On the use of evolutionary methods in metric theories of gravity V.

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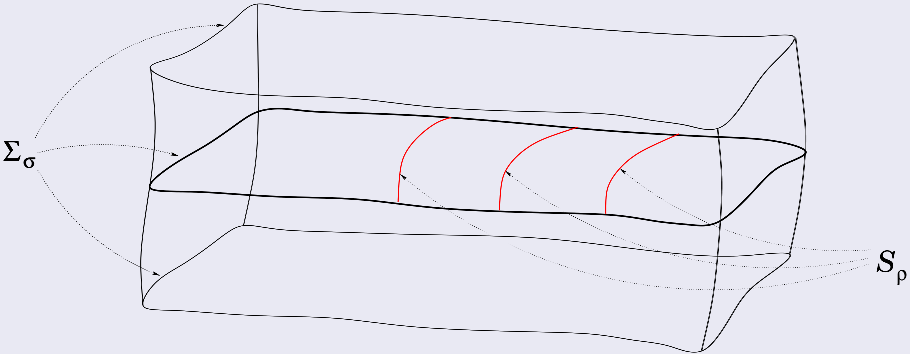
The constraint surface:

- **Considerations had been restricted to the case of foliations by** $\sigma = const$ **hypersurfaces**



- metric of Euclidean signature will be involved
- no gauge condition

  ... arbitrary choice of foliations & "time evolution" vector field

The program for the next two lectures:

Our slogan to remember:

some of the arguments and techniques developed originally and applied so far exclusively only in the Lorentzian case do also apply to Riemannian spaces

Plans:

- **Constraints as evolutionary systems**
  - open any textbook on GR: "the constrains are elliptic PDEs"
  - parabolic-hyperbolic system
    ... global solution to the involved parabolic equation
  - strongly hyperbolic system
    ... study of near Kerr configurations

References:

- I. Rácz: *Constraints as evolutionary systems*, CQG **33** 015014 (2016)
- I. Rácz: *Cauchy problem as a two-surface based 'geometrodynamics'*, CQG **33** 015006 (2015)
- I. Rácz and J. Winicour: *Black hole initial data without elliptic equations*, Phys. Rev. D **91**, 124013 (2015)
- I. Rácz and J. Winicour: *Toward computing gravitational initial data without elliptic solvers*, CQG **35** 135002 (2018)
- I. Rácz: *On the evolutionary form of the constraints in electrodynamics*, arXiv:1811.06873 [gr-qc] (2018)

The spaces:

- **The primary space:** (M, g_{ab})
  - M : $n+1$-dimensional ($n \geq 3$), smooth, paracompact, connected, orientable manifold
  - g_{ab}: smooth Lorentzian$_{(-,+,\ldots,+)}$ or Riemannian$_{(+,\ldots,+)}$ metric
- **Einstein's equations:** restricting the geometry

$$\boxed{G_{ab} - \mathscr{G}_{ab} = 0}$$

with source term $\mathscr{G}_{ab}$ having a vanishing divergence $\nabla^a \mathscr{G}_{ab} = 0$

  - or, in a more conventionally looking setup

$$\boxed{[R_{ab} - \tfrac{1}{2}\, g_{ab}\, R] + \Lambda\, g_{ab} = 8\pi\, T_{ab}}$$

with matter fields satisfying their field equations with energy-momentum tensor T_{ab} and with cosmological constant Λ

  - $$\boxed{\mathscr{G}_{ab} = 8\pi\, T_{ab} - \Lambda\, g_{ab}}$$

The explicit form of the constraints:

- The projections of $E_{ab} = G_{ab} - \mathscr{G}_{ab}$ determine the constraint expressions:

  $\Big[$ for the normals to the $\sigma = conts$ hypersurfaces $\boxed{n^e n_e = \epsilon}\Big]$

$$
\begin{aligned}
E^{(\mathcal{H})} &= n^e n^f E_{ef} = \tfrac{1}{2}\,\{-\epsilon\,{}^{(n)}\!R + (K^e{}_e)^2 - K_{ef}K^{ef} - 2\,\mathfrak{e}\} = 0 \\
E^{(\mathcal{M})}_a &= \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\,[D_e K^e{}_a - D_a K^e{}_e - \epsilon\,\mathfrak{p}_a] = 0
\end{aligned}
$$

- where D_a denotes the covariant derivative operator associated with h_{ab} and

$$\mathfrak{e} = n^e n^f \mathscr{G}_{ef}\,, \quad \mathfrak{p}_a = \epsilon\, h^e{}_a n^f \mathscr{G}_{ef}$$

- it is an underdetermined system: $n+1$ equations for the $n\,(n+1)$ variables

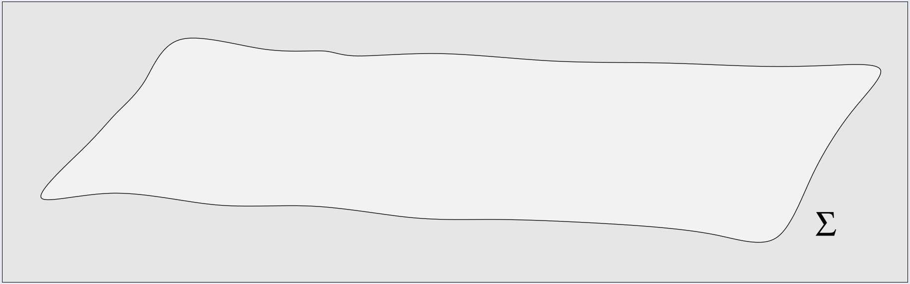
$$(h_{ij}, K_{ij})$$

A simple example:

Consider the underdetermined equation on $\Sigma \approx \mathbb{R}^2$ with coordinates (χ, ξ)

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

- it is an equation for the involved four variables $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ and $\boldsymbol{z}$ on $\Sigma \approx \mathbb{R}^2$
- in advance of solving it three of these variables have to be fixed on Σ

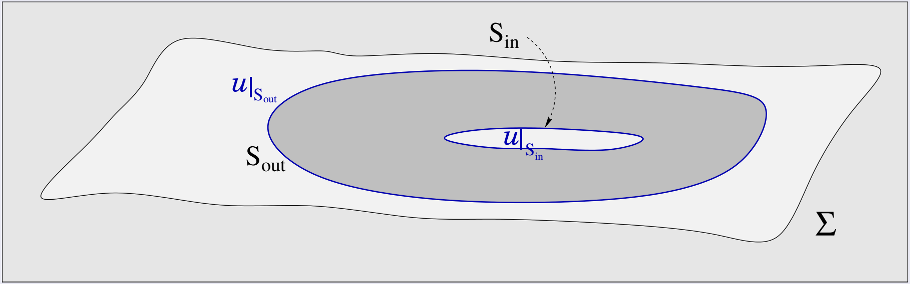


A simple example:

It is an elliptic equation for u on $\mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

- in solving this equation the variables v, w and z have to be specified on $\mathbb{R}^2$
- the variable u has also to be fixed at the boundaries $\mathrm{S_{out}}$ and $\mathrm{S_{in}}$

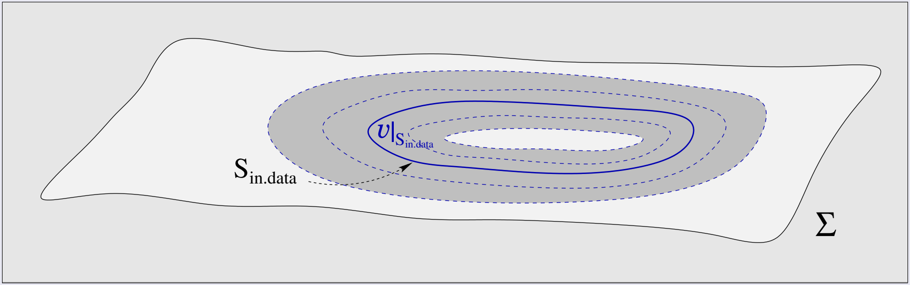


A simple example:

It is a hyperbolic equation for v on $\mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)\boldsymbol{w} + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}, \boldsymbol{w}$ and $\boldsymbol{z}$ have to be specified on $\mathbb{R}^2$
- the variable $\boldsymbol{v}$ has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$

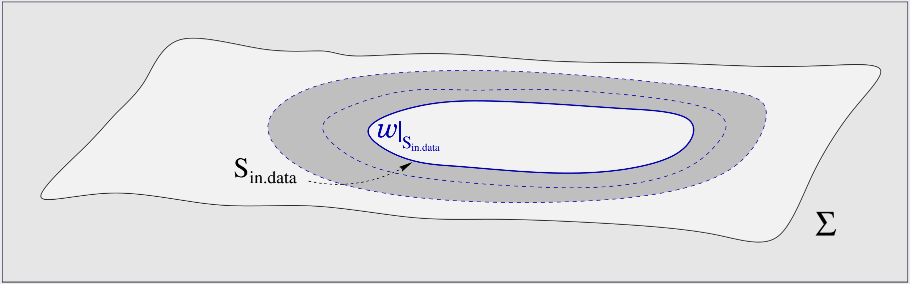


A simple example:

It is a parabolic equation for w on $\mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ have to be fixed on $\mathbb{R}^2$: $a > 0$
- the variable w has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$

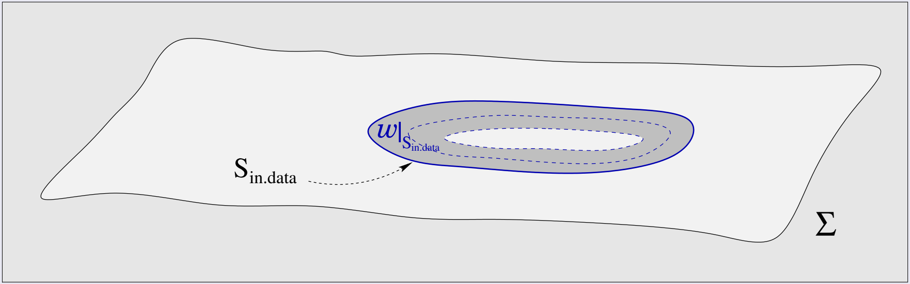


A simple example:

It is a parabolic equation for w on $\mathbb{R}^2$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi - \partial_\xi)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- in solving this equation the variables $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{z}$ have to be fixed on $\mathbb{R}^2$: $\boxed{a < 0}$
- the variable w has also to be fixed at the initial data surface $\mathrm{S}_{\mathrm{in.data}}$



A simple example:

It is an algebraic equation for $\boldsymbol{z}$:

$$(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi^2 - \partial_\xi^2)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w + \boldsymbol{z} = 0$$

- once the variables u, v, w are specified on $\mathbb{R}^2$ the solution is determined as

$$\boldsymbol{z} = -\left[(\partial_\chi^2 + \partial_\xi^2)\boldsymbol{u} + (\partial_\chi^2 - \partial_\xi^2)\boldsymbol{v} + (a\,\partial_\chi - \partial_\xi^2)w\right]$$

The conformal (elliptic) method:

Lichnerowicz A (1944) and **York J W (1972)**:

- replace

$$h_{ij} = \phi^{\frac{4}{n-2}}\,\widetilde{h}_{ij} \quad \text{and} \quad K_{ij} - \tfrac{1}{n}\,h_{ij}\,K^l{}_l = \phi^{-2}\,\widetilde{K}_{ij}$$

- using these variables the constraints are put into a **semilinear elliptic system**

$$\widetilde{D}^l\,\widetilde{D}_l\phi + \tfrac{n-2}{4\,(n-1)}\,\epsilon\,\widetilde{R}\,\phi + \tfrac{n-2}{4\,(n-1)}\,\widetilde{K}_{ij}\widetilde{K}^{ij}\,\phi^{\frac{2-3\,n}{n-2}} - \left[\tfrac{n-2}{4\,n}\,(K^l{}_l)^2 - \tfrac{n-2}{2\,(n-1)}\,\mathfrak{e}\right]\phi^{\frac{n+2}{n-2}} = 0$$

where $\widetilde{D}_l$, $\widetilde{R}$, ... $\widetilde{h}_{ij}$

- $\widetilde{K}_{ij} = \widetilde{K}^{[L]}_{ij} + \widetilde{K}^{[TT]}_{ij}$, where $\widetilde{K}^{[L]}_{ij} = \left(\widetilde{D}_iX_j + \widetilde{D}_jX_i - \tfrac{2}{n}\,\widetilde{h}_{ij}\widetilde{D}^l X_l\right)$

$$\widetilde{D}^l\,\widetilde{D}_l X_i + \tfrac{n-2}{n}\,\widetilde{D}_i(\widetilde{D}^l X_l) + \widetilde{R}_i{}^l X_l - \tfrac{n-1}{n}\,\phi^{\frac{2\,n}{n-2}}\,\widetilde{D}_i(K^l{}_l) + \epsilon\,\phi^{\frac{2\,(n+2)}{n-2}}\,\mathfrak{p}_i = 0$$

- $(h_{ij}, K_{ij}) \quad \longleftrightarrow \quad \left(\phi, \widetilde{h}_{ij}; K^l{}_l, X_i, \widetilde{K}^{[TT]}_{ij}\right)$

The conformal method:

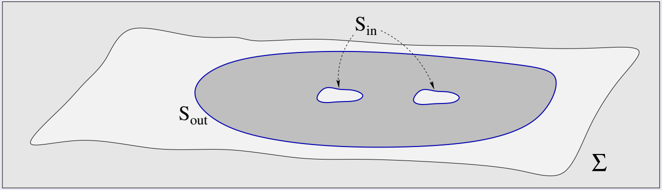
Impressive mathematical developments since 1944 but ...

- either "constancy" of $K^l{}_l$ or "smallness" of the TT part of $\widetilde{K}_{ij}$ is required
- it is highly implicit due to its elliptic character and the replacements

  $h_{ij} = \phi^{\frac{4}{n-2}}\,\widetilde{h}_{ij}$ and $K_{ij} = \frac{1}{n}\,\phi^{\frac{4}{n-2}}\,\widetilde{h}_{ij}\,K^l{}_l + \phi^{-2}\,\widetilde{K}_{ij}$ $\Longrightarrow$

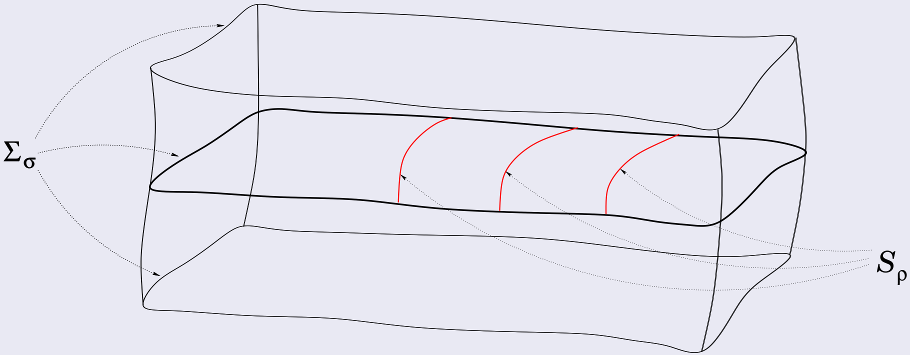
  - no direct control on the physical parameters of the initial data specifications

- **boundary conditions:**
  - are known to influence solutions everywhere in their domains
  - the inner boundary conditions—they are applied with **excision** in the black hole interior—cannot simply be supported by intuition (**trumpet data** ...)
  - Bowen-York type initial data: $\widetilde{h}_{ij}$ is flat $\widetilde{h}_{ij} = \delta_{ij}$ and $K^l{}_l = 0$ ~~Kerr BH~~ non-negligible spurious gravitational wave content of yielded time evolutions



The constraint surface:

- **Considerations will be restricted to a specific $\sigma = const$ hypersurface with some foliation**



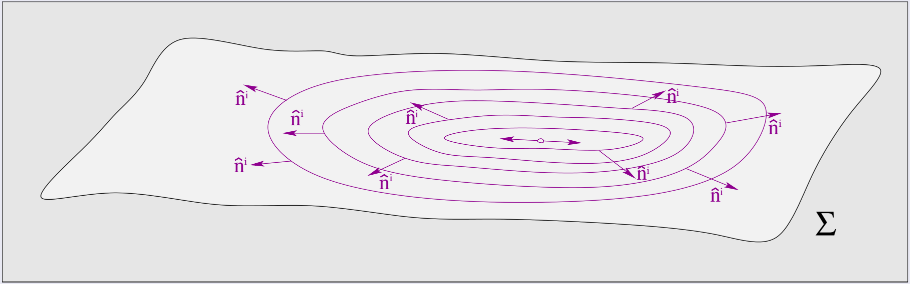
- metric of Euclidean signature will be involved
- no gauge condition

  ... arbitrary choice of foliations $\&$ "time evolution" vector field

The constraints as evolutionary systems:

Restrictions on the topology of Σ:

- Σ can be foliated by the $\rho = const$ level surfaces—by a one-parameter family of homologous codimension-one surfaces $\mathscr{S}_\rho$—such that (apart from possible critical points) the gradient $D_i\rho$ does not vanish on Σ. $\Longrightarrow$ $\boxed{\widehat{n}_i \sim D_i\rho}$



- **Assume** the existence of a smooth Morse function $\rho : \Sigma \to \mathbb{R}$ that possesses only isolated non-degenerate critical points on Σ each with index zero.

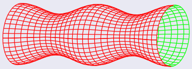
  [The critical points of a Morse function ρ (at which $D_i\rho = 0$) are know to be isolated and non-degenerate in the sense that the Hessian of ρ is non-singular at those points. The index of a critical point is the number of the negative eigenvalues of the Hessian there.]

How restrictive are these conditions?

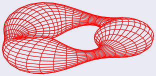
Examples:

- **All the product spaces** of the form $\mathbb{R} \times \mathscr{S}$ and $\mathbb{S}^1 \times \mathscr{S}$ **are allowed**, where the factor $\mathscr{S}$ is a codimension-one manifold in Σ **with arbitrary topology**. This product structure guarantee that the "height function" determined by the factor $\mathbb{R}$ (mod) on Σ will be a Morse function with no critical point.

$R \times S^{n-1}$



$S^1 \times S^{n-1}$



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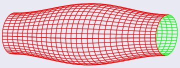
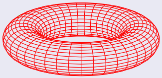


$S^1 \times S^{n-1}$

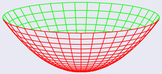


What happens if we close one or two ends of a cylinder?

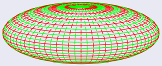
Examples:

- **All the product spaces** of the form $\mathbb{R} \times \mathscr{S}$ and $\mathbb{S}^1 \times \mathscr{S}$ **are allowed**, where the factor $\mathscr{S}$ is a codimension-one manifold in Σ **with arbitrary topology**.
  This product structure guarantee that the "height function" determined by the factor $\mathbb{R}$ (mod) on Σ will be a Morse function with no critical point.
  - $\Sigma = \mathbb{R}^n$
    - with $\mathscr{S} = \mathbb{R}^{n-1}$, or
    - with $\mathscr{S} = \mathbb{S}^{n-1}$: corresponding to the Morse function $\rho = \sum_{i=1}^{n}(x_i)^2$ with zero index at the origin in $\mathbb{R}^n$
  - n-dimensional sphere $\mathbb{S}^n$ foliated by codimension-one spheres $\mathbb{S}^{n-1}$
    - with height function $\rho(x_1, \ldots, x_{n+1}) \mapsto x_{n+1}$, where $\mathbb{S}^n = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} \mid \sum_{i=1}^{n+1}(x_i)^2 = \mathcal{R}^2\}$ and the critical points (with zero index) are the north and south poles are represented by the points located at $(0, \ldots, 0, \mathcal{R})$ and $(0, \ldots, 0, -\mathcal{R})$ in $\mathbb{R}^{n+1}$, respectively.

$R^n = R^+ \times S^{n-1}$



$S^n = [a,b] \times S^{n-1}$



New variables by applying $(n-1)+1$ decompositions:

Splitting of the metric h_{ij}:

assumed: $\Sigma \approx \mathbb{R} \times \mathscr{S}$ a.e. (almost everywhere)

Σ is smoothly foliated by a one-parameter family of codimension-one surfaces $\mathscr{S}_\rho$:
$\rho = const$ level surfaces of a smooth real function $\rho : \Sigma \to \mathbb{R}$ with $\partial_i \rho \neq 0$

$$\Longrightarrow \quad \widehat{n}_i \approx \partial_i \rho \; \ldots \; \& \ldots \; h^{ij} \longrightarrow \widehat{n}^i = h^{ij} \widehat{n}_j \longrightarrow \widehat{\gamma}^i{}_j = \delta^i{}_j - \widehat{n}^i \widehat{n}_j$$

- note that as h_{kl} is Riemannian no ϵ appears in $\widehat{n}^i \widehat{n}_i = 1$
- induced metric on the $\rho = const$ level surfaces

$$\widehat{\gamma}_{ij} = \widehat{\gamma}^k{}_i \, \widehat{\gamma}^l{}_j \, h_{kl}$$

- the metric h_{ij} can then be given as

$$h_{ij} = \widehat{\gamma}_{ij} + \widehat{n}_i \widehat{n}_j \quad \Longleftrightarrow \quad \{\widehat{n}_i, \widehat{\gamma}_{ij}\}$$

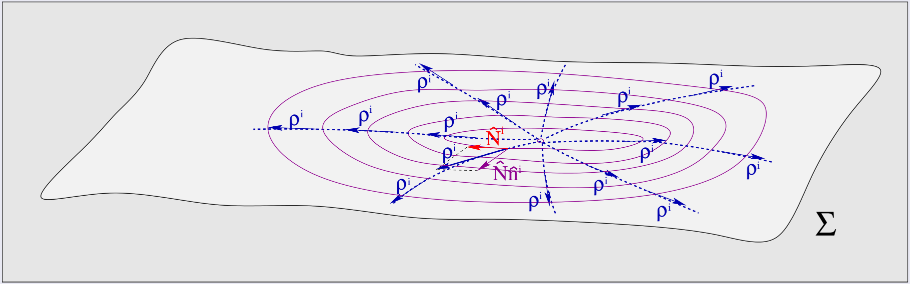
The "time evolution vector field":

The decomposition of the :

A vector field $\boxed{\rho^i}$ on Σ is a **flow**, or a **"time evolution vector field"** on Σ **IF**

- its integral curves intersecting each of the $\mathscr{S}_\rho$ level surfaces precisely once; yielding a $n-1$-parameter family of smooth curves
- and it is scaled such that $\boxed{\rho^j D_j \rho = 1}$
- the '**lapse**' and '**shift**' of $\boxed{\rho^i = (\partial_\rho)^i = \rho^i_\perp + \rho^i_\parallel = \widehat{N}\,\widehat{n}^i + \widehat{N}^i}$ where

$$\boxed{\widehat{N} = \rho^j \widehat{n}_j \text{ and } \widehat{N}^i = \widehat{\gamma}^i{}_j\, \rho^j\,, \quad !!! \quad h^{ij} = \widehat{\gamma}^{ij} + \widehat{n}^i \widehat{n}^j = \widehat{\gamma}^{ij} + \widehat{N}^{-2}(\rho^i - \widehat{N}^i)(\rho^j - \widehat{N}^j)}$$



Decompositions of a symmetric tensor field P_{ij} :

Consider an arbitrary symmetric tensor field P_{ij} defined on Σ :

- using $\widehat{n}^a$ and $\widehat{\gamma}^i{}_j$ it can be decomposed as

$$P_{ij} = \boldsymbol{\pi}\, \widehat{n}_i \widehat{n}_j + [\widehat{n}_i\, \mathbf{p}_j + \widehat{n}_j\, \mathbf{p}_i] + \mathbf{P}_{ij}$$

- where $\boldsymbol{\pi} = \widehat{n}^k \widehat{n}^l\, P_{kl}, \quad \mathbf{p}_i = \widehat{\gamma}^k{}_i \widehat{n}^l\, P_{kl}, \quad \mathbf{P}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j\, P_{kl}$

It is also rewarding to inspect the decomposition of the contraction $D^i P_{ij}$:

$$(D^l P_{lk})\, \widehat{n}^k = \mathscr{L}_{\widehat{n}} \boldsymbol{\pi} + \widehat{D}^l \mathbf{p}_l + [\boldsymbol{\pi}\, (\widehat{K}^l{}_l) - \mathbf{P}_{kl} \widehat{K}^{kl} - 2\, \dot{\widehat{n}}^l \mathbf{p}_l]$$
$$(D^l P_{lk})\, \widehat{\gamma}^k{}_i = \mathscr{L}_{\widehat{n}} \mathbf{p}_i + \widehat{D}^l \mathbf{P}_{li} + [(\widehat{K}^l{}_l)\, \mathbf{p}_i + \dot{\widehat{n}}_i\, \boldsymbol{\pi} - \dot{\widehat{n}}^l \mathbf{P}_{li}]$$

$$(D_k P_l{}^l)\, \widehat{n}^k = \mathscr{L}_{\widehat{n}} \boldsymbol{\pi} + \mathscr{L}_{\widehat{n}} \mathbf{P}_l{}^l$$
$$(D_k P_l{}^l)\, \widehat{\gamma}^k{}_i = \widehat{D}_i \boldsymbol{\pi} + \widehat{D}_i \mathbf{P}_l{}^l$$

, where $\dot{\widehat{n}}_i := \widehat{n}^l D_l \widehat{n}_i = -\widehat{D}_i \ln \widehat{N}$

and

$$\widehat{K}_{ij} = \widehat{\gamma}^l{}_i\, D_l\, \widehat{n}_j = \tfrac{1}{2}\, \mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij} \quad \text{and} \quad \widehat{K}^l{}_l = \widehat{\gamma}^{ij}\, \widehat{K}_{ij} = \tfrac{1}{2}\, \widehat{\gamma}^{ij}\, \mathscr{L}_{\widehat{n}} \widehat{\gamma}_{ij} = D_i\, \widehat{n}^i$$

The new variables:

The momentum constraint:

$$E_a^{(\mathcal{M})} = \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\, [D_e K^e{}_a - D_a K^e{}_e - \epsilon\, \mathfrak{p}_a] = 0$$

The splitting of the extrinsic curvature K_{ij}:

- $$K_{ij} = \boldsymbol{\kappa}\, \widehat{n}_i \widehat{n}_j + [\widehat{n}_i\, \mathbf{k}_j + \widehat{n}_j\, \mathbf{k}_i] + \mathbf{K}_{ij}$$

  where

  $$\boldsymbol{\kappa} = \widehat{n}^k \widehat{n}^l\, K_{kl}\,, \quad \mathbf{k}_i = \widehat{\gamma}^k{}_i \widehat{n}^l\, K_{kl} \quad \text{and} \quad \mathbf{K}_{ij} = \widehat{\gamma}^k{}_i \widehat{\gamma}^l{}_j\, K_{kl}$$

- the **trace** and **trace free** parts of $\mathbf{K}_{ij}$

  $$\mathbf{K}^l{}_l = \widehat{\gamma}^{kl}\, \mathbf{K}_{kl} \quad \text{and} \quad \mathring{\mathbf{K}}_{ij} = \mathbf{K}_{ij} - \tfrac{1}{n-1}\, \widehat{\gamma}_{ij} \mathbf{K}^l{}_l$$

- the independent components of (h_{ij}, K_{ij}) are represented by the variables

  $$(\widehat{N}, \widehat{N}^i, \widehat{\gamma}_{ij}; \boldsymbol{\kappa}, \mathbf{k}_i, \mathbf{K}^l{}_l, \mathring{\mathbf{K}}_{ij})$$

The momentum constraint:

The momentum constraint:

$$E_a^{(\mathcal{M})} = \epsilon\, h^e{}_a n^f E_{ef} = \epsilon\, [D_e K^e{}_a - D_a K^e{}_e - \epsilon\, \mathfrak{p}_a] = 0$$

The decomposition of the contraction $D^i K_{ij}$:

$$(D^l K_{lk})\,\widehat{n}^k = \mathscr{L}_{\widehat{n}} \boldsymbol{\kappa} + \widehat{D}^l \mathbf{k}_l + [\boldsymbol{\kappa}\,(\widehat{K}^l{}_l) - \mathbf{K}_{kl} \widehat{K}^{kl} - 2\,\dot{\widehat{n}}^l \mathbf{k}_l]$$

$$(D^l K_{lk})\,\widehat{\gamma}^k{}_i = \mathscr{L}_{\widehat{n}} \mathbf{k}_i + \widehat{D}^l \mathbf{K}_{li} + [(\widehat{K}^l{}_l)\,\mathbf{k}_i + \dot{\widehat{n}}_i\,\boldsymbol{\kappa} - \dot{\widehat{n}}^l \mathbf{K}_{li}]$$

$$(D_k K_l{}^l)\,\widehat{n}^k = \mathscr{L}_{\widehat{n}} \boldsymbol{\kappa} + \mathscr{L}_{\widehat{n}} \mathbf{K}_l{}^l$$

$$(D_k K_l{}^l)\,\widehat{\gamma}^k{}_i = \widehat{D}_i \boldsymbol{\kappa} + \widehat{D}_i \mathbf{K}_l{}^l$$

The principal parts of the decompositions of $D^l K_{kl} - D_k K^l{}_l$:

as $\mathbf{K}_{ij} = \mathring{\mathbf{K}}_{ij} + \frac{1}{n-1}\,\widehat{\gamma}_{ij} \mathbf{K}^l{}_l$

$$[\,D^l K_{lk} - D_k K_l{}^l\,]\,\widehat{n}^k = -\,\mathscr{L}_{\widehat{n}}\,\mathbf{K}\,_l{}^l + \widehat{D}^l \mathbf{k}_l + (l.o.t.)$$

$$[\,D^l K_{lk} - D_k K_l{}^l\,]\,\widehat{\gamma}^k{}_i = \mathscr{L}_{\widehat{n}}\,\mathbf{k}_i - \tfrac{n-2}{n-1}\,\widehat{D}_i \mathbf{K}_l{}^l - \widehat{D}_i \boldsymbol{\kappa} + \widehat{D}^l \mathring{\mathbf{K}}_{li} + (l.o.t.)$$

The momentum constraint:

First order symmetric hyperbolic system:

$$\mathscr{L}_{\hat n}\mathbf{k}_i - \tfrac{n-2}{n-1}\,\widehat D_i(\mathbf{K}^l{}_l) - \widehat D_i\boldsymbol{\kappa} + \widehat D^l\mathring{\mathbf{K}}_{li} + (\widehat K^l{}_l)\,\mathbf{k}_i + \boldsymbol{\kappa}\,\dot{\hat n}_i - \dot{\hat n}^l\,\mathbf{K}_{li} - \epsilon\,\mathfrak{p}_l\,\widehat\gamma^l{}_i = 0 \quad (1)$$

$$\mathscr{L}_{\hat n}(\mathbf{K}^l{}_l) - \widehat D^l\mathbf{k}_l - \boldsymbol{\kappa}\,(\widehat K^l{}_l) + \mathbf{K}_{kl}\widehat K^{kl} + 2\,\dot{\hat n}^l\,\mathbf{k}_l + \epsilon\,\mathfrak{p}_l\,\hat n^l = 0 \quad (2)$$

- notably, $\frac{n-1}{n-2}\,\widehat N\,\widehat\gamma^{ij}$ times of (1) and $\widehat N$ times of (2) when writing them out in (local) coordinates $(\rho, x^2, \dots, x^n)$, adopted to the foliation $\mathscr{S}_\rho$ and the vector field ρ^i,

$$\left\{\begin{pmatrix} \frac{n-1}{n-2}\,\widehat\gamma^{AB} & 0 \\ 0 & 1 \end{pmatrix}\partial_\rho + \begin{pmatrix} -\frac{n-1}{n-2}\,\widehat N^K\,\widehat\gamma^{AB} & -\widehat N\,\widehat\gamma^{AK} \\ -\widehat N\,\widehat\gamma^{BK} & -\widehat N^K \end{pmatrix}\partial_K\right\}\begin{pmatrix} \mathbf{k}_B \\ \mathbf{K}^E{}_E \end{pmatrix} + \begin{pmatrix} \mathscr{B}^A_{(\mathbf{k})} \\ \mathscr{B}_{(\mathbf{K})} \end{pmatrix} = 0$$

- indep. of ϵ: a first order **symmetric hyperbolic** system for the vector valued variable

$$(\mathbf{k}_B, \mathbf{K}^E{}_E)^T$$

  where **the 'radial coordinate' ρ plays the role of 'time'**.

- **HW (5)** ... with characteristic cone (apart from the surfaces $\mathscr{S}_\rho$ with $\hat n^i\xi_i = 0$)

$$\left[\,\widehat\gamma^{ij} - \left(\tfrac{n-1}{n-2}\right)\,\hat n^i\hat n^j\,\right]\,\xi_i\xi_j = \left[\,h^{ij} - \left(1 + \tfrac{n-1}{n-2}\right)\,\hat n^i\hat n^j\,\right]\,\xi_i\xi_j = 0$$

That is all for now...