widehat{K}^L{}_L) + \widehat{N} \left[\widehat{D}^L \mathbf{L}_L + \dot{\hat{n}}^L \mathbf{L}_L \right] = 0 \qquad (**)}$$

- use $\boxed{\mathbf{L}_L}$ on Σ as free data, as we know the metric h_{ij}, the variables $\boxed{\widehat{N}, \widehat{N}^I, \widehat{\gamma}_{IJ}}$ are also known everywhere on $\Sigma \Longrightarrow$ all the coefficients in $(**)$ can be evaluated there
- thus, **independent** of the choice of the foliation or of that of the time-evolution vector field, $(**)$ is a **linear hyperbolic equation** for $\boldsymbol{\lambda}$
- in general, in initial-boundary value problems, the **global existence** of solutions are guaranteed to these type of equations

$2+1$ decompositions of h_{ij} and K_{ij}:

- as we saw the metric h_{ij} can be given as

$$\boxed{h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i\widehat{n}_j} \quad \Longleftrightarrow \quad \boxed{\{\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}\}}$$

- $$\boxed{K_{ij} = \boldsymbol{\kappa}\,\widehat{n}_i\widehat{n}_j + [\widehat{n}_i\,\mathbf{k}_j + \widehat{n}_j\,\mathbf{k}_i] + \mathbf{K}_{ij}}$$

  where

$$\boxed{\boldsymbol{\kappa} = \widehat{n}^k\widehat{n}^l\,K_{kl}\,, \quad \mathbf{k}_i = \widehat{\gamma}^k{}_i\widehat{n}^l\,K_{kl} \quad \text{and} \quad \mathbf{K}_{ij} = \widehat{\gamma}^k{}_i\widehat{\gamma}^l{}_j\,K_{kl}}$$

- the **trace** and **trace free** parts of $\mathbf{K}_{ij}$

$$\boxed{\mathbf{K}^l{}_l = \widehat{\gamma}^{kl}\,\mathbf{K}_{kl} \quad \text{and} \quad \overset{\circ}{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{2}\,\widehat{\gamma}_{ij}\mathbf{K}^l{}_l}$$

The new variables:

- $$\boxed{(h_{ij}, K_{ij})} \quad \Longleftrightarrow \quad \boxed{(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \overset{\circ}{\mathbf{K}}_{ij})}$$
- these variables retain the physically distinguished nature of h_{ij} and K_{ij}

The constraints of GR:

The momentum constraint: (gets to be symmetrizable hyperbolic system !!!)

$$\dot{\widehat{n}}_i := \widehat{n}^l D_l \widehat{n}_i = -\widehat{D}_i \ln \widehat{N}$$

$$D_e K^e{}_a - D_a K^e{}_e = 0$$

$$\widehat{K}_{ij} = \tfrac{1}{2} \mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij}; \ \widehat{K}^l{}_l = \widehat{\gamma}^{ij} \widehat{K}_{ij}$$

$$\mathscr{L}_{\widehat{n}} \mathbf{k}_i - \tfrac{1}{2} \widehat{D}_i (\mathbf{K}^l{}_l) - \widehat{D}_i \boldsymbol{\kappa} + \widehat{D}^l \mathring{\mathbf{K}}_{li} + (\widehat{K}^l{}_l) \, \mathbf{k}_i + \boldsymbol{\kappa} \, \dot{\widehat{n}}_i - \dot{\widehat{n}}^l \, \mathbf{K}_{li} = 0$$

$$\mathscr{L}_{\widehat{n}} (\mathbf{K}^l{}_l) - \widehat{D}^l \mathbf{k}_l - \boldsymbol{\kappa} \, (\widehat{K}^l{}_l) + \mathbf{K}_{kl} \widehat{K}^{kl} + 2 \, \dot{\widehat{n}}^l \, \mathbf{k}_l = 0$$

The Hamiltonian constraint: (gets to be parabolic or algebraic !!!)

$$^{(3)}R + (K^e{}_e)^2 - K_{ef} K^{ef} = 0$$

- $$\widehat{R} - \left\{ 2 \, \mathscr{L}_{\widehat{n}} (\widehat{K}^l{}_l) + (\widehat{K}^l{}_l)^2 + \widehat{K}_{kl} \, \widehat{K}^{kl} + 2 \, \widehat{N}^{-1} \widehat{D}^l \widehat{D}_l \widehat{N} \right\} + 2 \, \boldsymbol{\kappa} \, \mathbf{K}^l{}_l + \tfrac{1}{2} (\mathbf{K}^l{}_l)^2 - 2 \, \mathbf{k}^l \mathbf{k}_l - \mathring{\mathbf{K}}_{kl} \, \mathring{\mathbf{K}}^{kl} = 0$$
- it is a **parabolic equation** for $\widehat{N}$ (the sign of $\widehat{K}^l{}_l$ plays a role)
- it is an **algebraic equation** for $\boldsymbol{\kappa}$ (what is if $\mathbf{K}^l{}_l$ vanishes somewhere?)

Summary:

Constrained dynamical system living in 4-dimensional **Lorentzian spaces** were considered. **!!!** $[n(\geq 4)$ & one can also allow **Riemannian spaces** for no cost$]$

1. it was demonstrated, by using specific examples of physical interest, that, as opposed to the conventional picture, **the constrains need not to be elliptic**
2. in particular, it was shown that the usual **divergence of a vector field** type **constraints** of electrodynamics can be solved as **linear hyperbolic systems**
3. concerning the constraint equations in Einstein's theory it was shown:
    - **momentum constraint** as a **first order symmetric hyperbolic system**
    - **the Hamiltonian constraint** as a **parabolic** or an **algebraic** equation
    - **in either case** the coupled constraint equations comprise **a well-posed evolutionary system:** a **parabolic-hyperbolic** or a **strongly hyperbolic**,
    - **(local) existence and uniqueness** of C^∞ solutions is guaranteed
4. **!!! no use** of gauge conditions
   regardless of your choice for some preferred coordinates or your preference for a distinguished one among those in any of your experimental setup **you are always guaranteed** to get these **evolution equations** for sure