On the use of evolutionary methods in metric theories of gravity III.

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The main message and the program for today:

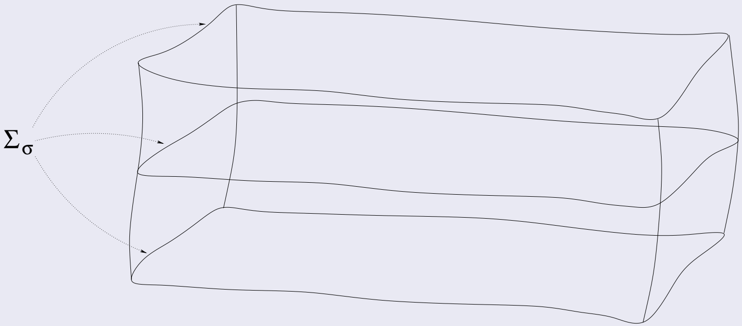
The main message:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

The program for today:

- **Kinematical background:** (M, g_{ab})
  - notations and conventions
  - the ambient space (M, g_{ab}) with no use of field equations
  - embedded codimension-one surfaces
  - the basic tools are $n+1$ decompositions
  - the decomposition of the ambient space Riemann tensor
  - the decomposition of the ambient space Ricci tensor
  - foliations of the ambient manifold by codimension-one surfaces
  - another alternative decomposition of $\nabla_a n_b$
  - some fundamental relations

The main areas where $3+1$ decompositions had been used:



- **initial value problem**

  Darmois (1923) [$N = 1, N^a = 0$, Gaussian normal system], Lichnerowicz (1939) [$N^a = 0$, a bit more flexible], Choque-Bruhat (1952) [the generic one],...

- **Hamiltonian formalism**

  Dirac (1959), Arnowitt-Deser-Misner (ADM), Wheeler (1960-1970), Moncrief (1975),...

Notations and conventions:

The generic setup:

The considered spaces: (M, g_{ab})

- M : $(n+1)$-dimensional, **smooth**, paracompact, connected, orientable manifold
- g_{ab}: **smooth**, Lorentzian$_{(-,+,+,+)}$ or Riemannian$_{(+,+,+,+)}$ metric

The abstract index notation

- A tensor of type (k, l) will be denoted by a letter followed by k contravariant and l covariant, lower case Latin indices: $T^{a_1 \dots a_k}{}_{b_1 \dots b_l}$
- Components relevant for particular choices of dual basis fields $\{\mathrm{v}_\nu\} \subset \mathcal{T}$, $\{\mathrm{v}^{*\nu}\} \subset \mathcal{T}^*$ will be indicated by using lower case Greek indices: $T^{\mu_1 \dots \mu_k}{}_{\nu_1 \dots \nu_l}$
- isomorphism between $\mathcal{T}$ and $\mathcal{T}^*$ provided by the metric: $v^a = g^{ab} v_b$ ...

Notation I.:

Curvature:

- The unique torsion free metric compatible covariant derivative operator ∇_a

$$\boxed{\nabla_a g_{bc} = 0}$$

- the action of the commutator can be expressed in terms of a tensor field $R_{abc}{}^{d}$ such that for an arbitrary form field ω_a

$$\boxed{(\nabla_a\nabla_b - \nabla_b\nabla_a)\,\omega_c = R_{abc}{}^{d}\,\omega_d}$$

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\; T^{c_1\dots c_k}{}_{d_1\dots d_l} = -\sum_{i=1}^{k} R_{abe}{}^{c_i}\; T^{c_1\dots \overset{i}{\smile}{e}\dots c_k}{}_{d_1\dots d_l} + $$

$$+ \sum_{j=1}^{l} R_{abd_j}{}^{e}\; T^{c_1\dots c_k}{}_{d_1\dots \overset{j}{\smile}{e}\dots d_l}$$

Key properties of the Riemann tensor:

If ∇_a is a torsion free covariant derivative operator and $R_{abc}{}^d$ its curvature

(1) $R_{abc}{}^d = -R_{bac}{}^d$,

(2) $R_{[abc]}{}^d = 0$,

(3) $R_{abcd} = -R_{abdc}$ (**if** ∇_a metric compatible),

(4) $\nabla_{[a}R_{bc]d}{}^e = 0$ [Bianchi-identity] $(\nabla_a R_{bcd}{}^e + \nabla_c R_{abd}{}^e + \nabla_b R_{cad}{}^e = 0)$.

Ricci tensor, scalar curvature, Einstein tensor:

- ! $R_{abc}{}^d$: curvature of a metric compatible ∇_a

  (1) and (3) $\Rightarrow$ $R_a{}^a{}_c{}^d = R_{abe}{}^e = 0$ BUT $R_{ab}{}^{ad}$ and $R_{abc}{}^b$ in general not $\Longrightarrow$

- **Ricci tensor**:

$$R_{ab} = R_{aeb}{}^e$$

symmetric $R_{ab} = R_{aeb}{}^e = R_{ea}{}^e{}_b = R_{eb}{}^e{}_a = R_{ba}$

- **scalar curvature:**

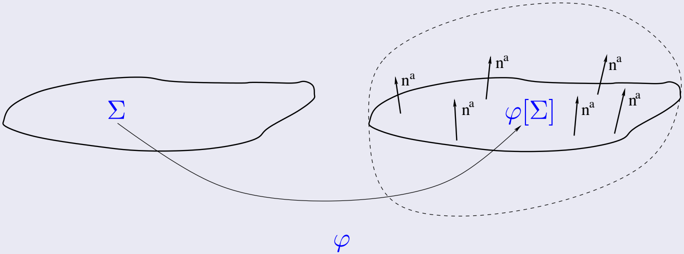
$$R = R_{ab}g^{ab} = R_a{}^a$$

- **Einstein tensor:**

$$G_{ab} = R_{ab} - \tfrac{1}{2}\, g_{ab}\, R\,, \qquad (\nabla^a G_{ab} = 0)\,.$$

Embeddings of codimension-one surfaces:

smooth codimension-one surface   submanifold in an ambient space (M, g_{ab})



φ is an embedding if $\varphi : \Sigma \to \varphi[\Sigma]$ is a homeomorphism

- self-intersection of $\varphi[\Sigma]$ is not allowed
- there exist linear maps $\varphi_* : \mathcal{T}(p) \to \mathcal{T}(\varphi(p))$ and $\varphi^* : \mathcal{T}^*(\varphi(p)) \to \mathcal{T}^*(p)$ relating the tangent and cotangent spaces of points $p \in \Sigma$ and $\varphi(p) \in M$, respectively
- these can be extended to $(k, 0)$ and $(0, l)$ type tensors but, as $\varphi[\Sigma] \subset M$ is a codimension-one surface in M (a proper one-dimension lower subset of M), there is no $\varphi*$ that could relate arbitrary (k, l) type tensors

Morse function:

- there exist a smooth function $\sigma : \mathscr{O}\ [\subset M] \to \mathbb{R}$ on a neighborhood $\mathscr{O}$ of $\varphi[\Sigma]$ such that $\partial_\alpha \sigma \neq 0$ (almost everywhere)

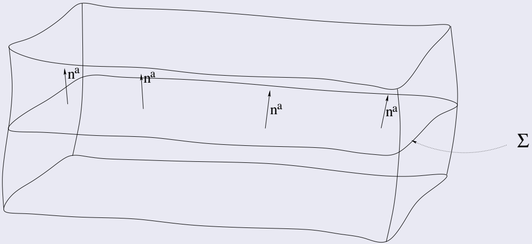
$$\boxed{\varphi[\Sigma] = \{\, p \in M \mid \sigma(p) = const \,\}}$$

- in mathematician's sayings: $\sigma : \mathscr{O}\ [\subset M] \to \mathbb{R}$ is usually assumed to be a Morse function such that it has only **non-degenerate** and **isolated critical points**
  - a point is **critical point** is where $\boxed{\partial_\alpha \sigma = 0}$
  - the critical point is **non-degenerate** if the Hessian of the map, i.e. the matrix $\boxed{\partial_\alpha \partial_\beta \sigma}$ is non-singular;
    the index of a critical point is the number of the negative eigenvalues
- in physicist's sayings: only regular origins may occur;
  only positive eigenvalues of the Hessian are allowed or, in other words, there exists a well defined tangent space there

The $n + 1$ decomposition:

The unit normals

- there exist a smooth function $\sigma : \mathscr{O}\ [\subset M] \to \mathbb{R}$ with non-vanishing gradient $\partial_a\sigma$ such that $\varphi[\Sigma]$—from now on (by standard abuse of notation) we denote it by Σ—is represented by a $\sigma = const$ level surface
- $$n_a \sim \partial_a\sigma \ \ldots \ \& \ldots \ g^{ab} \ \longrightarrow \ n^a = g^{ab} n_b$$



- n^a the 'unit norm' vector field that is normal to Σ

$$\boxed{n^a n_a = \epsilon} \qquad n_a|_{\mathscr{O}} = (\epsilon\, g^{ef} \partial_e\sigma\partial_f\sigma)^{-\frac{1}{2}} (\partial_a\sigma)$$

  - the sign is not fixed: ϵ takes the value -1 or $+1$ for Lorentzian or Riemannian metric g_{ab}, respectively.

Projections:

The projection operator:

- **the projection operator**

$$h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b \quad \text{with} \quad h_a{}^e h_e{}^b = h_a{}^b$$

  to the $\sigma = const$ level surface

- **the metric induced** on the $\sigma = const$ level surface

$$h_{ab} = h_a{}^e h_b{}^f g_{ef} = g_{ab} - \epsilon\, n_a n_b$$

- the covariant derivative operator D_a associated with h_{ab}: $\forall \omega_b$ on Σ

$$D_a \omega_b := h_a{}^d h_b{}^e \nabla_d\, \omega_e$$

$$D_a h_{bc} = h_a{}^d h_b{}^e h_c{}^f \nabla_d \left(g_{ef} - \epsilon\, n_e n_f \right) = 0$$

- the curvature of D_a is internal w.r.t. Σ as it is determined by h_{ab}

Decompositions of $\nabla_a n_b$ using $\delta_a{}^b = h_a{}^b + \epsilon\, n_a n^b$

The acceleration and the "extrinsic curvature":

- **a trivial decomposition**

$$\nabla_a n_b = \delta_a{}^e \delta_b{}^f \nabla_e n_f = (h_a{}^e + \epsilon\, n_a n^e)(h_b{}^f + \epsilon\, n_b n^f)\nabla_e n_f$$
$$= (h_a{}^e h_b{}^f \nabla_e n_f) + \epsilon (h_a{}^e n_b n^f + h_b{}^f n_a n^e)(\nabla_e n_f) + n_a n^e n_b n^f (\nabla_e n_f)$$

- with $n^f \nabla_e n_f = \frac{1}{2}\nabla_e (n^f n_f) = 0$
- the **acceleration** $\dot{n}_b := n^e \nabla_e n_b = h_b{}^f (n^e \nabla_e n_f)$ is tangential to Σ
- **the extrinsic curvature** on Σ $K_{ab} = h^e{}_a h^f{}_b \nabla_e n_f = h^e{}_a \nabla_e n_b$

1) K_{ab} is symmetric as $\nabla_e n_f = \nabla_{(e} n_{f)} + \nabla_{[e} n_{f]} = \nabla_{(e} n_{f)} + n_{[e} X_{f]}$

in the last step the Frobenius theorem was applied to the hypersurface orthogonal n^a, i.e. there exists a form field X_a such that $\nabla_{[e} n_{f]} = n_{[e} X_{f]}$

2) $K_{ab} = h_a{}^e h_b{}^f \nabla_{(e} n_{f)} = \frac{1}{2} h_a{}^e h_b{}^f \mathscr{L}_n g_{ef} = \frac{1}{2} h_a{}^e h_b{}^f \mathscr{L}_n h_{ef} = \frac{1}{2}\mathscr{L}_n h_{ab}$

- the foregoings also imply

$$\nabla_a n_b = (h_a{}^e h_b{}^f \nabla_e n_f) + \epsilon\, n_a h_b{}^f n^e (\nabla_e n_f) = K_{ab} + \epsilon\, n_a \dot{n}_b$$

The relations between the two Riemann tensors:

The Gauss relation:

Examples: $h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b$

- the only non-vanishing projections: $h_a{}^f h_b{}^g h_c{}^k h_j{}^e\, R_{fgk}{}^j$, $h_a{}^f h_b{}^g h_j{}^e n^k\, R_{fgk}{}^j$, $h_b{}^f h_e{}^d n^a n^c\, R_{afc}{}^e \cong n^a n^c\, R_{abc}{}^d$
- $$h_a{}^d h_b{}^f \nabla_d\, h_f{}^e = h_a{}^d h_b{}^f \nabla_d\,(g_f{}^e - \epsilon\, n_f n^e) = -\epsilon\, K_{ab} n^e$$
- $$h_a{}^d n^e \nabla_d\, \omega_e = \cancel{h_a{}^d \nabla_d\,(n^e \omega_e)} - [h_a{}^d \nabla_d\, n^e]\, \omega_e = -K_a{}^e \omega_e \quad (\forall\ \omega_e \in \mathcal{T}^*\Sigma)$$

now we are prepared to relate the curvatures of ∇_a and D_a $\quad (\forall\ \omega_e \in \mathcal{T}^*\Sigma)$

$$\begin{aligned} D_a D_b\, \omega_c = D_a(h_b{}^d h_c{}^e \nabla_d\, \omega_e) &= h_a{}^f h_b{}^g h_c{}^k \nabla_f\, (h_g{}^d h_k{}^e \nabla_d\, \omega_e) \\ &= h_a{}^f h_b{}^d h_c{}^e \nabla_f\, \nabla_d\, \omega_e - \epsilon\, h_c{}^e K_{ab}\, n^d \nabla_d\, \omega_e - \epsilon\, h_b{}^d K_{ac}\, n^e \nabla_d\, \omega_e \\ &= h_a{}^f h_b{}^d h_c{}^e \nabla_f\, \nabla_d\, \omega_e - \epsilon\, h_c{}^e K_{ab}\, n^d \nabla_d\, \omega_e + \epsilon\, K_{ac} K_b{}^e\, \omega_e \end{aligned}$$

by which, as ω_a is arbitrary, we get the **Gauss relation**:

$$^{(n)}R_{abc}{}^e = h_a{}^f h_b{}^g h_c{}^k h_j{}^e\, R_{fgk}{}^j + \epsilon\, [\, K_{ac} K_b{}^e - K_{bc} K_a{}^e\,]$$

The Codazzi relation:

Examples: $h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b$

- similarly, by definition $(\nabla_a \nabla_b - \nabla_b \nabla_a)\, n^d = -R_{abc}{}^d\, n^c$

  $\Rightarrow$ in determining the contraction $R_{abc}{}^d\, n^c h_e{}^a h_f{}^b h_d{}^g$ we need to evaluate $h_e{}^a h_f{}^b h_d{}^g (\nabla_a \nabla_b\, n^d)$

- which, by $\nabla_a n^b = \delta_a{}^e \nabla_e n^b = (h_a{}^e + \epsilon\, n_a n^e)\nabla_e n^b = K_a{}^b + \epsilon\, n_a \dot{n}^b$,

$$\begin{aligned} h_e{}^a h_f{}^b h_d{}^k (\nabla_a\, \nabla_b\, n^d) &= h_e{}^a h_f{}^b h_d{}^k (\nabla_a\, [K_b{}^d + \epsilon\, n_b \dot{n}^d]) \\ &= h_e{}^a h_f{}^b h_d{}^k (\nabla_a\, K_b{}^d) + \epsilon\, h_e{}^a h_f{}^b (\nabla_a\, n_b) \dot{n}^d \\ &= D_e\, K_f{}^k + \epsilon\, K_{ef} \dot{n}^k \end{aligned}$$

  from which we get the **Codazzi relation** as:

$$h_e{}^a h_f{}^b h_d{}^k\, n^c R_{abc}{}^d = -2 D_{[e}\, K_{f]}{}^k$$

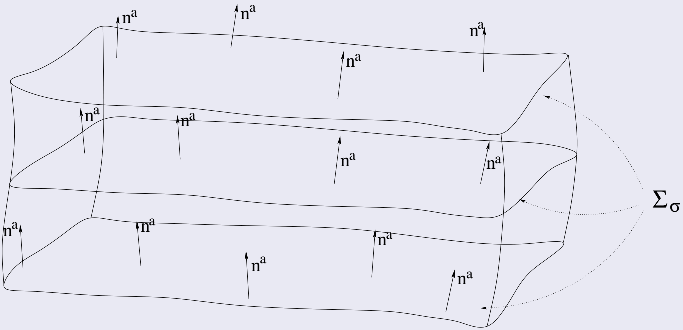
  contracting in f, g and using the symmetry of the Riemann tensor we get

$$h_e{}^a\, n^c R_{ac} = D_h\, K_e{}^h - D_e\, K_h{}^h$$

The 3^{rd} relation: $h_b{}^e h_f{}^d n^a n^c R_{aec}{}^f$

Requires derivatives non-tangential to a single hypersurface: foliations are needed

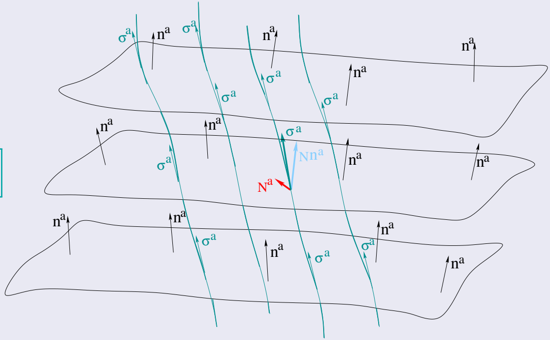
- **Assume:** M is foliated by a one-parameter family of homologous hypersurfaces, i.e. $M \simeq \mathbb{R} \times \Sigma$, for some n-dimensional manifold Σ.
  - known to **hold for globally hyperbolic spacetimes** (Lorentzian case)
  - **equivalent to** the existence of a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing gradient $\nabla_a \sigma$ such that the $\sigma = const$ level surfaces $\Sigma_\sigma = \{\sigma\} \times \Sigma$ comprise the one-parameter foliation of M.
  - $$n_a \sim \nabla_a \sigma \ \ \ldots \ \& \ldots \ g^{ab} \ \longrightarrow \ \ n^a = g^{ab} n_b$$



σ^a is "time evolution vector field" **if**:

- the integral curves of σ^a meet the $\sigma = const$ level surfaces precisely once
- $\sigma^e \nabla_e \sigma = 1$

$$\sigma^a = \sigma^a_\perp + \sigma^a_\parallel = N\, n^a + N^a$$



- where N and N^a denotes the **lapse** and **shift** of σ^a:

$$N = \epsilon\,(\sigma^e n_e) \quad \text{and} \quad N^a = h^a{}_e\, \sigma^e$$

The 3^{rd} relation: $h_b{}^e h_f{}^d n^a n^c\, R_{aec}{}^f \cong n^a n^c\, R_{abc}{}^d$

Examples: $h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b$

- we need some lemmas:

Lemma 1: $n^e \nabla_e\, n_a = -\epsilon\, D_a(\ln N)$

$\sigma^e \nabla_e \sigma = 1 \;\&\; \sigma^a = N\, n^a + N^a \Rightarrow N\, n^e \nabla_e \sigma = 1 \Rightarrow n_a = \epsilon\, N\, \nabla_a \sigma \Rightarrow$

$$\begin{aligned} n^e \nabla_e\, n_a &= n^e \nabla_e\, (\epsilon\, N\, \nabla_a \sigma) = \epsilon\, \left[(n^e \nabla_e\, N)\, (\epsilon\, N^{-1} n_a) + N\, n^e\, (\nabla_a \nabla_e \sigma) \right] \\ &= \epsilon\, \left[\epsilon\, n_a n^e \nabla_e\, (\ln N) + N \left\{ \nabla_a (n^e\, \nabla_e \sigma) - \cancel{(\nabla_a\, n^e)\, (n_e \epsilon\, N^{-1})} \right\} \right] \\ &= \epsilon\, [\, -\nabla_a (\ln N) + \epsilon\, n_a n^e \nabla_e\, (\ln N)\,] = -\epsilon\, D_a(\ln N) \end{aligned}$$

- the symbol $\cong$ in expressions indicates that the two sides get to be equal to each other once projections to $\sigma = conts$ level surfaces, in the free indexes, have been performed

- **Lemma 3:** $\mathscr{L}_n K_b{}^d \cong n^e \nabla_e\, K_b{}^d$ $\cancel{+ K_e{}^d (\nabla_b\, n^e) - K_b{}^e (\nabla_e\, n^d)}$ as

$$K_e{}^d (\nabla_b\, n^e) - K_b{}^e (\nabla_e\, n^d) = K_e{}^d (K_b{}^e + \epsilon\, n_b\, \dot{n}^e) - K_b{}^e (K_e{}^d + \epsilon\, n_e\, \dot{n}^d) \cong 0$$

- **Lemma 4:** $n^e \nabla_b\, K_e{}^d = \cancel{\nabla_b\, (n^e K_e{}^d)} - (\nabla_b\, n^e) K_e{}^d \cong -K_b{}^e K_e{}^d$

The 3^{rd} relation: $h_b{}^e h_f{}^d n^a n^c R_{aec}{}^f \cong n^a n^c R_{abc}{}^d$

Examples: $h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b$

using $\nabla_a n^b = \delta_a{}^e \nabla_e n^b = K_a{}^b + \epsilon\, n_a \dot{n}^b$ and $\dot{n}_a = -\epsilon\, D_a(\ln N)$ we get

$$\begin{aligned} n^a n^c R_{abc}{}^d &= -\, n^a (\nabla_a \nabla_b - \nabla_b \nabla_a)\, n^d \\ &= -\, n^a \left\{ \nabla_a \left[K_b{}^d + \epsilon\, n_b \dot{n}^d \right] - \nabla_b \left[K_b{}^d + \epsilon\, n_b \dot{n}^d \right] \right\} \\ &= -\, n^a \nabla_a K_b{}^d - \epsilon\, \dot{n}_b \dot{n}^d + n^a \nabla_b K_a{}^d + \epsilon\, \cancel{(n^a \nabla_b n_a)}\, \dot{n}^d \\ &\quad - \epsilon\, n_b\, (n^a \nabla_a \dot{n}^d) + \epsilon^2\, \nabla_b\, \dot{n}^d \\ &\cong -\, \mathscr{L}_n K_b{}^d - \epsilon\, D_b(\ln N) D^d(\ln N) - K_b{}^e K_e{}^d - \epsilon\, D_b(D^d(\ln N)) \\ &= -\, \mathscr{L}_n K_b{}^d - K_b{}^e K_e{}^d - \epsilon\, N^{-1} D_b D^d N \end{aligned}$$

- **The 3^{rd} relation:**

$$h_b{}^e h_f{}^d n^a n^c R_{aec}{}^f = -\mathscr{L}_n K_b{}^d - K_b{}^e K_e{}^d - \epsilon\, N^{-1} D_b D^d N$$

Projections of the Ricci tensor: $n^e n^f R_{ef}$, $h_a{}^e n^f R_{ef}$, $h_a{}^e h_b{}^f R_{ef}$

$h_a{}^e n^f R_{ef}$ and $n^e n^f R_{ef}$ $\quad (h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b)$

- $$h_a{}^e\, n^f R_{ef} = D_h\, K_a{}^h - D_a\, K_h{}^h$$
  has already been determined by the contracted Codazzi relation

- from the Gauss relation we get

$$^{(n)}R_{ac} = {}^{(n)}R_{aec}{}^e = h_a{}^f h_j{}^e h_c{}^k\, R_{fek}{}^j + \epsilon\, [\, K_{ac} K_e{}^e - K_{ce} K_a{}^e\,]$$

$$^{(n)}R = {}^{(n)}R_{ac} h^{ac} = h_j{}^e h^{fk}\, R_{fek}{}^j + \epsilon\, [\, (K_e{}^e)^2 - K_{ef} K^{ef}\,]$$

but

$$h_j{}^e h^{fk}\, R_{fek}{}^j = (g^{fk} - \epsilon\, n^f n^k)\, \left[R_{fk} - \epsilon\, n^e n^j R_{fekj}\right] = R - 2\, \epsilon\, n^f n^e\, R_{fe}$$

thereby

$$n^e n^f\, R_{ef} = \tfrac{1}{2}\, \epsilon\, \Big[(R - {}^{(n)}R) + \epsilon\, \{\, (K_e{}^e)^2 - K_{ef} K^{ef}\, \} \Big]$$

- **Theorema Egregium of Gauss** (“remarkable theorem”)

$$2\, \epsilon\, n^e n^f\, G_{ef} - R\, (1 - \epsilon) = -{}^{(n)}R + \epsilon\, \{\, (K_e{}^e)^2 - K_{ef} K^{ef}\, \}$$

Projections of the Ricci tensor: $n^e n^f R_{ef}$, $h_a{}^e n^f R_{ef}$, $h_a{}^e h_b{}^f R_{ef}$

$h_a{}^e h_b{}^f R_{ef}$ [$\Leftarrow$ the contracted Gauss relation & 3^{rd} relation]

$$
\begin{aligned}
h_b{}^e h_d{}^f R_{ef} &= h_b{}^e h_d{}^f g^{ac} R_{aecf} = h_b{}^e h_d{}^f h^{ac} R_{aecf} + \epsilon\, h_b{}^e h_d{}^f n^a n^c R_{aecf} \\
&= \left\{ {}^{(n)}R_{bd} - \epsilon \left[K_{bd} K_e{}^e - K_{be} K_d{}^e \right] \right\} \\
&\qquad + \epsilon \left\{ -\mathscr{L}_n K_{bd} + K_b{}^e K_{de} - \epsilon\, N^{-1} D_b D_d N \right\} \\
&= {}^{(n)}R_{bd} + \epsilon \left\{ -\mathscr{L}_n K_{bd} - K_{bd} K_e{}^e + 2 K_b{}^e K_{de} - \epsilon\, N^{-1} D_b D_d N \right\}
\end{aligned}
$$

- the contraction $h^{bd} h_b{}^e h_d{}^f R_{ef}$ yields

$$
\begin{aligned}
&R - \epsilon\, n^e n^f R_{ef} \\
&\qquad = {}^{(n)}R + \epsilon \left\{ -h^{bd} \mathscr{L}_n K_{bd} - (K_e{}^e)^2 + 2 K_{ef} K^{ef} - \epsilon\, N^{-1} D^e D_e N \right\}
\end{aligned}
$$

- thereby, using the previously derived expression $n^e n^f R_{ef}$, we get

$$
R = {}^{(n)}R + \epsilon \left\{ -2\, \mathscr{L}_n (K_{bd} h^{bd}) - (K_e{}^e)^2 - K_{ef} K^{ef} - 2\, \epsilon\, N^{-1} D^e D_e N \right\}
$$

where

$$
h^{bd} \mathscr{L}_n (K_{bd}) = \mathscr{L}_n (K_{bd} h^{bd}) - K_{bd} \mathscr{L}_n h^{bd} = \mathscr{L}_n (K_{bd} h^{bd}) + 2 K_{bd} K^{bd}
$$

That is all for now...