A new method of constructing binary black hole initial data

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Outline:

1. Motivations
2. The parabolic-hyperbolic form of the constrains
3. Kerr-Schild black holes and the superposed ones
4. Solving the constraints as an initial-boundary value problem
5. Input parameters and ADM charges
6. Summary

Motivations:

GW observations:

- inspiral and merger of binary black holes is of distinguished importance for the emerging field of gravitational wave astronomy
- non-linearities necessitate the use of accurate numerical approaches in determining the emitted waveforms
- precision of these simulations—in particular, their initializations—is of critical importance in enhancing the detection of gravitational wave signals

A new construction:

- I. Rácz: *Constraints as evolutionary systems*, Class. Quantum Grav. **33** 015014 (2016)
- I. Rácz: *A simple method of constructing binary black hole initial data*, arXiv:1605.01669
- I. Rácz: *On the ADM charges of multiple black holes* , arXiv:1608.02283
- A. Nakonieczna, L. Nakonieczny and I. Rácz: *Black hole initial data by numerical integration of the parabolic-hyperbolic form of the constraints*, arXiv:1712.00607

Intitialization:

The constraints:

- vacuum initial data: (h_{ij}, K_{ij}) on a 3-dimensional manifold Σ

$$
\begin{aligned}
{}^{(3)}R + (K^e{}_e)^2 - K_{ef}K^{ef} &= 0 \\
D_e K^e{}_a - D_a K^e{}_e &= 0
\end{aligned}
$$

where D_a denotes the covariant derivative operator associated with h_{ab}

- it is an underdetermined system: 4 equations for 12 variables

New variables by applying $2+1$ decompositions:

Splitting of the metric h_{ij}:

assume $\boxed{\Sigma \approx \mathbb{R} \times \mathscr{S}}$

Σ is smoothly foliated by a one-parameter family of two-surfaces $\mathscr{S}_\rho$:
$\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

$$\Longrightarrow \boxed{\partial_i \rho \;\; \& \;\; h^{ij} \;\longrightarrow\; \widehat{n}_i\,, \widehat{n}^i = h^{ij}\widehat{n}_j \;\; \ldots \;\; \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i \widehat{n}_j}$$

- choose ρ^i to be a vector field on Σ: the integral curves... & $\boxed{\rho^i \partial_i \rho = 1}$
- 'lapse' and 'shift' of ρ^i

$$\boxed{\rho^i = \widehat{N}\,\widehat{n}^i + \widehat{N}^i}$$

$$\boxed{\widehat{N} = \rho^i \widehat{n}_i\,, \quad \widehat{N}^i = \widehat{\gamma}^i{}_j\,\rho^j \quad \text{and} \quad \widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j h_{kl}}$$

- the metric h_{ij} can then be given as

$$\boxed{h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i \widehat{n}_j} \quad \Longleftrightarrow \quad \boxed{\{\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}\}}$$

$2+1$ decompositions:

Splitting of the symmetric tensor field K_{ij}:

- $$K_{ij} = \boldsymbol{\kappa}\,\widehat{n}_i\widehat{n}_j + [\widehat{n}_i\,\mathbf{k}_j + \widehat{n}_j\,\mathbf{k}_i] + \mathbf{K}_{ij}$$

  where

  $$\boldsymbol{\kappa} = \widehat{n}^k\widehat{n}^l\,K_{kl}\,, \quad \mathbf{k}_i = \widehat{\gamma}^k{}_i\widehat{n}^l\,K_{kl} \quad \text{and} \quad \mathbf{K}_{ij} = \widehat{\gamma}^k{}_i\widehat{\gamma}^l{}_j\,K_{kl}$$

- the **trace** and **trace free** parts of $\mathbf{K}_{ij}$

  $$\mathbf{K}^l{}_l = \widehat{\gamma}^{kl}\,\mathbf{K}_{kl} \quad \text{and} \quad \mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{2}\,\widehat{\gamma}_{ij}\mathbf{K}^l{}_l$$

The new variables:

- $$(h_{ij}, K_{ij}) \quad \Longleftrightarrow \quad (\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$$

- these variables retain the physically distinguished nature of h_{ij} and K_{ij}

The parabolic-hyperbolic form of the constraints:

An evolutionary system for the constrained fields $\widehat{N}, \mathbf{k}_i$ and $\mathbf{K}^l{}_l$:

$$\overset{\star}{K}\,[\,(\partial_\rho \widehat{N}) - \widehat{N}^l(\widehat{D}_l \widehat{N})\,] - \widehat{N}^2(\widehat{D}^l \widehat{D}_l \widehat{N}) - \mathcal{A}\,\widehat{N} + \mathcal{B}\,\widehat{N}^3 = 0$$

$$\mathscr{L}_{\widehat{n}}\mathbf{k}_i - \tfrac{1}{2}\,\widehat{D}_i(\mathbf{K}^l{}_l) - \widehat{D}_i\boldsymbol{\kappa} + \widehat{D}^l\overset{\circ}{\mathbf{K}}_{li} + \widehat{N}\,\overset{\star}{K}\,\mathbf{k}_i + [\,\boldsymbol{\kappa} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)\,]\,\dot{\widehat{n}}_i - \dot{\widehat{n}}^l\,\overset{\circ}{\mathbf{K}}_{li} = 0$$

$$\mathscr{L}_{\widehat{n}}(\mathbf{K}^l{}_l) - \widehat{D}^l\mathbf{k}_l - \widehat{N}\,\overset{\star}{K}\,[\,\boldsymbol{\kappa} - \tfrac{1}{2}\,(\mathbf{K}^l{}_l)\,] + \widehat{N}\,\overset{\circ}{\mathbf{K}}_{kl}\overset{\star}{K}{}^{kl} + 2\,\dot{\widehat{n}}^l\,\mathbf{k}_l = 0\,,$$

where $\widehat{D}_i$ denotes the covariant derivative operator associated with $\widehat{\gamma}_{ij}$

$$\overset{\star}{K} = \tfrac{1}{2}\,\widehat{\gamma}^{ij}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_j\widehat{N}^j$$

$$\overset{\star}{K}_{ij} = \tfrac{1}{2}\mathscr{L}_\rho\widehat{\gamma}_{ij} - \widehat{D}_{(i}\widehat{N}_{j)}\,, \qquad \dot{\widehat{n}}_k = \widehat{n}^l D_l\widehat{n}_k = -\widehat{D}_k(\ln \widehat{N})$$

$$\mathcal{A} = (\partial_\rho \overset{\star}{K}) - \widehat{N}^l(\widehat{D}_l\overset{\star}{K}) + \tfrac{1}{2}[\,\overset{\star}{K}{}^2 + \overset{\star}{K}_{kl}\overset{\star}{K}{}^{kl}\,]$$

$$\mathcal{B} = \tfrac{1}{2}\,[\,\widehat{R} + 2\,\boldsymbol{\kappa}\,(\mathbf{K}^l{}_l) + \tfrac{1}{2}\,(\mathbf{K}^l{}_l)^2 - 2\,\mathbf{k}^l\mathbf{k}_l - \overset{\circ}{\mathbf{K}}_{kl}\,\overset{\circ}{\mathbf{K}}{}^{kl}\,]$$

The parabolic-hyperbolic system:

The parabolic-hyperbolic system:

- no restriction applies to $\boxed{\widehat N^i, \widehat\gamma_{ij}, \boldsymbol\kappa \text{ and } \mathring{\mathbf{K}}_{ij}}$ $\Longrightarrow$ they are freely specifiable throughout Σ
- the parabolic equation is uniformly parabolic in those subregions of Σ, where $\overset{\star}{K}$ is either positive or negative
- $\overset{\star}{K}$ depends exclusively on the freely specifiable fields $\widehat\gamma_{ij}$ and $\widehat N^i$ $\Longrightarrow$ its sign can be tailored according to the desire of the investigated problem
- the combined evolutionary system is (locally) well-posed
  - once suitable initial values for the constrained fields $\boxed{\widehat N, \mathbf{k}_i \text{ and } \mathbf{K}^l{}_l}$ is chosen, on some level surface $\mathscr{S}_0$ in Σ, then, in the domain of dependence of $\mathscr{S}_0$, unique solution exists to the evolutionary system
  - the fields $\boxed{h_{ij} \text{ and } K_{ij}}$ that can be reconstructed from the freely specifiable and constrained variables do satisfy the Hamiltonian and momentum constraints

The Kerr black hole:

In Kerr-Schild form:

- $$g_{\alpha\beta} = \eta_{\alpha\beta} + 2H\ell_\alpha\ell_\beta$$
- inertial coordinates (t, x, y, z) adapted to the Minkowski background $\eta_{\alpha\beta}$

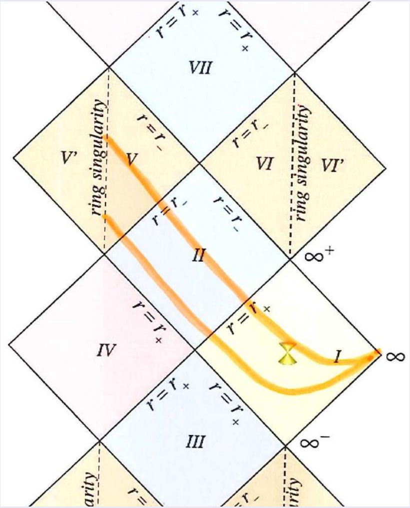
$$H = \frac{r^3 M}{r^4 + a^2 z^2}$$

$$\ell_\alpha = \left(1, \frac{r\,x + a\,y}{r^2 + a^2}, \frac{r\,y - a\,x}{r^2 + a^2}, \frac{z}{r}\right)$$

- the Boyer-Lindquist radial coordinate r is related to the spatial part of the inertial coordinates as

$$r^4 - (x^2 + y^2 + z^2 - a^2)\,r^2 - a^2\,z^2 = 0$$

$t = const$ slices in Kerr spacetime:



the $\theta = \frac{\pi}{2}$ section

$$\Sigma \approx \mathbb{R}^3 \setminus \{\text{"ring singularity"}\}$$

Generic Kerr-Schild black holes:

- the Kerr-Schild metrics are form-invariant under Lorentz transformations
  - if a Lorentz transformation $x'^{\alpha} = \Lambda^{\alpha}{}_{\beta}\, x^{\beta}$ is performed
  - the metric retains its distinguished Kerr-Schild form

$$g'_{\alpha\beta} = \eta_{\alpha\beta} + 2H'\ell'_{\alpha}\ell'_{\beta}$$

  - where $H' = H'(x'^{\alpha})$ and $\ell'_{\beta} = \ell'_{\beta}(x'^{\varepsilon})$ are given as

$$H' = H\left([\Lambda^{\alpha}{}_{\beta}]^{-1}x'^{\beta}\right),\ \ell'_{\beta} = \Lambda^{\alpha}{}_{\beta}\,\ell_{\alpha}\left([\Lambda^{\varepsilon}{}_{\varphi}]^{-1}x'^{\varphi}\right)$$

- boosts and spatial rotations are special Lorentz transformations $\Longrightarrow$ it is straightforward to construct models of moving and rotating black holes with preferably oriented speed and spin

Superposed Kerr-Schild black holes:

A binary system will be approximated by:

- $$g_{\alpha\beta} = \eta_{\alpha\beta} + 2H^{[1]}\ell_\alpha{}^{[1]}\ell_\beta{}^{[1]} + 2H^{[2]}\ell_\alpha{}^{[2]}\ell_\beta{}^{[2]} \qquad (*)$$
    - $H^{[n]}$ and $\ell_\alpha{}^{[n]}$ correspond to the Kerr-Schild data for individual black holes
- not a solution! **but** good approximation close to the individual black holes
- Einstein tensor falls off at the rate $\mathcal{O}(|\vec{x}|^{-4})$, where $|\vec{x}| = \sqrt{x^2+y^2+z^2}$

Choice for the free data:

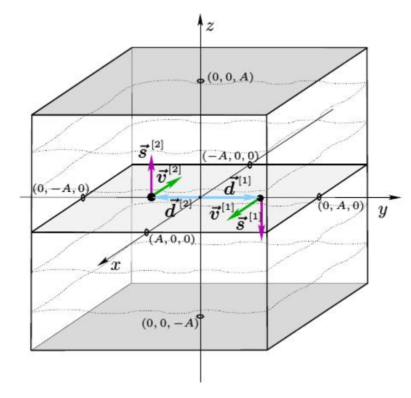
- $\widehat{N}^i, \widehat{\gamma}_{ij}, \boldsymbol{\kappa}$ and $\mathring{\mathbf{K}}_{ij}$ as if (*) solved the Einstein equations
- $\widehat{N}, \mathbf{K}^l{}_l$ and $\mathbf{k}_i$ on some level surface $\mathscr{S}_0$ in Σ deduced from (*) [only on $\mathscr{S}_0$!]

The initial-boundary value problem:

the $\mathscr{S}_\rho$ surfaces have tacitly been assumed to be compact without boundary:

- in numerical approaches Σ is chosen to be a large but bounded subset of $\mathbb{R}^3$
- the product structure $\Sigma \approx \mathbb{R} \times \mathscr{S}$ can be guaranteed by choosing the $\mathscr{S}_\rho$ leaves to be diffeomorphic to a closed disk in $\mathbb{R}^2$

choose Σ to be a cube centered at the origin in $\mathbb{R}^3$:



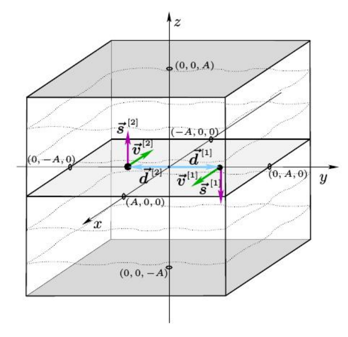
- for large enough value of A ...
- boundary of Σ:

  six squares each with edges of size $2A$
- the black holes are assumed to be located on the $z = 0$ plane
- speeds are parallel, spins are orthogonal to the $z = 0$ plane
- foliation by $z = const$ level surfaces
- deduce $\overset{\star}{K}$ from (*)

Solving the initial-boundary value problem:

The parabolic-hyperbolic system:

- $\overset{\star}{K}$ can be given as $\boxed{\overset{\star}{K} = -z \cdot \overset{+}{K}}$
- $\overset{\star}{K}$ is positive below the $z=0$ plane, while it is negative above that plane
- solved by propagating, along the z-streamlines, initial values specified on the horizontal $z=\pm A$ squares
- boundary values are to be given on the four vertical sides of the cube



- $\widehat{N}, \mathbf{K}^l{}_l$ and $\mathbf{k}_i$ are developed on Σ^+ and Σ^- separately
  - global existence and matching of these solutions at their common Cauchy horizon—at the $z=0$ plane—is of fundamental importance
- the auxiliary metric (*) possesses a $z \to -z$ reflection symmetry
- (apart from singularities) the existence of unique (at least) C^2 solutions with proper matching at the "common Cauchy horizon" can be verified

Input parameters and global ADM charges:

- **Input parameters:** the rest masses $M^{[n]}$, displacements $\vec{d}^{\,[n]}$, speeds $\vec{v}^{\,[n]}$ and spins $a^{[n]}\vec{s}_\circ^{[n]}$ of the involved black holes
  - essentially the same as used in post-Newtonian description of binaries **!!!**
- **Global ADM charges:** in terms of the input parameters
  - though (*) does not satisfy Einstein's equations it is asymptotically flat
  - constructed by adding contributions of individual black hole metrics to a Minkowski background
  - the ADM quantities are linear in deviation from flat Euclidean space at infinity

$$
\begin{aligned}
M^{ADM} &= \gamma^{[1]} M^{[1]} + \gamma^{[2]} M^{[2]} \\
M^{ADM} \vec{d}^{\,ADM} &= \gamma^{[1]} M^{[1]} \vec{d}^{\,[1]} + \gamma^{[2]} M^{[2]} \vec{d}^{\,[2]} \\
\vec{P}^{ADM} &= \gamma^{[1]} M^{[1]} \vec{v}^{\,[1]} + \gamma^{[2]} M^{[2]} \vec{v}^{\,[2]} \\
\vec{J}^{ADM} &= \gamma^{[1]} \left\{ M^{[1]} \vec{d}^{\,[1]} \times \vec{v}^{\,[1]} + M^{[1]} a^{[1]} \vec{s}_\circ^{[1]} \right\} \\
&\quad + \gamma^{[2]} \left\{ M^{[2]} \vec{d}^{\,[2]} \times \vec{v}^{\,[2]} + M^{[2]} a^{[2]} \vec{s}_\circ^{[2]} \right\}
\end{aligned}
$$

Summary:

1. a new method to initialize time evolution of binary black hole systems by applying
   - a parabolic-hyperbolic formulation of constraint equations
   - superposing Kerr-Schild black holes
2. the parabolic-hyperbolic equations solved as an initial-boundary value problem
3. existence of unique (at least) C^2 solutions is guaranteed (apart from singularities)
4. the input parameters—the rest masses, speeds, spins and displacements—are essentially the same as used in PN **!!!**
5. each of the ADM charges can be given in terms of the input parameters
6. no use of boundary conditions in the strong field regime (tidal deformations)