On the use of evolutionary methods in metric theories of gravity II.

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The main message and the program for today:

The main message:

- some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces
- **(!)** there will be a number of open research problems mentioned

The program for today:

- plans and aims for the rest of the course
- The Einstein-matter equations as non-linear wave equations: generalized harmonic gauge with matter fields
- symmetric hyperbolic systems: global existence and uniqueness to linear systems; uniqueness to generic systems

the viewgraphes will be uploaded time-to-time to the page
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Plans and aims for the rest of the course:

1. **Kinematical background:** (M, g_{ab}) **(!)** Lorentzian or Euclidean signature
   - notations and conventions
   - the basic tools are $n+1$ decompositions: no use of field equations

2. **The propagation of the constraints**
   - Einsteinian spaces: (M, g_{ab})
   - Bianchi identity
   - no gauge condition: arbitrary choice of foliations $\&$ "evolutionary" vector field

3-4 **Constraints as evolutionary systems**
   - conformal method: semilinear elliptic system
   - parabolic-hyperbolic system

     ... global solution to the involved parabolic equation
   - strongly hyperbolic system

     ... study of near Kerr configurations

5. **The construction of initial data for binary black holes**
   - parabolic-hyperbolic system
   - superposed Kerr-Schild, with initial-boundary value problem

     ... no use of ad hoc boundary conditions in the strong field regime

     ... there is an unprecedented full control of the ADM charges

Plans and Aims II.:

6-8 Time evolution and the degrees of freedom

- intimate relations between various parts of Einstein's equations
- partly and/or fully constrained evolutionary schemes
- hyperbolic-hyperbolic systems

  ... gauge choices

  ... the conformal structure

  ... gravitational degrees of freedom

9-10 Geroch's quasi-local argument and the positive mass theorems

- quasi-local quantities
- the Hawking-Geroch mass
- variation of the Geroch mass

  ... construction of initial data with non-decreasing Geroch mass

11-12 Dynamical horizons: black hole thermodynamics with 'dynamics'

- dynamical black holes
- dynamical horizons and their geometrical properties
- variation of physical and geometrical quantities

  ... vary them along dynamical horizons

  ... derive the dynamical first law of black hole thermodynamics

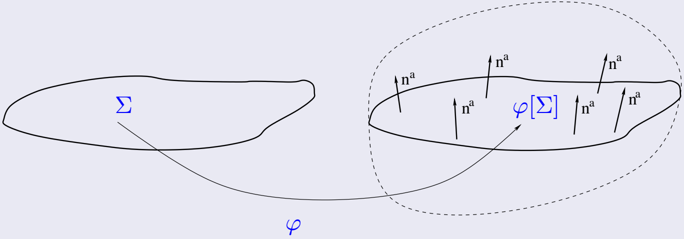
The main conceptual issue:

Assume that suitable initial data is given on some initial data surface Σ:

As a fixed background/arena does not exist in GR neither the base manifold M (where the solution manifest itself) nor the metric g_{ab} (satisfying the Einstein equations) is know in advance to solving the pertinent Cauchy problem

Initial data surface:
(Σ, h_{ij}, K_{ij})
(satisfying the constraints)

Spacetime:
(M, g_{ab})
(satisfying the Einstein equations)



$$(h_{ij}, K_{ij}) \quad \longrightarrow \quad \varphi_* \quad \longrightarrow \quad (\varphi_* h_{ij}, \varphi_* K_{ij})$$

(induced metric, extrinsic curvature)

Gravity-matter equations as non-linear wave equations:

Generalized harmonic gauge with matter fields I.

- Yvonne Choquet-Bruhat **1952** (..., James York, Helmut Friedrich,...)
- **spacetime:** (M, g_{ab}): now(!) Lorentzian signature $(-,+,\dots,+)$
- **matter fields:** as we have g_{ab} we may assume $\psi_{{}_{(i)}a\dots b}$, $i=1,\dots,I$, $(0,l_i)$ type tensor fields (shorthand: $\psi_{{}_{(i)}}$), satisfying

$$\boxed{\nabla^a\nabla_a\psi_{{}_{(i)}} = \mathcal{F}_{{}_{(i)}}\left(\psi_{{}_{(j)}}, \nabla_c\psi_{{}_{(j)}}, g_{ef}\right)}$$

  where $\mathcal{F}_{{}_{(i)}}$ are $(0,l_i)$ type tensorial expressions which depend smoothly on the indicated variables; e.g.: KG, Maxwell A_a, YMH, $\exists$...
- equations for the metric

$$\boxed{R_{ab} = \mathcal{R}_{ab}\left(\psi_{{}_{(i)}}, \nabla_c\psi_{{}_{(i)}}, g_{ef}\right)}$$

- $\mathcal{R}_{ab}$: $(0,2)$ type tensorial expressions which depend smoothly on the indicated variables & $\boxed{\nabla^a(\mathcal{R}_{ab} - \frac{1}{2}\,g_{ab}\mathcal{R}) = 0}$ (integ.cond.!)

Gravity-matter equations as non-linear wave equations:

Generalized harmonic gauge with matter fields II.

- special case: Einstein's equations with cosmological constant

$$\boxed{\mathcal{R}_{ab}\left(\psi_{(i)}, \nabla_c\psi_{(i)}, g_{ef}\right) = 8\pi\left(T_{ab} - \tfrac{1}{2}g_{ab}T\right) - \Lambda\, g_{ab}}$$

- solubility only for PDEs deduced from tensor equations ... how do they look like (?) in arbitrary local coordinates x^α
- the Ricci tensor

$$\boxed{R_{\alpha\beta} = -\tfrac{1}{2}g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} + g_{\delta(\alpha}\nabla_{\beta)}\Gamma^\delta + H'_{\alpha\beta}(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho})}$$

$\boxed{\Gamma^\mu = g^{\alpha\beta}\Gamma^\mu{}_{\alpha\beta}}$ (!transforms as vectors!) $\boxed{\nabla_\alpha\Gamma^\delta = \partial_\alpha\Gamma^\delta + \Gamma^\delta{}_{\alpha\varepsilon}\Gamma^\varepsilon}$

- $\nabla^a\nabla_a\psi_{(i)}$ in loc.coords. x^α contain $g^{\mu\nu}\nabla_\mu\Gamma^\gamma{}_{\nu\alpha}$, can be written as

$$\boxed{g^{\mu\nu}\nabla_\mu\Gamma^\gamma{}_{\nu\alpha} = R_\alpha{}^\gamma + \nabla_\alpha\Gamma^\gamma + H^{*\gamma}_\alpha(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho})}$$

where $H'_{\alpha\beta}$ and $H^{*\gamma}_\alpha$ are C^∞ functionals of indicated variables

Gravity-matter equations as non-linear wave equations:

Generalized harmonic gauge with matter fields III.

- matter equations

$$\nabla^\mu \nabla_\mu \psi_{(i)} = g^{\mu\nu}\partial_\mu\partial_\nu\psi_{(i)} - \sum_{k=1}^{l_i} \left(\psi_{(i)}\right)_\delta^{[\alpha_k]} \left(R_{\alpha_k}{}^\delta + \nabla_{\alpha_k}\Gamma^\delta\right) + \\ + \mathcal{H}'_{(i)}(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho}, \psi_{(j)}, \partial_\gamma \psi_{(j)})$$

where $(\psi_{(i)})_\delta^{[\alpha_k]}$ stands for $\psi_{(i)\,\alpha_1\ldots\alpha_{k-1}\delta\alpha_{k+1}\ldots\alpha_{l_i}}$ and $\mathcal{H}'_{(i)}$ ...

- equations for the coupled gravity-matter system

$$g^{\mu\nu}\partial_\mu\partial_\nu\psi_{(i)} = \sum_{k=1}^{l_i} \left(\psi_{(i)}\right)_\delta^{[\alpha_k]} \nabla_{\alpha_k}\Gamma^\delta + \mathcal{H}_{(i)}(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho}, \psi_{(j)}, \partial_\gamma \psi_{(j)})$$

$$g^{\mu\nu}\partial_\mu\partial_\nu g_{\alpha\beta} = 2 g_{\delta(\alpha}\nabla_{\beta)}\Gamma^\delta + H_{\alpha\beta}(g_{\varepsilon\rho}, \partial_\gamma g_{\varepsilon\rho}, \psi_{(j)}, \partial_\gamma \psi_{(j)})$$

- if we knew Γ^δ we would have a well-posed initial value problem ($\exists$, unique, continuous dep. on in.dat., causal) for $\psi_{(i)}$ és $g_{\alpha\beta}$

Gravity-matter equations as non-linear wave equations:

Generalized harmonic gauge with matter fields IV.

- **BUT** we do not know Γ^δ: **reduced** equations $\Gamma^\delta \to f^\delta : M \to \mathbb{R}$
  $\nabla_\alpha \Gamma^\delta \to \nabla_\alpha f^\delta$ $\quad \boxed{R^{(red.)}_{\alpha\beta} = R_{\alpha\beta} - g_{\delta(\alpha}\nabla_{\beta)}\left[\Gamma^\delta - f^\delta\right]}$
- f^δ is not completely arbitrary: given initial data on Σ for $[[g_{\alpha\beta}, \dot g_{\alpha\beta}]; \psi_{(i)}, \dot\psi_{(i)}]$, Γ^δ és $\partial_t \Gamma^\delta$ can be evaluated. **choose** f^δ :

$$\boxed{f^\delta = \Gamma^\delta \quad \text{and} \quad \partial_t f^\delta = \partial_t \Gamma^\delta}$$

- what does the relation $f^\delta = \Gamma^\delta$ mean?

$$\boxed{\nabla^\mu \nabla_\mu x^{\underline{\delta}} = g^{\mu\nu}\nabla_\mu(\partial_\nu x^{\underline{\delta}}) = g^{\mu\nu}\left[\,\partial_\mu(\delta_\nu{}^{\underline{\delta}}) - \Gamma^\varepsilon{}_{\mu\nu}(\delta_\nu{}^{\underline{\delta}})\,\right] = -\Gamma^{\underline{\delta}}}$$

- setting $f^\delta = \Gamma^\delta$ is equivalent to singling out specific local coord.s x^α (gen. harmonic): using initial data on Σ for $[x^\alpha, \dot x^\alpha]$ such that $\{dx^\alpha\}$ are linearly independent: $\quad \boxed{\nabla^\mu \nabla_\mu x^\delta = -f^\delta}$

Gravity-matter equations as non-linear wave equations:

Generalized harmonic gauge with matter fields V.

- **suppose** we have solutions $g_{\alpha\beta}$, $\psi_{(i)}$, x^α to the **reduced** equations
- recall $$\boxed{0 = R^{(red.)}_{\alpha\beta} - \mathcal{R}_{\alpha\beta} = \Big(R_{\alpha\beta} - g_{\delta(\alpha}\nabla_{\beta)}\Big)[\Gamma^\delta - f^\delta]\Big) - \mathcal{R}_{\alpha\beta}}$$
- introducing the variable $\mathcal{D}^\delta = \Gamma^\delta - f^\delta$

$$R_{\alpha\beta} - \mathcal{R}_{\alpha\beta} = g_{\delta(\alpha}\nabla_{\beta)}\mathcal{D}^\delta$$

$$\nabla^\alpha\nabla_\alpha\psi_{(i)} - \mathcal{F}_{(i)} = \sum_{k=1}^{l_i}\Big(\psi_{(i)}\Big)^{[\alpha_k]}_{\delta}\nabla_{\alpha_k}\mathcal{D}^\delta$$

- using the twice contracted Bianchi identity $\nabla^a\left[R_{ab} - \frac{1}{2}g_{ab}R\right] = 0$, and the integrability condition $\nabla^a\left[\mathcal{R}_{ab} - \frac{1}{2}g_{ab}\mathcal{R}\right] = 0$, we get for $\mathcal{D}^\delta = \Gamma^\delta - f^\delta$

$$\boxed{\nabla^\mu\nabla_\mu\mathcal{D}^\delta + R^\delta{}_\nu\mathcal{D}^\delta = 0}$$

the solution to the reduced equations is solution to the original problem □

Symmetric hyperbolic systems

- consider equations of the form

$$\boxed{\mathcal{A}^0(t,x,\mathbf{u})\,\partial_t\mathbf{u} + \mathcal{A}^i(t,x,\mathbf{u})\,\partial_i\mathbf{u} + \mathcal{B}(t,x,\mathbf{u}) = 0 \qquad (*)}$$

- it is a system of equations for $N \in \mathbb{N}$ real variables which are collected into a vector-valued function $\mathbf{u}$.
- these variables will be defined on appropriate subsets of $\mathbb{R} \times \mathbb{R}^n$. ($n$ will always stand for the 'spatial' dimension. 4-dim spacetime: $n = 3$.)
- a point of $\mathbb{R} \times \mathbb{R}^n$ will also be signified by the Cartesian coordinates $(t, x^1, \ldots, x^n)$, or in shorthand by (t, x).
- this system of equations is called *first order symmetric hyperbolic system* (FOSH) if the coefficient matrices $\mathcal{A}^\alpha$ are symmetric, and if $\mathcal{A}^0$ is positive definite.
- it is quasi-linear as it is linear in the first order derivatives of $\mathbf{u}$

First order **linear** symmetric hyperbolic systems:

Global existence and uniqueness

$$\boxed{\mathcal{A}^\alpha(t,x)\,\partial_\alpha \mathbf{u} + \mathcal{E}(t,x)\,\mathbf{u} + \mathbf{F}(t,x) = 0 \qquad (**) \quad (or)}$$

$$\boxed{(\mathcal{A}^\alpha(t,x))^{IJ}\,\partial_\alpha u_J + (\mathcal{E}(t,x))^{IJ}\,u_J + (F(t,x))^{I} = 0 \qquad (**)}$$

where

- $\boxed{(\mathcal{A}^\alpha(t,x))^{IJ}}$ and $\boxed{(\mathcal{E}(t,x))^{IJ}}$ are $N \times N$ matrices such that
- $\boxed{(\mathcal{A}^\alpha(t,x))^{IJ} = (\mathcal{A}^\alpha(t,x))^{JI}}$ are symmetric and
- $\boxed{(\mathcal{A}^0(t,x))^{IJ}}$ is positive definite, i.e. $\boxed{(\mathcal{A}^0(t,x))^{IJ} v_I\, v_J > 0 \text{ for } \forall\ v_I \neq 0}$

- **Conditions on the coefficients:**
  - $\boxed{(\mathcal{A}^0(t,x))^{IJ}}$ is **uniformly** positive definite: $\boxed{(\mathcal{A}^0(t,x))^{IJ} v_I\, v_J \geq C_0\,|\mathbf{v}|^2}$ for $\forall\ v_I \neq 0$ and for some $C_0 > 0$ constant
  - $\boxed{\partial_\alpha(\mathcal{A}^\alpha(t,x))^{IJ}}$ and $\boxed{(\mathcal{E}(t,x))^{IJ}}$ are **uniformly** bounded
  - $\boxed{(F(t,x))^{I}}$ is square integrable: on each $t = const$ level surface in $\mathbb{R} \times \mathbb{R}^n$

First order **linear** symmetric hyperbolic systems:

The strategy:

- energy of $\mathbf{u}$ at $t = const$

$$\boxed{\mathscr{E}(t,\mathbf{u}) := \tfrac{1}{2}\int_{\Sigma_t}[\,(\mathcal{A}^0(t,x))^{IJ}u_I\,u_J\,]\,\mathrm{d}^n x}$$

- **first** we show that $\mathscr{E}(t,\mathbf{u})$ satisfy the **energy equality**

$$\mathscr{E}(t,\mathbf{u}) = \mathscr{E}(t_0,\mathbf{u}) + \int_{t_0}^{t}\Big[\int_{\Sigma_{t'}}\Big(\tfrac{1}{2}(\partial_\alpha\mathcal{A}^{\alpha\,IJ})\,u_I u_J - (\mathcal{E}^{IJ})\,u_I u_J - (F^J)\,u_J\Big)\,\mathrm{d}^n x\Big]\,\mathrm{d}t'$$

- **second:** the energy equality implies an **integral energy ineq.**

$$\boxed{\mathscr{E}(t,\mathbf{u}) \leq \mathscr{E}(t_0,\mathbf{u}) + \int_{t_0}^{t}\Big[C_1(t')\,\mathscr{E}(t',\mathbf{u}) + C_1(t')\,(\mathscr{E}(t',\mathbf{u}))^{1/2}\Big]\,\mathrm{d}t'}$$

- **third** we show that the energy remains bounded

First order **linear** symmetric hyperbolic systems:

The energy equality:

- note first $\boxed{u_J\,\mathcal{A}^{\alpha\,IJ}(\partial_\alpha u_I) = u_I\,\mathcal{A}^{\alpha\,IJ}(\partial_\alpha u_J)}$ thereby

$$\begin{aligned} u_J\,\mathcal{A}^{\alpha\,IJ}(\partial_\alpha u_I) &= \tfrac{1}{2}\,\mathcal{A}^{\alpha\,IJ}\,\partial_\alpha(u_I u_J) \\ &= \tfrac{1}{2}\,\partial_\alpha(\mathcal{A}^{\alpha\,IJ} u_I u_J) - \tfrac{1}{2}\,u_I u_J\,(\partial_\alpha \mathcal{A}^{\alpha\,IJ}) \end{aligned}$$

- e.g. **if the initial data is of compact support** (fall off...) the integral if the total **spatial divergence** $\partial_i \mathcal{A}^{i\,IJ} u_I u_J$ vanishes

- using $\boxed{(\mathcal{A}^\alpha)^{IJ}\,\partial_\alpha u_J + (\mathcal{E})^{IJ}\,u_J + (F)^I = 0}$

$$\int_{t_0}^{t} \Big\{ \int_{\Sigma_{t'}} \tfrac{1}{2}\Big[\partial_0(u_I u_J \mathcal{A}^{0\,IJ}) - u_I u_J\,(\partial_\alpha \mathcal{A}^{\alpha\,IJ})\Big] - (\mathcal{E}^{\,IJ})\,u_I u_J - (F^{\,I})\,u_I\,\Big)\,\mathrm{d}^n x \Big\}\,\mathrm{d}t'$$

The energy inequality:

- **now** we have that $\mathscr{E}(t,\mathbf{u})$ satisfy the **energy equality**

$$\boxed{\begin{aligned}\mathscr{E}(t,\mathbf{u}) = \mathscr{E}(t_0,\mathbf{u}) + \int_{t_0}^{t}\Big[\int_{\Sigma_{t'}}\Big(\tfrac{1}{2}(\partial_\alpha \mathcal{A}^{\alpha\, IJ})\,u_I u_J \\ - (\mathcal{E}^{\,IJ})\,u_I u_J - (F^{\,J})\,u_J\Big)\,\mathrm{d}^n x\Big]\,\mathrm{d}t'\end{aligned}}$$

- $\boxed{\partial_\alpha(\mathcal{A}^\alpha(t,x))^{\,IJ}}$ and $\boxed{(\mathcal{E}(t,x))^{\,IJ}}$ are **uniformly** bounded and $\boxed{(F(t,x))^{\,J}}$ is square integrable: **(!)** on each $t = const$ surface
- Cauchy-Schwarz inequality

$$\boxed{\Big|\int_{\Sigma_{t'}}(F^{\,J})\,u_J\,\mathrm{d}^n x\Big| \leq \Big(\int_{\Sigma_{t'}}|\mathbf{u}|^2\,\mathrm{d}^n x\Big)^{1/2}\Big(\int_{\Sigma_{t'}}|\mathbf{F}|^2\,\mathrm{d}^n x\Big)^{1/2}}$$

- implies the **integral energy ineq.**

$$\boxed{\mathscr{E}(t,\mathbf{u}) \leq \mathscr{E}(t_0,\mathbf{u}) + \int_{t_0}^{t}\Big[C_1(t')\,\mathscr{E}(t',\mathbf{u}) + C_2(t')\,(\mathscr{E}(t',\mathbf{u}))^{1/2}\Big]\,\mathrm{d}t'}$$

First order **linear** symmetric hyperbolic systems:

The global boundedness of the energy:

- we have now the **integral energy inequality**

$$\boxed{\mathscr{E}(t,\mathbf{u}) \leq \mathscr{E}(t_0,\mathbf{u}) + \int_{t_0}^{t} \Big[C_1(t')\,\mathscr{E}(t',\mathbf{u}) + C_2(t')\,(\mathscr{E}(t',\mathbf{u}))^{1/2}\Big]\,\mathrm{d}t'}$$

- **Grönwall's lemma:** consider a differential equation $\boxed{\partial_t z(t) = f(t,z(t))}$, assume that $f(t,y)$ is function that is continuous (C^0) in t and Lipschitz type (C^{1-}) in y; **if** for a C^1 function $y(t)$ both of the inequalities hold

$$\boxed{\partial_t y \leq f(t,y), \quad y(t_0) \leq z(t_0)} \qquad \text{then} \qquad \boxed{y(t) \leq z(t)}$$

- this, in particular, implies that **if** for $y(t)$, with $C_1(t), C_2(t) \geq 0$,

$$\boxed{y(t) \leq y(t_0) + \int_{t_0}^{t} \Big[C_1(t')\,y(t') + C_2(t')\,(y(t'))^{1/2}\Big]\,\mathrm{d}t'}$$

  and $\boxed{y(t_0) \leq z(t_0)}$ **then** $\boxed{y(t) \leq z(t)}$, where $z(t)$ satisfies the corresponding equality

- applying this to $y(t) = \mathscr{E}(t,\mathbf{u})$ for the corresponding $z(t)$ we get $\quad$ (t dep. supp.)

$\boxed{z' = C_1\,z + C_2\,z^{1/2}}$ or by setting $\boxed{z = \zeta^2 \;\&\; \zeta_0 = y_0^{1/2}}$ we get $\boxed{2\,\zeta' = C_1\,\zeta + C_2}$

with solution that is finite for any $t = const$: **note:** uniqueness $C_2 = 0$ and $y_0 = 0$

Symmetric hyperbolic systems

Uniqueness of solutions in the generic case:

- $$\boxed{\mathcal{A}^0(t,x,\mathbf{u})\,\partial_0\mathbf{u} + \mathcal{A}^i(t,x,\mathbf{u})\,\partial_i\mathbf{u} + \mathcal{B}(t,x,\mathbf{u}) = 0 \qquad (*)}$$
- assume that $\mathbf{u}$ is a solution to (*):
  a hypersurface Σ is called to be **spacelike with respect to a solution $\mathbf{u}$** of (*) if for any co-normal one-form n_α that is orthogonal to Σ, i.e. $n_\alpha X^\alpha = 0$ for any any $X \in \mathcal{T}\Sigma$, the matrix $n_\alpha \mathcal{A}^\alpha$ is positive definite.
- this notion of being spacelike has a priori nothing to do with the usual one applied in general relativity. nevertheless, the two concepts can be shown to be closely related in the cases considered earlier: $\mathbf{u}$ ... '$\Box$'... $= g^{\mu\nu}\partial_\mu\partial_\nu \ldots$
- assume that $\mathbb{R}\times\mathbb{R}^n$ is foliated by 'spacelike' surfaces Σ_t
- **lemma:** to any C^1 function $\mathscr{F} : (\mathbb{R}\times\mathbb{R}^n)\times\mathbb{R}^N \to \mathbb{R}^N$ there always exist a C^0 function $\mathscr{H} : (\mathbb{R}\times\mathbb{R}^n)\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$ such that

  $$\boxed{\mathscr{F}(t,x,\mathbf{u}_1) - \mathscr{F}(t,x,\mathbf{u}_2) = (\mathbf{u}_1 - \mathbf{u}_2)\,\mathscr{H}(t,x,\mathbf{u}_1,\mathbf{u}_2)}$$

  for $N=1$: $F(u_1) - F(u_2) = (u_1 - u_2)\int_0^1 F'[\,t(u_1-u_2)+u_2\,]\,dt$,
  for general N: one may prove it by induction

Uniqueness of solutions:

- $$\boxed{\mathcal{A}^0(t,x,\mathbf{u})\,\partial_0\mathbf{u} + \mathcal{A}^i(t,x,\mathbf{u})\,\partial_i\mathbf{u} + \mathcal{B}(t,x,\mathbf{u}) = 0 \qquad (*)}$$
- **(!)** $\mathcal{A}^\alpha$ and $\mathcal{B}$ at least C^1 functions of their indicated variables
- assume that $\mathbf{u}_1$ and $\mathbf{u}_2$ are solutions to (*) such that the Σ_t surfaces are spacelike with respect to $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_1|_{\Sigma_0} = \mathbf{u}_2|_{\Sigma_0}$
- in virtue of the above lemma there should exist C^0 functions $\mathcal{C}^\alpha$ and $\mathcal{D}$ such that

$$\boxed{\begin{aligned}\mathcal{A}^\alpha(t,x,\mathbf{u}_1) - \mathcal{A}^\alpha(t,x,\mathbf{u}_2) &= \mathcal{C}^\alpha(t,x,\mathbf{u}_1,\mathbf{u}_2)\,(\mathbf{u}_1 - \mathbf{u}_2)\\ \mathcal{B}(t,x,\mathbf{u}_1) - \mathcal{B}(t,x,\mathbf{u}_2) &= \mathcal{D}(t,x,\mathbf{u}_1,\mathbf{u}_2)\,(\mathbf{u}_1 - \mathbf{u}_2)\end{aligned}}$$

- it follows then from the foregoings that for $\Delta\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$

$$\boxed{\mathcal{A}^\alpha(\mathbf{u}_1)\,\partial_\alpha(\Delta\mathbf{u}) + \left[\,\mathcal{C}^\alpha(\mathbf{u}_1,\mathbf{u}_2)\,(\partial_\alpha\mathbf{u}_2) + \mathcal{D}(\mathbf{u}_1,\mathbf{u}_2)\,\right](\Delta\mathbf{u}) = 0}$$

is a linear homogeneous first order system $\square$

That is all for now...