On the use of evolutionary methods in metric theories of gravity IV.

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Characteristic polynomials and directions

- consider now a First Order Symmetric Hyperbolic (FOSH) system **HW (2) !**

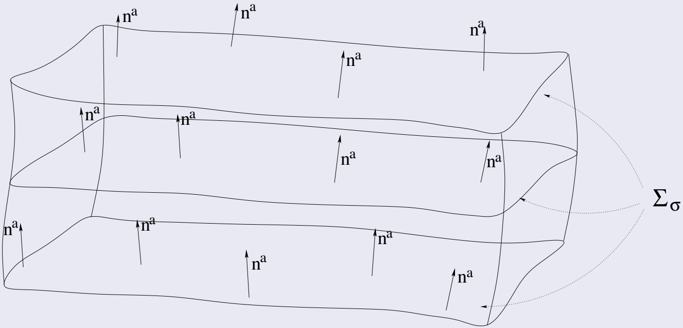
$$\mathcal{A}^\alpha(t,x)\,\partial_\alpha \mathbf{u} + \mathcal{B}(t,x,\mathbf{u}) = 0 \qquad (*)$$

- when the coefficients $\mathcal{A}^\alpha$ depend on the unknowns $\mathbf{u}$, all the definitions below can also be given but one has to be careful and one should refer to sections of fiber bundles (in fact a vector bundles) over the ambient manifold "$(t,\,x,\,\mathbf{u})$", this would require more structures to be used
- assume that ξ_α is a covector at a given point $p \in M$. **the principal symbol** of (*) at (p,ξ_α) is the matrix $\varsigma(p,\xi_\alpha) = \mathcal{A}^\alpha\,\xi_\alpha$ (could be viewed as a linear map)
- suppose that $\mathscr{H}$ is a hypersurface in a neighbourhood $\mathcal{U}$ of $p \in M$, i.e. there exist a function $\chi : \mathcal{U} \to \mathbb{R}$ such that $\mathscr{H} = \{\chi = const \;\&\; \partial_\alpha\chi \neq 0\}$
- $\mathscr{H}$ is said to be nowhere characteristic for (*) if $\det(\varsigma(p,\partial_\alpha\chi)) \neq 0$ for any $p \in \mathscr{H}$, whereas $\mathscr{H}$ is called to be **a characteristic hypersurface**, or simply **a characteristic** for (*), if $\det(\varsigma(p,\partial_\alpha\chi)) = 0$ for any $p \in \mathscr{H}$
  - if $\mathscr{H}$ is nowhere characteristic for (*) then if initial data ${}^0\mathbf{u}$ is given on $\mathscr{H}$ a formal expansion of a "to be solution" $\mathbf{u}$ can be given taking formal ∂_0^k-derivatives (*) and solving the yielded equations for $\partial_0^k\mathbf{u}$ on $\mathscr{H}$
  - if $\mathscr{H}$ is characteristic, the initial data cannot be prescribed freely on $\mathscr{H}$ $\det(\varsigma(p,\partial_\alpha\chi)) = 0$ induces relations among the initial data on $\mathscr{H}$: **inner equations** on $\mathscr{H}$; $\xi_\alpha = \partial_\alpha\chi$ is then a **characteristic direction**

Assumptions on the topology of the ambient manifold

Foliations

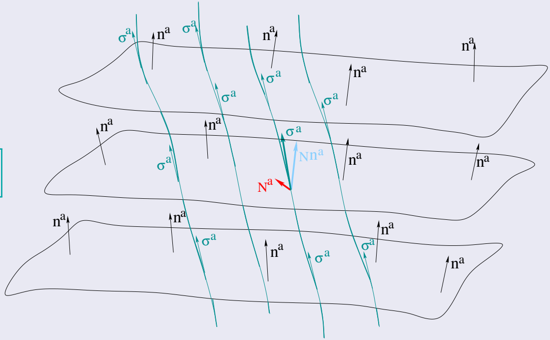
- **Assume:** M is foliated by a one-parameter family of homologous hypersurfaces, i.e. $M \simeq \mathbb{R} \times \Sigma$, for some n-dimensional manifold Σ.
  - known to **hold for globally hyperbolic spacetimes** (Lorentzian case)
  - **equivalent to** the existence of a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing gradient $\nabla_a\sigma$ such that the $\sigma = const$ level surfaces $\Sigma_\sigma = \{\sigma\} \times \Sigma$ comprise the one-parameter foliation of M.
  - $$n_a \sim \nabla_a\sigma \ \ldots \ \& \ldots \ g^{ab} \ \longrightarrow \ \ n^a = g^{ab} n_b$$



σ^a is "time evolution vector field" **if**:

- the integral curves of σ^a meet the $\sigma = const$ level surfaces precisely once
- $\boxed{\sigma^e \nabla_e \sigma = 1}$

$$\boxed{\sigma^a = \sigma^a_\perp + \sigma^a_\parallel = N\, n^a + N^a}$$



- where N and N^a denotes the **lapse** and **shift** of σ^a:

$$\boxed{N = \epsilon\,(\sigma^e n_e) \quad \text{and} \quad N^a = h^a{}_e\, \sigma^e}$$

The main creatures:

- n^a the 'unit norm' vector field that is normal to the Σ_σ level surfaces

$$\boxed{n^a n_a = \epsilon}$$

  - ϵ takes the value -1 or $+1$ for Lorentzian or Riemannian metric g_{ab}, resp.

- **the projection operator**

$$\boxed{h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b}$$

- **the metric induced**

$$\boxed{h_{ab} = h_a{}^e h_b{}^f\, g_{ef} = g_{ab} - \epsilon\, n_a n_b}$$

- the covariant derivative operator $\boxed{D_a}$ associated with h_{ab}: $\forall \omega_b$ on Σ

$$\boxed{D_a \omega_b := h_a{}^d h_b{}^e \nabla_d\, \omega_e}$$

- **the extrinsic curvature** on Σ (symmetric!)

$$\boxed{K_{ab} = h^e{}_a \nabla_e n_b = \tfrac{1}{2}\, \mathscr{L}_n h_{ab}}$$

- acceleration $\boxed{\dot n_a = h^f{}_a n^e \nabla_e n_f = n^e \nabla_e n_a}$

Summary of the principal relations:

Gauss relation: $${}^{(n)}R_{abc}{}^{e} = h_a{}^f h_b{}^g h_c{}^k h_j{}^e\, R_{fgk}{}^j + \epsilon\, [\, K_{ac}K_b{}^e - K_{bc}K_a{}^e\,]$$

Codazzi relation: $$h_e{}^a h_f{}^b h_d{}^g\, n^c R_{abc}{}^d = -2D_{[e}\, K_{f]}{}^g$$

The 3^{rd} relation: $$h_b{}^e h_f{}^d n^a n^c\, R_{aec}{}^f = -\mathscr{L}_n K_b{}^d - K_b{}^e K_e{}^d - \epsilon\, N^{-1} D_b D^d N$$

Various projections of the Ricci tensor:

$$n^e n^f\, R_{ef} = \tfrac{1}{2}\, \epsilon \left[(R - {}^{(n)}R) + \epsilon\, \{\, (K_e{}^e)^2 - K_{ef}K^{ef}\, \} \right]$$

$$h_a{}^e n^f R_{ef} = D_e\, K_{ae}{}^e - D_a\, K_e{}^e$$

$$h_b{}^e h_d{}^f R_{ef} = {}^{(n)}R_{bd} + \epsilon\, \left\{ -\mathscr{L}_n K_{bd} - K_{bd}K_e{}^e + 2K_b{}^e K_{de} - \epsilon\, N^{-1} D_b D_d N \right\}$$

relation of the scalar curvatures:

$$R = {}^{(n)}R + \epsilon\, \left\{ -2\, \mathscr{L}_n (K_{bd}h^{bd}) - (K_e{}^e)^2 - K_{ef}K^{ef} - 2\, \epsilon\, N^{-1} D^e D_e N \right\}$$

no field equation had been used yet !!!

More exercises:

Any symmetric tensor field P_{ab} can be decomposed

in terms of n^a and fields living on the $\sigma = const$ level surfaces as

$$P_{ab} = \boldsymbol{\pi}\, n_a n_b + [n_a\, \mathbf{p}_b + n_b\, \mathbf{p}_a] + \mathbf{P}_{ab}$$

where $$\boldsymbol{\pi} = n^e n^f\, P_{ef}, \quad \mathbf{p}_a = \epsilon\, h^e{}_a n^f\, P_{ef}, \quad \mathbf{P}_{ab} = h^e{}_a h^f{}_b\, P_{ef}$$

The projections of $\nabla^a P_{ab}$

$h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b$

$$\begin{aligned}
\nabla_e P_{ab} &= \nabla_e [\, \boldsymbol{\pi}\, n_a n_b + [n_a\, \mathbf{p}_b + n_b\, \mathbf{p}_a] + \mathbf{P}_{ab}\,] \\
&= (\nabla_e \boldsymbol{\pi})\, n_a n_b + \boldsymbol{\pi}\, (\nabla_e\, n_a) n_b + \boldsymbol{\pi}\, n_a (\nabla_e\, n_b) \\
&\quad + (\nabla_e\, n_a)\, \mathbf{p}_b + n_a (\nabla_e\, \mathbf{p}_b) + (\nabla_e\, \mathbf{p}_a)\, n_b + (\nabla_e\, n_b)\, \mathbf{p}_a + \nabla_e\, \mathbf{P}_{ab}
\end{aligned}$$

$$\begin{aligned}
h_f{}^b (\nabla^a P_{ab}) &= h_f{}^b\, [\, h^{ea} + \epsilon\, n^e n^a\,]\, \nabla_e P_{ab} = \boldsymbol{\pi}\, \dot{n}_f + (K_a{}^a)\, \mathbf{p}_f \\
&\quad + h_f{}^b\, [n^a \nabla_a\, \mathbf{p}_b + \mathbf{p}_a (h^{ea} \nabla_e\, n_b)] + D^a \mathbf{P}_{af} - \epsilon\, (n^e \nabla_e\, n^a)\, \mathbf{P}_{af} \\
&= \boldsymbol{\pi}\, \dot{n}_f + (K_a{}^a)\, \mathbf{p}_f + h_f{}^b\, \mathscr{L}_n\, \mathbf{p}_b + D^a \mathbf{P}_{af} - \epsilon\, \dot{n}^a \mathbf{P}_{af}
\end{aligned}$$

$$h_f{}^b (\nabla^a P_{ab}) = \mathscr{L}_n\, \mathbf{p}_f + D^a \mathbf{P}_{af} + [\, \boldsymbol{\pi}\, \dot{n}_f + (K_a{}^a)\, \mathbf{p}_f - \epsilon\, \dot{n}^a \mathbf{P}_{af}]$$

More exercises:

The projection $n^b(\nabla^a P_{ab})$ — $h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b$

$$\nabla_e P_{ab} = (\nabla_e \boldsymbol{\pi})\, n_a n_b + \boldsymbol{\pi}\,(\nabla_e\, n_a) n_b + \boldsymbol{\pi}\, n_a (\nabla_e\, n_b) + (\nabla_e\, n_a)\,\mathbf{p}_b + n_a(\nabla_e\,\mathbf{p}_b) + (\nabla_e\,\mathbf{p}_a)\, n_b + (\nabla_e\, n_b)\,\mathbf{p}_a + \nabla_e\,\mathbf{P}_{ab}$$

$$\begin{aligned} n^b(\nabla^a P_{ab}) &= n^b\,[\,h^{ea} + \epsilon\, n^e n^a\,]\,\nabla_e P_{ab} = \epsilon\,\mathscr{L}_n \boldsymbol{\pi} + \epsilon\,\boldsymbol{\pi}(K_e{}^e) + (n^a \nabla_a\,\mathbf{p}_b)\, n^b \\ &\quad + \epsilon\,[\,h^{ea} + \epsilon\, n^e n^a\,]\,\nabla_e\,\mathbf{p}_a - \mathbf{P}_{ab}[\,h^{ea} + \epsilon\, n^e n^a\,]\,(\nabla_e\, n^b) \\ &= \epsilon\,\mathscr{L}_n \boldsymbol{\pi} + \epsilon\,\boldsymbol{\pi}(K_e{}^e) - \dot{n}^a\,\mathbf{p}_a + \epsilon\,[\,D^a \mathbf{p}_a - \epsilon\,\dot{n}^a\,\mathbf{p}_a\,] \\ &\quad - \mathbf{P}_{ab}\, h^{ea}\,[\,K_e{}^b + \epsilon\, n_e \dot{n}^b\,] \\ &= \epsilon\,[\,\mathscr{L}_n \boldsymbol{\pi} + \boldsymbol{\pi}(K_e{}^e) + D^a \mathbf{p}_a\,] - 2\,\mathbf{p}_a \dot{n}^a - \mathbf{P}_{ab} K^{ab} \end{aligned}$$

$$\epsilon\, n^b(\nabla^a P_{ab}) = \mathscr{L}_n \boldsymbol{\pi} + D^e \mathbf{p}_e + [\,\boldsymbol{\pi}(K_e{}^e) - \epsilon\,\mathbf{P}_{ef} K^{ef} - 2\,\epsilon\,\mathbf{p}_e \dot{n}^e\,]$$

Simple projections $n^a(\nabla_a P_b{}^b)$ & $h_e{}^a(\nabla_a P_b{}^b)$ — $h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b$

as $P_b{}^b = \epsilon\,\boldsymbol{\pi} + \mathbf{P}_b{}^b$

$$n^a(\nabla_a P_b{}^b) = \epsilon\,\mathscr{L}_n \boldsymbol{\pi} + \mathscr{L}_n \mathbf{P}_b{}^b$$

$$h_e{}^a(\nabla_a P_b{}^b) = \epsilon\, D_e \boldsymbol{\pi} + D_e \mathbf{P}_b{}^b$$

Projections of divergences:

The projections of $\nabla^a P_{ab}$ $\qquad h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b$

$$\epsilon\, n^b(\nabla^a P_{ab}) = \mathscr{L}_n \boldsymbol{\pi} + D^e \mathbf{p}_e + [\, \boldsymbol{\pi}(K_e{}^e) - \epsilon\, \mathbf{P}_{ef} K^{ef} - 2\, \epsilon\, \mathbf{p}_e \dot{n}^e \,]$$

$$h_f{}^b(\nabla^a P_{ab}) = \mathscr{L}_n\, \mathbf{p}_f + D^a \mathbf{P}_{af} + [\, \boldsymbol{\pi}\, \dot{n}_f + (K_a{}^a)\, \mathbf{p}_f - \epsilon\, \dot{n}^a \mathbf{P}_{af}]$$

- assume $\nabla^a P_{ab} = 0$, and that $\dot{n}_a = 0,\ K^e_e = 0,\ \mathbf{P}_{ef} K^{ef} = 0$
- then the above relations reduce to

$$\mathscr{L}_n \boldsymbol{\pi} + D^e \mathbf{p}_e = 0$$
$$\mathscr{L}_n\, \mathbf{p}_b + D^a \mathbf{P}_{ab} = 0$$

- though P_{ab} is arbitrary the above relations look very much like the balance relation in fluid dynamics $\partial_t \rho + \partial_{\bar\alpha}(\rho\, \mathrm{v}^{\bar\alpha}) = 0$

  and the Euler equation $\rho\, [\partial_t \mathrm{v}^{\bar\alpha} + \mathrm{v}^{\bar\varepsilon} \partial_{\bar\varepsilon} \mathrm{v}^{\bar\alpha}] = -h^{\bar\alpha\bar\varepsilon} \partial_{\bar\varepsilon} P$

The program for the present lecture:

The main message:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

Plans and Aims:

The propagation of the constraints

- Einsteinian spaces: (M, g_{ab})
- Bianchi identity
- no gauge condition

... arbitrary choice of foliations $\&$ "evolutionary" vector field

Reference:

- I. Rácz: *Is the Bianchi identity always hyperbolic?*, CQG **31** 155004 (2014)

The considered Einsteinian spaces:

- **The ambient spaces:** (M, g_{ab})
  - M : $n+1$-dimensional, smooth, paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian$_{(-,+,\dots,+)}$ or Riemannian$_{(+,+,\dots,+)}$ metric

- **Einstein's equations:**

$$\boxed{G_{ab} - \mathscr{G}_{ab} = 0} \quad \text{with source term:} \quad \boxed{\nabla^a \mathscr{G}_{ab} = 0}$$

  - in a more familiar setup: **Einstein's equations** with cosmological constant Λ

$$\boxed{[R_{ab} - \tfrac{1}{2}\, g_{ab}\, R] + \Lambda\, g_{ab} = 8\pi\, T_{ab}}$$

    with matter fields satisfying their Euler-Lagrange equations
  -

$$\boxed{\mathscr{G}_{ab} = 8\pi\, T_{ab} - \Lambda\, g_{ab}}$$

Decompositions of various fields:

- the metric

$$g_{ab} = \epsilon\, n_a n_b + h_{ab}$$

- the "source term"

$$\mathscr{G}_{ab} = n_a n_b\, \mathfrak{e} + [n_a\, \mathfrak{p}_b + n_b\, \mathfrak{p}_a] + \mathfrak{S}_{ab}$$

where the energy, momentum and stress densities are defined as

$$\mathfrak{e} = n^e n^f\, \mathscr{G}_{ef}, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f\, \mathscr{G}_{ef}, \quad \mathfrak{S}_{ab} = h^e{}_a h^f{}_b\, \mathscr{G}_{ef}$$

- projections of the divergence $\nabla^a \mathscr{G}_{ab}$

$$\epsilon\, n^b (\nabla^a \mathscr{G}_{ab}) = \mathscr{L}_n \mathfrak{e} + D^e \mathfrak{p}_e + [\, \mathfrak{e}(K_e{}^e) - \epsilon\, \mathfrak{S}_{ef} K^{ef} - 2\, \epsilon\, \mathfrak{p}_e \dot{n}^e\,]$$
$$h_f{}^b (\nabla^a \mathscr{G}_{ab}) = \mathscr{L}_n\, \mathfrak{p}_f + D^a \mathfrak{S}_{af} + [\, \mathfrak{e}\, \dot{n}_f + (K_a{}^a)\, \mathfrak{p}_f - \epsilon\, \dot{n}^a \mathscr{G}_{af}]$$

- as $\nabla^a \mathscr{G}_{ab} = 0$, assuming that $\dot{n}_a = 0,\ K^e_e = 0,\ \mathfrak{S}_{ef} K^{ef} = 0$

$$\mathscr{L}_n \mathfrak{e} + D^e \mathfrak{p}_e = 0$$
$$\mathscr{L}_n\, \mathfrak{p}_f + D^a \mathfrak{S}_{af} = 0$$

Decompositions of various fields:

- **Einstein's equations:**

$$\boxed{G_{ab} - \mathscr{G}_{ab} = 0} \quad \text{with source term:} \quad \boxed{\nabla^a \mathscr{G}_{ab} = 0}$$

- r.h.s. of Einstein's equation: $E_{ab} = G_{ab} - \mathscr{G}_{ab}$

$$E_{ab} = n_a n_b \, E^{(\mathcal{H})} + [n_a \, E^{(\mathcal{M})}_b + n_b \, E^{(\mathcal{M})}_a] + (E^{(\mathcal{EVOL})}_{ab} + h_{ab} \, E^{(\mathcal{H})})$$

$$E^{(\mathcal{H})} = n^e n^f \, E_{ef} \,, \quad E^{(\mathcal{M})}_a = \epsilon \, h^e{}_a n^f \, E_{ef} \,, \quad E^{(\mathcal{EVOL})}_{ab} = h^e{}_a h^f{}_b \, E_{ef} - h_{ab} \, E^{(\mathcal{H})}$$

The decomposition of the covariant divergence $\nabla^a \, E_{ab} = 0$ of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$:

$$\mathscr{L}_n \, E^{(\mathcal{H})} + D^e E^{(\mathcal{M})}_e + [\, E^{(\mathcal{H})} \, (K^e{}_e) - 2 \, \epsilon \, (\dot{n}^e \, E^{(\mathcal{M})}_e) - \epsilon \, K^{ae} \, (E^{(\mathcal{EVOL})}_{ae} + h_{ae} \, E^{(\mathcal{H})}) \,] = 0$$

$$\mathscr{L}_n \, E^{(\mathcal{M})}_b + D^a (E^{(\mathcal{EVOL})}_{ab} + h_{ab} \, E^{(\mathcal{H})}) + [\, E^{(\mathcal{H})} \, \dot{n}_b + (K^e{}_e) \, E^{(\mathcal{M})}_b - \epsilon \, (E^{(\mathcal{EVOL})}_{ab} + h_{ab} \, E^{(\mathcal{H})}) \, \dot{n}^a \,] = 0$$

The decomposition of the covariant divergence $\nabla^a E_{ab} = 0$ of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$:

$$\mathscr{L}_n E^{(\mathcal{H})} + D^e E^{(\mathcal{M})}_e + [\, E^{(\mathcal{H})} (K^e{}_e) - 2\,\epsilon\,(\dot{n}^e E^{(\mathcal{M})}_e) - \epsilon\, K^{ae}\,(E^{(\mathcal{EVOL})}_{ae} + h_{ae}\, E^{(\mathcal{H})})\,] = 0$$

$$\mathscr{L}_n E^{(\mathcal{M})}_b + D^a(E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})}) + [\, E^{(\mathcal{H})}\, \dot{n}_b + (K^e{}_e)\, E^{(\mathcal{M})}_b - \epsilon\,(E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})\,\dot{n}^a\,] = 0$$

1st order symmetric hyperbolic system: linear and homogeneous in $(E^{(\mathcal{H})}, E^{(\mathcal{M})}_i)^T$:

- $N \times$ "(1)" and $N h^{ij} \times$ "(2)" in local coordinates $(\sigma, x^1, x^2, \dots, x^n)$ adopted to the vector field $\sigma^a = N\, n^a + N^a$: $\sigma^e \nabla_e \sigma = 1$ and the foliation $\{\Sigma_\sigma\}$, read as

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & h^{ij} \end{pmatrix} \partial_\sigma + \begin{pmatrix} -N^k & N\, h^{ik} \\ N\, h^{jk} & -N^k\, h^{ij} \end{pmatrix} \partial_k \right\} \begin{pmatrix} E^{(\mathcal{H})} \\ E^{(\mathcal{M})}_i \end{pmatrix} = \begin{pmatrix} \mathscr{E} \\ \mathscr{E}^j \end{pmatrix}$$

  where the source terms $\mathscr{E}$ and $\mathscr{E}^j$ are linear and homogeneous in $E^{(\mathcal{H})}$ and $E^{(\mathcal{M})}_i$

- $$\mathcal{A}^\mu\, \partial_\mu \mathbf{u} + \mathcal{B}\, \mathbf{u} = 0 \quad \text{with} \quad \mathbf{u} = (E^{(\mathcal{H})}, E^{(\mathcal{M})}_i)^T$$

HW (+): determine the characteristic directions for this equation.

The main result:

Theorem

Let (M, g_{ab}) *be an Einsteinian space as specified and assume that the metric* h_{ab} *induced on the* $\sigma = const$ *level surfaces is Riemannian. Then,* **regardless whether** g_{ab} **is of Lorentzian or Euclidean signature**, *any solution to the reduced equations* $E^{(\mathcal{EVOL})}_{ab} = 0$ *is also a solution to the full set of field equations* $G_{ab} - \mathscr{G}_{ab} = 0$ *provided that the constraint expressions* $E^{(\mathcal{H})}$ *and* $E^{(\mathcal{M})}_{a}$ *vanish on one of the* $\sigma = const$ *level surfaces.*

- no gauge condition was used anywhere in the above analyze !
  - it applies regardless of the choice of the foliation, $\boxed{\Sigma_\sigma}$, of M and for any choice of the evolution vector field, $\boxed{\sigma^a \ (N, N^a)}$.

The decomposition of the covariant divergence $\nabla^a E_{ab} = 0$ of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$:

$$\mathscr{L}_n E^{(\mathcal{H})} + D^e E^{(\mathcal{M})}_e + [\, E^{(\mathcal{H})} (K^e{}_e) - 2\,\epsilon\,(\dot n^e E^{(\mathcal{M})}_e) - \epsilon\, K^{ae}\,(E^{(\mathcal{EVOL})}_{ae} + h_{ae}\, E^{(\mathcal{H})})\,] = 0$$

$$\mathscr{L}_n E^{(\mathcal{M})}_b + D^a(E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})}) + [\, E^{(\mathcal{H})}\, \dot n_b + (K^e{}_e)\, E^{(\mathcal{M})}_b - \epsilon\,(E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})\,\dot n^a\,] = 0$$

What is if the constraints hold on each $\sigma = const$ hypersurface?

- $E^{(\mathcal{H})} = 0$ and $E^{(\mathcal{M})}_a = 0$

$$K^{ae}\, E^{(\mathcal{EVOL})}_{ae} = 0$$

$$D^a E^{(\mathcal{EVOL})}_{ab} - \epsilon\, E^{(\mathcal{EVOL})}_{ab}\, \dot n^a = 0$$

- homework **HW (3)**:
  - show that the constraints holds **for any foliations** of the ambient manifold then the evolution equations follow

The explicit forms I.:

using the **projections of the Ricci tensor:**

$$n^e n^f R_{ef} = \tfrac{1}{2}\,\epsilon\left[(R - {}^{(n)}\!R) + \epsilon\,\{\,(K_e{}^e)^2 - K_{ef}K^{ef}\,\}\right]$$

$$h_a{}^e n^f R_{ef} = D_e\,K_{ae}{}^e - D_a\,K_e{}^e$$

$$h_b{}^e h_d{}^f R_{ef} = {}^{(n)}\!R_{bd} + \epsilon\,\left\{-\mathscr{L}_n K_{bd} - K_{bd}K_e{}^e + 2K_b{}^e K_{de} - \epsilon\,N^{-1}D_b D_d N\right\}$$

along with the **relation of the scalar curvatures:**

$$R = {}^{(n)}\!R + \epsilon\,\left\{-2\,\mathscr{L}_n(K^e_e) - (K_e{}^e)^2 - K_{ef}K^{ef} - 2\,\epsilon\,N^{-1}D^e D_e N\right\}$$

and that of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$, the following explicit forms can be verified:

The explicit forms II.:

$$E^{(\mathcal{H})} = n^e n^f E_{ef} = \frac{1}{2}\left\{-\epsilon\,{}^{(n)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\right\},$$

$$E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\,[\,D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a\,]$$

$$E^{(\mathcal{EVOL})}_{ab} = {}^{(n)}\!R_{ab} + \epsilon\,\left\{-\mathscr{L}_n K_{ab} - (K^e{}_e)K_{ab} + 2\,K_{ae}K^e{}_b - \epsilon\,N^{-1}\,D_a D_b N\right\}$$

$$- [\mathfrak{S}_{ab} - \mathfrak{e}\,h_{ab}] - \frac{1}{2}\,h_{ab}\,\Big\{(1-\epsilon)\,{}^{(n)}\!R - 2\,\epsilon\,\mathscr{L}_n(K^e{}_e)$$

$$+(1-\epsilon)\,(K^e{}_e)^2 - (1+\epsilon)\,K_{ef}K^{ef} - 2\,N^{-1}\,D^e D_e N\Big\}$$

where

$$\mathfrak{e} = n^e n^f\,\mathscr{G}_{ef},\quad \mathfrak{p}_a = \epsilon\,h^e{}_a n^f\,\mathscr{G}_{ef},\quad \mathfrak{S}_{ab} = h^e{}_a h^f{}_b\,\mathscr{G}_{ef}$$

The explicit forms II.:

The reduced evolutionary expression $E_{ab}^{(\mathcal{EVOL})}$ looks pretty complicated. Therefore, in certain cases (in particular, whenever $\epsilon = -1$) it is rewarding to introduce

$$\widetilde{E}_{ab}^{(\mathcal{EVOL})} = E_{ab}^{(\mathcal{EVOL})} - \tfrac{1}{n-1}\, h_{ab} \left(E_{ef}^{(\mathcal{EVOL})} h^{ef} \right)$$

It can also be seen **HW (4)** that $\widetilde{E}_{ab}^{(\mathcal{EVOL})} = 0 \quad \Leftrightarrow \quad E_{ab}^{(\mathcal{EVOL})} = 0$ and

$$\widetilde{E}_{ab}^{(\mathcal{EVOL})} = h^e{}_a h^f{}_b \left[R_{ab} - \left(\mathscr{G}_{ab} - \frac{1}{n-1}\, g_{ab} \left[\mathscr{G}_{ef}\, g^{ef} \right] \right) \right] + \frac{1+\epsilon}{n-1}\, h_{ab}\, E^{(\mathcal{H})}$$

In virtue of the above relations we have

$$\begin{aligned} \widetilde{E}_{ab}^{(\mathcal{EVOL})} = {}^{(n)}R_{ab} + \epsilon\, \{ & -\mathscr{L}_n K_{ab} - (K^e{}_e) K_{ab} + 2\, K_{ae} K^e{}_b - \epsilon\, N^{-1}\, D_a D_b N \} \\ & - \left(\mathfrak{S}_{ab} - \frac{1}{n-1}\, h_{ab} \left[\mathfrak{S}_{ef}\, h^{ef} + \epsilon\, \mathfrak{e} \right] \right) \\ & + \frac{1+\epsilon}{2\,(n-1)}\, h_{ab} \left\{ -\epsilon\, {}^{(n)}R + (K^e{}_e)^2 - K_{ef} K^{ef} - 2\, \mathfrak{e} \right\} \end{aligned}$$

That is all for now...