On the use of evolutionary methods in metric theories of gravity IX.

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Plans and Aims:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

- **Time evolution and the degrees of freedom**
  - intimate relations between various parts of Einstein's equations
  - fully constrained evolutionary schemes
  - evolutionary-evolutionary systems

    ... gauge choices
    ... the conformal structure
    ... gravitational degrees of freedom

Based on some recent papers

- I. Rácz: *Is the Bianchi identity always hyperbolic?*, Class. Quantum Grav. 31 (2014) 155004
- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, Class. Quantum Grav. 32 (2015) 015006
- I. Rácz: *Dynamical determination of the gravitational degrees of freedom*, arXiv:1412.0667 (2015)
- I. Rácz: *Constraints as evolutionary systems*, Class. Quantum Grav. **33** 015014 (2016)

Janus-faced GR:

The arena and the phenomena :

All the pre-GR physical theories provide a distinction between the **arena** in which physical phenomena take place and the **phenomena** themselves.

	arena:	phenomena:
classical mechanics	phase space: δ_{ab}	dynamical trajectories
electrodynamics	Minkowski spacetime: η_{ab}	evolution of F_{ab}
general relativity	curved spacetime: g_{ab}	evolution of g_{ab}

Such a clear distinction between the arena and the phenomenon is simply not available in general relativity

the metric plays both roles.

- GR is more than merely a field theoretic description of gravity.
  It is a certain body of universal rules:
  - modeling the space of events by a four-dimensional differentiable manifold
  - the use of tensor fields and tensor equations to describe physical phenomena
  - use of the (otherwise dynamical) metric in measuring of distances, areas, volumes, angles ...

The degrees of freedom:

What are the degrees of freedom?

- in a theory possessing an initial value formulation: "degrees of freedom" is a synonym of "how many" distinct solutions to the basic equations exist
- (for historical reasons) in ordinary particle mechanics: the number of degrees of freedom is equal to $\dim(Q)$: the number of quantities that must be specified as "initial data" divided by two

The degrees of freedom in linearized GR:

Einstein (1916, 1918): the field equations involve **two degrees of freedom per spacetime point** when studying linearized theory

Is the full nonlinear theory characterized by two degrees of freedom?

Darmois (1923): probably the earliest **answer in the confirmatory** while investigating the Cauchy problem (the initial value problem) in GR

The degrees of freedom in GR:

What are the main issues?

- "... **no way singles** out precisely which functions (i.e., which of the 12 metric or extrinsic curvature components or functions of them) can be freely specified, which functions are determined by the constraints, and which functions correspond to gauge transformations. Indeed, **one of the major obstacles to developing a quantum theory of gravity** is the inability to single out the physical degrees of freedom of the theory. "

  R.M. Wald: General Relativity, Univ. Chicago Press, (1984)

- How these two degrees of freedom may be expressed in terms of the components of the metric tensor and its derivatives (or such combinations of these as, e.g. the Riemann tensor)?
  - Notably, there may be **a lot of possible representations**.
- accordingly, the main issue is **not to find the only legitimate quantities** representing the gravitational degrees of freedom
- rather, **finding a particularly convenient embodiment** of this information which could be handy in solving various problems of physical interest

The outline of the current and the next lectures:

1. **Euclidean and Lorentzian signature** Einsteinian spaces of $\mathbf{n+1}$-dimension $(\mathbf{n \geq 3})$, satisfying some mild topological assumptions, had been considered.
   - smoothly foliated by a one-parameter family of homologous codimension-one surfaces
   - one of these level surfaces is smoothly foliated by a one-parameter family of homologous codimension-two surfaces

     $\Longrightarrow$
   - smoothly foliated by a **two-parameter family of codimension-two-surfaces**:
2. before making an attempt to solve the **Cauchy problem** we better **explore relations** of various subsets of the basic field equations
   - the **Bianchi identity** can be used to do so
3. introduction of a new method: **fully constrained evolution**
4. a **mixed initial value problem** for the Einstein's equations
5. it will also be demonstrated that the "**conformal structure**" provides a **convenient embodiment** of the degrees of freedom of GR

Assumptions:

- **The primary space:** (M, g_{ab})
  - M : $n+1$-dim. $(n \geq 3)$, smooth, paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian$_{(-,+,\dots,+)}$ or Riemannian$_{(+,\dots,+)}$ metric
- **Einsteinian space:** Einstein's equation restricting the geometry

$$\boxed{G_{ab} - \mathscr{G}_{ab} = 0}$$

with source term $\mathscr{G}_{ab}$ having a vanishing divergence, $\nabla^a \mathscr{G}_{ab} = 0$.

  - or, in a more conventionally looking setup

$$\boxed{[R_{ab} - \tfrac{1}{2}\, g_{ab}\, R] + \Lambda\, g_{ab} = 8\pi\, T_{ab}}$$

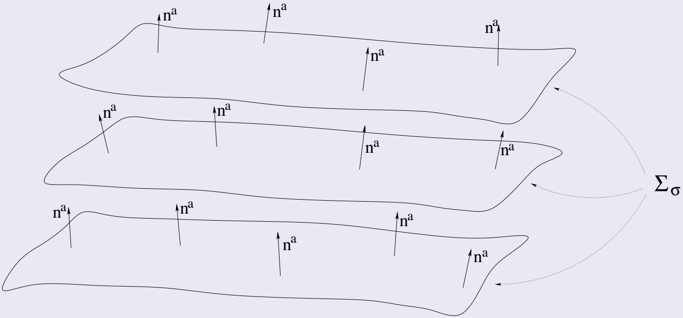
  - with matter fields satisfying their individual field equations with energy-momentum tensor T_{ab} and with cosmological constant Λ

$$\boxed{\mathscr{G}_{ab} = 8\pi\, T_{ab} - \Lambda\, g_{ab}}$$

The primary $n+1$ splitting:

No restriction on the topology by Einstein's equations! (local PDEs)

- **Assume:** M is foliated by a one-parameter family of homologous hypersurfaces, i.e. $M \simeq \mathbb{R} \times \Sigma$, for some codimension-one manifold Σ.
  - known to **hold for globally hyperbolic spacetimes** (Lorentzian case)
  - **equivalent to** the existence of a smooth function $\sigma : M \to \mathbb{R}$ with non-vanishing gradient $\nabla_a \sigma$ such that the $\sigma = const$ level surfaces $\Sigma_\sigma = \{\sigma\} \times \Sigma$ comprise the one-parameter foliation of M.



The main creatures:

- n^a the 'unit norm' vector field that is normal to the Σ_σ level surfaces

$$\boxed{n^a n_a = \epsilon}$$

  - ϵ takes the value -1 or $+1$ for Lorentzian or Riemannian metric g_{ab}, resp.

- **the projection operator**

$$\boxed{h_a{}^b = \delta_a{}^b - \epsilon\, n_a n^b}$$

- **the metric induced**

$$\boxed{h_{ab} = h_a{}^e h_b{}^f\, g_{ef} = g_{ab} - \epsilon\, n_a n_b}$$

- the covariant derivative operator $\boxed{D_a}$ associated with h_{ab}: $\forall\ \omega_b$ on Σ

$$\boxed{D_a \omega_b := h_a{}^d h_b{}^e \nabla_d\, \omega_e}$$

- **the extrinsic curvature** on Σ (symmetric!)

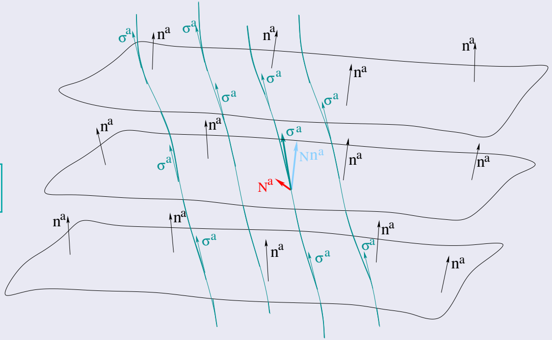
$$\boxed{K_{ab} = h^e{}_a \nabla_e n_b = \tfrac{1}{2}\, \mathscr{L}_n h_{ab}}$$

Assumptions on the topology of the ambient manifold:

FLOW or "time evolution vector field":

- σ^a is a flow or "time evolution vector field": the integral curves of σ^a meet the $\sigma = const$ level surfaces precisely once & it is scaled: $\boxed{\sigma^e \nabla_e \sigma = 1}$

$$\boxed{\sigma^a = \sigma^a_\perp + \sigma^a_\parallel = N\, n^a + N^a}$$



- where N and N^a denotes the 'lapse' and 'shift' of σ^a:

$$\boxed{N = \epsilon\,(\sigma^e n_e) \quad \text{and} \quad N^a = h^a{}_e\, \sigma^e}$$

Decompositions of various fields:

Any symmetric tensor field P_{ab} on M can be decomposed

in terms of n^a and fields living on the $\sigma = const$ level surfaces as

$$\boxed{P_{ab} = \boldsymbol{\pi}\, n_a n_b + [n_a\, \mathbf{p}_b + n_b\, \mathbf{p}_a] + \mathbf{P}_{ab}}$$

where $\boxed{\boldsymbol{\pi} = n^e n^f\, P_{ef}\,, \quad \mathbf{p}_a = \epsilon\, h^e{}_a n^f\, P_{ef}\,, \quad \mathbf{P}_{ab} = h^e{}_a h^f{}_b\, P_{ef}}$

It is also rewarding to inspect the decomposition of the contraction $\nabla^a P_{ab}$:

$$\begin{aligned} \epsilon\,(\nabla^a P_{ae})\, n^e &= \mathscr{L}_n \boldsymbol{\pi} + D^e \mathbf{p}_e + [\boldsymbol{\pi}\,(K^e{}_e) - \epsilon\, \mathbf{P}_{ef} K^{ef} - 2\,\epsilon\, \dot{n}^e \mathbf{p}_e] \\ (\nabla^a P_{ae})\, h^e{}_b &= \mathscr{L}_n \mathbf{p}_b + D^e \mathbf{P}_{eb} + [(K^e{}_e)\, \mathbf{p}_b + \dot{n}_b\, \boldsymbol{\pi} - \epsilon\, \dot{n}^e \mathbf{P}_{eb}] \end{aligned}$$

$\boxed{K_{ab} = h^e{}_a \nabla_e n_b = \tfrac{1}{2}\, \mathscr{L}_n h_{ab}}$ $\quad$ $\boxed{\dot{n}_a := n^e \nabla_e n_a = -\epsilon\, D_a \ln N}$

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Decompositions of various fields:

Examples:

- the metric

$$\boxed{g_{ab} = \epsilon\, n_a n_b + h_{ab}}$$

- the "source term"

$$\boxed{\mathscr{G}_{ab} = n_a n_b\, \mathfrak{e} + [n_a\, \mathfrak{p}_b + n_b\, \mathfrak{p}_a] + \mathfrak{S}_{ab}}$$

where $\boxed{\mathfrak{e} = n^e n^f\, \mathscr{G}_{ef}\,, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f\, \mathscr{G}_{ef}\,, \quad \mathfrak{S}_{ab} = h^e{}_a h^f{}_b\, \mathscr{G}_{ef}}$

- r.h.s. of Einstein's equation: $E_{ab} = G_{ab} - \mathscr{G}_{ab}$

$$\boxed{E_{ab} = n_a n_b\, E^{(\mathcal{H})} + [n_a\, E^{(\mathcal{M})}_b + n_b\, E^{(\mathcal{M})}_a] + (E^{(\mathcal{EVOL})}_{ab} + h_{ab}\, E^{(\mathcal{H})})}$$

$$\boxed{E^{(\mathcal{H})} = n^e n^f\, E_{ef}\,, \quad E^{(\mathcal{M})}_a = \epsilon\, h^e{}_a n^f\, E_{ef}\,, \quad E^{(\mathcal{EVOL})}_{ab} = h^e{}_a h^f{}_b\, E_{ef} - h_{ab}\, E^{(\mathcal{H})}}$$

The decomposition of the covariant divergence $\nabla^a E_{ab} = 0$ of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$:

$$\mathscr{L}_n E^{(\mathcal{H})} + D^e E^{(\mathcal{M})}_e + \big[E^{(\mathcal{H})} (K^e{}_e) - 2\,\epsilon\,(\dot{n}^e E^{(\mathcal{M})}_e) - \epsilon\, K^{ae}\,(E^{(\mathcal{EVOL})}_{ae} + h_{ae} E^{(\mathcal{H})}) \big] = 0$$

Div

$$\mathscr{L}_n E^{(\mathcal{M})}_b + D^a (E^{(\mathcal{EVOL})}_{ab} + h_{ab} E^{(\mathcal{H})}) + \big[(K^e{}_e) E^{(\mathcal{M})}_b + E^{(\mathcal{H})} \dot{n}_b - \epsilon\,(E^{(\mathcal{EVOL})}_{ab} + h_{ab} E^{(\mathcal{H})})\,\dot{n}^a \big] = 0$$

Theorem

Let (M, g_{ab}) be as specified above and assume that the metric h_{ab} induced on the $\sigma = const$ level surfaces is Riemannian. Then, **regardless whether g_{ab} is of Lorentzian or Euclidean signature**, *any solution to the reduced equations $E^{(\mathcal{EVOL})}_{ab} = 0$ is also a solution to the full set of field equations $G_{ab} - \mathscr{G}_{ab} = 0$ provided that the constraint expressions $E^{(\mathcal{H})}$ and $E^{(\mathcal{M})}_a$ vanish on one of the $\sigma = const$ level surfaces.*

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- no gauge condition was used anywhere in the above analyze !
  - it applies regardless of the choice of the foliation, $\boxed{\Sigma_\sigma}$, of M and for any choice of the evolution vector field, $\boxed{\sigma^a \ (N, N^a)}$.

Relations between various parts of the basic equations:

$$\mathscr{L}_n \, E^{(\mathcal{H})} + D^e E^{(\mathcal{M})}_e + [\, E^{(\mathcal{H})} \, (K^e{}_e) - 2\, \epsilon \, (\dot{n}^e \, E^{(\mathcal{M})}_e) - \epsilon \, K^{ae} \, (E^{(\mathcal{EVOL})}_{ae} + h_{ae} \, E^{(\mathcal{H})})\,] = 0$$

Div

$$\mathscr{L}_n \, E^{(\mathcal{M})}_b + D^a (E^{(\mathcal{EVOL})}_{ab} + h_{ab} \, E^{(\mathcal{H})}) + [\,(K^e{}_e) \, E^{(\mathcal{M})}_b + E^{(\mathcal{H})} \, \dot{n}_b - \epsilon \, (E^{(\mathcal{EVOL})}_{ab} + h_{ab} \, E^{(\mathcal{H})}) \, \dot{n}^a \,] = 0$$

Corollary

If the constraint expressions $E^{(\mathcal{H})}$ and $E^{(\mathcal{M})}_a$ vanish on the $\sigma = const$ level surfaces then the relations

$$\begin{aligned} K^{ab} \, E^{(\mathcal{EVOL})}_{ab} &= 0 \\ D^a E^{(\mathcal{EVOL})}_{ab} - \epsilon \, \dot{n}^a \, E^{(\mathcal{EVOL})}_{ab} &= 0 \end{aligned}$$

hold for the evolutionary expression $E^{(\mathcal{EVOL})}_{ab}$.

$$h^e{}_a h^f{}_b \, E_{ef} = E^{(\mathcal{EVOL})}_{ab} + \cancel{h_{ab} \, E^{(\mathcal{H})}}$$

The explicit forms:

Expressions in the $n+1$ decomposition:

$$
\begin{aligned}
E^{(\mathcal{H})} &= n^e n^f E_{ef} = \tfrac{1}{2}\,\{-\epsilon\,{}^{(n)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} \\
E^{(\mathcal{M})}_a &= \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\,[\, D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a\,] \\
\widetilde{E}^{(\mathcal{EVOL})}_{ab} &= {}^{(n)}\!R_{ab} + \epsilon\,\{-\mathscr{L}_n K_{ab} - (K^e{}_e)K_{ab} + 2\,K_{ae}K^e{}_b - \epsilon\,N^{-1}D_aD_bN\} \\
&\quad + \tfrac{1+\epsilon}{(n-1)}\,h_{ab}\,E^{(\mathcal{H})} - \Big(\mathfrak{S}_{ab} - \tfrac{1}{n-1}\,h_{ab}\,[\mathfrak{S}_{ef}\,h^{ef} + \epsilon\,\mathfrak{e}]\Big)
\end{aligned}
$$

where

$$
\mathfrak{e} = n^e n^f \mathscr{G}_{ef}, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f \mathscr{G}_{ef}, \quad \mathfrak{S}_{ab} = h^e{}_a h^f{}_b\,\mathscr{G}_{ef}
$$

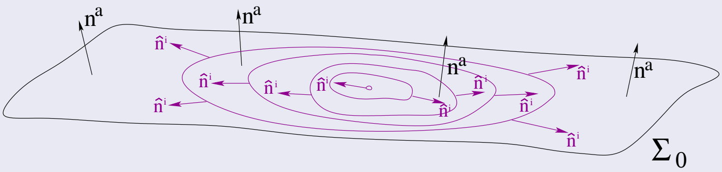
$$
\widetilde{E}^{(\mathcal{EVOL})}_{ab} = E^{(\mathcal{EVOL})}_{ab} - \tfrac{1}{n-1}\,h_{ab}\left(E^{(\mathcal{EVOL})}_{ef}\,h^{ef}\right)
$$

$$
\widetilde{E}^{(\mathcal{EVOL})}_{ab} = 0 \quad \Longleftrightarrow \quad E^{(\mathcal{EVOL})}_{ab} = 0
$$

The secondary $[n-1]+1$ splitting:

Assume that on one of the $\sigma = const$ level surfaces—say on Σ_0, for some $\sigma = \sigma_0\,(\in \mathbb{R})$,—there exists a smooth function $\rho : \Sigma_0 \to \mathbb{R}$, with (a.e.—almost everywhere) non-vanishing gradient such that:

- the $\rho = const$ level surfaces $\mathscr{S}_\rho$ provide a one-parameter foliation of Σ_0



- The metric h_{ij} on Σ_0 can be decomposed as

$$\boxed{h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i \widehat{n}_j}$$

in terms of the positive definite metric $\widehat{\gamma}_{ij}$, induced on the $\mathscr{S}_\rho$ hypersurfaces,

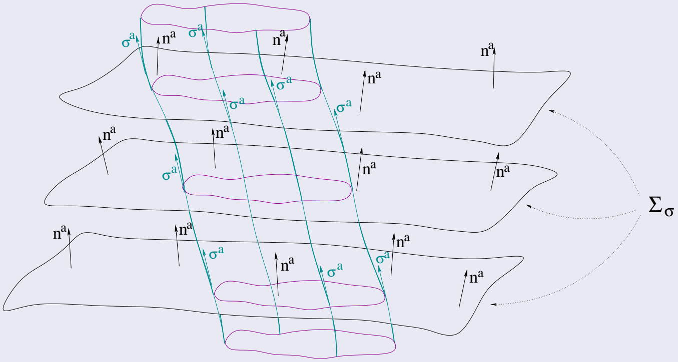
- and the unit norm field

$$\boxed{\widehat{n}^i = \widehat{N}^{-1}\,[\,\rho^i - \widehat{N}^i\,]}$$

normal to the $\mathscr{S}_\rho$ hypersurfaces on Σ_0, where $\widehat{N}$ and $\widehat{N}^i$ denotes the 'laps' and 'shift' of a **flow** (or 'evolution') vector field ρ^i on Σ_0.

The two-parameter foliations:

The Lie drag this foliation of Σ_0 along the integral curves of the vector field σ^a yields then a two-parameter foliation $\mathscr{S}_{\sigma,\rho}$:



- the fields $\widehat{n}^i$, $\widehat{\gamma}_{ij}$ and the projection $\boxed{\widehat{\gamma}^k{}_l = h^k{}_l - \widehat{n}^k \widehat{n}_l}$, to the codimension-two surfaces $\mathscr{S}_{\sigma,\rho}$, get to be well-defined on each of the individual $\sigma = const$ hypersurfaces

The equations on the $\sigma = const$ hypersurfaces:

Some important relations we learned while studying the kinematical background:

using

$$\boxed{h_b{}^e h_d{}^f R_{ef} = {}^{(n)}\!R_{bd} + \epsilon \left\{-\mathscr{L}_n K_{bd} - K_{bd}K_e{}^e + 2K_b{}^e K_{de} - \epsilon\, N^{-1} D_b D_d N\right\}}$$

$$\boxed{R = {}^{(n)}\!R + \epsilon \left\{-2\,\mathscr{L}_n(K_{bd}h^{bd}) - (K_e{}^e)^2 - K_{ef}K^{ef} - 2\,\epsilon\, N^{-1} D^e D_e N\right\}}$$

one gets

$$\boxed{h_b{}^e h_d{}^f E_{ef}} = h_b{}^e h_d{}^f \left\{\left[\,R_{ef} - \tfrac{1}{2}\,g_{ef}R\right] - \mathscr{G}_{bd}\right\} = h_b{}^e h_d{}^f \left\{\left[\,R_{ef} - \tfrac{1}{2}\,h_{ef}R\right] - \mathscr{G}_{bd}\right\}$$
$$= \left[\,{}^{(n)}\!R_{bd} - \tfrac{1}{2}\,h_{ef}\,{}^{(n)}\!R\right] - {}^{(n)}\mathscr{G}_{bd} = {}^{(n)}G_{bd} - {}^{(n)}\mathscr{G}_{bd} = \boxed{{}^{(n)}E_{bd}}$$

where

$${}^{(n)}\mathscr{G}_{ab} = \mathfrak{S}_{ab} - \epsilon \Big\{ - \mathscr{L}_n K_{ab} - (K^e{}_e)K_{ab} + 2\,K_{ae}K^e{}_b - \epsilon\, N^{-1}\, D_a D_b N$$
$$+ h_{ab}\left[\mathscr{L}_n(K^e{}_e) + \tfrac{1}{2}\,(K^e{}_e)^2 + \tfrac{1}{2}\,K_{ef}K^{ef} + \epsilon\, N^{-1}\, D^e D_e N\,\right]\Big\}$$

The explicit forms:

Expressions in the $[n-1]+1$ decomposition:

$$\boxed{{}^{(n)}\!E_{ij} = \widehat{E}^{(\mathcal{H})}\,\widehat{n}_i\widehat{n}_j + [\widehat{n}_i\widehat{E}^{(\mathcal{M})}_j + \widehat{n}_j\widehat{E}^{(\mathcal{M})}_i] + (\widehat{E}^{(\mathcal{EVOL})}_{ij} + \widehat{\gamma}_{ij}\widehat{E}^{(\mathcal{H})})}$$

$$\boxed{\widehat{E}^{(\mathcal{H})} = \widehat{n}^e\widehat{n}^f\,{}^{(n)}\!E_{ef}\,,\quad \widehat{E}^{(\mathcal{M})}_i = \widehat{\gamma}^e{}_j\widehat{n}^f\,{}^{(n)}\!E_{ef}\,,\quad \widehat{E}^{(\mathcal{EVOL})}_{ij} = \widehat{\gamma}^e{}_i\widehat{\gamma}^f{}_j\,{}^{(n)}\!E_{ef} - \widehat{\gamma}_{ij}\widehat{E}^{(\mathcal{H})}}$$

$$\begin{aligned}
\widehat{E}^{(\mathcal{H})} =&\ \tfrac{1}{2}\,\{-\widehat{R} + (\widehat{K}^l{}_l)^2 - \widehat{K}_{kl}\widehat{K}^{kl} - 2\,\widehat{\mathfrak{e}}\}\,,\\
\widehat{E}^{(\mathcal{M})}_i =&\ \widehat{D}^l\widehat{K}_{li} - \widehat{D}_i\widehat{K}^l{}_l - \widehat{\mathfrak{p}}_i\,,\\
\widehat{E}^{(\mathcal{EVOL})}_{ij} =&\ \widehat{R}_{ij} - \mathscr{L}_{\widehat{n}}\widehat{K}_{ij} - (\widehat{K}^l{}_l)\widehat{K}_{ij} + 2\,\widehat{K}_{il}\widehat{K}^l{}_j - \widehat{N}^{-1}\,\widehat{D}_i\widehat{D}_j\widehat{N}\\
&+ \widehat{\gamma}_{ij}\{\mathscr{L}_{\widehat{n}}\widehat{K}^l{}_l + \widehat{K}_{kl}\widehat{K}^{kl} + \widehat{N}^{-1}\widehat{D}^l\widehat{D}_l\widehat{N}\} - [\widehat{\mathfrak{S}}_{ij} - \widehat{\mathfrak{e}}\,\widehat{\gamma}_{ij}]
\end{aligned}$$

where $\widehat{D}_i, \widehat{R}_{ij}$ and $\widehat{R}$ denote ....

$$\boxed{\widehat{\mathfrak{e}} = \widehat{n}^k\widehat{n}^l\,{}^{(n)}\!\mathscr{G}_{kl}\,,\quad \widehat{\mathfrak{p}}_i = \widehat{\gamma}^k{}_i\,\widehat{n}^l\,{}^{(n)}\!\mathscr{G}_{kl}\quad \text{and}\quad \widehat{\mathfrak{S}}_{ij} = \widehat{\gamma}^k{}_i\widehat{\gamma}^l{}_j\,{}^{(n)}\!\mathscr{G}_{kl}}$$

and

$$\boxed{\widehat{K}_{ij} = \widehat{\gamma}^l{}_i\,D_l\,\widehat{n}_j = \tfrac{1}{2}\,\mathscr{L}_{\widehat{n}}\widehat{\gamma}_{ij}}$$

Relations between various parts of the basic equations I.:

Substituting the $[n-1]+1$ splitting of ${}^{(n)}E_{ij}$:

$$\boxed{\begin{aligned} K^{ab}\,{}^{(n)}E_{ab} &= 0 \\ D^a[{}^{(n)}E_{ab}] - \epsilon\,\dot n^a\,{}^{(n)}E_{ab} &= 0 \end{aligned}}$$

as

$$\boxed{{}^{(n)}E_{ab} = h^e{}_a h^f{}_b\,E_{ef} = E^{(\mathcal{EVOL})}_{ab} + \cancel{h_{ab}\,E^{(\mathcal{H})}}}$$

$$K^{ab\,(n)}E_{ab} = \boldsymbol{\kappa}\,\widehat{E}^{(\mathcal{H})} + 2\,\mathbf{k}^e\,\widehat{E}^{(\mathcal{M})}_e + \mathbf{K}^{ef}\,\cancel{\widehat{E}^{(\mathcal{EVOL})}_{ef}} + (\mathbf{K}^e{}_e)\,\widehat{E}^{(\mathcal{H})}$$

$$\dot n^{a\,(n)}E_{ab} = [(\widehat{n}_a\dot n^a)\widehat{E}^{(\mathcal{H})} + (\dot n^a\,\widehat{E}^{(\mathcal{M})}_a)]\widehat{n}_b + (\widehat{n}_a\dot n^a)\widehat{E}^{(\mathcal{M})}_b + \dot n^a\,[\cancel{\widehat{E}^{(\mathcal{EVOL})}_{ab}} + \widehat{\gamma}_{ab}\,\widehat{E}^{(\mathcal{H})}]$$

$$\widehat{n}^e D^a[{}^{(n)}E_{ae}] = \mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{H})} + \widehat{D}^e\widehat{E}^{(\mathcal{M})}_e + (\widehat{K}^e{}_e)\,\widehat{E}^{(\mathcal{H})} - [\cancel{\widehat{E}^{(\mathcal{EVOL})}_{ef}} + \widehat{\gamma}_{ef}\,\widehat{E}^{(\mathcal{H})}]\,\widehat{K}^{ef} - 2\,\dot{\widehat{n}}^e\widehat{E}^{(\mathcal{M})}_e$$

$$\widehat{\gamma}^e{}_b D^a[{}^{(n)}E_{ae}] = \mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{M})}_b + \widehat{D}^e[\cancel{\widehat{E}^{(\mathcal{EVOL})}_{eb}} + \widehat{\gamma}_{eb}\,\widehat{E}^{(\mathcal{H})}] + (\widehat{K}^e{}_e)\,\widehat{E}^{(\mathcal{M})}_b - \dot{\widehat{n}}^e\,\cancel{\widehat{E}^{(\mathcal{EVOL})}_{eb}}$$

$$\boxed{\begin{aligned} \mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{H})} + \widehat{\gamma}^{ef}\,\widehat{D}_e\widehat{E}^{(\mathcal{M})}_f &= \widehat{\mathscr{E}} \\ \mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{M})}_b + \widehat{D}_b\widehat{E}^{(\mathcal{H})} &= \widehat{\mathscr{E}}_b \end{aligned}}$$

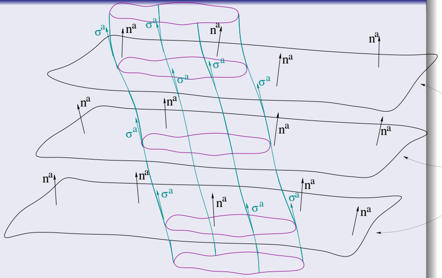
$\Longrightarrow$ **IF** $\widehat{E}^{(\mathcal{EVOL})}_{ef} = 0$ holds: a **linear and homogeneous FOSH** for $(\widehat{E}^{(\mathcal{H})}, \widehat{E}^{(\mathcal{M})}_i)^T$

What do the above observations imply?

Theorem (I.)

- *Assume that the primary constraint expressions* $E^{(\mathcal{H})}$ *and* $E^{(\mathcal{M})}_a$ *vanish on the* $\sigma = const$ *level surfaces, also that*
- *the secondary constraint expressions* $\widehat{E}^{(\mathcal{H})}$ *and* $\widehat{E}^{(\mathcal{M})}_a$ *vanish along the hypersurface yielded by the Lie dragging,* $\mathscr{W}_{\rho_0} = \Phi_\sigma[\mathscr{S}_{\rho_0}]$*, of one of the level surfaces* $\mathscr{S}_{\rho_0}$ *foliating* Σ_0*.*
- $\Longrightarrow$

  Then, to get solutions to the full set of Einstein's equations $G_{ab} - \mathscr{G}_{ab} = 0$ *it suffices*—**regardless whether the primary metric g_{ab} is Riemannian or Lorentzian**—*to solve, in addition, only the secondary reduced equations* $\widehat{E}^{(\mathcal{EVOL})}_{ij} = 0$*.*



Remark (i).: *the Lie dragging is done by using the one-parameter group of diffeomorphisms,* Φ_σ*, associated by the "time evolution vector field"* σ^a *— could be only a world-line*

Remark (ii): *if one wants to setup an initial-boundary value problem on either side of the hypersurface* $\mathscr{W}_{\rho_0}$ *the previous theorem provides a clear mean to identify the geometrical freedom we have on* $\mathscr{W}_{\rho_0}$

Relations between various parts of the basic equations II.:

$$\boxed{\begin{aligned} K^{ab}\,{}^{(n)}\!E_{ab} &= 0 \\ D^a[{}^{(n)}\!E_{ab}] - \epsilon\,\dot n^a\,{}^{(n)}\!E_{ab} &= 0 \end{aligned}}$$

$$K^{ab\,(n)}\!E_{ab} = \boldsymbol{\kappa}\,\widehat{E}^{(\mathcal{H})} + 2\,\mathbf{k}^e\,\widehat{E}^{(\mathcal{M})}_e + \mathbf{K}^{ef}\,\widehat{E}^{(\mathcal{EVOL})}_{ef} + (\mathbf{K}^e{}_e)\,\widehat{E}^{(\mathcal{H})}$$

$$\dot n^{a\,(n)}\!E_{ab} = [(\widehat{n}_a\dot n^a)\widehat{E}^{(\mathcal{H})} + (\dot n^a\,\widehat{E}^{(\mathcal{M})}_a)]\widehat{n}_b + (\widehat{n}_a\dot n^a)\widehat{E}^{(\mathcal{M})}_b + \dot n^a\,[\widehat{E}^{(\mathcal{EVOL})}_{ab} + \widehat{\gamma}_{ab}\,\widehat{E}^{(\mathcal{H})}]$$

$$\widehat{n}^e D^a[{}^{(n)}\!E_{ae}] = \mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{H})} + \widehat{D}^e\widehat{E}^{(\mathcal{M})}_e + (\widehat{K}^e{}_e)\,\widehat{E}^{(\mathcal{H})} - [\widehat{E}^{(\mathcal{EVOL})}_{ef} + \widehat{\gamma}_{ef}\,\widehat{E}^{(\mathcal{H})}]\,\widehat{K}^{ef} - 2\,\dot{\widehat{n}}^e\widehat{E}^{(\mathcal{M})}_e$$

$$\widehat{\gamma}^e{}_b D^a[{}^{(n)}\!E_{ae}] = \mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{M})}_b + \widehat{D}^e[\widehat{E}^{(\mathcal{EVOL})}_{eb} + \widehat{\gamma}_{eb}\,\widehat{E}^{(\mathcal{H})}] + (\widehat{K}^e{}_e)\,\widehat{E}^{(\mathcal{M})}_b - \dot{\widehat{n}}^e\,\widehat{E}^{(\mathcal{EVOL})}_{eb}$$

$\Longrightarrow$ if the trace free part of $\widehat{E}^{(\mathcal{EVOL})}_{ef}$ vanishes:

$$\boxed{\overset{\circ}{\widehat{E}}{}^{(\mathcal{EVOL})}_{ef} = \widehat{E}^{(\mathcal{EVOL})}_{ef} - \tfrac{1}{n-1}\,\widehat{\gamma}_{ef}\,\Big[\widehat{\gamma}^{kl}\,\widehat{E}^{(\mathcal{EVOL})}_{kl}\Big] = 0}$$

$$\boxed{\begin{aligned} &K^{ab}E^{(\mathcal{EVOL})}_{ab} = (\boldsymbol{\kappa} + \mathbf{K}^e{}_e)\,\widehat{E}^{(\mathcal{H})} + 2\,\mathbf{k}^e\,\widehat{E}^{(\mathcal{M})}_e + \tfrac{1}{n-1}\,(\mathbf{K}^e{}_e)\,(\widehat{\gamma}^{kl}\,\widehat{E}^{(\mathcal{EVOL})}_{kl}) = 0 \\ &\mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{H})} + \widehat{\gamma}^{ef}\,\widehat{D}_e\widehat{E}^{(\mathcal{M})}_f = \widehat{\mathscr{E}} \\ &\mathscr{L}_{\widehat{n}}\,\widehat{E}^{(\mathcal{M})}_b - (\mathbf{K}^e{}_e)^{-1}\,[\,\boldsymbol{\kappa}\,\widehat{D}_b\widehat{E}^{(\mathcal{H})} + 2\,\mathbf{k}^e\,\widehat{D}_b\widehat{E}^{(\mathcal{M})}_e] = \widehat{\mathscr{E}}_b \end{aligned}}$$

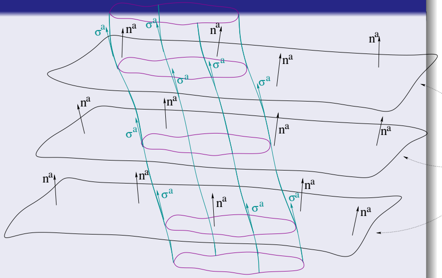
and $\boxed{\boldsymbol{\kappa}\cdot\mathbf{K}^e{}_e < 0 \quad !!!}$ it is a **linear and homogeneous strongly hyperbolic system**

What is the meaning?

Theorem (II.)

- *Assume that the primary constraint expressions* $E^{(\mathcal{H})}$ *and* $E_a^{(\mathcal{M})}$ *vanish on the* $\sigma = const$ *level surfaces, also that*
- $\boldsymbol{\kappa} \cdot \mathbf{K}^e{}_e < 0$ *(on all* $\mathscr{S}_{\sigma,\rho}$*) and the secondary constraint expressions* $\widehat{E}^{(\mathcal{H})}$ *and* $\widehat{E}_a^{(\mathcal{M})}$ *vanish along the Lie dragging*$^{(*)}$, $\mathscr{W}_{\rho_0} = \Phi_\sigma[\mathscr{S}_{\rho_0}]$, *of one of the level surfaces* $\mathscr{S}_{\rho_0}$ *foliating* Σ_0.

$^{(*)}$ *w.r.t. the one-parameter group of diffeomorphisms,* Φ_σ, *associated by the "time evolution vector field"* σ^a



Remark: *initial-boundary value problem on either side of the hypersurface* $\mathscr{W}_{\rho_0}$ *.....*
— could be only a world-line

- $\Longrightarrow$ *Then, to get solutions to the full set of the primary Einstein's equations* $G_{ab} - \mathscr{G}_{ab} = 0$ *it suffices*—**regardless whether the primary metric** g_{ab} **is Riemannian or Lorentzian**—*to solve, in addition, only the trace free part of the secondary reduced equations*

$$\boxed{\overset{\circ}{\widehat{E}}{}_{ef}^{(\mathcal{EVOL})} = \widehat{E}_{ef}^{(\mathcal{EVOL})} - \frac{1}{n-1}\,\widehat{\gamma}_{ef}\,\left[\widehat{\gamma}^{kl}\,\widehat{E}_{kl}^{(\mathcal{EVOL})}\right] = 0}$$

That is all for now...